STABILITY STRUCTURES ON LIE ALGEBRAS, AFTER KONTSEVICH AND SOIBELMAN

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These are notes for a talk on parts of Section 2 in [1]. We follow [1] quite closely but add some arguments, mostly completely elementary, which may be helpful. Many thanks to Jacopo Stoppa for his help in preparing this talk.

1. Nilpotent Lie algebras

We will consider the following situation: $\mathfrak{g} \subset \mathfrak{gl}(V)$ a sub Lie algebra such that all $X \in \mathfrak{g}$ are nilpotent endomorphisms of V. The ground field is an arbitrary field of characteristic zero, but the vector space V is allowed to be of infinite dimension.

Recall that for $A \in \mathfrak{gl}(V)$ with $A^k = 0$, one has $\mathrm{ad}(A)^{2k-1} = 0$ in $\mathfrak{gl}(\mathfrak{gl}(V))$. So for all $X \in \mathfrak{g} \subset \mathfrak{gl}(V)$ not only X as an endomorphism of V is nilpotent, but also $\mathrm{ad}(X)$ as an endomorphism of $\mathfrak{g}$.

Remark 1.1. i) If $\mathfrak{g}$ is a (finite dimensional) nilpotent Lie algebra, then the image of the adjoint representation $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, i.e. the adjoint Lie algebra $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ satisfies our condition. Passing from $\mathfrak{g}$ to $\mathrm{ad}(\mathfrak{g})$ we only loose the center $\mathfrak{c}(\mathfrak{g}) \subset \mathfrak{g}$ of $\mathfrak{g}$ which is an abelian Lie algebra.

ii) If $\mathfrak{g} \subset \mathfrak{gl}(V)$ contains only nilpotent endomorphisms and $\mathfrak{g}$ is finite dimensional, then $\mathfrak{g}$ is a nilpotent Lie algebra by the theorem of Engel.

Proposition 1.2. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a a sub Lie algebra containing only nilpotent endomorphisms of V. Then $G := \{\exp(X) \mid X \in \mathfrak{g}\} \subset \mathrm{Gl}(V)$ is a subgroup. Moreover,*

i) $\exp : \mathfrak{g} \xrightarrow{\sim} G$ *and*

ii) $\exp(X) \cdot \exp(Y) = \exp(X * Y)$ *with $X * Y$ given by the Campbell–Hausdorff formula.*

Proof. First note that $\exp(X) = \sum_0^\infty \frac{X^n}{n!}$ is actually a finite sum for any nilpotent $X \in \mathfrak{gl}(V)$, so in particular for any $X \in \mathfrak{g}$. Since $\exp(X) \cdot \exp(-X) = 1$ (easy), this finite sum defines an element in $\mathrm{Gl}(V)$.

The inverse map is given by $\log(A) = \sum_0^\infty \frac{(-1)^k}{k+1}(A-1)^{k+1}$, which is also finite for every $A = \exp(X)$ with $X \in \mathfrak{gl}(V)$ nilpotent.

The fact that G is a group follows from the Campbell–Hausdorff formula in ii).

For later use we recall: $X * Y = X + Y + (1/2)[X, Y] + (1/12)[X, [X, Y]] + (1/12)[Y, [Y, X]] + (1/24)[Y, [X, [Y, X]]] + \ldots$ or in closed form

$$X * Y = X + Y + \sum_{k=1}^{\infty} \frac{(-1)^k}{k+1} \sum \frac{1}{\ell_1 + \ldots \ell_k + 1} \left(\frac{\mathrm{ad}(X)^{\ell_1}}{\ell_1!} \frac{\mathrm{ad}(Y)^{m_1}}{m_1!} \cdots \frac{\mathrm{ad}(X)^{\ell_k}}{\ell_k!} \frac{\mathrm{ad}(X)^{m_k}}{m_k!} \right)(X),$$

where the second sum runs over all $\ell_1, m_1, \ldots, \ell_k, m_k \geq 0$ with $\ell_i + m_i > 0$. One can show that for X and Y nilpotent, the sum is again finite. This shows that $\exp(X) \cdot \exp(Y)$ is again in the image of $\exp : \mathfrak{g} \longrightarrow \mathrm{Gl}(V)$. $\square$

The group G is also called the adjoint group $\mathrm{Ad}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$.

The proposition holds for any finite dimensional nilpotent Lie algebra. This can then be generalized to the infinite dimensional case if $\mathfrak{g}$ comes with the structure of a pro Lie algebra. Since this is, as far as I can see, not assumed in [1], we make the additional assumption that $\mathfrak{g} \subset \mathfrak{gl}(V)$ is as above. By the proposition, the group G does not depend on the choice of V.

In the next step we will deal with Lie algebras which are not nilpotent but can be written as the inverse limit of nilpotent Lie algebras. So we will consider (infinite dimensional) Lie algebras of the form $\lim \mathfrak{g}_i$ with $\mathfrak{g}_i$ (possibly infinite dimensional) Lie algebras satisfying the assumption of the proposition. If G_i denote the adjoint groups of the $\mathfrak{g}_i$, then we let G be the group $\lim G_i$, which comes with a bijective exponential $\exp : \lim \mathfrak{g}_i \xrightarrow{\sim} \lim G_i = G$ and the limit of the Campbell–Hausdorff formula.

Let us begin with an arbitrary graded Lie algebra $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$, where $\Gamma \simeq \mathbb{Z}^d$. So in particular for $a(\gamma_i) \in \mathfrak{g}_{\gamma_i}$, one has $[a(\gamma_1), a(\gamma_2)] \in \mathfrak{g}_{\gamma_1+\gamma_2}$.

