Errata to Model Categories
by Mark Hovey

Thanks to Georges Maltsiniotis, maltsin@math.jussieu.fr, for catching most of these errors. The one he did not catch, on the non-smallness of topological spaces, was caught by Mike Cole and fixed by Don Stanley.

1. In the intro, I need to thank Georges Maltsiniotis for finding so many errors. Replace last paragraph of p. xii with the following:

   I would like to acknowledge the help of several people in the course of writing this book. I went from knowing very little about model categories to writing this book in the course of about two years. This would not have been possible without the patient help of Phil Hirschhorn, Dan Kan, Charles Rezk, Brooke Shipley, and Jeff Smith, experts in model categories all. I wish to thank John Palmieri for countless conversations about the material in this book. Thanks are also due Gaunce Lewis for help with compactly generated topological spaces, and Mark Johnson for comments on early drafts of this book. I also thank Georges Maltsiniotis for his careful reading of the first edition of this book. And I wish to thank my family, Karen, Grace, and Patrick, for the emotional support so necessary in the frustrating enterprise of writing a book.
2. On p. 2, l. -6, the definition 1.1.1 (2) of functorial factorization is not as strong as I intended, nor as strong as the small object argument implies. Here is the fix:

   Given a category $\mathcal{C}$, we can form the category $\operatorname{Map}\mathcal{C}$ whose objects are morphisms of $\mathcal{C}$ and whose morphisms are commutative squares. Note that there are domain and codomain functors $d, c\colon \operatorname{Map}\mathcal{C} \to \mathcal{C}$.

   Definition 1.1.1. Suppose $\mathcal{C}$ is a category.
   1. A map f in $\mathcal{C}$ is a *retract* of a map $g \in \mathcal{C}$ if f is a retract of g as objects of $\operatorname{Map}\mathcal{C}$. That is, f is a retract of g if and only if there is a commutative diagram of the following form,

      $$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & D & \longrightarrow & B \end{array}$$

      where the horizontal composites are identities.
   2. A *functorial factorization* is an ordered pair (α, β) of functors $\operatorname{Map}\mathcal{C} \to \operatorname{Map}\mathcal{C}$ such that $d \circ \alpha = d$, $c \circ \alpha = d \circ \beta$, $c \circ \beta = c$, and $f = \beta(f) \circ \alpha(f)$ for all $f \in \operatorname{Map}\mathcal{C}$. In particular, a commutative square of the following form

      $$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow[g]{} & D \end{array}$$

induces the following commutative square.

$$\begin{array}{ccccc} A & \xrightarrow{\alpha(f)} & (c\circ\alpha)(f) & \xrightarrow{\beta(f)} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle (c\circ\alpha)(u,v)} & & \downarrow{\scriptstyle v} \\ C & \xrightarrow[\alpha(g)]{} & (c\circ\alpha)(g) & \xrightarrow[\beta(g)]{} & D \end{array}$$

3. On p. 11, l.3, it should be "trivial fibration" instead of trivial cofibration. So the top paragraph on p. 11 should be:

   Now, we have a map $C \xrightarrow{K} X$ induced by H and H', such that $Kj_0 = f$ and $Kj_1 = g$. This is not a left homotopy, but it can be made into one by factoring $j_0 + j_1$ into a cofibration followed by a trivial fibration $C' \to C$. Then C' is a cylinder object for A, and the composite $C' \to C \xrightarrow{K} X$ is a left homotopy between f and h.
4. On p.12, line -7, I don't make it clear that the cylinder object taken for A should be both cofibrant and fibrant. The proof should read:

