Under Spec $\mathbb{Z}$

A reader's companion

Timothy Hosgood

Submitted as a
PART C DISSERTATION (COD)
for the
FINAL HONOUR SCHOOL OF MATHEMATICS

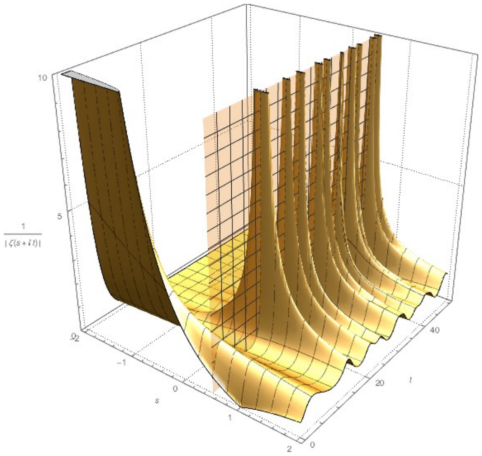
University of Oxford
Hilary Term 2016

Abstract

Relative algebraic geometry is an approach to algebraic geometry using category theory. This allows us to generalise algebraic geometry to many different settings. This project will cover basic notions from category theory, symmetric monoidal categories, Grothendieck topologies, algebraic geometry relative to a symmetric monoidal category and the example of classical algebraic geometry and monoid algebraic geometry which is a version of the field with one element.

One very important paper in this area is [TV07], which is written in French, and translating the first few sections of it into English would open this paper up to a whole new audience. Although mathematical French is in general not entirely impenetrable when one is armed with a good dictionary or glossary, a lot of the language found in this paper is hard to find in other sources. Further, when we are dealing with such abstract mathematics, grammar and semantics are of the utmost importance, and small variations can change the meaning wildly, making 'on-the-fly' translation tricky.

The aims of this project are: to translate the first few sections (those dealing with establishing the formalities of the subject) of [TV07] into English; to provide ample editorial commentary concerning the translation and historical context; and to comment on the mathematics in the paper, providing enough background information for the new reader to be able to follow the main ideas – the main emphasis is placed on this last point.



Thanks

Many thanks to give;
word limit too tight – I swim
in deep gratitude.

Firstly, thanks to [Lan12] for creating such an invaluable mathematical French dictionary. Secondly, thanks to Kobi Kremnitzer and Christopher Hollings for their excellent supervision, and for not complaining about the vast number of questions I asked. Lastly, thanks to all my family and friends, without whom I would not be lucky enough to be at university, let alone be able write this dissertation.

Point n'est besoin d'espérer pour entreprendre, ni de réussir pour persévérer
– Willem van Oranje Nassau

Contents

1 Introduction

1.1 Formatting

This paper consists of translated sections from [TV07] as well as the author's own writing. The general layout and order tries to follow that of the original paper as much as possible, including section numbering and naming.

Longer sections of translated text will be boxed, left aligned, sans serif, and ended with a small black square (■). At the start of such sections there will also be a reference to the location of the source text. Shorter 'quotations' will simply be in italics and referenced afterwards. Theorems (and definitions) that have been translated will not be boxed off or in quotes, but there will be a reference to the original theorem (or definition) after the theorem (or definition) number. Hopefully it will be largely self-explanatory, but here are a few guidelines that the author has tried to adhere to:

- Usually they will be given as a section and a paragraph, e.g. (§1 ¶2), where the paragraphs are counted by looking at indentations.
- Negative paragraph numbers indicate counting from the end of the section (as given), with the last paragraph being the -1st. For example, (§2.1 ¶-2) would indicate the penultimate paragraph of section 2.1. (Luckily there are no subsubsections, so we don't have to worry about things getting any more complex.)

Sometimes a section that we wish to translate will contain some reference to a theorem or definition in the original paper, and this might have a different numbering in *this* paper. Because of this, all such references (e.g. *voir définition 2.12*) from the original will be suitably replaced with numbering relevant to *this* paper (e.g. *voir* [...] *(Lemma 1.3.1.3)*).

If there are any references to a certain section or paragraph without specifying from which paper, then they are to [TV07]. Similarly, if any lemma (or theorem) is stated without a proof then a proof can be found in the referenced lemma (or theorem) in [TV07].

Finally, all footnotes, in translated sections or not, are by the author and *not* from [TV07].

1.1.1 Conventions

We retain the following conventions from [TV07]:

§ 1
¶ -2

All the monoids and monoidal categories considered will be unital and associative, and all modules over a monoid will be unital. We will ignore all set-theoretic problems to do with the choice of universe; the reader can consult [TV05; TV08] to find a method to resolve them. ■

We also impose the following conventions ourselves, which are always assumed (unless otherwise stated):

- all algebras and rings are unital and associative;
- k is an algebraically closed field;
- $0 \in \mathbb{N}$;
- for a ring R we write $R^\times$ to mean the group of multiplicative units in R;
- $\mathbb{G}_m = k^\times = k \setminus \{0\}$;
- for $n \in \mathbb{N} \setminus \{0\}$ we write μ_n to mean the cyclic group of order n;
- given a category $\mathcal{C}$ we write $x \in \mathcal{C}$ to mean $x \in \mathrm{ob}(\mathcal{C})$;
- 'presheaf' means a Set-valued presheaf.

We usually use 'functor' to mean 'covariant functor'.

1.2 Overview

In this paper we will summarise some of the results of [TV07], providing background definitions along the way, as well as filling in some of the proofs that are omitted or only sketched. All of the pictures, as well as Sections 1.3 and 4, are entirely original and aim to complement the main results (though the pictures are *not* to be taken too literally – they often illustrate simple cases, such as when $\mathcal{C} = \mathsf{Op}(T)$). There are also explanations of motivation (e.g. Section 3.1) and historical notes (e.g. Section 4.2) that are original. This is why this paper is subtitled '*a readers' guide*', and not simply '*a translation*'.

The results of [TV07] are many, and we will not have time to cover most of the later sections; we will focus largely on the first three[*] sections. Because of this, for us, the introduction of [TV07] summarises the purpose of the paper better than the abstract.

§ 1
¶ 1 The aim of this paper is to construct several categories of *schemes* that are defined over bases found *under* $\operatorname{Spec}\mathbb{Z}$. Of course, since $\mathbb{Z}$ is the initial object in the category of commutative rings, it is vital to leave the usual framework of rings and permit the use of more general objects, but only objects that resemble commutative rings enough such that the notion of a scheme can still be defined. Our approach to this problem is based on the theory of relative algebraic geometry, largely inspired by [Hak72]. It comes from remarking that a commutative ring is nothing but a commutative monoid in the monoidal category of $\mathbb{Z}$-modules, and that, in general, for a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$, the commutative monoids in $\mathcal{C}$ can be thought of as models for the *affine schemes relative to* $\mathcal{C}$. It is remarkable that such a general (simplistic, even) approach allows us to actually define the notion of schemes, and moreover in a functorial way in $\mathcal{C}$. So, in choosing $\mathcal{C}$ equipped with a sensible symmetric monoidal functor $\mathcal{C} \to \mathbb{Z}\text{-}\mathsf{Mod}$, we find a notion of schemes relative to $\mathcal{C}$ and a base-change functor to $\mathbb{Z}$-schemes, and thus a notion of schemes under $\operatorname{Spec}\mathbb{Z}$. ■

---

[*] Not including the introduction, so sections 2 (*Géométrie algébrique relative*), 3 (*Trois exemples de géométries relatives*), and 4 (*Quelques exemples de schémas au-dessous de* $\operatorname{Spec}\mathbb{Z}$).

1.3 Background knowledge

This section acts as a prelude to [TV07], containing a few motivating examples and prerequisite definitions and lemmas. Before diving straight into the abstract definitions, we give an example of why we might think to try a category-theoretic approach to algebraic geometry.

1.3.1 Motivating example

Let A be a finitely-generated commutative k-algebra. Then we can write

$$A = \frac{k[x_1,\ldots,x_n]}{(f_1,\ldots,f_m)}$$

for some $m,n \in \mathbb{N}$ and $f_i \in k[x_1,\ldots,x_n]$. If B is another commutative k-algebra (not necessarily finitely generated) then the collection of algebra morphisms $A \to B$ is in bijection with points of B^n that vanish on all of the f_i, since a morphism is determined entirely by where it sends each of the x_i whilst satisfying $0 \mapsto 0$. So, letting $\mathsf{CommAlg}_k$ denote the category of commutative k-algebras,

$$\mathrm{Hom}_{\mathsf{CommAlg}_k}(A,B) \cong \{b \in B^n \mid f_1(b) = \ldots = f_m(b) = 0\} \tag{1.3.1.1}$$

where, as usual, we evaluate $f_i(b)$ inside B.

Equation (1.3.1.1) implies that we should maybe think of $\mathrm{Hom}(A,B)$ as some variety inside B^n determined by A, for general $A,B \in \mathsf{CommAlg}_k$, and so we might be able to recover a lot of algebraic geometry from studying these $\mathrm{Hom}(A,B)$. In fact, thinking of $\mathrm{Hom}(A,-)$ as a functor $\mathsf{CommAlg}_k \to \mathsf{Set}$ which takes an algebra B to a *variety* (a set of points) inside B^n, we are led to the more general idea of studying *all* functors $\mathsf{CommAlg}_k \to \mathsf{Set}$, and calling such functors *spaces*.

Before formalising this, we first recall a few things from category theory.

Definition 1.3.1.2 [Presheaves]
Let $\mathcal{C}$ be a category. The category of *presheaves on* $\mathcal{C}$ is defined as the functor category $\mathsf{PSh}(\mathcal{C}) = \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$, whose objects are (covariant) functors $\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ and morphisms are natural transformations $F \Rightarrow G$ between such functors. ⌟

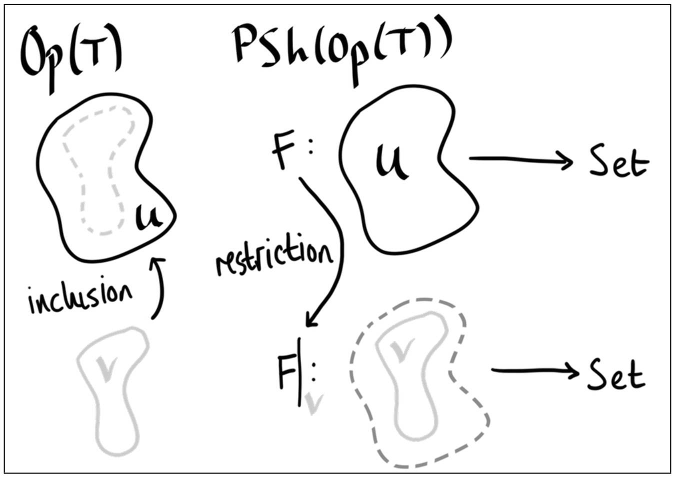


Figure 1: Presheaves on $\mathsf{Op}(T)$ (see paragraph after Definition 2.1.1.5) – inclusion of open sets corresponds to restriction of presheaves

Lemma 1.3.1.3 [Yoneda lemma]
Let $\mathcal{C}$ be a locally small category.[*] *Define the* downwards Yoneda functor[†] *by*

$$\begin{aligned} Y_{(-)}\colon \mathcal{C} &\to \mathsf{PSh}(\mathcal{C}) \\ A &\mapsto \operatorname{Hom}_{\mathcal{C}}(-,A), \end{aligned}$$

which is well defined, since $\operatorname{Hom}(-,A)\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ *covariantly. Then, for any* $A \in \mathcal{C}$ *and* $F \in \mathsf{PSh}(\mathcal{C})$*,*

$$\operatorname{Hom}_{\mathsf{PSh}(\mathcal{C})}(Y_A, F) \cong F(A) \tag{1.3.1.4}$$

via the canonical restriction map. Further, Y is fully faithful. ⌟

Proof. See [Mac78, §III.2, p.59–62]. □

Since $\mathsf{CommAlg}_k$ is locally small,[‡] we can make the definitions in Table 1.1, where Y is the Yoneda functor from Lemma 1.3.1.3.[§]

---

[*] That is, the hom-sets $\operatorname{Hom}(A,B)$ are actual sets for all $A, B \in \mathcal{C}$.

[†] This is not at all common terminology. It is often called the *contravariant Yoneda functor*: it maps an object $A \in \mathcal{C}$ to the *contravariant* functor $\operatorname{Hom}_{\mathcal{C}}(-,A)\colon \mathcal{C} \to \mathsf{Set}$. But the functor $\mathcal{C} \to \mathsf{PSh}(\mathcal{C})$ itself is *covariant*, so we use 'downwards' to avoid confusion. The dual 'covariant' (*upwards*, in our terminology) functor is $Y^{(-)}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Fun}(\mathcal{C},\mathsf{Set})$ given by $\operatorname{Spec} A \mapsto Y^A = \operatorname{Hom}_{\mathcal{C}}(A,-)$, where we write $\operatorname{Spec} A \in \mathcal{C}^{\mathrm{op}}$ to be the object corresponding to $A \in \mathcal{C}$. Then the statement $\operatorname{Hom}_{\mathsf{Fun}(\mathcal{C},\mathsf{Set})}(Y^A, G) \cong G(A)$ holds.

[‡] As in [TV07], we try to ignore such set-theoretic issues, but here we have the reasonable explanation that $\mathsf{CommAlg}_k$ is a concrete category, and thus locally small.

[§] In the definition of Spec we use the fact that any covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a contravariant functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, and vice versa.

Name	Notation	Definition
the category of *affine schemes over* k	Aff_k	${\mathsf{CommAlg}_k}^{\mathrm{op}}$
the category of k-*spaces*	Sp_k	$\mathsf{PSh}(\mathsf{Aff}_k)$
the *spectrum functor*	Spec	$Y\colon \mathsf{CommAlg}_k \to \mathsf{Sp}_k$

Table 1.1: Categorical approach to algebraic geometry with $\mathsf{CommAlg}_k$

We've already given a reason for calling objects of the functor category $\mathsf{Fun}(\mathsf{CommAlg}_k, \mathsf{Set})$ *spaces*, and we call objects of ${\mathsf{CommAlg}_k}^{\mathrm{op}}$ *schemes* because we know that the Yoneda lemma (Lemma 1.3.1.3) will give us a way of viewing Aff_k as sitting inside of Sp_k.

Lemma 1.3.1.5 [Yoneda embedding]
The category Aff_k *is equivalent to the essential image of the Yoneda functor* $Y\colon \mathsf{Aff}_k \to \mathsf{Sp}_k$. ⌟

Proof. Here we use the fact that a functor gives an equivalence of categories if and only if it is fully faithful and essentially surjective.* Lemma 1.3.1.3 tells us that Y is fully faithful so it forms an equivalence of categories between Aff_k and the essential image of Y in Sp_k. □

So Lemma 1.3.1.5 lets us imagine Aff_k as sitting inside Sp_k. This mirrors classical algebraic geometry where, loosely speaking, given some commutative ring R we define the space $\mathrm{Spec}\, R$ of prime ideals of R endowed with the Zariski topology. We can then give $\mathrm{Spec}\, R$ some extra structure to make it an affine scheme. Then Spec is a map taking commutative rings to affine schemes, which form a subclass of the objects (schemes) in which we're interested.

Studying algebraic geometry this way is called the *functor of points* approach,† because Lemma 1.3.1.5 says that describing some affine scheme $X \in \mathsf{Aff}_k$ is exactly the same as describing its functor of points $X(-) \in \mathsf{Sp}_k$ under the Yoneda embedding. Sometimes this latter method is far easier, as the following example shows.

Example 1.3.1.6 [GL_n]
Given $A \in \mathsf{CommAlg}_k$ we can define $\mathrm{GL}_n(A)$ to be the group of $n \times n$ invertible matrices over A. This induces a functor $\mathrm{GL}_n(-)\colon {\mathsf{Aff}_k}^{\mathrm{op}} \to \mathsf{Set}$, so $\mathrm{GL}_n(-) \in \mathsf{Sp}_k$. We claim that this functor is in the essential image of the Yoneda (spectrum) functor $Y\colon \mathsf{CommAlg}_k \to \mathsf{Sp}_k$, and is thus represented by an affine scheme. This is not obvious a priori. To prove this, we need to find

---

[*] [Mac78, Theorem 1, §IV.4]

[†] As opposed to the traditional *ringed space* approach.

an isomorphism

$$\mathrm{GL}_n(-) \cong \mathrm{Hom}_{\mathsf{CommAlg}_k}(R, -) = \mathrm{Spec}\, R$$

for some $R \in \mathsf{CommAlg}_k$, where we mean $\mathrm{Spec}\, R$ as defined in Table 1.1.[*] By definition, this is equivalent to finding a bijection of sets $\mathrm{GL}_n(A) \cong \mathrm{Hom}_{\mathsf{CommAlg}_k}(R, A)$ that transforms naturally in A for each $A \in \mathsf{CommAlg}_k$.

Note that an element of $\mathrm{GL}_n(A)$ is a choice of $x_{11}, x_{21}, \ldots, x_{nn} \in A$ such that $\det(x_{ij})$ is invertible. That is, there exists some $y \in A$ such that $y \det(x_{ij}) = 1$. Hence

$$R = \frac{k[x_{11}, x_{21}, \ldots, x_{nn}, y]}{(y \det(x_{ij}) - 1)}$$

gives us the desired result, and so $GL_n = R \in \mathsf{Aff}_k$ is an affine scheme. ⌟

1.3.2 Preliminary definitions

We now give some definitions of which [TV07] assumes prior knowledge. The motivation for them usually comes from taking $\mathcal{C} = \mathsf{Set}$, and their application to algebraic geometry can be better understood[†] by taking $\mathcal{C} = \mathbb{Z}\text{-}\mathsf{Mod} = \mathsf{Ab}$. We assume that the reader is familiar with notions such as categories, functors, natural transformations, and functor categories, but not too much else.