Example 1.3. Let us consider the simplest case $d = 1$, i.e. $\mathfrak{g} = \bigoplus_{-\infty}^{\infty} \mathfrak{g}_i$. Note that $\mathfrak{g}_{>0} := \bigoplus_1^\infty \mathfrak{g}_i \subset \mathfrak{g}$ is a sub Lie algebra (to which we usually restrict) and $J_d := \bigoplus_{i>d} \mathfrak{g}_i \subset \mathfrak{g}_{>0}$ for $d > 0$ is an ideal. Moreover, $\mathfrak{g}_{\leq d} := \mathfrak{g}_{>0}/J_d$ is a Lie algebra with $\mathrm{ad}(X)$ nilpotent for all $X \in \mathfrak{g}_{\leq d}$. Tacitly we will assume that $\mathfrak{g}_{\leq d}$ satisfies the assumption of the proposition or that it is finite dimensional. (For the latter case see [2].) In any case, we let $G_{\leq d}$ be the adjoint group of $\mathfrak{g}_{\leq d}$. The natural Lie algebra homomorphisms $\mathfrak{g}_{\leq d_2} \twoheadrightarrow \mathfrak{g}_{\leq d_1}$ for $d_2 \geq d_1$ give rise to the limit $\lim_d \mathfrak{g}_{\leq d}$ and the group $G := \lim G_{\leq d}$.

A similar construction can be performed for $\mathfrak{g}_{<0}$, but note that $\mathfrak{g}_0$ may destroy the nilpotence of the truncations.

If $\Gamma \simeq \mathbb{Z}^d$ with $d > 1$, then a similar construction can be performed once certain cones and their truncations have been fixed. In the situation we are interested in, they depend on additional data that will be discussed next.

2. Strict Cones and triangles

We will consider cones in $\Gamma_{\mathbb{R}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^d$ and in $\mathbb{R}^2$ which will be related via certain additive maps $Z : \Gamma \longrightarrow \mathbb{C} = \mathbb{R}^2$ (or rather their $\mathbb{R}$-linear extensions $Z : \Gamma_{\mathbb{R}} \longrightarrow \mathbb{R}^2$).

Cones in $\mathbb{R}^n$ need not be open or closed, they can be degenerate (i.e. of strictly smaller dimension), but we will always assume that the origin is not contained in the cone. Cones in $\Gamma_{\mathbb{R}}$ are usually denoted by $C \subset \Gamma_{\mathbb{R}}$ where as in $\mathbb{R}^2$ they are called $V \subset \mathbb{R}^2$.

In [1] a cone is called *strict* if no line (without the origin) is contained in it. It seems that implicitly a slightly stronger notion is used for cones $V \subset \mathbb{R}^2$ by requiring that the angle is strictly smaller than π. So the upper half plane would be strict in the general sense, but as a cone in the target of an additive map $Z : \Gamma \longrightarrow \mathbb{R}^2$ we would want to exclude it. (There may be a way around this problem, but we leave this for later.) In [1] cones that are strict are called strict sectors.

In particular a ray in $\mathbb{R}^2$ is a strict cone and those will be denoted $\ell = \mathbb{R}_{>0}z$ with $z \in \mathbb{C}^*$.

If $V \subset \mathbb{R}^2$ is a strict sector, then by $\Delta \subset V$ one denotes a bounded region in V cut by an affine line. For obvious reasons, they are called *triangle* and V can be seen as a direct limit of its triangles. (Here we use the stronger version of 'strict'.)

In order to associate to a strict cone $V \subset \mathbb{R}^2$ a strict cone in $\Gamma_\mathbb{R}$ we will need an additive map $Z : \Gamma \longrightarrow \mathbb{R}^2$ and a quadratic form Q on $\Gamma_\mathbb{R}$. (Often, only the $\mathbb{R}$-linear extension $Z : \Gamma_\mathbb{R} \longrightarrow \mathbb{R}^2$ is really used.)

Definition 2.1. For a strict cone $V \subset \mathbb{R}^2$ let

$$C(V, Z, Q) \subset \Gamma_\mathbb{R}$$

be the cone generated by the set $\{0 \neq x \in \Gamma_\mathbb{R} \mid Z(x) \in V, Q(x) \geq 0\}$.

Clearly, $Z(C(V, Z, Q)) \subset V$ and the next observation is

Lemma 2.2. *$C(V, Z, Q)$ is strict.*

Proof. Let us first consider the special case $V = \ell$.

If $Z(\Gamma_\mathbb{R}) \cap \ell = \emptyset$, then $C(V, Z, Q) = \emptyset$. In the other case, pick $x \in \Gamma_\mathbb{R}$ with $Z(x) \in \ell$ and consider $x\mathbb{R} \oplus \ker(Z) \subset \Gamma_\mathbb{R}$. Then $Z^{-1}(\ell) = \mathbb{R}_{>0}x \oplus \ker(Z)$. Any element $y \in C(\ell, Z, Q)$ can be written as $y = \sum(\lambda_i x + y_i)$ with $\lambda_i > 0$ and $y_i \in \ker(Z)$. Hence $-y$ cannot be of the same form.

The same proof works when V is not just a ray but $Z(\Gamma_\mathbb{R})$ is one-dimensional.

If $Z(\Gamma_\mathbb{R}) = \mathbb{R}^2$, then we pick $x_1, x_2 \in \Gamma_\mathbb{R}$ such that $\overline{V}$ is the sector between $\mathbb{R}_{>0}Z(x_1)$ and $\mathbb{R}_{>0}Z(x_2)$. Then elements $y \in C(V, Z, Q)$ are of the form $\sum(\lambda_i x_1 + \mu_i x_2 + y_i)$ with $\lambda_i, \mu_i \geq 0$ and $\lambda_i + \mu_i > 0$ and $y_i \in \ker(Z)$. Which excludes the possibility that also $-y \in C(V, Z, Q)$. $\square$

Note that so far we have not made any assumption on the restriction of Q to $\ker(Z)$.