   Proof. We show that $\mathcal{C}_{cf}/\sim$ has the same universal property that $\operatorname{Ho}\mathcal{C}_{cf}$ enjoys (see Lemma 1.2.2). The functor δ takes homotopy equivalences to isomorphisms, and hence takes weak equivalences to isomorphisms by Proposition 1.2.8. Now suppose $F\colon \mathcal{C}_{cf} \to \mathcal{D}$ is a functor that takes weak equivalences to isomorphisms. Suppose $f, g\colon A \to B$ are homotopic maps with $A, B \in \mathcal{C}_{cf}$. Then there is a left homotopy $H\colon A \times I \to B$ from f to g, where $A \amalg A \xrightarrow{i_0+i_1} A \times I \xrightarrow{s} A$ is the functorial cylinder object on A, by Corollary 1.2.6. Note that $A \times I$ is still cofibrant and fibrant. Then $si_0 = si_1 = 1_A$, and so, since s is a weak equivalence, we have $Fi_0 = Fi_1$. Thus $Ff = (FH)(Fi_0) = (FH)(Fi_1) = Fg$, and so F identifies homotopic maps. Thus there is a unique functor $G\colon \mathcal{C}_{cf}/\sim \to \mathcal{D}$ such that $G\delta = F$. Indeed, G is the identity on objects and takes the equivalence class of a map f to Ff. Lemma 1.2.2 then completes the proof. □

5. On p.21, there is some ambiguity in the phrase "reflects weak equivalences between cofibrant objects". We should replace the paragraph above Corollary 1.3.16 by the following.

   We now give the most useful criterion for checking when a given Quillen adjunction is a Quillen equivalence. Recall that a functor F is said to *reflect* some property of morphisms if, given a morphism f, if Ff has the property so does f. More precisely, we say that F *reflects weak equivalences between cofibrant objects* if, whenever $f\colon A \to B$ is a map between cofibrant objects such that Ff is a weak equivalence, then f is a weak equivalence. We can replace cofibrant by fibrant, of course.
6. Typo on p.21, line 27, where I have $f \to X \to Y$. The paragraph should read:

   Suppose first that F is a Quillen equivalence. We have already seen in Proposition 1.3.13 that the map $X \to URFX$ is a weak equivalence for all cofibrant X and that the map $FQUY \to Y$ is a weak equivalence for all fibrant Y. Now suppose $f\colon X \to Y$ is a map between cofibrant objects such that Ff is a weak equivalence. Then, since F preserves weak equivalences between cofibrant objects, FQf is also a weak equivalence. Thus $(LF)f$ is an isomorphism. Since LF is an equivalence of categories, this implies that f is an isomorphism in the homotopy category, and hence a weak equivalence. Thus F reflects weak equivalences between cofibrant objects. The dual argument implies that U reflects weak equivalences between fibrant objects. Thus (a) implies both (b) and (c).

7. The fourth paragraph on p. 49 is wrong and should be replaced by:

   Unlike the categories of sets, R-modules, and chain complexes of R-modules, not every object in **Top** is small. In fact, the two point space $X = \{0, 1\}$ with the indiscrete topology is not small in **Top**, as was pointed out to the author by Don Stanley. To see this, given a limit ordinal λ and $\alpha < \lambda$, define $X_\alpha = [\alpha, \lambda) \times X$, with topology consisting of the sets V_β for $\beta \in [\alpha, \lambda)$ together with the empty set. Here $V_\beta = [\alpha, \lambda) \times \{0\} \cup [\beta, \lambda) \times \{1\}$. For $\alpha < \alpha'$, there is a continuous map $X_\alpha \to X_{\alpha'}$ that sends (β, x) to (α', x) if $\beta \leq \alpha'$ and sends (β, x) to itself otherwise. The colimit of the X_α consists of the two points $(\lambda, 0)$ and $(\lambda, 1)$ with the indiscrete topology, so is homeomorphic to X, but there is no continuous map $X \to X_\alpha$. Thus X is not small.