Definition 1.3.2.1 [Monoidal category]
A *monoidal category* consists of the following data:

- a category $\mathcal{C}$;
- an object $1 \in \mathcal{C}$, which we call the *unit* or *identity*;
- a bifunctor $(- \otimes -)\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the *monoidal* or *tensor product*;
- natural isomorphisms α (the *associator*), λ (the *left unitor*), and ρ (the *right unitor*), constructed from morphisms

$$\begin{array}{rrcl} \alpha_{ABC}\colon & (A \otimes B) \otimes C & \xrightarrow{\sim} & A \otimes (B \otimes C) \\ \lambda_A\colon & 1 \otimes A & \xrightarrow{\sim} & A \\ \rho_A\colon & A \otimes 1 & \xrightarrow{\sim} & A \end{array}$$

for all $A, B, C \in \mathcal{C}$.

---

[*] That is, identify $\mathrm{Spec}\, R \in \mathsf{Aff}_k$ with $\mathrm{Spec}\, R := \mathrm{Hom}_{\mathsf{CommAlg}_k{}^{\mathrm{op}}}(-, R)$.

[†] Since [...] *a commutative ring is nothing but a commutative monoid in the monoidal category of $\mathbb{Z}$-modules* [...] (§1 p.2 ¶1).

Further, the three natural isomorphisms are subject to the following coherence conditions*:

- (*unit associativity*) for all $A, B \in \mathcal{C}$ the following commutes

$$\begin{tikzcd}
(A \otimes 1) \otimes B \arrow[rr, "\alpha_{A1B}"] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"'] & & A \otimes (1 \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

- (*4-associativity*) for all $A, B, C, D \in \mathcal{C}$ the following commutes

$$\begin{tikzcd}
& (A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A(B\otimes C)D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[dr, "\mathrm{id}_A \otimes \alpha_{BCD}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{ABC} \otimes \mathrm{id}_D"] \arrow[drr, "\alpha_{(A\otimes B)CD}"'] & & & & A \otimes (B \otimes (C \otimes D)) \\
& & (A \otimes B) \otimes (C \otimes D) \arrow[urr, "\alpha_{AB(C\otimes D)}"'] & &
\end{tikzcd}$$

⌟

Definition 1.3.2.2 [Symmetric monoidal category]
A monoidal category $(\mathcal{C}, \otimes, 1)$ is *symmetric* if it can be equipped with a *maximally-symmetric brading* γ. That is, for all $A, B \in \mathcal{C}$, there exists an isomorphism

$$\gamma_{AB} \colon A \otimes B \xrightarrow{\sim} B \otimes A$$

that is natural in both A and B, and also subject to the following coherence conditions†:

- (*unit associativity*) for all $A \in \mathcal{C}$ the following commutes:

$$\begin{tikzcd}
1 \otimes A \arrow[rr, "\gamma_{1A}"] \arrow[dr, "\lambda_A"'] & & A \otimes 1 \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

---

*These diagrams simply say that $\otimes$ is associative in all the ways that you might expect.

†These diagrams simply say that $\otimes$ is commutative in all the ways you might expect.

- (*3-associativity*) for all $A, B, C \in \mathcal{C}$ the following commutes:

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\gamma_{AB} \otimes \mathrm{id}_C"] \arrow[d, "\alpha_{ABC}"'] & (B \otimes A) \otimes C \arrow[d, "\alpha_{BAC}"] \\
A \otimes (B \otimes C) \arrow[d, "\gamma_{A(B \otimes C)}"'] & B \otimes (A \otimes C) \arrow[d, "\mathrm{id}_B \otimes \gamma_{AC}"] \\
(B \otimes C) \otimes A \arrow[r, "\alpha_{BCA}"] & B \otimes (C \otimes A)
\end{tikzcd}$$

- (*maximal symmetry*) for all $A, B \in \mathcal{C}$ the following commutes:

$$\begin{tikzcd}
& B \otimes A \arrow[dr, "\gamma_{BA}"] & \\
A \otimes B \arrow[ur, "\gamma_{AB}"] \arrow[rr, "\mathrm{id}_{A \otimes B}"'] & & A \otimes B
\end{tikzcd}$$

⌟

Definition 1.3.2.3 [Closed symmetric monoidal category*]
A symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ is *closed* if, for all $A \in \mathcal{C}$, the functor $- \otimes A \colon \mathcal{C} \to \mathcal{C}$ has a right adjoint, written $(A \Rightarrow -)$. This means that

$$\mathrm{Hom}(X \otimes A, B) \cong \mathrm{Hom}(X, A \Rightarrow B)$$

naturally in X and B for all $A, B, X \in \mathcal{C}$. The object $(A \Rightarrow B) \in \mathcal{C}$ is called the *internal Hom*.† ⌟

Lemma 1.3.2.4
Let $(\mathcal{C}, \otimes, 1)$ be a closed symmetric monoidal category. Then the bifunctor

$$(- \otimes -) \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

commutes with colimits in both of its arguments. ⌟

Proof. Since $(- \otimes A)$ has a right adjoint, it commutes with colimits (see Lemma 2.1.1.3). This gives us commutativity in the first argument. As for commutativity in the second argument, we claim that

$$A \otimes (\operatorname{colim} X_i) \overset{(1)}{\cong} (\operatorname{colim} X_i) \otimes A \overset{(2)}{\cong} \operatorname{colim}(X_i \otimes A) \overset{(3)}{\cong} \operatorname{colim}(A \otimes X_i).$$

---

*See [TV07, Hypothese 2.6, p.14]

†[TV07] uses the notation $\underline{\mathrm{Hom}}(A, B)$.

(1) follows since $(\mathcal{C}, \otimes, 1)$ is symmetric;

(2) follows from our first statement;

(3) requires a bit of work, but using the isomorphisms $X_i \otimes A \cong A \otimes X_i$ and the fact that they are *natural* in both A and X_i we can show that $\operatorname{colim}(X_i \otimes A)$ satisfies the universal property required to be the colimit of $\{A \otimes X_i\}$, giving us the required isomorphism. □

Definition 1.3.2.5 [Cosmos]
A (*Bénabou*[*]) *cosmos*[†] is a bicomplete[‡] closed symmetric monoidal category.

⌟

Definition 1.3.2.6 [Commutative monoid in $(\mathcal{C}, \otimes, 1)$]
A *commutative monoid* (A, μ, η) in a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ is an object $A \in \mathcal{C}$ along with morphisms

- $\mu\colon A \otimes A \to A$ (*multiplication*);
- $\eta\colon 1 \to A$ (*unit*),

such that the following commute:

- (*associativity*)

$$\begin{tikzcd}
 & A \otimes (A \otimes A) \arrow[r, "\mathrm{id}_A \otimes \mu"] & A \otimes A \arrow[dr, "\mu"] & \\
(A \otimes A) \otimes A \arrow[ur, "\alpha_{AAA}"] \arrow[drr, "\mu \otimes \mathrm{id}_A"'] & & & A \\
 & & A \otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}$$

- (*left and right unity*)

$$\begin{tikzcd}
1 \otimes A \arrow[r, "\eta \otimes \mathrm{id}_A"] \arrow[dr, "\lambda_A"'] & A \otimes A \arrow[d, "\mu"] & A \otimes 1 \arrow[l, "\mathrm{id}_A \otimes \eta"'] \arrow[dl, "\rho_A"] \\
 & A &
\end{tikzcd}$$

---

[*] After the French mathematician Jean Bénabou.

[†] Possible etymology: *catégorie monoïdale symétrique* gets initialised to *CMS* which gets pronounced ‘acronymically’ as *cosmos*.

[‡] All small limits and small colimits exist.

- (*commutativity*)

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\gamma_{AA}"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}$$

We denote the category of all such objects by Comm $(\mathcal{C})$, where the morphisms are morphisms $f\colon A \to A'$ in $\mathcal{C}$ such that everything transfers nicely (that is, $\eta' = f \circ \eta\colon 1 \to A'$ and $f \circ \mu = \mu' \circ (f \otimes f)\colon A \otimes A \to B$). ⌟

Definition 1.3.2.7 [Module over a monoid]
Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category and $A \in$ Comm $(\mathcal{C})$ a commutative monoid in $\mathcal{C}$. A *module*[*] (M, σ) *over* A is an object $M \in \mathcal{C}$ with a morphism $\sigma\colon A \otimes M \to M$ such that the following diagrams commute[†]:

- (*compatibility with* μ)

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "\mathrm{id}_A \otimes \sigma"] \arrow[d, "\mu \otimes \mathrm{id}_M"'] & A \otimes M \arrow[d, "\sigma"] \\
A \otimes M \arrow[r, "\sigma"] & M
\end{tikzcd}$$

- (*unity*)

$$\begin{tikzcd}
1 \otimes M \arrow[rr, "\eta \otimes \mathrm{id}_M"] \arrow[dr, "\lambda_M"'] & & A \otimes M \arrow[dl, "\sigma"] \\
& M &
\end{tikzcd}$$

We denote the category of all such objects by A-Mod, where the morphisms are morphisms $f\colon M \to M'$ in $\mathcal{C}$ such that everything transfers nicely (that is, $f \circ \sigma = \sigma' \circ (\mathrm{id}_A \otimes f)\colon A \otimes M \to M'$). ⌟

---

[*] Really this is the definition for a *left* A-module, but since A is commutative the notions of right and left modules coincide.

[†] That is, σ is an *action*.

2 Relative algebraic geometry

§ 2 ¶ 1 The aim of this first part is to present the idea of a scheme relative to a symmetric monoidal category $\mathcal{C}$. We start with a general process of construction of Grothendieck topologies from prestacks in categories satisfying certain conditions. This allows us to then define the *faithfully flat and quasi-compact topology*, as well as the *Zariski topology* in the very general setting. We will afterwards define the idea of a relative scheme by gluing back together affine objects with the help of the Zariski topology. ■

2.1 Constructing Grothendieck topologies

2.1.1 Exactness and pullbacks

Definition 2.1.1.1 [Conservative functor]
Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then F is *conservative* if, for all morphisms f in $\mathcal{C}$, whenever the morphism $F(f)$ in $\mathcal{D}$ is an isomorphism then so too is f. ⌟

Definition 2.1.1.2 [Exact functor]
Let F be a functor. We say that F is *left exact* if it commutes with finite limits. Dually, F is *right exact* if it commutes with finite colimits. If F is both left and right exact, then we say that it is *exact*. ⌟

If $F\colon \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ both have zero objects then, loosely speaking, F being conservative is saying that* $[F(c) = 0 \implies c = 0]$, and F being exact is saying that $F(0) = 0$.

Lemma 2.1.1.3
Let $(F \dashv G)$ be an adjunction of functors. Then F commutes with colimits and G commutes with limits. ⌟

Proof. [Mac78, §V.5, p.118] □

Corollary 2.1.1.4
Left-adjoint functors are right exact; right-adjoint functors are left exact. ⌟

---

*If we wanted to really abuse notation then we would write this as $F^{-1}(0) = \{0\}$.

Definition 2.1.1.5 [Pullbacks]
Let $\mathcal{C}$ be a category, $X, X', Y \in \mathcal{C}$ objects, and $f\colon X \to Y$, $g\colon X' \to Y$ morphisms:

$$\begin{tikzcd}
X \arrow[dr, "f"] & \\
& Y \\
X' \arrow[ur, "g"'] &
\end{tikzcd}$$

Then the *pullback* (or *fibred product*) *of* Y *(along* f *and* g*)* is the limit of this diagram (if it exists) and is written as $X\,{}^{f}\!\times^{g}_{Y} X'$ (or just $X \times_Y X'$ when no confusion may arise). The commutative diagram

$$\begin{tikzcd}
& X \arrow[dr, "f"] & \\
X \times_Y X' \arrow[ur, "\pi_X"] \arrow[dr, "\pi_{X'}"'] & & Y \\
& X' \arrow[ur, "g"'] &
\end{tikzcd}$$

is also called a *cartesian square*. ⌟

Since the pullback is a limit, if our category $\mathcal{C}$ has finite limits then it has pullbacks. In Set, the pullback $X \times_A Y$ is given by 'intersecting' the images of X and Y in A:

$$X \times_A Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}.$$

Working in $\mathsf{Op}(T)$ for some topological space T – the category whose objects are open sets of T and whose morphisms are the inclusion maps of the open sets – pullbacks correspond to intersections (Figure 2). So pullbacks generalise intersection, but also *fibres*: let Y, T be topological spaces, $p \in T$ some point with inclusion map $\iota\colon \{p\} \hookrightarrow T$, and $f\colon Y \to T$ continuous. Then the pullback is the *fibre* (or *preimage*) *of* p *under* f:

$$Y \times_T \{p\} = \{y \in Y \mid f(y) = p\}.$$

(Figure 3). So when we have a continuous $\tau\colon S \to T$ we can think of $Y \times_T S$ as the *fibre of the points of* $\tau(S) \subset T$ *under* f.

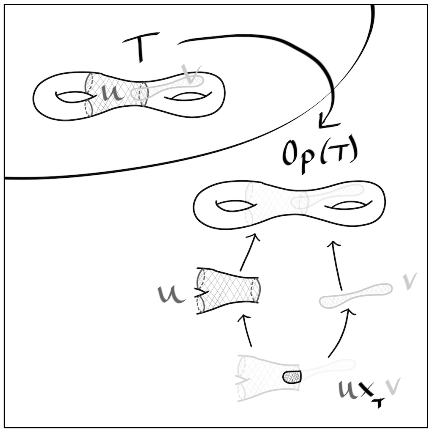


Figure 2: Pullbacks in $\mathsf{Op}(T)$

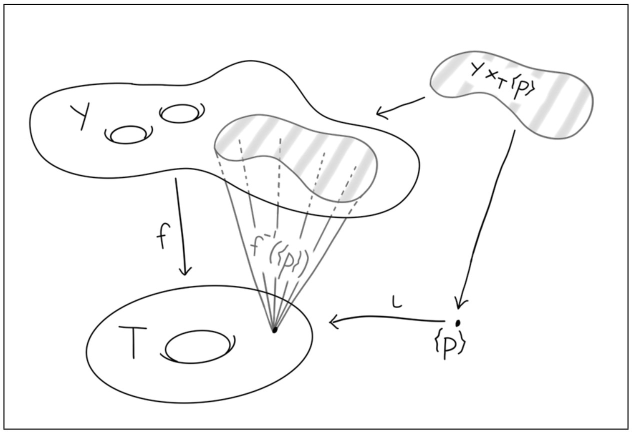


Figure 3: Pullbacks as fibres

2.1.2 Grothendieck pseudofunctor

Definition 2.1.2.1 [Grothendieck pseudofunctor* (Hypothèse 2.1, §2.1, p.8)] Let $\mathcal{D}$ be a category that has finite limits and $M\colon \mathcal{D}^{\mathrm{op}} \to \mathsf{Cat}$ a pseudofunctor.† Then M is a *Grothendieck pseudofunctor* if it satisfies the following conditions:

(i) for each X in $\mathcal{D}$, the category $M(X)$ has all limits and colimits;

(ii) for each $p\colon X' \to X$ in $\mathcal{D}$, the functor $M(p) = p^*\colon M(X) \to M(X')$ has a right adjoint $p_*\colon M(X') \to M(X)$ that is conservative;

(iii) (*the Beck-Chevalley condition*) for all pullbacks

$$\begin{tikzcd}
 & Y \arrow[dr, "q"] & \\
X \times_Z Y \arrow[ur, "p'"] \arrow[dr, "q'"'] & & Z \\
 & X \arrow[ur, "p"'] &
\end{tikzcd}$$

in $\mathcal{D}$, the natural transformation $p^*q_* \Rightarrow q'_*(p')^*$ (called the *change of base*‡) is an isomorphism. (*See [TV07, §2, ¶3] for how this transformation is constructed*). ⌟

---

[*] This is not standard terminology, but aims to hint at the links to *Grothendieck fibrations* and the *Grothendieck construction*. We should probably instead call M a *$\mathcal{D}$-indexed category with $\mathcal{D}$-indexed coproducts* (see [PS12, Definitions 2.1, 3.1]).

[†] That is, a 'not-quite-functor' $\mathcal{D} \to \mathsf{Cat}$ (the category of small categories), in the sense that it doesn't necessarily preserve composition of morphisms and the identity morphism *exactly*, but only up to *coherent isomorphism*. Roughly speaking, we require a natural isomorphism $F(g \circ f) \cong F(g) \circ F(f)$ such that we can 'evaluate' $F(h) \circ F(g) \circ F(f)$ in an associative way. There is a similar condition concerning the identity morphism. For more details see [Bor94, Definition 7.5.1, §7.5, p.296].

[‡] See Note 2.2.3.3 for information on this terminology.

The motivating example for such a pseudofunctor M is when $\mathcal{D}$ *is the category of affine schemes and* [. . .] $M(X)$ *is the category of quasi-coherent sheaves on* $X \in \mathcal{D}$ (Remarque 2.2, §2.1, p.9).

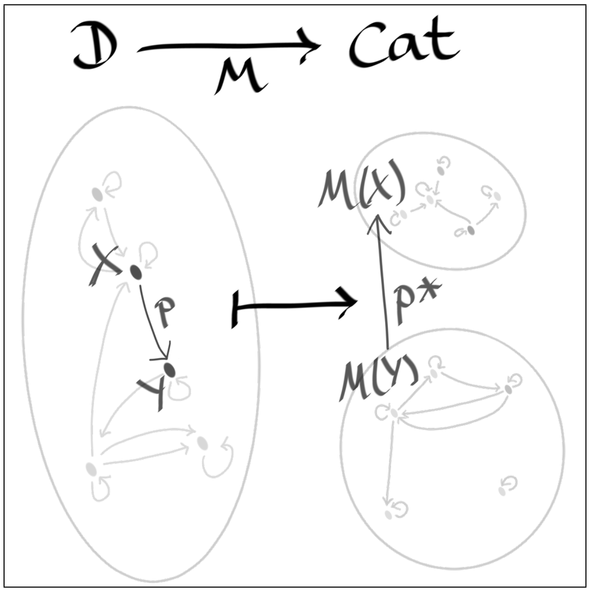


Figure 4: Definition 2.1.2.1 – note that this is a picture of a contravariant $M\colon \mathcal{D} \to \mathsf{Cat}$ rather than a covariant $M\colon \mathcal{D}^{\mathrm{op}} \to \mathsf{Cat}$.