Remark 2.3. Warning: $C(V, Z, Q)$ may contain elements $x \in \Gamma_\mathbb{R}$ with $Q(x) < 0$ and this can even happen for a ray $V = \ell$. As an example consider $\Gamma = \mathbb{Z}^2$ and Z such that $Z(e_2) = Z(-2e_2 + e_1) \in \mathbb{R}^2$. If $Q(x) = x_2^2 - x_1^2$ and $V = \mathbb{R}_{>0}Z(e_2)$, then $e_2, -2e_2 + e_1 \in C(V, Z, Q)$ and hence also $e_1 \in C(V, Z, Q)$ which has negative square. However in this example Q is not negative definite on the $\ker(Z)$, which is a requirement later.

But if Q is assumed to be negative definite on $\ker(Z)$, then any $x \in C(V, Z, Q)$ satisfies $Q(x) \geq 0$. Let us prove this for $V = \ell$. Then either $Z^{-1}(\ell\mathbb{R}) = \ker(Z)$, and then $C(\ell, Z, Q) = \emptyset$, or there exists a Q-orthogonal decomposition $Z^{-1}(\ell\mathbb{R}) = \mathbb{R}x_0 \oplus \ker(Z)$ with $Z(x_0) \in \ell$. In the latter case, $C(\ell, Z, Q)$ is either empty (if $Q(x_0) < 0$) or spanned by elements $x = \lambda x_0 \oplus y$ of positive square with $y \in \ker(Z)$ and $\lambda > 0$. If $Q(x_0) = 0$, then $C(\ell, Z, Q) = \mathbb{R}_{>0}x_0$. Otherwise, Q on $Z^{-1}(\ell\mathbb{R})$ is a quadratic form of signature $(1, n)$ and therefore the set of elements of positive square has two connected components. The condition $\lambda > 0$ ensures that $C(\ell, Z, Q)$ is spanned by elements $x = \lambda x_0 \oplus y$ all contained in one of the two connected components and hence $C(\ell, Z, Q)$ is contained in this component too. In particular $Q(x) > 0$ for all $x \in C(\ell, Z, Q)$.

Let us now consider a strict sector $V \subset \mathbb{R}^2$ and a triangle $\Delta \subset V$. For a given Z and Q consider

$$\mathfrak{g}_{V,Z,Q} := \prod_{\gamma \in \Gamma \cap C(V,Z,Q)} \mathfrak{g}_\gamma \ \text{ and } \ J_\Delta := \prod_{\gamma \in \Gamma \cap C(V,Z,Q), Z(\gamma) \notin \Delta} \mathfrak{g}_\gamma \subset \mathfrak{g}_{V,Z,Q}.$$

Then let

$$\mathfrak{g}_\Delta := \mathfrak{g}_{V,Z,Q}/J_\Delta.$$

Lemma 2.4. *$\mathfrak{g}_{V,Z,Q}$ is a Lie algebra and $J_\Delta \subset \mathfrak{g}_{V,Z,Q}$ is an ideal. The induced Lie algebra structure on $\mathfrak{g}_\Delta$ is nilpotent, i.e. for any $X \in \mathfrak{g}_\Delta$ the endomorphism* $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g}_\Delta)$ *is nilpotent.*

Proof. The first two assertions follow immediately from the fact that $C(V,Z,Q)$ is strict. (Note that we really use the boundedness of Δ.)

For $\mathfrak{g}_\Delta$ being nilpotent it is crucial that $\mathfrak{g}_0$ is not contained in $\mathfrak{g}_{V,Z,Q}$. The assertion follows from the observation that for $\gamma \in C(V,Z,Q)$, $X \in \mathfrak{g}_\gamma$, and $k \gg 0$ the endomorphism $\mathrm{ad}(X)^k$ sends $\mathfrak{g}_{V,Z,Q}$ to J_Δ. $\square$

As in the case $\Gamma \simeq \mathbb{Z}$ we will tacitly assume that $\mathfrak{g}_\Delta$ (which is the analogue of $\mathfrak{g}_{\leq d}$) is finite dimensional or that it satisfies the assumption of Proposition 1.2.

If $\Delta_1 \subset \Delta_2$, then $J_{\Delta_2} \subset J_{\Delta_1}$ and the induced map $\mathfrak{g}_{\Delta_2} \twoheadrightarrow \mathfrak{g}_{\Delta_1}$ is a Lie algebra homomorphism. Clearly $\lim_\Delta \mathfrak{g}_\Delta \simeq \mathfrak{g}_{V,Z,Q}$. But viewing $\mathfrak{g}_{V,Z,Q}$ as the limit of nilpotent Lie algebras allows one to associate the group $G_{V,Z,Q} := \lim G_\Delta$, where $G_\Delta = \exp(\mathfrak{g}_\Delta)$. Recall that also for the limit the exponential map $\exp : \mathfrak{g}_{V,Z,Q} \xrightarrow{\sim} G_{V,Z,Q}$ is bijective with the inverse given by the logarithm.

3. Stability structures

Let $\mathfrak{g} = \bigoplus_\gamma \mathfrak{g}_\gamma$ be a graded Lie algebra with $\Gamma \simeq \mathbb{Z}^d$. We shall write $\Gamma^*_{\mathbb{C}} := \mathrm{Hom}(\Gamma, \mathbb{C})$ for the set of additive maps $Z : \Gamma \longrightarrow \mathbb{C}$. We fix a norm $\| \ \|$ on $\Gamma_{\mathbb{R}}$, but the notion of a stability structure will turn out to be independent of it.