8. The definition of weak equivalence in Definition 2.4.3 is not quite right because it fails when X is the empty set. Replace it by:

   Definition 2.4.3. A map $f\colon X \to Y$ in **Top** is a *weak equivalence* if X is nonempty and

   $$\pi_n(f, x)\colon \pi_n(X, x) \to \pi_n(Y, f(x))$$

   is an isomorphism for all $n \geq 0$ and all $x \in X$, or if both X and Y are empty. Define the set of maps I' to consist of the boundary inclusions $S^{n-1} \to D^n$ for all $n \geq 0$, and define the set J to consist of the inclusions $D^n \to D^n \times I$ that take x to $(x, 0)$ for $n \geq 0$. Define the map f to be a *cofibration* if it is in I'-cof, and define f to be a *fibration* if it is in J-inj.

9. Lemma 3.1.4 on p.76 is wrong. A counterexample is provided by $\Delta[2]$ modulo its boundary, which has only two-nondegenerate simplices, a 0-simplex and a 2-simplex. The colimit in question is $\Delta[2]$ modulo its 0-skeleton, which is obviously not K. So we need to consider *regular* simplicial sets K, where K is regular if for every non-degenerate n-simplex the induced map $\Delta[n] \to K$ is injective. This will not affect the only time this lemma is used, in 5.4.1, where we apply it to the boundary of $\Delta[n]$ or a horn.

   So we should replace the last paragraph of p. 75 through the end of Lemma 3.1.4 on p.76 with the following:

   The advantage of this description of the category of simplices is that it is functorial in the simplicial set K. However, if one is working with a *regular* simplicial set K, it is often more helpful to consider the category of non-degenerate simplices $\Delta' K$. Here a simplicial set K is regular if and only if, for every non-degenerate simplex of K, the induced map $\Delta[n] \to K$ is injective. For example, $\Delta[n]$ itself is regular, and every subsimplicial set of a regular simplicial set is regular. Thus both $\partial\Delta[n]$ and $\Lambda^r[n]$ are regular.

   An object of $\Delta' K$ is a map $\Delta[n] \xrightarrow{f} K$ such that fi_n is non-degenerate. A morphism is an *injective* order-preserving map $[k] \to [n]$ making the obvious triangle commutative. We then have the following lemma, whose proof we leave to the reader.

Lemma 3.1.4. *Let K be a regular simplicial set. Then a colimit of the functor $\Delta' K \to$ **SSet** that takes $f\colon \Delta[n] \to K$ to $\Delta[n]$ is K itself.*

10. On p.82, line -1, I made a bad typo. It should read:

    We can then give an alternative characterization of anodyne extensions. Let J' denote the set of maps $I \,\Box\, f$, where f is one of the maps $\Lambda^{\varepsilon}[1] \to \Delta[1]$.

11. p. 102, Definition 4.1.1: This definition of monoidal category is not optimal. We only need the first two coherence diagrams. So we should replace Definition 4.1.1 with the following:

    Definition 4.1.1. A *monoidal* structure on a category $\mathcal{C}$ is a tensor product bifunctor $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$, a unit object $S \in \mathcal{C}$, a natural associativity isomorphism $a\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, a natural left unit isomorphism $\ell\colon S \otimes X \to X$, and a natural right unit isomorphism $r\colon X \otimes S \to X$ such that two coherence diagrams are commutative. These coherence diagrams can be found in any reference on category theory, such as [ML71]. There is a pentagon for fourfold associativity and a triangle equating the two different ways to get from $(X \otimes S) \otimes Y$ to $X \otimes Y$ using the associativity and unit isomorphisms. A *monoidal category* is a category together with a monoidal structure on it.

    Note that one usually assumes as well that r and ℓ agree on the unit S. This follows from the previous diagrams [JS93, Proposition 1.1].

12. There is also a superfluity of coherence diagrams in Definition 4.1.4, the definition of a symmetric monoidal category. All we need is the fact that $T^2 = 1$ and the compatibility of the associativity and commutativity isomorphisms.