Definition 2.1.2.2 [M-faithfully flat (Définition 2.3, §2.1, p.9)]
Let $\mathcal{D}$ and M be as in Definition 2.1.2.1, and let $\{p_i\colon X_i \to X\}_{i\in I}$ be a family of morphisms in $\mathcal{D}$. Then $\{p_i\colon X_i \to X\}_{i\in I}$ is

(i) M*-covering* if there exists a finite (non-empty) subset $J \subset I$ such that the family of functors

$$\{p_j^*\colon M(X) \to M(X_j)\}_{j\in J}$$

is conservative;

(ii) M*-flat* if all the functors $p_i^*\colon M(X) \to M(X_i)$ are left exact[*];

(iii) M*-faithfully flat* if it is both M-covering and M-flat. ⌟

The reason for the name 'M-faithfully flat' comes from the fact that *if* p *is* M*-faithfully flat then* p^* *is faithful*[†] (§2.1 ¶5).

---

[*] Note that, by Definition 2.1.2.1, a morphism $p\colon X' \to X$ is M-flat if and only if the functor p^* is *exact*. This is because left adjoint functors are right exact (Corollary 2.1.1.4) and we have assumed that p^* *has* a right adjoint (Definition 2.1.2.1).

[†] We provide here a quick proof. Since $M(X)$ is bicomplete we have a *zero object* and

2.1.3 Grothendieck topologies

Classically, we use a topology on a commutative k-algebra (or commutative ring) to define the notion of a *sheaf*. One of the pivotal moments for algebraic geometry (and mathematics as a whole) was Groethendieck's generalisation of this idea in 1958, rephrased in terms of category theory (which was itself only around 13 years old). The foundational definition was that of a *site*, which is a category endowed with a *Grothendieck topology*[*] – a structure that mirrored that of open sets of a topological space.[†] There is also a *Grothedieck pretopology*, which is slightly less strict, but can be used to construct a topology. We formalise all of this below.

Definition 2.1.3.1 [Grothendieck pretopology ([Sch10b])]
Let $\mathcal{C}$ be a category with pullbacks. A *Grothendieck pretopology on* $\mathcal{C}$ is an assignment, to each object $X \in \mathcal{C}$, of a collection $C(X)$ of families of morphisms to X, called *covering families*, satisfying the following conditions for all $X \in \mathcal{C}$:

(i) (*isomorphisms cover*) for every isomorphism $Y \xrightarrow{\sim} X$ in $\mathcal{C}$, the singleton family $\{Y \xrightarrow{\sim} X\}$ is in $C(X)$;

(ii) (*stability*) the collection $C(X)$ is stable under pullback (or change of base): if $\{X_i \to X\} \in C(X)$ and $f\colon Y \to X$ is some arbitrary morphism in $\mathcal{C}$, then $\{f^*X_i \to Y\}_{i\in I} \in C(Y)$, where $f^*X_i = (Y \times_X X_i)$;

(iii) (*transitivity*) if $\{X_i \to X\}_{i\in I} \in C(X)$ and there is a covering family $\{X_{i,j} \to X_i\}_{j\in J_i} \in C(X_i)$ for each $i \in I$, then the family of composites $\{X_{i,j} \to X_i \to X\}_{i\in I, j\in J_i}$ is also in $C(X)$. ⌟

The previously-introduced notion of 'M-faithfully flat' now comes into use: such families can be used to construct a pretopology.

Lemma 2.1.3.2 [(Proposition 2.4, §2.1, p.9])
Let $\mathcal{D}$ and M be as in Definition 2.1.2.1. Then the M-faithfully flat families

---

the notion of an *equaliser* (dual to Definition 2.2.1.2). By definition, two morphisms f, g in $M(X)$ are equal if and only if their equaliser is zero. Now p^* is exact, so it commutes with limits. Thus $\mathrm{eq}(p^*(f) \rightrightarrows p^*(g)) \cong \mathrm{eq}(p^*(f \rightrightarrows g))$. Further, since p^* is exact, $p^*(0) = 0$. Also, since p^* is conservative, it reflects limits, so $[p^*(c) = 0 \implies c = 0]$. Thus $p^*(c) = 0$ if and only if $c = 0$. Putting these facts together we see that $f = g$ if and only if $p^*(f) = p^*(g)$. That is, p^* is faithful.

[*] We sometimes refer to this simply as a *topology*, but only when it is obvious what we mean.

[†] Though there is some confusion with this nomenclature; [Joh02] suggests the name *Grothendieck coverage*, but we stick with 'topology' for simplicity. See [Ško11].

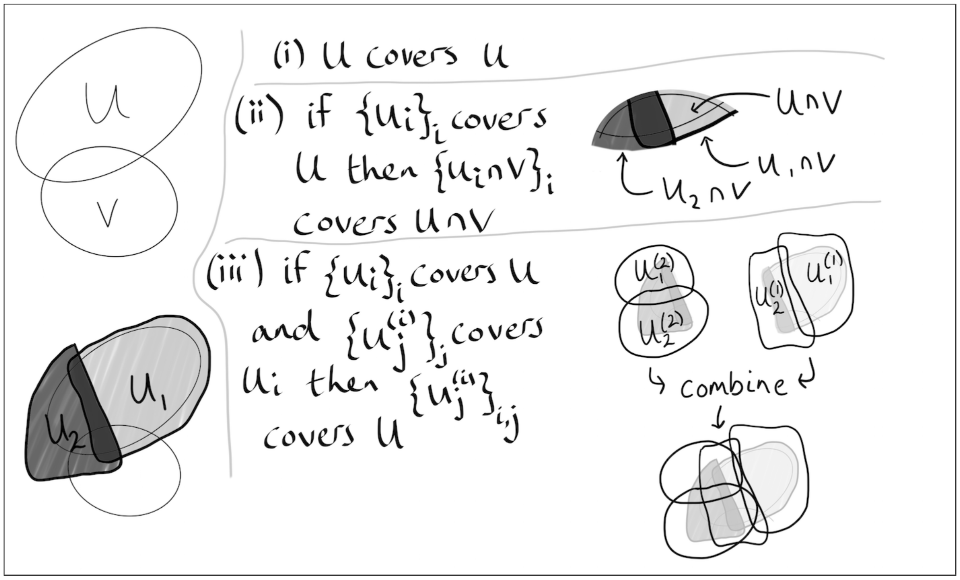


Figure 5: Definition 2.1.3.1 with $\mathcal{C} = \mathsf{Op}(T)$

define[*] *a Grothendieck pretopology on $\mathcal{D}$.* ⌟

Proof.

(i) (*Isomorphisms cover*) Let $p\colon Y \xrightarrow{\sim} X$ be some isomorphism in $\mathcal{D}$. We want $p^*\colon M(X) \to M(Y)$ to be conservative and left exact. If we have $p^* \cong (p^{-1})_*$ then left exactness follows from being a right adjoint, and being conservative follows from Definition 2.1.2.1.

Since M is a pseudofunctor, we have natural isomorphisms

$$(p^{-1})_* p_* \cong (pp^{-1})_* \cong \mathrm{id}_{M(X)}$$
$$\mathrm{id}_{M(Y)} \cong (p^{-1}p)_* \cong p_*(p^{-1})_*$$

Thus $(p^{-1})_* \dashv p_*$ and so $(p^{-1})_* \cong p^*$ as required.

(ii) (*Stability*) Let $\{p_i\colon X_i \to X\}$ be M-faithfully flat and $f\colon Y \to X$. Define $Y_i = Y \times_X X_i$ with morphisms

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ Y_i \arrow[ur, "q_i"] \arrow[dr, "f_i"'] & & X \\ & X_i \arrow[ur, "p_i"'] & \end{tikzcd}$$

We want to show that $\{q_i\colon Y_i \to Y\}$ is M-faithfully flat.

---

[*] To each $X \in \mathcal{D}$ we assign the collection $C(X)$ of M-faithfully flat families $\{X_i \to X\}$.

Firstly, to be M-covering we want

$$\left(\prod_{j\in J} q_j^*\right) : M(Y) \to \prod_{j\in J} M(Y_j)$$

to be conservative for some finite $J \subset I$. We claim that the same finite $J \subset I$ that makes $\{p_i\}$ M-covering works. By Definition 2.1.2.1, $(f_i)_*$ is conservative, and so the above morphism is conservative if and only if

$$\left(\prod_{j\in J} (f_j)_* q_j^*\right) : M(Y) \to \prod_{j\in J} M(X_j)$$

is conservative. Now, $p_i^* f_* \cong (f_i)_* q_i^*$, so the above morphism is conservative if and only if

$$\left(\prod_{j\in J} p_i^* f_*\right) : M(Y) \to \prod_{j\in J} M(X_j)$$

is conservative. But both f_* and $\{p_j^*\}_{j\in J}$ are conservative, so we are done.

Secondly, to be M-flat we want q_i^* to be left exact for all $i \in I$. This follows straight from $p_i^* f_* \cong (f_i)_* q_i^*$, since both f_* and $(f_i)_*$ are right adjoints, thus left exact, and p_i^* is left exact by hypothesis.

(iii) (*Transitivity*) Let $\{p_i \colon X_i \to X\}_{i\in I}$ be M-faithfully flat. Suppose we also have M-faithfully flat $\{q_j^{(i)} \colon Y_j^{(i)} \to X_i\}_{j\in J}$ for all $i \in I$. The fact that

$$\{p_i q_j^{(i)} \colon Y_j^{(i)} \to X\}_{(i,j)\in I\times J}$$

is M-covering follows from the fact that the product of two finite sets is finite; that it is M-flat follows from the natural isomorphisms $(p_i q_j^{(i)})^* \cong (q_j^{(i)})^* p_i^*$ and the fact that the composition of two left-exact functors is again left exact. □

Definition 2.1.3.3 [Sieves on an object]
Let $X \in \mathcal{C}$ be an object in some category. A *sieve on* X is a subset[*] $S \subset \mathcal{C}/X$ of the objects of the slice category that is *saturated* (i.e. closed under precomposition). That is, if $(Y \to X) \in S$ and $Y' \to Y$ is some arbitrary morphism in $\mathcal{C}$, then the composition $Y' \to Y \to X$ is also in S. ⌟

If we have any collection C of morphisms to some fixed object $X \in \mathcal{C}$ then we can *saturate* the collection to obtain a sieve on X – we can extend the collection to include all precompositions by arbitrary morphisms to any object that is the source of a morphism in C.

---

[*] Again, we are assuming local smallness here for simplicity.

Definition 2.1.3.4 [Pullback sieve]
Let $\mathcal{C}$ be some category, S a sieve on $X \in \mathcal{C}$, and $f\colon Y \to X$ some morphism in $\mathcal{C}$. Then the *pullback of S along f*, written f^*S, is the sieve on Y defined by[*]

$$f^*S = \{g \in \mathrm{Hom}(Y', Y) \mid Y \in \mathcal{C}, fg \in S\}. \qquad \lrcorner$$

As to why this construction is called a pullback, we recall Definition 2.1.1.5 and claim that f^*S is the image[†] of the projection

$$S \times_{\mathrm{Hom}(-,X)} \mathrm{Hom}(-,Y) \to \mathrm{Hom}(-,Y)$$

where we take the pullback in Set, and our morphisms are the inclusion $S \hookrightarrow \mathrm{Hom}(-,X)$ and post composition $(f \circ -)\colon \mathrm{Hom}(-,Y) \to \mathrm{Hom}(-,X)$. It follows from Definition 2.1.1.5 that, in Set, the pullback is given by

$$X\, {}^{\varphi}\!\times^{\psi}_{Z} Y = \{(x,y) \in X \times Y \mid \varphi(x) = \psi(y)\}.$$

So here,

$$S \times_{\mathrm{Hom}(-,X)} \mathrm{Hom}(-,Y) = \{(g,h) \in S \times \mathrm{Hom}(-,Y) \mid g = f \circ h\}$$

and projecting this to $\mathrm{Hom}(-,Y)$ gives the set of all morphisms h such that $f \circ h = g$ for some $g \in S$. This is exactly what Definition 2.1.3.4 says.

Definition 2.1.3.5 [Grothendieck topology[‡]]
Let $\mathcal{C}$ be some category. A *Grothendieck topology on $\mathcal{C}$* is an assignment, to each object $X \in \mathcal{C}$, of a collection $J(X)$ of sieves on X, called *covering sieves*, such that the following conditions are satisfied for all $X \in \mathcal{C}$:

(i) (*base change*) if $S \in J(X)$ and $f\colon Y \to X$ is some arbitrary morphism in $\mathcal{C}$, then the pullback sieve satisfies $f^*S \in J(Y)$;

(ii) (*maximal sieve*) $\mathrm{Hom}(-,X) \in J(X)$;

(iii) (*intersections*)[§] $S, T \in J(X)$ if and only if $S \cap T \in J(C)$;

(iv) (*transitivity*) if $S \in J(X)$ is such that $T_S \in J(X)$, where

$$T_S = \bigcup_{Y \in \mathcal{C}} \{f\colon Y \to X \mid f^*S \text{ is covering on } Y\},$$

then $S \in J(X)$. $\lrcorner$

---

[*] Roughly speaking, take all morphisms to X that factor through Y along f and 'divide them by f'.

[†] Which we can avoid defining category-theoretically here since we have a natural underlying set structure.

[‡] As in [Sch09].

[§] This condition is actually redundant; see the comments after [MM92, Definition 1, §III.2].

Definition 2.1.3.6 [Site]
Let $\mathcal{C}$ be a category and J a Grothendieck topology on $\mathcal{C}$. Then the pair $(\mathcal{C}, J)$ is called a *site*. ⌟

Recall: a pretopology on a category $\mathcal{C}$ consists of, for each $X \in \mathcal{C}$, a collection $C(X)$ of covering families $\{X_j \to X\}$. We can use these to pick certain sieves on X that we wish to be in $J(X)$, and then see that this choice satisfies the conditions of Definition 2.1.3.5. The actual method is quite simple: given some sieve $S = \{S_i \to X\}$ on X, we say that $S \in J(X)$ if and only if it contains some covering family $\{X_j \to X\} \in C(X)$. More details on this construction can be found in [MM92, §III.2].

Note 2.1.3.7
There is an important result ([TV07, Théorème 2.5, §2.1, p.11]) that we don't cover here because it deals with the notion of *stacks*. We do not have the space in this paper to cover the background needed to talk about stacks; we refer the interested reader back to [TV07]. From now on we will skip over stack-theoretic theorems without mentioning them. ⌟

2.2 The faithfully flat topology

Throughout this section $(\mathcal{C}, \otimes, 1)$ is a cosmos[*] *and we use $\mathcal{D}$ to refer to an arbitrary category. The structure of this section is largely based on [Mar09, §1.2, 1.3] (where all proofs can be found) and [TV08, §1.1].*

Definition 2.2.0.1 [Affine schemes over $\mathcal{C}$ (Définition 2.7, §2.2, p.14)]
The category of *affine schemes over $\mathcal{C}$* is the opposite category

$$\mathsf{Aff}_{\mathcal{C}} = \mathsf{Comm}\,(\mathcal{C})^{\mathrm{op}}.$$

For an object $A \in \mathsf{Comm}\,(\mathcal{C})$ we write $\operatorname{Spec} A$ to denote the corresponding object in $\mathsf{Aff}_{\mathcal{C}}$. ⌟

---

[*] In keeping with Hypothèse 2.6, §2.2, p.14; recalling Definition 1.3.2.5.

2.2.1 Commutative algebras

Definition 2.2.1.1 [Commutative A-algebras]
Let $A \in \mathsf{Comm}\,(\mathcal{C})$. The objects of $A/\mathsf{Comm}\,(\mathcal{C})$ are called *commutative A-algebras,* and we write A-$\mathsf{CommAlg}$ to mean the category of such objects. ⌟

This slightly opaque definition is best explained after showing how we can endow A-Mod with a symmetric monoidal structure.

Definition 2.2.1.2 [Coequaliser]
The *coequaliser* $\operatorname{coeq}(X \rightrightarrows Y)$ of two morphisms $f, g\colon X \to Y$ in $\mathcal{D}$ is defined by the universal property of being a pair (Q, q), where $Q \in \mathcal{D}$ and $q\colon Y \to Q$, such that $q \circ f = q \circ g$, and for any other such pair (R, r) there exists a unique morphism $\varphi\colon Q \to R$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[rd, "r"] & Q \arrow[d, dotted, "\varphi"] \\
& & R
\end{tikzcd}$$

⌟

Definition 2.2.1.3 [Symmetric monoidal structure on A-Mod]
Let $A \in \mathsf{Comm}\,(\mathcal{C})$. Define the bifunctor $(-\otimes_A -)\colon A\text{-}\mathsf{Mod} \times A\text{-}\mathsf{Mod} \to A\text{-}\mathsf{Mod}$ by the object of the coequaliser

$$X \otimes_A Y = \operatorname{coeq}(X \otimes A \otimes Y \rightrightarrows X \otimes Y),$$

where the morphisms are the natural ones, namely

$$X \otimes (A \otimes Y) \xrightarrow{\mathrm{id}_X \otimes \sigma_Y} X \otimes Y$$
$$(X \otimes A) \otimes Y \xrightarrow{(\sigma_X \circ \gamma_{XA}) \otimes \mathrm{id}_Y} X \otimes Y.$$

Then $(A\text{-}\mathsf{Mod}, \otimes_A, A)$ is a symmetric monoidal category.* ⌟

Lemma 2.2.1.4 [Equivalent definitions of A-$\mathsf{CommAlg}$]
Let $A \in \mathsf{Comm}\,(\mathcal{C})$. Then we have the equivalence of categories

$$\mathsf{Comm}\,(A\text{-}\mathsf{Mod}) \equiv A\text{-}\mathsf{CommAlg} = A/\mathsf{Comm}\,(\mathcal{C})$$

(using the symmetric monoidal structure from Definition 2.2.1.3). ⌟

Proof. [Mar09, Proposition 1.2.15, §1.2, p.14]. ☐

---

* [Mar09, Proposition 1.2.15, §1.2, p.14]

Because of this equivalence we sometimes write objects of A-$\mathsf{CommAlg}$ as X for some $X \in \mathsf{Comm}\,(A\text{-}\mathsf{Mod})$, and sometimes as $(A \to X)$ for some $X \in \mathsf{Comm}\,(\mathcal{C})$. Object-wise, we can think of the structures we have defined as follows:

$$\mathsf{Comm}\,(A\text{-}\mathsf{Mod}) \equiv A\text{-}\mathsf{CommAlg} \subset A\text{-}\mathsf{Mod} \subset \mathsf{Comm}\,(\mathcal{C}) \subset \mathcal{C}.$$

Corollary 2.2.1.5
Let $A \in \mathsf{Comm}\,(\mathcal{C})$. Then

(i) $\mathsf{Comm}\,(\mathcal{C})$ *and* A-$\mathsf{CommAlg}$ *are bicomplete;*

(ii) $(A\text{-}\mathsf{Mod}, \otimes_A, A)$ *is a cosmos.* ⌟

Proof. Lemma 2.2.1.4 and [Mar09, Propositions 1.2.14, 1.2.17 §1.2]. ☐

Lemma 2.2.1.6
Let $A \in \mathsf{Comm}\,(\mathcal{C})$ and $X, Y \in \mathsf{Comm}\,(A\text{-}\mathsf{Mod})$. Then

$$X \otimes_A Y \cong X \sqcup_A Y$$

where the coproduct is taken in $\mathsf{Comm}\,(\mathcal{C})$. ⌟

Proof. [Mar09, Proposition 1.2.6, §1.2, p.16] ☐

2.2.2 Change of base for modules over a monoid

Assume we have some morphism $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ in $\mathsf{Aff}_{\mathcal{C}}$. We claim that this induces a forgetful functor B-$\mathsf{Mod} \to A$-Mod. That is, on the level of objects, B-$\mathsf{Mod} \subset A$-Mod.