Definition 3.1. (Stability structure, algebra version) $\mathrm{Stab}(\mathfrak{g})$ is the set of pairs (Z, a) with

$$Z \in \Gamma^*_{\mathbb{C}} \ \text{ and } \ a = (a(\gamma)) \in \prod_{\gamma \neq 0} \mathfrak{g}_\gamma$$

satisfying the *support property* (SP):
There exists a constant $C > 0$ such that for any $a(\gamma) \neq 0$, one has

$$\|\gamma\| \leq C \cdot |Z(\gamma)|.$$

Remark 3.2. Here are a few consequences of the support property.
i) If $Z(\gamma) = 0$ for $\gamma \neq 0$, then $a(\gamma) = 0$.
ii) For a strict sector $V \subset \mathbb{R}^2 = \mathbb{C}$ and a triangle $\Delta \subset V$ the set

$$\{Z(\gamma) \mid \gamma \in \Gamma, Z(\gamma) \in \Delta, a(\gamma) \neq 0\}$$

is finite. Indeed, since Δ is bounded, it suffices to show that this set is discrete. Suppose there is an accumulation point $Z(\gamma_i) \longrightarrow z$. Then for this sequence the norm $|Z(\gamma_i)|$ is bounded from above, say by D, and the support property shows that $\|\gamma_i\| \leq C \cdot |Z(\gamma_i)| \leq C \cdot D$ and hence the set $\{\gamma_i\}$ is bounded. But a bounded set in Γ must be finite.

Observe that it is easy to construct examples with $Z(\Gamma)\cap\Delta$ not finite. The inequality in the support property, implied by the additional condition $a(\gamma)\neq 0$, is really needed.

iii) For general $Z\in\Gamma_{\mathbb{C}}^*$ any strict sector V, which is not just a ray, will contain infinitely many rays of the form $\mathbb{R}_{>0}Z(\gamma)$. Even only allowing those with $a(\gamma)\neq 0$ for a given $(Z,a)\in\mathrm{Stab}(\mathfrak{g})$ will not change this. However, due to ii) there are only finitely many rays in V spanned by $Z(\gamma)$ which are contained in a fixed triangle $\Delta\subset V$ and such that $a(\gamma)\neq 0$.

Lemma 3.3. *The support property for (Z,a) is equivalent to the following condition: There exists a quadratic form Q on $\Gamma_{\mathbb{R}}$ which is negative definite on $\ker(Z)$ and for which $a(\gamma)\neq 0$ implies $Q(\gamma)\geq 0$.*

Proof. As a first check, one observes that this condition also implies i) in the above remark.

If (Z,a) satisfies the support property, then define Q by $Q(\gamma):=-\|\gamma\|^2+C^2\cdot|Z(\gamma)|^2$. Clearly with this definition Q is negative definite on $\ker(Z)$ and for $a(\gamma)\neq 0$ the support property shows $\|\gamma\|^2\leq C^2\cdot|Z(\gamma)|^2$, i.e. $Q(\gamma)\geq 0$.

For the converse suppose Q is given. Then write $\Gamma_{\mathbb{R}}=\ker(Z)\oplus W$ with $W:=\ker(Z)^{\perp}$ (use that Q is non-degenerate on $\ker(Z)$). Next choose a norm $\|\ \|$ on $\Gamma_{\mathbb{R}}$ such that $Q(\gamma)=-\|\gamma\|^2$ for all $\gamma\in\ker(Z)$ (using that Q is negative definite on $\ker(Z)$) and such that the direct sum decomposition is also orthogonal with respect to $\|\ \|$. Since Z is injective on W, there exists a constant C^2 such that $Q(\gamma)+\|\gamma\|^2\leq C^2\cdot|Z(\gamma)|^2$ for all $\gamma\in W$. If $\gamma=\gamma_1\oplus\gamma_2$ with $Z(\gamma_1)=0$ and $\gamma_2\in W$, then $Q(\gamma)=-\|\gamma_1\|^2+Q(\gamma_2)\leq-\|\gamma_1\|^2+C^2\cdot|Z(\gamma_2)|^2-\|\gamma_2\|^2$. Thus, $a(\gamma)\neq 0$, which by assumption on Q implies $Q(\gamma)\geq 0$, also yields $\|\gamma\|\leq C\cdot|Z(\gamma)|$. Therefore, the support property holds. $\square$

Frequently, both descriptions of the SP will be used simultaneously. E.g. the inequality in the SP then holds for any element γ with $Q(\gamma)\geq 0$.

The subset of all $(Z,a)\in\mathrm{Stab}(\mathfrak{g})$ for which a fixed Q satisfies the condition in the lemma is denoted $\mathrm{Stab}_Q(\mathfrak{g})$. So,

$$\mathrm{Stab}(\mathfrak{g})=\bigcup\mathrm{Stab}_Q(\mathfrak{g}).$$

Remark 3.4. Let $(Z,a)\in\mathrm{Stab}_Q(\mathfrak{g})$. Consider another quadratic form Q' which is negative definite on $\ker(Z)$ and satisfies $Q\leq Q'$. Then $(Z,a)\in\mathrm{Stab}_{Q'}(\mathfrak{g})$.

Remark 3.5. Let us rephrase the finiteness in Remark 3.2, ii) in terms of Q. For $(Z,a)\in\mathrm{Stab}_Q(\mathfrak{g})$ the set $\{Z(\gamma)\in\Delta\mid\gamma\in\Gamma,Q(\gamma)\geq 0\}$ is finite. (Obvious, since the inequality $Q(\gamma)\geq 0$ implies the inequality in the support property for well chosen C and $\|\ \|$.)