    So we should replace Definition 4.1.4 with the following:

    Definition 4.1.4. A *symmetric monoidal structure* on a category $\mathcal{C}$ is a monoidal structure and a natural commutativity isomorphism $T_{X,Y}\colon X \otimes Y \to Y \otimes X$ satisfying two additional coherence diagrams. One of these says that $T_{Y,X} \circ T_{X,Y} = 1_{X \otimes Y}$, and the other is a hexagon equating the two different ways of getting from $(X \otimes Y) \otimes Z$ to $Y \otimes (Z \otimes X)$ using the associativity and commutativity isomorphism. A category with a symmetric monoidal structure is a *symmetric monoidal category.*

    One usually also requires that $r_X = \ell_X \circ T_{X,S}$ and $T_{S,S} = 1_S$, but these both follow from the coherence diagrams above [JS93, Proposition 2.1]. Note that if we drop the condition that $T_{Y,X} \circ T_{X,Y} = 1_{X \otimes Y}$ and add the dual hexagon that equates the two ways of getting from $X \otimes (Y \otimes Z)$ to $(Z \otimes X) \otimes Y$ using a^{-1} and T, we get the notion of a *braided monoidal category* [JS93].

13. In Definition 4.1.5, I want to add a line about braided monoidal functors, as follows:

    Definition 4.1.5. Given symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$, a *symmetric monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a monoidal functor (F, m, α) such that the following diagram is commutative.

$$\begin{array}{ccc} FX \otimes FY & \xrightarrow{m} & F(X \otimes Y) \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle F(T)} \\ FY \otimes FX & \xrightarrow{m} & F(Y \otimes X) \end{array}$$

    The definition of a braided monoidal functor between braided monoidal categories is exactly the same.

14. There is also a superfluity of coherence diagrams in Definition 4.1.6, the definition of a $\mathcal{C}$-module. Definition 4.1.6 should be replaced by the following.

    Definition 4.1.6. Suppose $\mathcal{C}$ is a monoidal category. A *right* $\mathcal{C}$*-module structure* on a category $\mathcal{D}$ is a triple $(\otimes, a, r)$, where $\otimes\colon \mathcal{D}\times\mathcal{C}\to\mathcal{D}$ is a functor, a is a natural isomorphism $(X\otimes K)\otimes L\to X\otimes(K\otimes L)$, and r is a natural isomorphism $X\otimes S\to X$ making two coherence diagrams commutative. One of these is the four-fold associativity pentagon, and the other is the unit triangle equating the two ways to get from $X\otimes(S\otimes K)$ to $X\otimes K$. A *right* $\mathcal{C}$*-module* is a category equipped with a right $\mathcal{C}$-module structure.

    It follows from these coherence diagrams that the triangle relating the two ways to get from $X\otimes(K\otimes S)$ to $X\otimes K$ using the unit isomorphisms of $\mathcal{C}$ and $\mathcal{D}$ is also commutative. The proof of this is the same as the proof of the corresponding fact in a monoidal category [JS93, Proposition 1.1].
15. On the top of p.105, the induction functor from S-modules to T-modules can not be a $\mathcal{C}$-algebra functor unless S and T are commutative. So replace that paragraph with the following:

    For example, let $\mathcal{C}$ be the category of left R-modules for a commutative ring R, and suppose we have a map of commutative R-algebras $S\to T$. Then we get a $\mathcal{C}$-algebra functor F from the category of left S-modules to the category of left T-modules that takes the S-module M to $T\otimes_S M$. Note that in this case, F does not preserve the map i on the nose, since $Fi(M)=T\otimes_S(S\otimes_R M)$, which is canonically isomorphic, but not equal, to $T\otimes_R M$.
16. The definition 4.1.10 of a central $\mathcal{C}$-algebra on p. 105 obviously makes no sense since t is not composable with itself. Here is the correct definition:

    It is more interesting to consider central $\mathcal{C}$-algebras, but to do so we must first recall the definition of the *center* of a monoidal category $\mathcal{D}$ from [JS93, Example 2.3]. The objects of the center of $\mathcal{D}$ are pairs (X,u), where X is an object of $\mathcal{D}$ and u is a natural isomorphism $X\otimes - \to -\otimes X$, such that $r_X=\ell_X\circ u_S$ and the hexagon of Definition 4.1.4 commutes with u in place of T. A morphism from (X,u) to (Y,v) is a morphism $f\colon X\to Y$ such that $v_Z\circ(f\otimes 1)=(1\otimes f)\circ u_Z$ for all Z. Then the center of $\mathcal{D}$ is a braided monoidal category, where we define $(X\otimes Y,w)$ to be the unique map making the dual hexagon commutative, where the dual hexagon equates the two different ways of getting from $X\otimes(Y\otimes Z)$ to $(Z\otimes X)\otimes Y$. The braiding is defined by u_Y; the opposite braiding is defined by v_X^{-1}. Note that there is an obvious monoidal forgetful functor j from the center of $\mathcal{D}$ to $\mathcal{D}$.

    Definition 4.1.10. Suppose $\mathcal{C}$ is a symmetric monoidal category. Then a *central* $\mathcal{C}$*-algebra structure* on a monoidal category $\mathcal{D}$ is a braided monoidal functor i from $\mathcal{C}$ to the center of $\mathcal{D}$. A *central* $\mathcal{C}$*-algebra* is a category equipped with a central $\mathcal{C}$-algebra structure.

    The forgetful functor j makes a central $\mathcal{C}$-algebra into a $\mathcal{C}$-algebra in a canonical way. A central $\mathcal{C}$-algebra structure on $\mathcal{D}$ is equivalent to a $\mathcal{C}$-algebra structure on $\mathcal{D}$ together with a natural isomorphism $t_{X,Y}\colon iX\otimes Y\to Y\otimes iX$ satisfying various coherence diagrams.
17. p. 106, add comment after Definition 4.1.12:

    Note that the choice of subscript for Hom_r is obviously arbitrary, and some authors may make the opposite choice.

18. On p.109, Lemma 4.2.7, I messed up the adjointness, so actually conditions (b) and (b') should be switched.

    Lemma 4.2.7. *Suppose* $\mathcal{C}$ *is a closed monoidal category that is also a model category. Then the following are equivalent.*
    (a) *The map* $QS \otimes X \to X$ *is a weak equivalence for all cofibrant* X*.*
    (b) *The map* $X \to \mathrm{Hom}_\ell(QS, X)$ *is a weak equivalence for all fibrant* X*.*
    Similarly, the following are equivalent.
    (a') *The map* $X \otimes QS \to X$ *is a weak equivalence for all cofibrant* X*.*
    (b') *The map* $X \to \mathrm{Hom}_r(QS, X)$ *is a weak equivalence for all fibrant* X*.*

19. p. 124, Definition 5.2.1: The degree function is not a functor on the Reedy category.

    Definition 5.2.1. A *Reedy category* is a triple $(\mathcal{B}, \mathcal{B}_+, \mathcal{B}_-)$ consisting of a small category $\mathcal{B}$ and two subcategories $\mathcal{B}_+$, and $\mathcal{B}_-$, such that there exists a function $d\colon \mathrm{ob}\mathcal{B} \to \lambda$, called a *degree function*, for some ordinal λ, such that every nonidentity map in $\mathcal{B}_+$ raises the degree, every nonidentity map in $\mathcal{B}_-$ lowers the degree, and every map $f \in \mathcal{B}$ can be factored uniquely as $f = gh$, where $h \in \mathcal{B}_-$ and $g \in \mathcal{B}_+$. In particular, $\mathcal{B}_+$ is a direct category and $\mathcal{B}_-$ is an inverse category. By abuse of notation, we often say $\mathcal{B}$ is a Reedy category, leaving the subcategories implicit.

20. The bibliography needs one additional item:

References

[JS93] André Joyal and Ross Street, *Braided tensor categories*, Adv. Math. **102** (1993), no. 1, 20–78.