Take some object $X \in B$-Mod. This comes with a B-action $\sigma_B\colon B \otimes X \to X$. We can compose f and σ_B to obtain

$$\sigma_A\colon A \otimes X \xrightarrow{f \otimes \mathrm{id}_X} B \otimes X \xrightarrow{\sigma_B} X.$$

We claim that this is an A-action on X. This just means showing that both of the diagrams in Definition 1.3.2.7 commute. Here we show just one since the other is equally straightforward. Note that

$$\begin{tikzcd} 1 \otimes X \arrow[rr, "\eta_A \otimes \mathrm{id}_X"] \arrow[dr, "\lambda_X"'] & & A \otimes X \arrow[dl, "\sigma_A"] & = & 1 \otimes X \arrow[rr, "(f \circ \eta_A) \otimes \mathrm{id}_X"] \arrow[dr, "\lambda_X"'] & & B \otimes X \arrow[dl, "\sigma_B"] \\ & X & & & & X & \end{tikzcd}$$

But f is a morphism in $\mathsf{Comm}\,(\mathcal{C})$, so $f \circ \eta_A = \eta_B$ (recall Definition 1.3.2.6), and hence the second diagram commutes. Thus so too does the first diagram. We also have an induced forgetful functor B-$\mathsf{CommAlg} \to A$-$\mathsf{CommAlg}$, since if $(B \to X) \in B$-$\mathsf{CommAlg}$ then, composing with f, we get $(A \to B \to X) \in A$-$\mathsf{CommAlg}$.

We mention the forgetful functor B-$\mathsf{Mod} \to A$-Mod because it has a left-adjoint, namely

$$(- \otimes_A B)\colon A\text{-}\mathsf{Mod} \to B\text{-}\mathsf{Mod}$$

(which is well defined: f lets us consider B, which is a B-module, as an A-module). This tells us that $(- \otimes_A B)$ is right exact.

2.2.3 fpqc topology

Now we apply the results from Sections 2.2.1 and 2.2.2. Let $\mathcal{D} = \mathsf{Aff}_\mathcal{C}$ and, using $(\mathsf{Aff}_\mathcal{C})^{\mathrm{op}} = \mathsf{Comm}\,(\mathcal{C})$, define the pseudofunctor

$$\begin{aligned} M\colon (\mathsf{Aff}_\mathcal{C})^{\mathrm{op}} &\to \mathsf{Cat} \\ A &\mapsto A\text{-}\mathsf{Mod} \\ (A \to B) &\mapsto (- \otimes_A B\colon A\text{-}\mathsf{Mod} \to B\text{-}\mathsf{Mod}). \end{aligned} \tag{2.2.3.1}$$

We claim that $\mathcal{D}$ and M satisfy the conditions of Definition 2.1.2.1, and so generate a Grothendieck topology on $\mathsf{Aff}_\mathcal{C}$. There are four conditions that we need to check*:

(i) $\mathsf{Aff}_\mathcal{C}$ *has finite limits*: Corollary 2.2.1.5 (recall that $\mathsf{Aff}_\mathcal{C} = \mathsf{Comm}\,(\mathcal{C})^{\mathrm{op}}$);

(ii) A-Mod *is bicomplete*: Corollary 2.2.1.5;

(iii) $(- \otimes_A B)$ *has a conservative right adjoint*: the right adjoint is the forgetful functor B-$\mathsf{Mod} \to A$-Mod, and this is conservative since it gives us B-Mod as a full subcategory of A-Mod;

(iv) *the Beck-Chevalley condition*†: say we have the following pullback diagram

$$\begin{tikzcd} & \operatorname{Spec} B \arrow[dr, "q"] & \\ \operatorname{Spec} X \arrow[ur, "p'"] \arrow[dr, "q'"'] & & \operatorname{Spec} C \\ & \operatorname{Spec} A \arrow[ur, "p"'] & \end{tikzcd}$$

in $\mathsf{Aff}_\mathcal{C}$ (i.e. $\operatorname{Spec} X = \operatorname{Spec} B \times_{\operatorname{Spec} C} \operatorname{Spec} A$ is the pullback). Then $X = B \sqcup_C A$ is the pushout in $\mathsf{Comm}\,(\mathcal{C})$, and $A, B \in \mathsf{Comm}\,(C\text{-}\mathsf{Mod}) \equiv$

---

*Really, we also need to check that M actually *is* a pseudofunctor, but since we only gave a rough definition of pseudofunctors we simply claim that this is, in fact, true.

†This is proved (much more generally) in [Shu13, Example 2.35].

C-CommAlg. Lemma 2.2.1.6 then tells us that $X \cong B \otimes_C A$. Thus,[*] for $M \in B$-Mod,

$$M \otimes_B X \cong M \otimes_B (B \otimes_C A) \cong (M \otimes_B B) \otimes_C A \cong M \otimes_C A,$$

and so $p^* q_* = (- \otimes_C A) \cong (- \otimes_B X) = (q')_* (p')^*$.

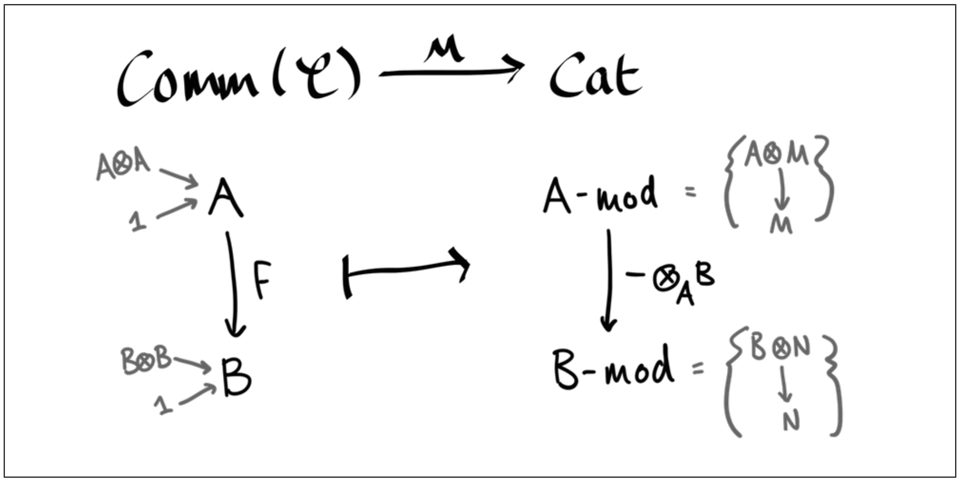


Figure 6: Constructing the fpqc topology – compare with Figure 4

Definition 2.2.3.2 [Faithfully flat and quasi-compact topology (Définition 2.8, §2.2, p.15)]
With definitions as in Equation (2.2.3.1), the M-faithfully flat topology on Aff_C is called the *faithfully flat and quasi-compact* (or simply *fpqc*,[†] or even just *flat*) topology. ⌟

Note 2.2.3.3 [Change of base]
For a morphism $f\colon X \to Y$ in some category $\mathcal{D}$ we have the notion of the *change of base functor* $f^*\colon \mathcal{D}/Y \to \mathcal{D}/X$ given by the pullback along f. This is *not* quite what we mean when we call $(- \otimes_A B)$ a *change of base*. We are instead applying the terminology from Definition 2.1.2.1: say we have

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[dr, "p"] & \\
& \operatorname{Spec} A \\
\operatorname{Spec} B' \arrow[ur, "q"'] &
\end{tikzcd}$$

---

[*] The second isomorphism comes from associativity; the third comes from the fact that $M \otimes_B B \cong M$. Both these facts mirror the usual case of tensor products over a commutative ring, and can be proved by using Lemma 2.2.1.6: $X \otimes_A Y \cong X \sqcup_A Y$.

[†] In keeping with most current literature, we choose to keep the original French initialism rather than using the English *ffqc*.

Then $q_*\colon B'\text{-Mod} \to A\text{-Mod}$ is the forgetful functor, and

$$p^* = (- \otimes_A B)\colon A\text{-Mod} \to B\text{-Mod}.$$

So $p^*q_* = (- \otimes_A B)\colon B'\text{-Mod} \to B\text{-Mod}$ is what we call, in line with Definition 2.1.2.1, a change of base. ⌟

In Section 2.3 we will actually redefine the fpqc topology by spelling out explicitly what it means to be M-faithfully flat with M as in Equation (2.2.3.1). We do this because it makes it easier to define the *Zariski topology*.

2.3 The Zariski topology

Yet again, throughout this section we assume that $(\mathcal{C}, \otimes, 1)$ *is a cosmos and use* $\mathcal{D}$ *to refer to an arbitrary category.*

2.3.1 Zariski covers

Our next goal is to define the *Zariski topology on* $\mathsf{Aff}_{\mathcal{C}}$. Once again, we start by defining a certain type of morphism in $\mathsf{Aff}_{\mathcal{C}}$ and saying when a collection of such morphisms gives an *open cover*.

Definition 2.3.1.1 [Zariski open (Définition 2.9, §2.3, p.15)]
Let $f\colon A \to B$ in $\mathsf{Comm}(\mathcal{C})$. Then f is

(i) *flat* if the functor

$$(- \otimes_A B)\colon A\text{-Mod} \to B\text{-Mod}$$

is exact[*];

(ii) an *epimorphism* if, for all $X \in \mathsf{Comm}(\mathcal{C})$, the map

$$(- \circ f)\colon \operatorname{Hom}(B, X) \to \operatorname{Hom}(A, X)$$

is injective;

---

[*] We already know that this functor commutes with colimits, since $(A\text{-Mod}, \otimes_A, A)$ is closed (Corollary 2.2.1.5), so it is exact if and only if it commutes with finite limits as well, i.e. if and only if it is left exact.

(iii) a *finite presentation* if, for every filtered diagram* of objects

$$\{X_i \in A/\mathsf{Comm}\,(\mathcal{C})\}_{i\in I},$$

the natural morphism

$$\operatorname{colim}_i \operatorname{Hom}_{A/\mathsf{Comm}(\mathcal{C})}(B, X_i) \longrightarrow \operatorname{Hom}_{A/\mathsf{Comm}(\mathcal{C})}(B, \operatorname{colim}_i X_i)$$

is an isomorphism.

We say that $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ in $\mathsf{Aff}_{\mathcal{C}}$ is *Zariski open* (or an *open Zariski immersion*) if the corresponding morphism $f\colon A \to B$ in $\mathsf{Comm}\,(\mathcal{C})$ is a flat epimorphism of finite presentation. ⌟

Definition 2.3.1.2 [fpqc and Zariski covers (Définition 2.10, §2.3, p.16)]
A family of morphisms $\{\operatorname{Spec} A_i \to \operatorname{Spec} A\}_{i\in I}$ in $\mathsf{Aff}_{\mathcal{C}}$ is

(i) an *fpqc cover* (or simply a *flat cover*) if

   (a) for all $i \in I$, the morphism $\operatorname{Spec} A_i \to \operatorname{Spec} A$ is flat;

   (b) there exists some finite subset $J \subset I$ such that the functor

$$\prod_{j\in J}(- \otimes_A A_j)\colon A\text{-}\mathsf{Mod} \to \prod_{j\in J}(A_j\text{-}\mathsf{Mod})$$

   is conservative†;

(ii) a *Zariski cover* if

   (a) it is an fpqc cover;

   (b) for all $i \in I$, the morphism $\operatorname{Spec} A_i \to \operatorname{Spec} A$ is Zariski open. ⌟

We are interested primarily in the Zariski topology, and the fpqc topology will be used essentially only for [. . .] *(Lemma 2.3.2.4)* (§2.3 p.16 ¶3).

Saying that a family of morphisms is a flat cover, in the sense of the above definition, is just another way‡ of saying it is M-faithfully flat, in the sense

---

*That is, our diagram is *non-empty* and such that

(i) for any two objects x, y in the diagram there exists some object k and arrows $x \to k \leftarrow y$;

(ii) for any parallel arrows $f, g\colon x \to y$ there exists some object k and an arrow $a\colon y \to k$ such that $af = ag$.

†By definition, this is just asking that $(- \otimes_A A_j)$ be conservative for each $j \in J$.

‡The only real difference between being a flat cover and being M-faithfully flat is that the former requires exactness of the change of base functor $(- \otimes_A A_i)$ whereas the latter requires only left exactness, but we know that this functor is right exact anyway, since it is left adjoint to the forgetful functor A_i-$\mathsf{Mod} \to A$-Mod. Thus these two requirements coincide.

of Section 2.2.3. So we already know that flat covers give rise to a topology: the fpqc (or flat) topology. To show that the same is true for Zariski covers we need to show that the property of being Zariski open is preserved by pullbacks and is associative (in the sense of Definition 2.1.3.1), and also that isomorphisms are Zariski open. However, Definition 2.3.1.1 is phrased *not* in terms of $\operatorname{Spec} B \to \operatorname{Spec} A$, but instead in terms of the corresponding $A \to B$. So we need to check that these properties of morphisms of *commutative monoids* are stable under *pushouts*. Stability of finite presentation follows from a lengthy definition chase, and epimorphisms are always stable under pushouts.[*] As for flatness, the fpqc topology already tells us that flatness is preserved.

Definition 2.3.1.3 [Zariski topology]
The topology on $\mathsf{Aff}_{\mathcal{C}}$ generated[†] by Zariski covers is called the *Zariski topology*. ⌟

Note 2.3.1.4
Because we require both fpqc and Zariski covers to be finitely conservative, a sieve S on $\operatorname{Spec} A$ is in the generated topology if and only if it contains some cover if and only if it contains the finite conservative subset of that cover. This means that these pretopologies are *quasi-compact*, i.e. generated by finite covering families. For more, see the beginning of [MM92, § IX.11]. ⌟

2.3.2 Sheaves

Just as in classical algebraic geometry, now that we have some topology we can introduce the idea of 'gluing together' affine schemes to obtain *schemes*. Fundamental to this idea is the definition of a *sheaf*.

Definition 2.3.2.1 [Sheaves on a site]
Let $(\mathcal{D}, J)$ be a site[‡] and $F \in \mathsf{PSh}(\mathcal{D})$ some presheaf.[§] We say that F is a *J-sheaf* (or simply a *sheaf* if the context is clear) if, for all objects $X \in \mathcal{D}$ and covering sieves $S \in J(X)$ on X, the natural map

$$\operatorname{Hom}(\operatorname{Hom}(-, X), F) \longrightarrow \operatorname{Hom}(S, F)$$

induced by the inclusion $S \hookrightarrow \operatorname{Hom}(-, X)$ is a bijection.

We write $\mathsf{Sh}^J(\mathcal{D})$ to be the category of sheaves on $(\mathcal{D}, J)$, a full subcategory of $\mathsf{PSh}(\mathcal{D})$. ⌟

---

[*] [Bor94, Proposition 2.5.3(1)]

[†] As in Lemma 2.1.3.2.

[‡] Definition 2.1.3.6

[§] Definition 1.3.1.2

If we have a pretopology then we can rewrite this definition in a way which is sometimes easier to apply in practice, and also gives a better geometric intuition.

Lemma 2.3.2.2 [Sheaves on a presite]
Let $(\mathcal{D}, C)$ be a presite,[*] *$(\mathcal{D}, J)$ the site that it generates, and $F \in \mathsf{PSh}(\mathcal{D})$. Then $F \in \mathsf{Sh}^J(\mathcal{D})$ if and only if, for all objects $X \in \mathcal{D}$ and covering families $\{X_i \to X\}_{i \in I} \in C(X)$ of X,*

$$F(X) = \operatorname{eq}\left(\prod_{i \in I} F(X_i) \rightrightarrows \prod_{j,k \in I} F(X_j \times_X X_k)\right)$$

(where the equaliser eq *is the dual notion to the coequaliser*[†]*).* ⌟

Proof. [MM92, §III.4, Proposition 1] □

As for how this provides us with some intuition, let us return to the example of $\mathsf{Op}(T)$. In this category, pullbacks correspond to intersection, and we think of presheaves as being functions on the open sets that take inclusion maps to restriction maps. Lemma 2.3.2.2 says that F is a sheaf if and only if, when we piece together all of the $F(X_i)$ that agree on overlaps (this is the equaliser term), we get exactly $F(X)$, for any cover X_i of X. This is just the (classical) presheaf condition – see Figure 7.