Definition 3.6. (Stability structure, group version) For a given quadratic form Q on $\Gamma_{\mathbb{R}}$ the set $\widehat{\mathrm{Stab}}_Q(\mathfrak{g})$ consists of all pairs (Z,A) with

$$Z\in\Gamma_{\mathbb{C}}^*\quad\text{and}\quad A=(A_V\in G_{V,Z,Q}))$$

with V running through all strict sectors in $\mathbb{R}^2$ and such that Q is negative definite on $\ker(Z)$ and for a disjoint decomposition of a strict sector $V=V_1\coprod V_2$ into strict sectors in clockwise order one has the *factorization property* (FP):

$$A_V=A_{V_1}\cdot A_{V_2}.$$

The last equation takes place in $G_{V,Z,Q}$, where we consider the natural inclusions $G_{V_i,Z,Q} \subset G_{V,Z,Q}$. If $\Delta \subset V$ is a triangle, then we will denote the projection of A_V to G_Δ by A_Δ.

Part of the data of $(Z, A) \in \widehat{\mathrm{Stab}}_Q(\mathfrak{g})$ are the group elements $A_\ell \in G_{\ell,Z,Q}$ for any ray $\ell \subset \mathbb{R}^2$. As will been shown next, they determine all the others.

Lemma 3.7. *Suppose $\Delta \subset V$ is such that there is exactly one ray $\ell = \mathbb{R}_{>0}Z(\gamma) \subset V$ with $a(\gamma) \neq 0$ (or $Q(\gamma) \geq 0$) and $Z(\gamma) \in \Delta$. Then the images of A_V and A_ℓ in $G_{\ell\cap\Delta} \subset G_\Delta$ coincide.*

Proof. (Note that we do not exclude that some small multiples of γ may also be mapped to Δ.) One again applies the factorization property. Decompose $V = V_1 \bigsqcup \ell \bigsqcup V_2$ clockwise, so V_1 and V_2 are the parts of V left respectively right from ℓ. (Draw a picture!) Then $A_V = A_{V_1} \cdot A_\ell \cdot A_{V_2}$. For $\Delta_i := \Delta \cap V_i$ consider $A_{\Delta_i} \in G_{\Delta_i}$. Now $G_{\Delta_i} = \exp(\mathfrak{g}_{\Delta_i})$ with $\mathfrak{g}_{\Delta_i} = \prod \mathfrak{g}_\delta$ with the product over all $\delta \in C(V_i, Z, Q) \cap Z^{-1}(\Delta_i) \cap \Gamma$, which is empty. Hence $A_{\Delta_i} = 1$ and hence $A_\Delta = A_{\ell\cap\Delta}$, which proves the claim. □

Let now $V \subset \mathbb{R}^2$ be an arbitrary strict sector and let $\Delta \subset V$ be a triangle. Suppose $\ell_i = \mathbb{R}_{>0}Z(\gamma_i)$, $i = 1, \ldots, k$ are the only rays in V with $Z(\gamma_i) \in \Delta$ and with $Q(\gamma_i) \geq 0$. See Remark 3.5 for the finiteness. Then decompose $V = \bigsqcup_1^{k+1} V_i$ clockwise such that $\ell_i \subset V_i$ (e.g. requiring that the V_i, $i = 1, \ldots, k$, are closed on the right with boundary ray ℓ_i).

Then the factorization property yields $A_V = A_{V_1} \cdot \ldots \cdot A_{V_k} \cdot A_{V_{k+1}}$ in $G_{V,Z,Q}$. This implies $A_\Delta = A_{\Delta_1} \cdot \ldots \cdot A_{\Delta_{k+1}}$ in G_Δ for any triangle $\Delta \subset V$, where $\Delta_i := \Delta \cap V_i$. Now apply the lemma to see that the Δ-truncations of A_V and the product $A_{\ell_1} \cdot \ldots \cdot A_{\ell_k}$ coincide.

Remark 3.8. The factorization property comes also in an infinite version. If $V = \bigsqcup_{i\in I} V_i$ is an arbitrary union indexed by a totally ordered set I such that $V_i < V_j$ (in clockwise order) if $i < j$, then

$$A_V = \prod\nolimits^{\rightarrow} A_{V_i}. \tag{3.1}$$

Let us explain what this means. For any triangle $\Delta \subset V$ let $\Delta_i := \Delta \cap V_i$ be the induced triangle in V_i. Due to Remark 3.5 only finitely many of the Δ_i will contain an element of the form $Z(\gamma)$, $\gamma \in \Gamma$ with $Q(\gamma) \geq 0$. Thus, in the product $\prod^{\rightarrow} A_{\Delta_i}$, which is taken clockwise, only finitely many terms contribute. So (3.1) means that for any Δ one has

$$A_\Delta = \prod\nolimits^{\rightarrow} A_{\Delta_i}.$$

Typically this is applied to a decomposition of a strict sector V such that each V_i contains exactly one ray of the form $\mathbb{R}_{>0}Z(\gamma)$, $\gamma \in \Gamma$ with $Q(\gamma) \geq 0$. This is then written as

$$A_V = \prod\nolimits_{\ell\subset V}^{\rightarrow} A_\ell,$$

where of course only those rays $\ell \subset V$ contribute that are of the form $\mathbb{R}_{>0}Z(\gamma)$, with $\gamma \in \Gamma$ and $Q(\gamma) \geq 0$.

The lemma also shows

Corollary 3.9. *If $(Z, A) \in \widehat{\mathrm{Stab}}_Q(\mathfrak{g})$, then the A_ℓ determine the A_V uniquely.* □

Taking logarithm of the A_ℓ yields

$$\log(A_\ell) \in \mathfrak{g}_{\ell,Z,Q} = \prod_{\gamma \in \Gamma \cap C(\ell,Z,Q)} \mathfrak{g}_\gamma \subset \prod_{Z(\gamma)\in \ell} \mathfrak{g}_\gamma$$

which shall be written as an infinite sum $\sum a(\gamma)$ with $\gamma \in \Gamma \cap C(\ell, Z, Q)$. Since for fixed Z each γ determines the ray $Z(\gamma)$, this defines $a = (a(\gamma)) \in \prod_{\gamma\neq 0} \mathfrak{g}_\gamma$.

Lemma 3.10. *If* $(Z, A) \in \widehat{\mathrm{Stab}}_Q(\mathfrak{g})$, *then* $(Z, a) \in \mathrm{Stab}_Q(\mathfrak{g})$.

Proof. We have to verify the support property or, equivalently (see Lemma 3.3), that $a(\gamma) \neq 0$ implies $Q(\gamma) \geq 0$. If $a(\gamma) \neq 0$, then the γ-component of $\log(A_\ell)$ is non-trivial which by construction of $\log(A_\ell)$ as an element in $\mathfrak{g}_{\ell,Z,Q}$ implies $\gamma \in C(\ell, Z, Q)$. Thus $Q(\gamma) \geq 0$ (see Remark 2.3). □

Proposition 3.11. *Associating* (Z, a) *to* $(Z, A) \in \widehat{\mathrm{Stab}}_Q(\mathfrak{g})$ *as described above yields a natural bijection*

$$\widehat{\mathrm{Stab}}_Q(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Stab}_Q(\mathfrak{g}).$$

Proof. We have seen that $(Z, A) \longmapsto (Z, a)$ defines a map $\widehat{\mathrm{Stab}}_Q(\mathfrak{g}) \longrightarrow \mathrm{Stab}_Q(\mathfrak{g})$. Since $A_\ell = \exp\left(\sum_{\gamma\in C(\ell,Z,Q)} a(\gamma)\right)$, the image determines the A_ℓ and hence, by Corollary 3.9, also A itself. This proves the injectivity.

For the inverse of this map recall that for $(Z, a) \in \mathrm{Stab}_Q(\mathfrak{g})$ the inequality $a(\gamma) \neq 0$ implies $Z(\gamma) \neq 0$ and $Q(\gamma) \geq 0$ (see Lemma 3.3). In particular, if $a(\gamma) \neq 0$, then $\gamma \in C(\mathbb{R}_{>0}Z(\gamma), Z, Q)$. Hence for any ray ℓ the infinite sum $\sum_{Z(\gamma)\in\ell} a(\gamma)$ is contained in $\mathfrak{g}_{\ell,Z,Q}$ and we let $A_\ell \in G_{\ell,Z,Q}$ be its exponential. Now use Lemma 3.7 and its generalization to conclude. Explicitly, any A_V is determined by its truncations A_Δ. Each Δ can be decomposed into finitely many Δ_i, $i = 1, \ldots, k$, each meeting only one ray of the form $\mathbb{R}_{>0}Z(\gamma_i)$ with $Z(\gamma_i) \in \Delta$ and $a(\gamma_i) \neq 0$ (or $Q(\gamma_i) \geq 0$). Then one sets $A_\Delta := \prod_1^k A_{\ell_i\cap\Delta}$, where $\ell_i = \mathbb{R}_{>0}Z(\gamma_i)$. Clearly, the A_Δ form an inverse system and thus really determine an element A_V. The factorization property of the A_V follows from the construction. □

The bijections $\widehat{\mathrm{Stab}}_Q(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Stab}_Q(\mathfrak{g})$ yield a bijection

$$\widehat{\mathrm{Stab}}(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Stab}(\mathfrak{g}),$$

where $\widehat{\mathrm{Stab}}(\mathfrak{g})$ is obtained by gluing the $\widehat{\mathrm{Stab}}_Q(\mathfrak{g})$ via an equivalence relation $\sim$, that is defined as follows. One says $(Z, A) \in \widehat{\mathrm{Stab}}_Q(\mathfrak{g})$ and $(Z', A') \in \widehat{\mathrm{Stab}}_{Q'}(\mathfrak{g})$ are equivalent if $Z = Z'$ and there exists a quadratic form Q_0 negative definite on $\ker(Z) = \ker(Z')$ such that $Q, Q' \leq Q_0$ and for all strict sectors V one has $A_V = A'_V$ in G_{V,Z,Q_0}. (For the latter one uses the natural inclusion $G_{V,Z,Q} \subset G_{V,Z,Q_0}$.) It is easy to see that with this definition $(Z, A) \sim (Z', A')$ if and only if $(Z, a) = (Z', a')$ in $\mathrm{Stab}(\mathfrak{g})$.

4. Topology on Stab

Proposition 4.1. *There is a natural topology on* $\mathrm{Stab}(\mathfrak{g})$ *such that a map*

$$\varphi : X \longrightarrow \mathrm{Stab}(\mathfrak{g}),\ \ x \longmapsto \varphi(x) = (Z_x, a_x) = (Z_x, A_x)$$

from a topological space X is continuous if and only if the following conditions are satisfied:
i) The composition $X \longrightarrow \mathrm{Stab}(\mathfrak{g}) \longrightarrow \Gamma^*_{\mathbb{C}}$ is continuous.
ii) $\varphi^{-1}(\mathrm{Stab}_Q(\mathfrak{g})) \subset X$ is open for all Q.
iii) If for a closed strict sector $V \subset \mathbb{R}^2$ and a point $x_0 \in X$ one has $Z_{x_0}\{\gamma \mid a_{x_0}(\gamma) \neq 0\} \cap \partial V = \emptyset$, then $x \longmapsto \log(A_{xV}) \in \prod \mathfrak{g}_\gamma$ is continuous in x_0.