Since the Zariski topology is coarser (see Figure 8) than the fpqc topology, the collection of fpqc covering sieves on some object is at least as large as the collection of Zariski covering sieves. This means that asking for the map induced by inclusion in Definition 2.3.2.1 to be a bijection for all covering sieves on an object is a stricter condition in the fpqc topology than in the Zariski topology. So being an fpqc sheaf implies being a Zariski sheaf, but the converse doesn't necessarily hold. Thus we have the subcategories

$$\mathsf{Sh}^{\mathrm{fpqc}}(\mathsf{Aff}_{\mathcal{C}}) \subset \mathsf{Sh}^{\mathrm{Zar}}(\mathsf{Aff}_{\mathcal{C}}) \subset \mathsf{PSh}(\mathsf{Aff}_{\mathcal{C}}).$$

Note 2.3.2.3
As already stated, our primary interest is the Zariski topology, so whenever we speak of sheaves without reference to a specific topology it is assumed that we mean Zariski sheaves. Similarly, we write $\mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}}) = \mathsf{Sh}^{\mathrm{Zar}}(\mathsf{Aff}_{\mathcal{C}})$. ⌟

---

[*] This is not standard terminology, but we define a *presite* as a pair $(\mathcal{D}, C)$ consisting of a category $\mathcal{D}$ and a Grothendieck pretopology C on $\mathcal{D}$. Given such a pair, we can talk about the *site that it generates* by endowing $\mathcal{D}$ with the Grothendieck topology generated by C.

[†] Definition 2.2.1.2

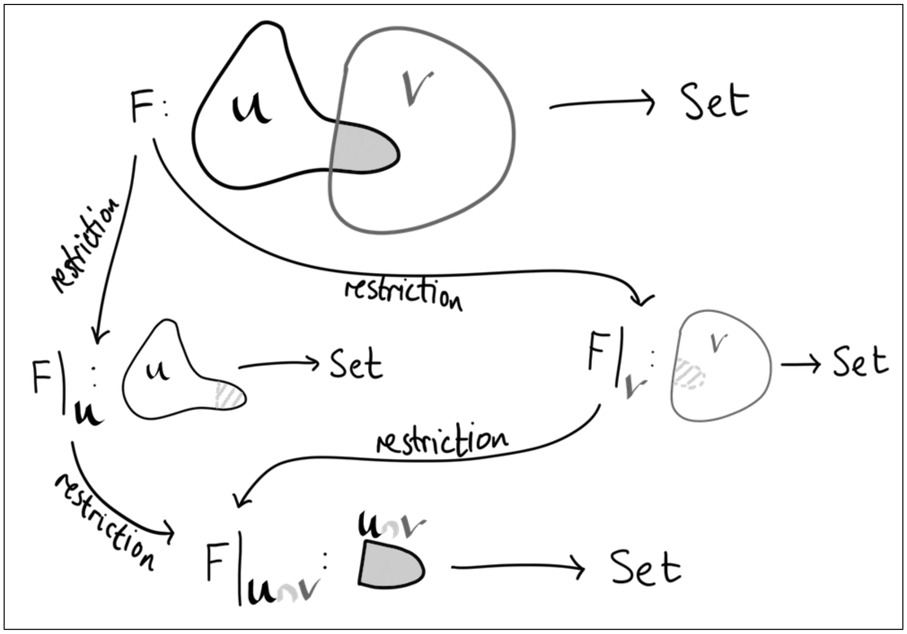


Figure 7: The fact that we always have a morphism from $F(X)$ to the equaliser in Lemma 2.3.2.2 corresponds to the classical presheaf condition: for presheaves on a topological space it doesn't matter whether we restrict $U \cup V$ to $U \cap V$ via U or via V

In Section 1.3.1 we identified Aff_k with its essential image in $\mathsf{PSh}(\mathsf{Aff}_k)$ under the Yoneda embedding, and we can do the same here: identify $\mathsf{Aff}_\mathcal{C}$ with its essential image in $\mathsf{PSh}(\mathsf{Aff}_\mathcal{C})$ under the Yoneda embedding. It turns out, however, that we can actually come up with a stronger result.

Lemma 2.3.2.4 [(Corollaire 2.11, §2.3, p.17)]
For all $X \in \mathsf{Aff}_\mathcal{C}$ the presheaf Y_X coming from the Yoneda embedding is an fpqc sheaf.* ⌟

Proof. The proof in [TV07] uses stacks; see Note 2.1.3.7. □

Firstly, this exactly says that the fpqc topology (and thus the Zariski topology, since it is coarser) is *subcanonical*: all representable presheaves are fpqc sheaves. Secondly, this means we have the equivalence of categories

$$\mathsf{Aff}_\mathcal{C} \equiv \mathsf{AffSch}(\mathcal{C}) \tag{2.3.2.5}$$

where we define the full subcategory $\mathsf{AffSch}(\mathcal{C}) \subset \mathsf{Sh}(\mathsf{Aff}_\mathcal{C})$ by

$$\mathsf{AffSch}(\mathcal{C}) = \{X \in \mathsf{Sh}(\mathsf{Aff}_\mathcal{C}) \mid X \cong \mathrm{Hom}_{\mathsf{Aff}_\mathcal{C}}(-, \operatorname{Spec} A) \text{ for some } \operatorname{Spec} A \in \mathsf{Aff}_\mathcal{C}\}.$$

We call the objects of $\mathsf{AffSch}(\mathcal{C})$ *affine schemes*.

---

*Lemmas 1.3.1.3 and 1.3.1.5

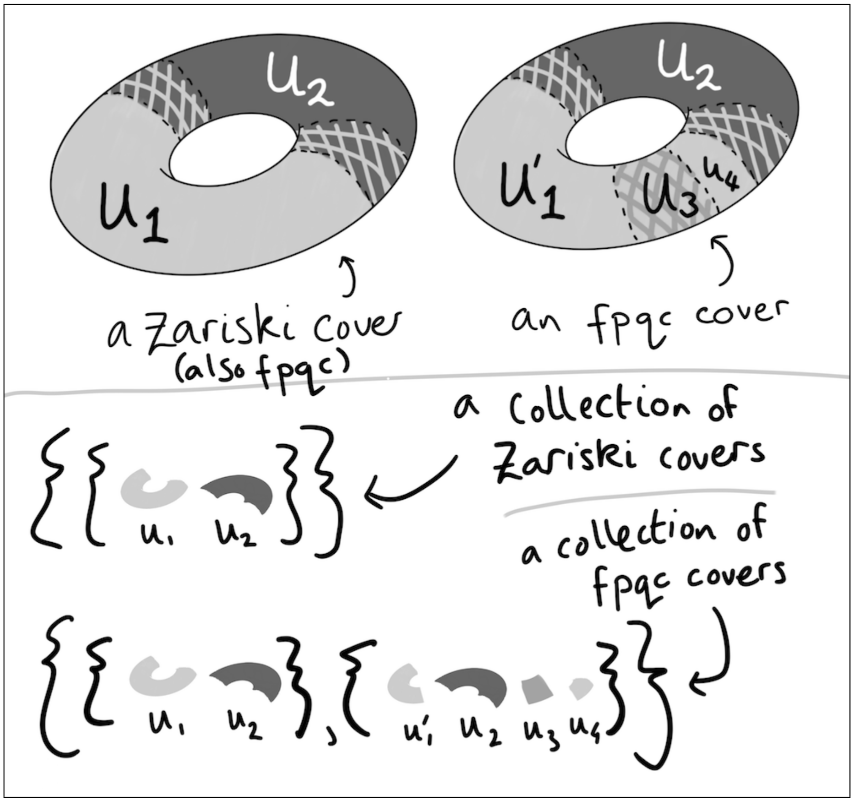


Figure 8: The Zariski topology is *coarser* than the fpqc topology, i.e. it has fewer open sets, so every Zariski cover is also an fpqc cover – the sheaf condition 'reverses' this inclusion: there are *more* fpqc covers than Zariski, hence *fewer* fpqc sheaves than Zariski (and **maybe** hence *more* fpqc schemes (something we don't define) than Zariski; see Figure 12 for a similar conjecture)

Definition 2.3.2.6 [Affine schemes over $\mathcal{C}$]
From now on we use the phrase 'affine scheme' interchangeably, to mean an object of either $\mathsf{Aff}_{\mathcal{C}}$ or $\mathsf{AffSch}(\mathcal{C})$. We often use the same notation for both as well (so for $\operatorname{Spec} A \in \mathsf{Aff}_{\mathcal{C}}$ we also write $\operatorname{Spec} A \in \mathsf{AffSch}(\mathcal{C})$ to mean $Y_A = \operatorname{Hom}(-, \operatorname{Spec} A)$, and vice versa). ⌟

2.4 Schemes

> *In this section we present the main definition of this paper, that of a* scheme over $\mathcal{C}$. (§2.4, ¶1)

2.4.1 Using sheaves

In light of Equation (2.3.2.5), we need to rephrase Definitions 2.3.1.1 and 2.3.1.2 in terms of sheaves.

Definition 2.4.1.1 [Zariski open in $\mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}})$ (Définition 2.12, §2.4, p.18)]

(i) Let $X \in \mathsf{AffSch}(\mathcal{C})$ and $F \subset X$ a subsheaf of X. We say that F is *Zariski open in* X if there exists a Zariski-open (in the sense of Definition 2.3.1.1*) family $\{X_i \to X\}_{i\in I}$ in $\mathsf{Aff}_{\mathcal{C}}$ (where I is *not* necessarily finite†) and a sheaf morphism $\coprod_{i\in I} X_i \to X$ whose image in X is F.

(ii) A morphism $f\colon F \to G$ in $\mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}})$ is *Zariski open* (or an *open Zariski immersion*) if, for all affine schemes X and morphisms $X \to G$, the induced morphism
$$F \times_G X \to X$$
is a monomorphism whose image is Zariski open in X. ⌟

By definition, Zariski-open morphisms are stable under a change of base and also under composition in $\mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}})$. Further, *it can be easily checked that Zariski-open morphisms are monomorphisms in* $\mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}})$. (§2.4, ¶2) When we introduce schemes we will see that this point about monomorphisms is an example of a property holding on all affine schemes in a cover implying that the same property holds for the whole scheme, i.e. 'locally a monomorphism implies globally a monomorphism'.

At the moment we have two definitions for what it means for a morphism of affine schemes to be Zariski open: Definition 2.3.1.1 for $\mathsf{Aff}_{\mathcal{C}}$; and Definition 2.4.1.1 for $\mathsf{AffSch}(\mathcal{C})$. The following lemma shows that the two definitions are indeed equivalent.

---

[*] Recall Definition 2.3.2.6: X is not necessarily in $\mathsf{Aff}_{\mathcal{C}}$, but we know that we can find some $\operatorname{Spec} B \in \mathsf{Aff}_{\mathcal{C}}$ such that $X \cong \operatorname{Hom}(-, \operatorname{Spec} B)$. Similarly we can find $\operatorname{Spec} B_i \in \mathsf{Aff}_{\mathcal{C}}$ such that $X_i \cong \operatorname{Hom}(-, \operatorname{Spec} B_i)$. Then we want the family $\{\operatorname{Spec} B_i \to \operatorname{Spec} B\}_{i\in I}$ to be Zariski open in the sense of Definition 2.3.1.1.

[†] See the paragraph accompanying Figures 10 and 11, just before Section 2.4.2.

Lemma 2.4.1.2 [Lemme 2.14, §2.4, p.18]
Let $f\colon Z \to Y$ be a morphism of affine schemes. Then f is Zariski open in the sense of Definition 2.3.1.1 if and only if it is Zariski open in the sense of Definition 2.4.1.1. ⌟

Proof. This proof is largely unpacking definitions, and is given in [TV07]. □

Definition 2.4.1.3 [Scheme relative to $\mathcal{C}$ (Définition 2.15, §2.4, p.19)]
A sheaf $F \in \mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}})$ is a *scheme relative to $\mathcal{C}$* (or simply a *scheme* if the context is clear) if there exists a family $\{X_i\}_{i\in I}$ of affine schemes and a morphism

$$p\colon \left(\coprod_{i\in I} X_i\right) \to F$$

satisfying the following two conditions:

(i) p is an epimorphism of sheaves;

(ii) for all $i \in I$ the morphism[*] $X_i \to F$ is an open Zariski immersion.

We define the category $\mathsf{Sch}(\mathcal{C})$ to be the full subcategory of $\mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}})$ consisting of such sheaves, and call the family $\{X_i \to X\}$ an *affine Zariski cover of F*.

⌟

At the moment, we simply know that a scheme is in some sense 'covered' by affine schemes – we do not yet know exactly how these affine schemes 'fit together' (we find out in Lemma 2.4.3.4).

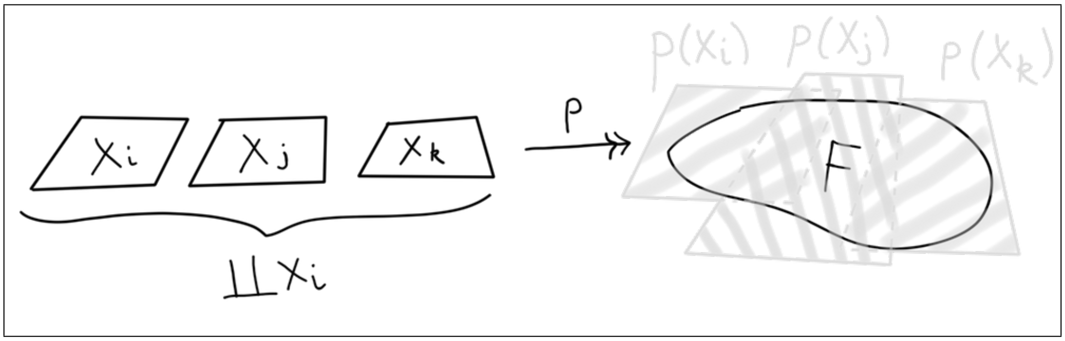


Figure 9: Definition 2.4.1.3 – schemes

When we defined Zariski covers in Definition 2.3.1.2 we required them to be finitely conservative, but here we don't have any finiteness conditions – we use the word 'cover' in two slightly different senses. Here it is not

---

[*] That is, the induced morphism $X_i \to \coprod X_i \to F$.

so much a topological cover, as a scheme-theoretic cover. Locally, schemes 'look like' affine schemes, which have this finitely conservative (i.e. *quasi-compact*) property, but globally they are not under such tight restrictions. See Figures 10 and 11.

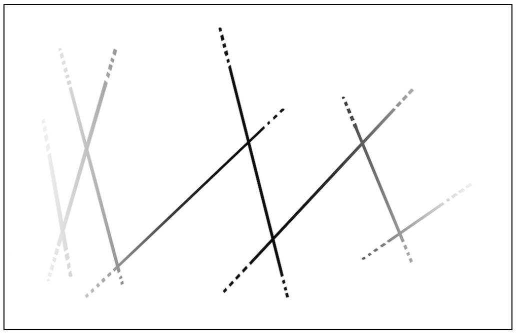


Figure 10: Consider infinitely many lines glued together pairwise at a point – this is a scheme, since we can take the X_i to be the lines, then $\{X_i\}$ is an *infinite* affine Zariski cover

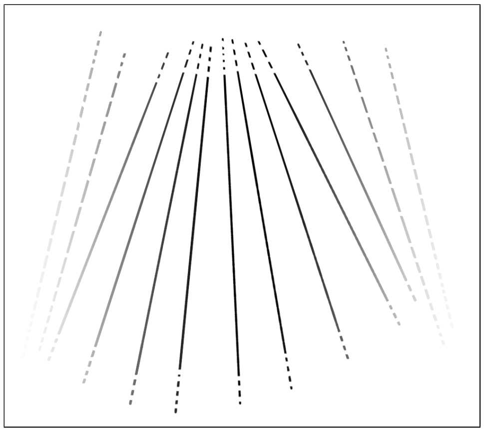


Figure 11: The trivial gluing (coproduct) of infinitely many lines; here we can take the X_i to be the lines and p to simply be the identity

2.4.2 Partial summary

Before moving on to talk about the properties schemes, we provide a brief summary of what we have done so far.

In Section 2.3.1 we defined two topologies on $\mathsf{Aff}_{\mathcal{C}} = \mathsf{Comm}\,(\mathcal{C})^{\mathrm{op}}$ by defining certain types of *open* morphisms: *fpqc* and *Zariski*. After this, in Section 2.3.2, we looked at a more abstract situation: using the topologies we just defined to consider fpqc and Zariski *sheaves*.[*] Then, with the Yoneda embedding, we considered $\mathsf{Aff}_{\mathcal{C}}$ as sitting inside $\mathsf{Sh}^{\mathrm{fpqc}}(\mathsf{Aff}_{\mathcal{C}}) \subset \mathsf{PSh}(\mathsf{Aff}_{\mathcal{C}})$, and called the essential image of this embedding $\mathsf{AffSch}(\mathcal{C})$, whose objects are *affine schemes*. Next, in Definition 2.4.1.1, we extended our definition of Zariski open morphisms to general sheaf morphisms and showed that it agreed with our previous definition. We used this to define a *scheme* as a sheaf that was covered by affine schemes, with each affine scheme embedding into the sheaf via a Zariski open immersion. By definition, every affine scheme is also a scheme (just as we would expect) and every scheme is a sheaf. Figure 12 shows how all these full subcategories overlap inside $\mathsf{PSh}(\mathsf{Aff}_{\mathcal{C}})$.

---

[*] Then agreeing, from now on, to say *sheaf* to mean a *Zariski* sheaf.