Here $\prod \mathfrak{g}_\gamma$ is endowed with the product of the discrete topologies on the $\mathfrak{g}_\gamma$.

Remark 4.2. The first condition says that $\mathrm{Stab}(\mathfrak{g}) \longrightarrow \Gamma^*_{\mathbb{C}}$ is continuous and the second that the sets $\mathrm{Stab}_Q(\mathfrak{g}) \subset \mathrm{Stab}(\mathfrak{g})$ are open.

A map $\psi : X \longrightarrow \prod \mathfrak{g}_\gamma$ is continuous in x_0 if the composition with each of the projections $p_\gamma : \prod \mathfrak{g}_\gamma \longrightarrow \mathfrak{g}_\gamma$ is locally constant. It does not mean that ψ itself is locally constant. In other words, the open neighbourhood of x_0 on which $p_\gamma \circ \psi$ is constant will in general depend on γ. In particular this shows that we cannot hope to phrase condition iii) as a continuity condition for A_x, we need to pass to $\log(A_x)$. On the other hand, iii) can neither be phrased purely in terms of the $a_x(\gamma)$. Indeed, we do have $\log(A_{x\ell}) = \sum_{Z(\gamma)\in\ell} a_x(\gamma)$, but $V = \ell = \mathbb{R}_{>0}Z(\gamma)$ is not even considered in iii). For those strict sectors $V \subset \mathbb{R}^2$ that are considered in iii) we know that $\log(A_{xV})$ can also be expressed in terms of the $a_x(\gamma)$ with $Z(\gamma) \in V$, but the expression involves the Campbell–Hausdorff formula and condition iii) says that the part of the Campbell–Hausdorff formula in all $a_x(\gamma_i)$ contributing to the $\mathfrak{g}_\gamma$-component is locally constant in x_0. See also the comment at the end of Example 4.4, ii).

The proof of the proposition is omitted. In fact any given class of maps from topological spaces into a given set defines a topology on the latter. Usually, the class of continuous maps gets larger in the process, but it is not difficult to check that this does not happen here. Actually, the assertion that a topology can be defined in this way is of no practical interest. What really matters is which maps from topological spaces into $\mathrm{Stab}(\mathfrak{g})$ one wants to allow and this is stated explicitly in the proposition. Later only continuous maps from intervals in $\mathbb{R}$ or from open sets in $\mathbb{R}^n$ will be needed.

Remark 4.3. It is not difficult to check that $\mathrm{Stab}(\mathfrak{g})$ is Hausdorff. For this, consider a sequence $(Z_n, a_n) = (Z_n, A_n) \in \mathrm{Stab}(\mathfrak{g})$ converging to two distinct points $(Z, a) = (Z, A), (Z', a') = (Z', A') \in \mathrm{Stab}(\mathfrak{g})$. Since $\mathrm{Stab}(\mathfrak{g}) \longrightarrow \Gamma^*_{\mathbb{C}}$ is continuous and $\Gamma^*_{\mathbb{C}}$ is Hausdorff, one has $Z = Z'$. If $(Z, a) \in \mathrm{Stab}_Q(\mathfrak{g})$ and $(Z', a') = (Z, a') \in \mathrm{Stab}_{Q'}(\mathfrak{g})$, then there exists a quadratic form Q_0 such that Q_0 is negative definite on $\ker(Z) = \ker(Z')$ and $Q, Q' \leq Q_0$. (Take the maximum of Q and Q' on $\ker(Z)$ and add a very positive form on its complement.) Then by Remark 3.4 $(Z, a), (Z', a') \in \mathrm{Stab}_{Q_0}(\mathfrak{g})$.

For any given $\gamma \in \Gamma$ we have to show that $a(\gamma) = a'(\gamma)$. Fix a strict sector V and a triangle $\Delta \subset V$ such that rays of the form $\mathbb{R}_{>0}Z(\delta)$, with $Z(\delta) \in \Delta$, $\delta \in \Gamma$, and $a(\gamma) \neq 0$, all coincide with $\ell := \mathbb{R}_{>0}Z(\gamma)$. Moreover, we assume $Z(\gamma) \in \Delta$. Then the Δ-truncations of A_V and A_ℓ coincide (cf. Lemma 3.7). Similarly for A'_V and A'_ℓ. By assumption, $\log(A_V)$ and $\log(A'_V)$ are both limits of $\log(A_{nV})$ and hence their γ-components coincide, for $\mathfrak{g}_\gamma$ is Hausdorff. Therefore, also the γ-components of $\log(A_\ell)$ and $\log(A'_\ell)$ are equal, i.e. $a(\gamma) = a'(\gamma)$.

Example 4.4. To make the comments in Remark 4.2 more transparent, let us study particular cases.

i) First, consider the case that $\ell_x := \mathbb{R}_{>0}Z_x(\Gamma)$ is a ray for x close to x_0. If V is a closed sector with ℓ_x in its interior, then iii) in Proposition 4.1 says $a_x(\gamma)$ for all γ is locally constant in x_0.

ii) Now consider the case $\Gamma \simeq \mathbb{Z}^2 = \gamma_1\mathbb{Z} \oplus \gamma_2\mathbb{Z}$ and a continuous path of stability structures $(Z_t, a_t) = (Z_t, A_t)$, $t \in [0, \varepsilon)$. Let $\gamma = \gamma_1 + \gamma_2$ and assume $Z_t(\gamma) \neq 0$ for all t, i.e. $\ell_t = \mathbb{R}_{>0}Z_t(\gamma)$ is a ray.