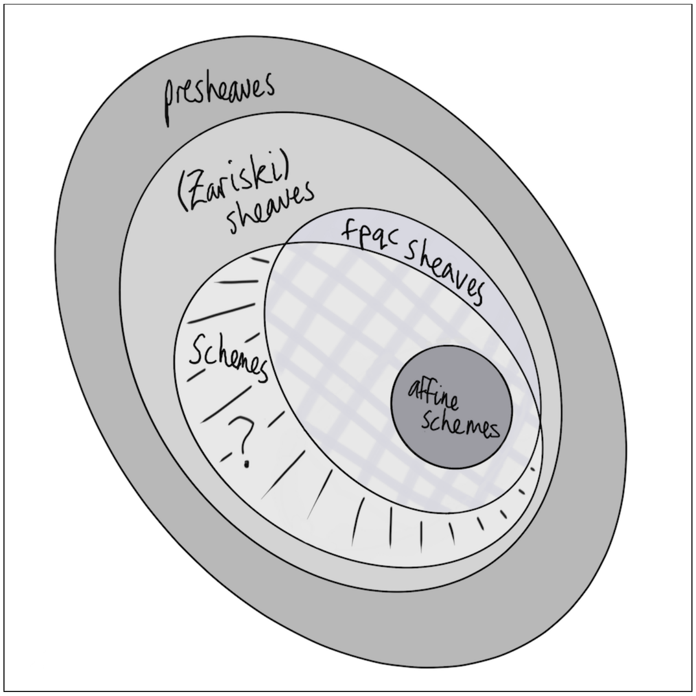


Figure 12: The hierarchy, considering objects up to isomorphism – really there are 'lots more' fpqc sheaves than schemes: $\mathsf{Sh}^{\mathsf{fpqc}}(\mathsf{Aff}_{\mathcal{C}})$ is bicomplete but $\mathsf{Sch}(\mathcal{C})$ does *not* have all colimits, so we can construct $X \in \mathsf{Sh}^{\mathsf{fpqc}}(\mathsf{Aff}_{\mathcal{C}}) \setminus \mathsf{Sch}(\mathcal{C})$ by talking a sufficiently nasty colimit
Note: it seems possible (hence the question mark) that $\mathsf{Sch}(\mathcal{C}) \subset \mathsf{Sh}^{\mathsf{fpqc}}(\mathsf{Aff}_{\mathcal{C}})$, since Zariski open morphisms are also flat morphisms, and we have an 'fpqc-sheafification' functor $\mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}}) \to \mathsf{Sh}^{\mathsf{fpqc}}(\mathsf{Aff}_{\mathcal{C}})$ that is a left adjoint (and so preserves colimits) – this is not mentioned in [TV07] and we don't have the space here to go any further

2.4.3 Properties of schemes

We now state some fundamental properties of schemes, and refer the reader to [TV07] for proofs of each one. However, there are diagrams (at the end of this section) of sketches for most of the proofs, which should hopefully convey the main ideas reasonably well. The conventions used in these diagrams is explained in Figure 14. We recall that we can think of pullbacks as a generalisation of fibres; that pullbacks of epimorphisms are epimorphisms; and that pullbacks of Zariski open morphisms are Zariski open.

Lemma 2.4.3.1 [Gluing and affine schemes (Proposition 2.6, §2.4, p.20)]

(i) *Let A, B be affine schemes, G a sheaf, and $G \to A \leftarrow B$ morphisms of sheaves. If G is a scheme then $F = B \times_A G$ is also a scheme.*

(ii) *Let A be an affine scheme, F a sheaf, and $F \to A$ a morphism of sheaves. If there exists an affine Zariski cover $\{A_i \to A\}$ such that each $F \times_A A_i$ is a scheme then F is a scheme.* ⌟

Lemma 2.4.3.2 [Proposition 2.17, §2.4, p.21]

(i) Let F be a scheme and $F_0 \subset F$ be Zariski open in the sense of Definition 2.4.1.1. Then F_0 is a scheme.

(ii) Let $f\colon F \to G$ be a morphism between schemes. Then f is Zariski open in the sense of Definition 2.4.1.1 if and only if f satisfies the following two conditions:

  (a) f is a monomorphism;

  (b) there exists an affine Zariski cover $\{X_i \to F\}$ such that each morphism $X_i \to G$ given by composition with f is Zariski open. ⌟

Definition 2.4.3.3 [Congruence on an object]
For $X \in \mathcal{D}$, a congruence R on X is[*] a monomorphism

$$R \overset{(p_1,p_2)}{\hookrightarrow} X \times X$$

equipped with the following morphisms:

(i) (*reflexivity*) $r\colon X \to R$ such that $p_1 \circ r = p_2 \circ r = \mathrm{id}_X$;

(ii) (*symmetry*) $s\colon R \to R$ such that $p_1 \circ s = p_2$ and $p_2 \circ s = p_1$;

(iii) (*transitivity*) $t\colon R \times_X R \to R$ such that

$$\begin{tikzcd}
R \times_X R \arrow[r, "\pi_1"'] \arrow[r, bend left, "t"] \arrow[d, "\pi_2"] \arrow[d, bend right, "t"'] & R \arrow[d, "p_1"] \\
R \arrow[r, "p_2"'] & X
\end{tikzcd}$$

commutes (where π_1, π_2 are the pullback morphisms).

Given such an R, we define the *quotient object* X/R as

$$X/R = \mathrm{coeq}(R \overset{p_1}{\underset{p_2}{\rightrightarrows}} X).$$ ⌟

Lemma 2.4.3.4 [Stability; gluing affine schemes (Proposition 2.18, §2.4, p.21)]

(i) The subcategory $\mathsf{Sch}(\mathcal{C}) \subset \mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}})$ is stable under disjoint unions and pullbacks.

---

[*] Up to some notion of isomorphism between morphisms – see [Ško09].

(ii) *A sheaf $F \in \mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}})$ is a scheme if and only if there exists some congruence R on some sheaf $X \in \mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}})$ where the following four conditions are satisfied:*

(a) $X \cong \coprod_{i\in I} U_i$ *for some affine schemes U_i;*

(b) *for all $(i,j) \in I^2$, the subsheaf $R_{i,j} \subset U_i \times U_j$ given by the pullback*[*]

$$\begin{tikzcd} R_{i,j} \arrow[r] \arrow[d] & U_i \times U_j \arrow[d] \\ R \arrow[r] & X \times X \end{tikzcd}$$

is such that each induced morphism

$$R_{i,j} \to U_i$$

is Zariski open;

(c) *for each $i \in I$ the subobject $R_{i,i} \subset U_i \times U_i$ is equal to the image of the diagonal morphism $U_i \to U_i \times U_i$;*

(d) $F \cong X/R$. ⌟

It is possible to rephrase Lemma 2.4.3.4(ii) in terms of pushouts. Say we have affine schemes A, X, Y and Zariski-open immersions $A \to X$, $A \to Y$. Then $X \coprod_A Y$ is the presheaf obtained by gluing X and Y along the images of A (see Figure 13). The above lemma then says that this presheaf is also actually a scheme.

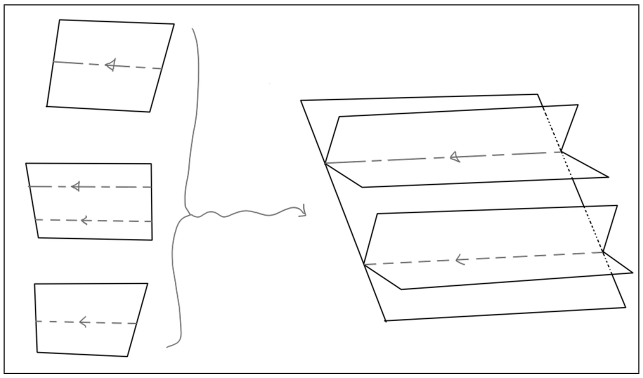


Figure 13: Rephrasing Lemma 2.4.3.4(ii) in familiar 'gluing' terms – if we take some affine schemes with Zariski-open affine subschemes and glue along these subschemes in a 'sufficiently nice' way, then we end up with a scheme – further, every scheme is obtained in exactly this way – *schemes are affine schemes glued together*

---

[*] 'Intersecting down' the congruence: $R \cong \coprod_{i,j} R_{i,j}$. The morphisms $U_i \times U_j \to X \times X$ are those induced by the $U_i \to \coprod U_i \xrightarrow{\sim} X$.

2.4.4 Another view

Here we briefly discuss how the functor of points approach that we have been using (Section 1.3) coincides with the classical ringed space approach. What follows is a sketch of the story – in particular we make claims without stating proofs – and we refer the reader to the end (the last four paragraphs) of [TV07, §2.4, p.24] as well as [MM92, §IX.1–3] for the details.

Given some topological space T we can view the category $\mathsf{Op}(T)$ as a lattice, with partial order given by inclusion. Generally, define a *frame* to be any lattice X that behaves suitably like $\mathsf{Op}(T)$: having arbitrary joins and finite meets, and meets distributing over arbitrary joins. The morphisms between frames are maps of partially-ordered sets preserving arbitrary joins and finite meets. This defines a category Fra of frames. We define the category of *locales*[*] as $\mathsf{Loc} = \mathsf{Fra}^{\mathrm{op}}$, and for $f\colon X \to Y$ in Loc we write $f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ to be the corresponding morphism of objects in Fra.

Next, given a locale X we can define *points of* X as locale morphisms $1 \to X$, where $1 \in \mathsf{Loc}$ is the terminal locale. We say that a locale *has enough points* if elements of the lattice can be distinguished by a single point.[†] That is, for any distinct $U, V \in \mathcal{O}(X)$, there exists $p\colon \{*\} \to X$ such that $p^{-1}(U) \neq p^{-1}(V)$. It can be shown[‡] that if a locale X has enough points then there exists some topological space $|X|$ such that $\mathcal{O}(X) \cong \mathsf{Op}(|X|)$.

Finally, given $X \in \mathsf{Sch}(\mathcal{C})$, we define $\mathsf{Zar}(X)$ as the full subcategory of $\mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}})/X$ consisting of $u\colon Y \to X$ such that Y is a scheme and u is an open Zariski immersion. It turns out that $\mathsf{Zar}(X)$ is a locale, and it has an induced topology coming from the canonical topology on $\mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}})$. If we define $\mathsf{AffZar}(X)$ to be the full subcategory of $\mathsf{Zar}(X)$ consisting of $Y \to X$ with Y an affine scheme, and endow this with the same restricted topology, then we have the equivalence of categories

$$\mathsf{Sh}(\mathsf{Zar}(X)) \equiv \mathsf{Sh}(\mathsf{AffZar}(X)),$$

so we write $\mathsf{Sh}(X_{\mathsf{Zar}})$ to mean either (under this identification). The topology on $\mathsf{Zar}(X)$ is generated by a quasi-compact pretopology (i.e. finite covering families), namely $\mathsf{AffZar}(X)$. This means that $\mathsf{Zar}(X)$ has enough points,[§] and so, by the above, $\mathsf{Zar}(X) \equiv \mathsf{Op}(|X|)$ for some topological space $|X|$. This induces the equivalence

$$\mathsf{Sh}(X_{\mathsf{Zar}}) \equiv \mathsf{Sh}(|X|).$$

---

[*] Translation note: locales are called *lieux* in French.

[†] [MM92, §IX.2] provides a nice way of thinking of this in terms of frames.

[‡] [MM92, Corollary 4, §IX.3]

[§] More generally, *Deligne's theorem* ([MM92, Corollary 3, §IX.11]) tells us that any *coherent topos* has enough points.

Now let $Y = (\operatorname{Spec} A \to X) \in \mathsf{AffZar}(X)$. We can associate to Y the object $A \in \mathsf{Comm}\,(\mathcal{C})$; letting Y vary over $\mathsf{AffZar}(X)$ induces a functor

$$\begin{aligned} \mathcal{O}_X \colon \mathsf{AffZar}(X)^{\mathrm{op}} &\to \mathsf{Comm}\,(\mathcal{C}) \\ (X \to \operatorname{Spec} A) &\mapsto A. \end{aligned}$$

Then $\mathcal{O}_X$ is a sheaf,[*] and the pair $(|X|, \mathcal{O}_X)$ acts as in a $\mathcal{C}$-ringed space approach. Replacing $\mathcal{C}$ with Ab we recover the classical ringed space approach.

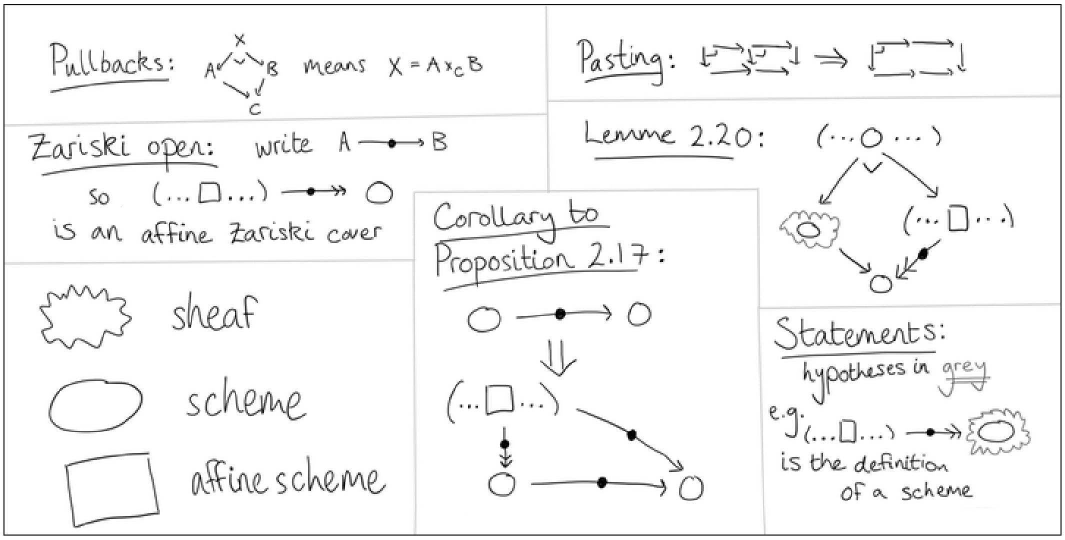


Figure 14: The motivation of the notation is the naive motto ‘straight lines are simpler than curved ones’: affine schemes are our building blocks, schemes are slightly more complicated, and sheaves are the least well behaved of all – *the choice of graphical notation is not meant to be read into too deeply*

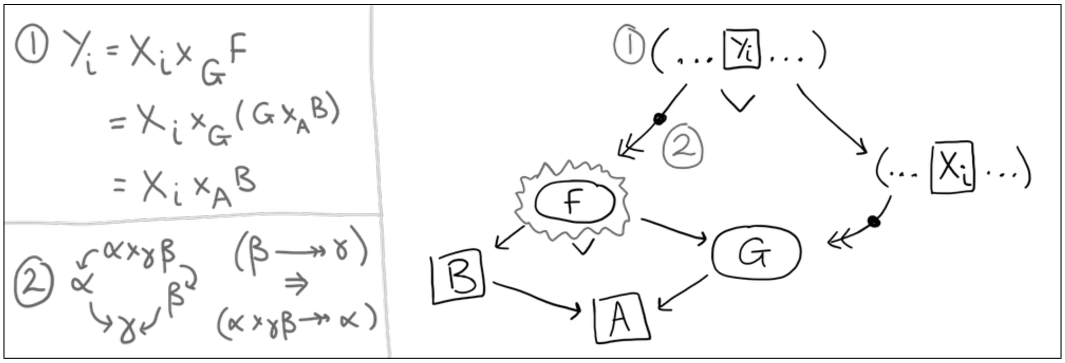


Figure 15: Lemma 2.4.3.1(i) – the proof of ① uses *pasting of pullbacks* and tells us that the Y_i are affine schemes, because $\operatorname{Spec}\alpha \times_{\operatorname{Spec}\beta} \operatorname{Spec}\gamma \cong \operatorname{Spec}(\alpha \coprod_\beta \gamma)$; the proof of ② simply says that the pullback of an epimorphism is also an epimorphism

---

[*] Lemma 2.3.2.4

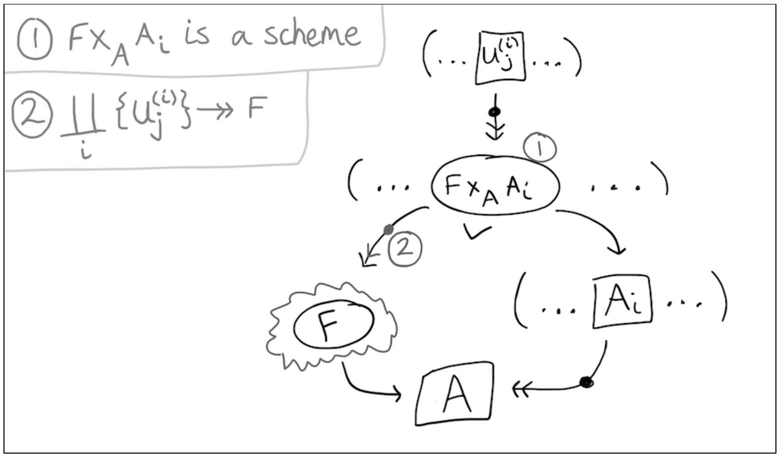


Figure 16: Lemma 2.4.3.1(ii) – a special case of [TV07, Lemme 2.20, §2.4]

For ① the previous part of the lemma tells us that $F \times_A A_i$ is a scheme; ② says that if we take affine covers for all of the $F \times_A A_i$ then together they cover F

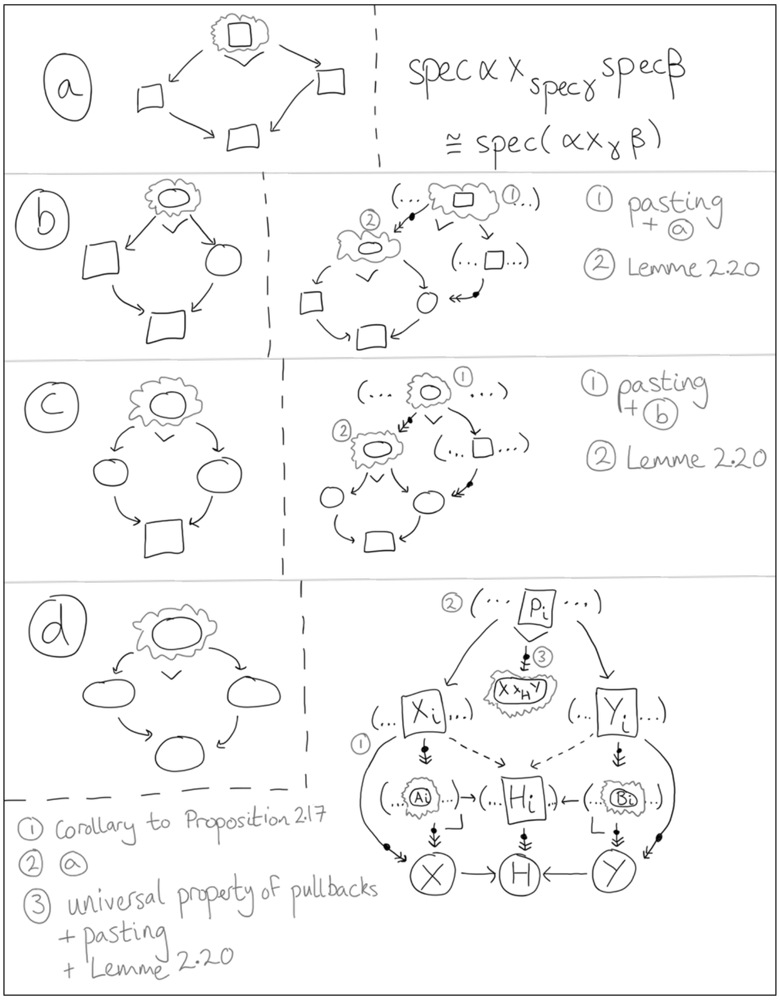


Figure 17: Lemma 2.4.3.4(i) – the statement concerning pullbacks

We build up to this proof in successive steps, from ⓐ to ⓓ – the statements are on the left and the proofs on the right

2.5 Changes of base

The aim of this section is to provide some technical machinery that we will use in Section 3, so we only state the main result of [TV07, §2.5] and refer the reader back to there for more details.

Theorem 2.5.0.1 [(Corollaire 2.22, §2.5, p.27)]
Let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ and $(\mathcal{D}, \odot, 1_{\mathcal{D}})$ be cosmoses, and $f\colon \mathcal{C} \to \mathcal{D}$ be a strong symmetric monoidal functor[*] *with left adjoint $g\colon \mathcal{D} \to \mathcal{C}$. Define*

$$\begin{aligned} f_!^\sim \colon \mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}}) &\to \mathsf{Sh}(\mathsf{Aff}_{\mathcal{D}}) \\ F &\mapsto (a \circ F \circ g) \end{aligned}$$

where a is the sheafification[†] *functor. Suppose that $g\colon \mathcal{D} \to \mathcal{C}$ is conservative and commutes with filtered colimits, and that, for every flat morphism $A \to B$ in* $\mathsf{Comm}\,(\mathcal{C})$ *and every $N \in f(A)$-*Mod*, the natural morphism*

$$g(N) \otimes_A B \to g(N \odot_{f(A)} f(B))$$

is an isomorphism in B-Mod*. Then*

(i) $f\colon \mathsf{Aff}_{\mathcal{C}} \to \mathsf{Aff}_{\mathcal{D}}$ is continuous in the Zariski topology;

(ii) the functor $f_!^\sim\colon \mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}}) \to \mathsf{Sh}(\mathsf{Aff}_{\mathcal{D}})$ preserves the subcategories of schemes, and induces a functor (called the change of base functor*)*

$$\begin{aligned} \mathsf{Sch}(\mathcal{C}) &\to \mathsf{Sch}(\mathcal{D}) \\ X &\mapsto X \otimes \mathcal{D} := f_!^\sim(X); \end{aligned}$$

(iii) we have an isomorphism

$$f_!^\sim(X) \cong f(X)$$

for every $X \in$ $\mathsf{AffSch}(\mathcal{C})$*.* ⌟

---

[*] That is, a functor that respects the symmetric monoidal structure of both $\mathcal{C}$ and $\mathcal{D}$. To be slightly more precise, the functor comes with an isomorphism $f(X) \odot f(Y) \xrightarrow{\sim} f(X \otimes Y)$, natural in X and Y, which respects the symmetry of $\otimes$ and $\odot$, and an isomorphism $1_{\mathcal{D}} \xrightarrow{\sim} f(1_{\mathcal{C}})$ satisfying certain coherence conditions. [TV07] and many others use the name *monoidal functor* to refer to what we (and some others) call a *strong monoidal functor*. The choice of nomenclature is only important insofar as consistency; we agree to use 'strong'.

[†] Similar to classical algebraic geometry: the inclusion functor $\iota\colon \mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}}) \to \mathsf{PSh}(\mathsf{Aff}_{\mathcal{C}})$ admits a left adjoint $a\colon \mathsf{PSh}(\mathsf{Aff}_{\mathcal{C}}) \to \mathsf{Sh}(\mathsf{Aff}_{\mathcal{C}})$ which we call the *sheafification functor*. For (vastly many) more details, see [MM92, §III.5, Theorem 1].

Note 2.5.0.2 [Notation]
Say $(\mathcal{T}, \otimes, 1)$ is a cosmos and $A, B \in \mathsf{Comm}(\mathcal{T})$. Then, with $\mathcal{C} = A\text{-}\mathsf{Mod}$ and $\mathcal{D} = B\text{-}\mathsf{Mod}$, we write

$$F_!^\sim(-) = (- \otimes_A B)\colon \mathsf{Sch}(A\text{-}\mathsf{Mod}) \to \mathsf{Sch}(B\text{-}\mathsf{Mod}),$$

extending the definition of $(- \otimes_A B)$ from Section 2.2.1. ⌟

As we would (very much) hope, the change of base functor is functorial: it doesn't matter in which order we change base and sheafify. In particular, if $X = \operatorname{Spec} A \in \mathsf{Aff}_\mathcal{C}$ for some $A \in \mathsf{Comm}(\mathcal{C})$ then Theorem 2.5.0.1 (iii) tells us[*] that

$$f_!^\sim(\operatorname{Spec} A) \cong \operatorname{Spec} f(A).$$

Thus, using the Hom-set isomorphism from the adjunction $(f \dashv g)$,

$$\begin{aligned} f_!^\sim(\operatorname{Spec} A)\colon \mathsf{Comm}(\mathcal{D}) &\to \mathsf{Set} \\ B &\mapsto \operatorname{Hom}(f(A), B) \cong \operatorname{Hom}(A, g(B)). \end{aligned}$$

We think of $\operatorname{Spec} f(A)$ as[†]

$$\begin{aligned} \operatorname{Spec} f(A) \quad `=\text{'} \quad & \operatorname{Hom}_{\mathsf{Aff}_\mathcal{D}}(-, \operatorname{Spec} f(A)) \\ = & \operatorname{Hom}_{\mathsf{Comm}(\mathcal{D})}(f(A), -). \end{aligned}$$

So, remembering Section 1.3.1, $f_!^\sim(\operatorname{Spec} A)$ gives the *functor of points of* A.

---

[*] [TV07, §2.5, ¶-1]

[†] Again, recall Definition 2.3.2.6.

3 Three examples of relative geometry

We now present our first three examples of categories of relative schemes. (§3 ¶1)

Note 3.0.0.1
As stated in Section 1.1.1, we adopt the convention that $0 \in \mathbb{N}$. ⌟

3.1 Under $\operatorname{Spec}\mathbb{Z}$

Instead of just restating the results of [TV07, §3.1–3.3, 4] we try here to provide some motivation for these choices of examples.

We know that $\mathbb{Z}$ is the initial object in the category CRing of commutative rings; we said in Section 1.2 that we would have to leave CRing in order to find schemes over bases that lie *under* $\operatorname{Spec}\mathbb{Z}$. If we remove all algebraic structure from CRing we end up with the category Set of sets, and we might worry that we have discarded too much structure to be able to define schemes any more. But $(\mathsf{Set}, \times, \{*\})$, where $\times$ is the cartesian product and $\{*\}$ is a singleton, *is* a cosmos,[*] and so we can define schemes over Set. Since Set is the prototype of a concrete category we can't really go down any further without becoming too far removed from our usual concept of algebraic structures. So we look at what (commutative associative and unital) structures we can find in between Set and CRing.

We can endow a set with some commutative associative binary operation, which we might as well call addition, and pick an object to be the additive identity – this results in a *commutative monoid*. Then we could go one of two ways: introducing additive inverses to our commutative monoid, making it an *abelian group*; or introducing some commutative multiplication with identity, making it a *commutative semiring*. Either way, the only thing (of interest to us now) sitting above these two structure comes from combining the two, resulting in a *commutative ring* – this is where we stop, since classical algebraic geometry can deal with things from here.

Note 3.1.0.1
A very important thing to be aware of (that notation doesn't make entirely clear) is that when we speak about A-Mod for $A \in \mathsf{Comm}(\mathcal{C})$, *this category depends on $\mathcal{C}$*. For example, if we take $\mathbb{Z} \in \mathsf{Comm}(\mathsf{Ab}) = \mathsf{CRing}$ then $\mathbb{Z}$-Mod

---

[*] It is arguably the most tautological example of a cosmos.

is the category of abelian groups, but if we consider $\mathbb{Z} \in \mathsf{Comm}(\mathsf{Set}) = \mathsf{CMon}$ then $\mathbb{Z}$-Mod is the category of *monoid actions*, or *$\mathbb{Z}$-actions* (sometimes confusingly written $\mathbb{Z}$-Set). *To avoid confusion in this section, we adopt the following (non-standard) notation:* we write A-$\mathsf{Mod}_{\mathcal{C}}$ to emphasise that $A \in \mathsf{Comm}(\mathcal{C})$. ⌟

Recalling Definitions 1.3.2.6 and 1.3.2.7, the following pattern emerges:

$$\begin{tikzcd}
1\text{-}\mathsf{Mod}_{\mathcal{C}} & (1,\mathcal{C}) & \mathsf{Comm}\,(1\text{-}\mathsf{Mod}_{\mathcal{C}}) \\
\mathsf{Ab} & (\mathbb{Z},\mathsf{Ab}) & \mathsf{CRing} \\
\mathsf{CMon} \arrow[u] & (\mathbb{N},\mathsf{CMon}) \arrow[u, "\text{introduce additive inverses}" description] & \mathsf{CSemiring} \arrow[u] \\
\mathsf{Set} \arrow[u] & (\{*\},\mathsf{Set}) \arrow[u, "\text{introduce addition}" description] & \mathsf{CMon} \arrow[u]
\end{tikzcd}$$

It is true that $\{*\}$-$\mathsf{Mod}_{\mathsf{Set}} = \mathsf{Set}$, but here lies the subtle issue: when we introduce addition we need our singleton to contain an additive identity *and* an additive generator for $\mathbb{N}$, but by definition these things must be distinct. That is, we want $0, 1 \in \{*\}$ with $0 \neq 1$. Clearly, such a singleton doesn't exist, but in [Tit57] Jacques Tits introduced the idea of the *field with one element*: $\mathbb{F}_1$. Although it is not a well-defined mathematical object,* it is interesting to put aside such concerns and try to glean as much information from it as possible, especially when it arises in such a natural way as it does here.†

By definition, 1 is initial in $\mathsf{Comm}(\mathcal{C})$ and so $\operatorname{Spec} 1$ is terminal in $\mathsf{Aff}_{\mathcal{C}}$. Thus we can think of schemes relative to 1-$\mathsf{Mod}_{\mathcal{C}}$ as being schemes *over* $\operatorname{Spec} 1$. It can be shown‡ that $(\mathsf{Ab}, \otimes, \mathbb{Z})$, $(\mathsf{CMon}, \otimes, \mathbb{N})$, and $(\mathsf{Set}, \times, \mathbb{F}_1)$ are all cosmoses. In [TV07, §3.1–3.3], the authors work up to defining $\mathbb{F}_1$-schemes, as well as a base change to $\mathbb{Z}$-schemes:

$$(- \otimes_{\mathbb{F}_1} \mathbb{Z}) \colon \mathbb{F}_1\text{-}\mathsf{Sch} \to \mathbb{Z}\text{-}\mathsf{Sch} \tag{3.1.0.2}$$

which acts on affine schemes by

$$\begin{aligned}
(- \otimes_{\mathbb{F}_1} \mathbb{Z}) \colon \mathsf{Aff}_{\mathbb{F}_1\text{-}\mathsf{Mod}} &\to \mathsf{Aff}_{\mathbb{N}\text{-}\mathsf{Mod}} &&\to \mathsf{Aff}_{\mathbb{Z}\text{-}\mathsf{Mod}} \\
\mathsf{CMon}^{\mathrm{op}} &\to \mathsf{CSemiring}^{\mathrm{op}} &&\to \mathsf{CRing}^{\mathrm{op}} \\
M &\mapsto \mathbb{N}[M] &&\mapsto \mathbb{Z}[M].
\end{aligned} \tag{3.1.0.3}$$

---

[*] At least, not within our current definitions of algebraic objects.

[†] In fact, studying this $\mathbb{F}_1$ is one of the main motivations for developing all of this abstract theory – see Section 4.2.

[‡] These are relatively standard facts – they can be found, for example, on the nLab.

3.2 Diagonalisable group schemes

With Equation (3.1.0.2) we can try to generalise some constructions of classical schemes to $\mathbb{N}$-Sch and $\mathbb{F}_1$-Sch. We now look at one such example, straight from [TV07, §4].

[Mil12, §XIV.3, p.217] provides a general introduction to diagonalisable group schemes. The basic idea is to define a functor

$$\begin{aligned}\mathbb{D}\colon \mathsf{Ab} &\to \mathsf{Fun}(\mathsf{CRing}, \mathsf{Grp})\\ M &\mapsto \mathrm{Hom}_{\mathsf{Grp}}(M, -^{\times})\end{aligned}$$

Then $\mathbb{D}(M)\colon R \mapsto \mathrm{Hom}_{\mathsf{Grp}}(M, R^{\times})$ *can be viewed as an 'affine group'.*

An abelian group is also a commutative monoid, and $\mathsf{Aff}_{\mathbb{F}_1\text{-}\mathsf{Mod}} = \mathsf{CMon}^{\mathrm{op}}$. So for $M \in \mathsf{Ab}$ we can define

$$\mathbb{D}_{\mathbb{F}_1}(M) = \operatorname{Spec} M \in \mathbb{F}_1\text{-}\mathsf{Sch}.$$

Now [Mil12, Proposition 3.3, §XIV.3, p.217] tells us two things:

(i) $\mathbb{D}(M) \cong \operatorname{Spec} \mathbb{Z}[M]$;

(ii) if M is finitely generated then $\mathbb{D}(M)$ is isomorphic to a finite product of copies of $\mathbb{G}_m$ and μ_n (for various $n \in \mathbb{N}$).

If we define the affine $\mathbb{F}_1$-scheme

$$\operatorname{Spec}(\mathbb{F}_{1^n}) = \mathbb{D}_{\mathbb{F}_1}(\mathbb{Z}/n\mathbb{Z})$$

then, using the above results and Equation (3.1.0.3), we see that

(i) $\mathbb{D}_{\mathbb{F}_1}(M) \otimes_{\mathbb{F}_1} \mathbb{Z} \cong \mathbb{D}(M)$;

(ii) $\operatorname{Spec}(\mathbb{F}_{1^n}) \otimes_{\mathbb{F}_1} \mathbb{Z} \cong \mu_n \cong \mathbb{D}(\mathbb{Z}/n\mathbb{Z})$.

So we can generalise $\mathbb{D}(-)$ to $\mathbb{D}_{\mathbb{F}_1}(-)$ to obtain *diagonalisable group schemes over* $\mathbb{F}_1$ that, after a change of base $\mathbb{F}_1 \to \mathbb{Z}$, agree with our existing notions of diagonalisable group schemes. By using a change of base $\mathbb{F}_1 \to \mathbb{N}$ we can also recover a definition for $\mathbb{N}$-schemes.

This idea of $\operatorname{Spec} \mathbb{F}_{1^n}$ plays a major role in Section 4.2.

4 Further applications

We now look at some applications that are not covered in [TV07].

4.1 Day convolution

The theory developed in [TV07] is very powerful, but relies on working with a *cosmos* – we assume, in particular, that our category is bicomplete. However, many categories with which we would like to work are *not* bicomplete. If this is the case then there are two reasonably natural approaches towards a solution:

(i) throw away some morphisms – the fewer the morphisms the 'easier' it is for a morphism to be universal in some way;

(ii) add in some objects – 'bicomplete' this category in some way.

An example of the first approach is when we define the category of Banach spaces to be Ban_1, whose morphisms are weak contractions, instead of the more general Ban, whose morphisms are any bounded linear maps – Ban_1 is bicomplete[*] whereas Ban isn't. This approach forms the beginning of the study of *categorical Banach space theory*: see e.g. [CLM79] for explicit descriptions of certain objects and functors (such as the change of base); [Cru08] for an enriched-category view; and [Cas10] for a general survey.

As for the second approach, it is a standard fact[†] that $\mathsf{PSh}(\mathcal{D})$ is bicomplete for any category $\mathcal{D}$, so here we consider $\mathsf{PSh}(\mathsf{Ban})$. But now the issue is giving $\mathsf{PSh}(\mathsf{Ban})$ a closed symmetric monoidal structure. A method to do this was given in [Day70] in the form of the eponymous *Day convolution*.[‡] We can sometimes use a simpler method though, if we start with a category $\mathcal{D}$ that is *closed* symmetric monoidal (for example, Ban). Then the tensor product commutes with colimits, since it is left adjoint to the internal Hom. Since every presheaf is (canonically, in fact) a colimit of representable presheaves[§] we can define a tensor product on $\mathsf{PSh}(\mathcal{D})$ by simply writing all presheaves as such colimits and then using our tensor product from $\mathcal{D}$. It turns out, however, that this gives exactly the same structure as Day convolution does, but is described in a much simpler fashion.

---

[*] [Yua12]

[†] Since (co)limits are computed pointwise in Set, and Set is bicomplete.

[‡] Most of the results we quote in this section are actually far more powerful than we need; they can be found in all their generality in [Day70] and [MMSS99], for example.

[§] [Mac78, Theorem 1, §III.7]

Definition 4.1.0.1 [Day convolution]
Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category, and $F, G \in \mathsf{PSh}(\mathcal{C})$. Define the *Day convolution product* as the *coend*

$$F \star G = \int^{d \in \mathcal{C}} \int^{c \in \mathcal{C}} F(c) \times G(d) \times \mathrm{Hom}_{\mathcal{C}}(-, c \otimes d)$$

where, for $S\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ with $\mathcal{D}$ cocomplete, we define the coend by

$$\int^{c \in \mathcal{C}} S(c, c) = \mathrm{coeq}\left(\prod_{c \to c'} S(c', c) \rightrightarrows \prod_{c \in \mathcal{C}} S(c, c) \right)$$

with the morphisms on the right coming from $S(c', c) \to S(c, c)$ and $S(c', c) \to S(c', c')$. ⌟

Definition 4.1.0.2
Let $\mathcal{A}$ be a small* symmetric monoidal category and $\mathcal{D}$ be a cosmos. Define the symmetric monoidal category

$$\{\mathcal{A}, \mathcal{D}\} = \big(\mathsf{Fun}(\mathcal{A}, \mathcal{D}), \star, \mathrm{Hom}_{\mathcal{A}}(1_{\mathcal{A}}, -)\big)$$

where $\star$ is the Day convolution. Also define

$$\mathsf{Fun}_{\mathrm{LSM}}(\mathcal{A}, \mathcal{D}) \subseteq \mathsf{Fun}(\mathcal{A}, \mathcal{D})$$

to be the full subcategory whose objects are lax† symmetric monoidal functors $\mathcal{A} \to \mathcal{D}$. ⌟

It turns out that we can characterise the objects of $\mathsf{Comm}\left(\{\mathcal{A}, \mathcal{D}\}\right)$ when $\mathcal{D} = \mathsf{Set}$ in more explicit terms, and then use this to better understand $\mathsf{AffSch}(\{\mathcal{A}, \mathcal{D}\})$.

Lemma 4.1.0.3

(i) $\{\mathcal{A}, \mathcal{D}\}$ *is a cosmos;*

(ii) $\mathsf{Comm}\left(\{\mathcal{A}, \mathcal{D}\}\right) \equiv \mathsf{Fun}_{\mathrm{LSM}}(\mathcal{A}, \mathcal{D})$. ⌟

Proof. For (i), see [Sch10a]; for (ii) see [Day70, Example 3.2.2] or [MMSS99, Proposition 22.1]. □

---

*Following [TV07], we ignore issues of universe.

†That is, a strong symmetric monoidal functor $F\colon \mathcal{A} \to \mathcal{D}$ but where the morphisms $f(X) \odot f(Y) \to f(X \otimes Y)$ and $1_{\mathcal{D}} \to f(1_{\mathcal{A}})$ are not necessarily isomorphisms (but still satisfy the coherence conditions).

Lemma 4.1.0.4
Let $\mathrm{Spec}\, A, \mathrm{Spec}\, B \in \mathsf{Aff}_{\mathcal{C}}$. *Then*

(i) $Y_A \in \mathsf{Aff}_{\{\mathcal{C}^{\mathrm{op}},\mathsf{Set}\}}$*;*

(ii) $Y_{Y_A} \in \mathsf{AffSch}(\{\mathcal{C}^{\mathrm{op}}, \mathsf{Set}\})$*;*

(iii) $\mathrm{Hom}(Y_{Y_A}, Y_{Y_B}) \cong \mathrm{Hom}(A, B)$. ⌟

Proof. Claims (ii) and (iii) follow straight from Yoneda's lemma; claim (i) is the only one that we need to explain. By Lemma 4.1.0.3 we need to show that Y_A is lax symmetric monoidal.* One of the morphisms we need to provide is

$$\mu_{X,Y} \colon \mathrm{Hom}(X, A) \times \mathrm{Hom}(Y, A) \to \mathrm{Hom}(X \otimes Y, A).$$

All we generally have is a morphism

$$(- \otimes -) \colon \mathrm{Hom}(X, A) \times \mathrm{Hom}(Y, A) \to \mathrm{Hom}(X \otimes Y, A \otimes A),$$

but if we have $\mu \colon A \otimes A \to A$ coming from the commutative monoid structure of A then we can compose the two to obtain $\mu_{X,Y} = \mu \circ (- \otimes -)$. Some diagram chasing (omitted here) shows that this $\mu_{X,Y}$ satisfies the required conditions, and that the conditions concerning units and symmetry also hold. □

Unfortunately we don't have the space to discuss this further, but this approach could lead to some very interesting research projects by using what seems like a new combination of techniques.

---

*We can think of $Y_A \in \mathsf{Comm}\left(\{\mathcal{C}^{\mathrm{op}}, \mathsf{Set}\}\right)$ since $\mathsf{Fun}(\mathcal{D}, \mathcal{E})^{\mathrm{op}} \equiv \mathsf{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}})$.

4.2 The Riemann hypothesis

One of the main motivations for trying to study $\mathbb{F}_1$ is the hope that it will lead to a proof of the Riemann hypothesis. An excellent history of $\mathbb{F}_1$, starting from Tit's original idea of 'interpreting S_n as the Chevalley group over the field of characteristic one' in [Tit57] (aiming to explain an analogy from [Ste51]), and ending with the relatively recent paper [CC09] (which deals with this zeta function approach) can be found in the introduction of [PL09].

In [KS96], treating $\mathbb{F}_1$ as any other finite field $\mathbb{F}_p$ for p prime, we think of its extension of degree n, denoted $\mathbb{F}_{1^n}$, coming from adjoining the nth roots of unity. This idea is built upon in [Sou08, §2.4], where it is conjectured that

$$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[T]/(T^n - 1).$$

Comparing this to the results in Section 3.2, we see that our definitions match, as we would hope.

The Riemann zeta function, defined, for $s \in \mathbb{C}$ with $\Re(s) > 1$, as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is a particular example of an *L-function*.[*] In fact, it is really the *generalised Riemann hypothesis* – which states that the non-trivial zeros of global[†] L-functions lie on the line $\Re(z) = 1/2$ in the complex plane – that has profound implications across the whole of mathematics.[‡] The link between a conjecture by Emil Artin on Artin L-functions [Art23] and the Riemann hypothesis was pointed out by André Weil in a letter to Artin written on July 10th 1942,[§] and was later mentioned more concretely in [Wei47, p. 4][¶]:

> *The Riemann hypothesis, after having lost hope of proving it by methods of the theory of functions, appears to us today in a new light, that shows it inseparable from the Artin conjecture on L-functions, these two problems being two sides of the same arithmetic-algebraic question, where the simultaneous study of all the cyclotomic extensions of a given number field will certainly play a vital role.*

---

[*] Certain meromorphic functions on the complex plane (conjecturally) coming from the analytic continuation of an infinite series called an *L-series*, which is associated to some mathematical object (for example, an *Artin L-function* is associated to a linear representation of a Galois group). *Dirichlet L-series* are of the form $\sum_{n=1}^{\infty} a_n/n^s$ where $(a_n)_{n\in\mathbb{N}}$ is a complex sequence and $s \in \mathbb{C}$.

[†] Defined as *Euler products* of *local zeta functions*.

[‡] Although nobody would likely turn their nose up at a proof of the Riemann hypothesis.

[§] Listed as [1942] in the bibliography of [Wei09].

[¶] What follows is a translation by the author – the original is in French.

Prior to this, in 1940,* Weil proved the *Riemann hypothesis for curves over finite fields* by taking a smooth curve C over a finite field $\mathbb{F}_p$ and looking at the diagonal of $C \times_{\mathbb{F}_p} C$. If we could think of $\mathbb{Z}$ as a smooth curve over some finite field (which seems natural since $\mathbb{Z}$ is of dimension one) then Weil's proof could hopefully be extended to a proof of the Riemann hypothesis. But $\mathbb{Z}$ is not an algebra over any field. However, one of the conjectured properties of $\mathbb{F}_1$ is that $\mathbb{Z}$ is an $\mathbb{F}_1$-algebra, and so we would be able to construct $\mathbb{Z} \times_{\mathbb{F}_1} \mathbb{Z}$.

Building on this idea, as well as previous conjectures by Artin [Art24], led Weil in 1949 to the famous *Weil conjectures* [Wei49], the proof of which provided the main impetus for Alexander Grothendieck's two decades of work building upon that of Jean-Pierre Serre. There were four conjectures: the *rationality conjecture*, proved by Bernand Dwork in 1960 [Dwo60]; the *functional equation* and the *Betti number* conjectures, proved by Grothendieck† in 1965 [Gro65b; Gro65a]; and the *Riemann hypothesis analogue*, proved by Deligne in 1974 [Del74]. This last conjecture implies that, if X is a smooth projective variety of dimension n over $\mathbb{F}_q$, then its *local zeta function*

$$\zeta_X(s) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m q^{-sm}}{m}\right)$$

(where N_m is the number of points of X defined over the degree-m extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$) is such that its zeros and poles lie on the lines $\Re(s) = j/2$ for $j = 1, 2, \ldots, 2n$. So, if we could realise $X = \operatorname{Spec}\mathbb{Z}$ as a smooth projective variety of dimension 1 over $\mathbb{F}_1$, then $\zeta_X(s) =$ would be the Riemann zeta function, and a proof of the Riemann hypothesis would almost fall straight out, or so we might hope.

Obviously though, there are some obstructions, otherwise the Riemann hypothesis would have been solved by now. The main problem is that we have many definitions for $\mathbb{F}_1$, and there have been many different ideas for what $\mathbb{F}_1$-schemes should be (see [PL09]), but none of them have all of the properties that we need to prove the Riemann hypothesis. Even if we *did* find some perfect definition, we would need to come up with new cohomology theorems, just as Grothendieck, Deligne, et al. did to solve the Weil conjectures. In fact, it seems as if solving the Riemann hypothesis is more a question of doing *analytic* geometry over $\mathbb{F}_1$ rather than *algebraic* geometry over $\mathbb{F}_1$. So where should we go from here?

---

[*] [Wei40], though he later showed that his result was independent of *this "transcendental" theory* in [Wei41].

[†] Together with Michael Artin, Pierre Deligne, Michel Raynaud, Jean Girard, and many others.

References

[Art23] Emil Artin. “Über eine neue art von L-Reihen”. German. In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 3.1 (1923), pp. 89–108. DOI: `10.1007/BF02954618`.

[Art24] Emil Artin. “Quadratische Körper im Gebiete der höheren Kongruenzen. II.” German. In: *Mathematische Zeitschrift* 19.1 (1924), pp. 207–246. DOI: `10.1007/BF01181075`.

[Bor94] Francis Borceux. *Handbook of categorical algebra: Volume 1, Basic Category Theory*. English. Cambridge University Press, Aug. 1994. ISBN: 978-0-521-44178-0.

[Cas10] Jesus M.F. Castillo. “The Hitchhiker Guide to Categorical Banach Space Theory. Part I”. English. In: *Extracta Mathematicae* 25.2 (2010), pp. 103–149.

[CC09] Alain Connes and Caterina Consani. “Schemes over $\mathbb{F}_1$ and zeta functions”. English. In: *arXiv.org* (July 2009). URL: `http://arxiv.org/pdf/0903.2024v3.pdf`.

[CLM79] Johann Cigler, Viktor Losert, and Peter Michor. *Banach modules and functors on categories of Banach spaces*. English. Marcel Dekker, Inc., 1979. ISBN: 978-0-8247-6867-6.

[Cru08] GSH Cruttwell. “Normed Spaces and the Change of Base for Enriched Categories”. English. PhD thesis. Dec. 2008. URL: `http://www.mta.ca/~gcruttwell/publications/thesis4.pdf`.

[Day70] Brian Day. “Construction of biclosed categories”. English. PhD thesis. 1970. URL: `http://maths.mq.edu.au/~street/DayPhD.pdf`.

[Del74] Pierre Deligne. “La conjecture de Weil. I”. French. In: *Publications mathématiques de l’I.H.É.S.* 43 (1974), pp. 273–307.

[Dwo60] Bernard Dwork. “On the rationality of the zeta function of an algebraic variety”. English. In: *American Journal of Mathematics* 82.3 (1960), pp. 631–684.

[Gro65a] Alexander Grothendieck. “Cohomologie l-adique et Fonctions L”. French. In: *Séminaire de Géométrie Algébrique du Bois Marie*. 1965.

[Gro65b] Alexander Grothendieck. “Formule de Lefschetz et rationalité des fonctions L”. French. In: *Séminaire Bourbaki*. 1965, pp. 41–55. URL: `http://archive.numdam.org/article/SB_1964-1966__9__41_0.pdf`.

[Hak72] Monique Hakim. *Topos annelés et schémas relatifs*. French. Springer-Verlag, 1972. ISBN: 978-3-540-05573-0.

[Joh02] Peter Tennant Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*. English. Clarendon Press, 2002. ISBN: 978-0-19-851598-2.

[KS96] M Kapranov and A Smirnov. *Cohomology determinants and reciprocity laws: number field case*. English. Unpublished, 1996. URL: `http://www.neverendingbooks.org/DATA/KapranovSmirnov.pdf`.

[Lan12] Kai-Wen Lan. *French Glossary*. English, French. Aug. 2012. URL: `http://www.math.umn.edu/~kwlan/documents/french-glossary.pdf`.

[Mac78] Saunders Mac Lane. *Categories for the Working Mathematician*. English. Second edition. Springer-Verlag, 1978. ISBN: 978-1-4419-3123-8. DOI: `10.1234/12345678`.

[Mar09] Florian Marty. "Des Ouverts Zariski et des Morphisms Lisses en Géométrie Relative". French. PhD thesis. July 2009. URL: `http://thesesups.ups-tlse.fr/540/1/Marty_Florian.pdf`.

[Mil12] James S Milne. *Basic Theory of Affine Group Schemes*. English. Mar. 2012. URL: `http://www.jmilne.org/math/CourseNotes/AGS.pdf`.

[MM92] Ieke Moerdijk and Saunders Mac Lane. *Sheaves in geometry and logic: A first introduction to topos theory*. English. Springer-Verlag, 1992. ISBN: 978-0-387-97710-2.

[MMSS99] M A Mandell, J P May, S Schwede, and B Shipley. *Model Categories of Diagram Spectra*. English. Dec. 1999. URL: `http://www.math.uchicago.edu/~may/PAPERS/mmssLMSDec30.pdf`.

[PL09] Javier López Peña and Oliver Lorscheid. "Mapping $\mathbb{F}_1$-land". English. In: *arXiv.org* (Sept. 2009). arXiv: `0903.2024 [math.AG]`. URL: `http://arxiv.org/pdf/0909.0069v1.pdf`.

[PS12] Kate Ponto and Michael Shulman. "Duality and traces for indexed monoidal categories". English. In: *Theory and Applications of Categories* 26.23 (2012), pp. 582–659.

[Sch09] Urs Schreiber. *Grothendieck topology*. English. Jan. 2009. URL: `https://ncatlab.org/nlab/show/Grothendieck+topology`.

[Sch10a] Urs Schreiber. *Day convolution*. English. May 2010. URL: `https://ncatlab.org/nlab/show/Day+convolution`.

[Sch10b] Urs Schreiber. *Grothendieck pretopology*. English. May 2010. URL: `https://ncatlab.org/nlab/show/Grothendieck+pretopology`.

[Shu13] Michael Shulman. “Enriched Indexed Categories”. English. In: *Theory and Applications of Categories* 28.21 (2013), pp. 616–695.

[Ško09] Zoran Škoda. *congruence*. English. Apr. 2009. URL: `https://ncatlab.org/nlab/show/congruence`.

[Ško11] Zoran Škoda. *historical note on Grothendieck topology*. English. May 2011. URL: `https://ncatlab.org/nlab/show/historical+note+on+Grothendieck+topology`.

[Sou08] Christophe Soulé. “Les variétés sur le corps à un élément”. French. In: *arXiv.org* (Feb. 2008). URL: `http://arxiv.org/pdf/math/0304444v1.pdf`.

[Ste51] Robert Steinberg. “A geometric approach to the representaitons of the full linear group over a Galois field”. English. In: *Transactions of the American Mathematical Society* 71 (1951), pp. 274–282.

[Tit57] Jacques Tits. “Sur les analogues algébriques des groupes semi-simples complexes”. French. In: *Colloque d’algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, Centre Belge de Recherches Mathématiques Établissements Ceuterick, Louvain*. Paris, 1957, pp. 261–289.

[TV05] Bertrand Toën and Gabriele Vezzosi. “Homotopical algebraic geometry I”. English. In: *Advances in Mathematics* 193 (June 2005), pp. 257–372. DOI: `10.1016/j.aim.2004.05.004`.

[TV07] Bertrand Toën and Michel Vaquié. “Under Spec Z”. French. In: *arXiv.org* (2005, revised in 2007). arXiv: `math/0509684v4 [math.AG]`. URL: `http://arxiv.org/abs/math/0509684v4`.

[TV08] Bertrand Toën and Gabriele Vezzosi. *Homotopical Algebraic Geometry II: Geometric Stacks and Applications*. English. American Mathematical Society, 2008. ISBN: 978-0-8218-4099-3.

[Wei09] André Weil. *Oeuvres Scientifiques / Collected Papers: Volume 2 (1951-1964)*. Springer Science & Business Media, 2009. ISBN: 978-3-540-87735-6.

[Wei40] André Weil. “Sur les fonctions algébriques à corps de constantes fini”. French. In: *Comptes Rendus Hebdomadaires des Séances de L’Academie des Sciences* 210 (1940), pp. 592–594.

[Wei41] André Weil. “On the Riemann hypothesis in function-fields”. English. In: *Proceedings of the National Academy of Sciences*. July 1941, pp. 345–347.

[Wei47] André Weil. "L'avenir des mathématiques". French. In: *Les grands courants de la pensée mathématique*. Ed. by Francois Le Lionnais. 1947, pp. 307–320.

[Wei49] André Weil. "Numbers of solutions of equations in finite fields". English. In: *Bulletin of the American Mathematical Society* 55.5 (1949), pp. 497–508.

[Yua12] Qiaochu Yuan. *Banach spaces (and Lawvere metrics, and closed categories)*. English. June 2012. URL: `https://qchu.wordpress.com/2012/06/23/banach-spaces-and-lawvere-metrics-and-closed-categories/`.