Suppose $Z_0(\gamma_i) \in \ell_0$, but $\mathbb{R}_{>0}Z_t(\gamma_1) < \mathbb{R}_{>0}Z_t(\gamma_2)$ (in the clockwise order) for $t \neq 0$. It should be helpful to write down explicitly how the $a_t(\gamma)$ changes.

So let V be a small strict sector with ℓ_0 in its interior. Moreover, fix a triangle $\Delta \subset V$ such that the only elements in Γ with image under Z_t (for small t) contained in Δ are $\gamma_1, \gamma_2, \gamma = \gamma_1 + \gamma_2$. (This is not quite realistic, as it seems that certain small multiples of γ_1 or γ_2 may also be mapped into Δ, but as one will see this does not affect the calculation.)

Then the truncation $A_{t\Delta}$ of A_{tV} is constant for small $0 \leq t$. Moreover, for $t = 0$ one has $A_{0\Delta} = A_{0\ell_0\cap\Delta}$ whose logarithm in $\mathfrak{g}_{\ell\cap\Delta}$ is $a_0(\gamma_1) + a_0(\gamma_2) + a_0(\gamma_1 + \gamma_2)$.

On the other hand, one can decompose V clockwise as $V = V_{1t} \bigsqcup \ell_t \bigsqcup V_{2t}$ and consider the induced triangles $\Delta_{it} := \Delta \cap V_{it}$. Then

$$A_{0\Delta} = A_{t\Delta} = A_{t\Delta_{1t}} \cdot A_{t\ell_t\cap\Delta} \cdot A_{t\Delta_{2t}}.$$

For $t > 0$, $A_{t\Delta_{it}}$ is the truncation of $A_{t\ell_{it}}$, where $\ell_{it} = \mathbb{R}_{>0}Z(\gamma_i)$, and, moreover, $A_{t\Delta_{1t}} = \exp(a_t(\gamma_i))$. The remaining factor $A_{t\ell_t\cap\Delta}$ for $t > 0$ is $\exp(a_t(\gamma))$.

The Campbell–Hausdorff formula then shows

$$a_0(\gamma_1) + a_0(\gamma_2) + a_0(\gamma) = a_t(\gamma_1) + a_t(\gamma_2) + a_t(\gamma) + (1/2)[a_t(\gamma_1), a_t(\gamma_2)].$$

In other words, for $t > 0$ one has $a_0(\gamma_i) = a_t(\gamma_i)$ and

$$a_0(\gamma) = a_t(\gamma) + (1/2)[a_t(\gamma_1), a_t(\gamma_2)] = a_t(\gamma) + (1/2)[a_0(\gamma_1), a_0(\gamma_2)]. \tag{4.1}$$

In particular, the $a_t(\gamma_i)$ are locally constant in $t = 0$, but $a_t(\gamma)$ is not. Its jump is expressed by the Lie bracket $[a_0(\gamma_1), a_0(\gamma_2)]$.

If one follows the path for $t < 0$ and assumes that then $\mathbb{R}_{>0}Z(\gamma_2) < \mathbb{R}_{>0}Z(\gamma_1)$, then $a_0(\gamma_1)$ and $a_0(\gamma_2)$ get interchanged in (4.1). This then yields

$$a_{>0}(\gamma) = a_{<0}(\gamma) + [a_0(\gamma_1), a_0(\gamma_2)],$$

where $a_{<0}(\gamma_i) = a_{>0}(\gamma_i) = a_0(\gamma)$.

(The Campbell–Hausdorff formula a priori also produces terms of the form $[a_t(\gamma_1), a_t(\gamma)] \in \mathfrak{g}_{\gamma_1+\gamma}$ but those are trivial in the truncation $\mathfrak{g}_\Delta$ and in any case do not contribute to the γ-component.)

The last example shows that the $a_t(\gamma)$ may not be locally constant in a given point $t = 0$, at least if the map $Z_t : \Gamma \longrightarrow \mathbb{C}$ changes its behavior, e.g. if its rank drops. This is made more precise by the concepts of walls. In the following a quadratic form Q is fixed and only linear functions Z are considered for which Q is negative definite on $\ker(Z)$.

For two linearly independent $\gamma_1, \gamma_2 \in \Gamma$ with $Q(\gamma_i) \geq 0$ one considers the *wall*

$$W_{\gamma_1,\gamma_2} := \{Z \in \Gamma^*_{\mathbb{C}} \mid \mathbb{R}_{>0}Z(\gamma_1) = \mathbb{R}_{>0}Z(\gamma_2)\},$$

which is of real codimension one. Sometimes it is convenient to think of walls as associated to sublattices $\Gamma_0 \subset \Gamma$ with $\mathrm{rk}(\Gamma_0) = 2$. Then the wall would be the set of those Z such that $\mathrm{rk}Z(\Gamma_0) = 1$.

Proposition 4.5. *Suppose $(Z_t, a_t) \in \mathrm{Stab}(\mathfrak{g})$, $t \in [0,1]$, is a continuous path. Consider $\gamma \in \Gamma$ and assume that $Z_t \notin \bigcup_{\gamma_1+\gamma_2=\gamma} W_{\gamma_1,\gamma_2}$ for all t. Then $a_t(\gamma)$ is constant.*

Note that in Example 4.4, iii) the assumption does not hold, as $Z_0 \in W_{\gamma_1,\gamma_2}$.

Proof. A wall W_{γ_1,γ_2} with $\gamma_1 + \gamma_2 = \gamma$ will be called a γ-*wall*. So the assumption says that Z_t are not contained in any γ-wall.

We prove that $a_t(\gamma)$ is locally constant in $t = 0$.

First fix Q such that $(Z_t, a_t) \in \mathrm{Stab}_Q(\mathfrak{g})$ for all t close to 0. If $Q(\gamma) < 0$, then $a_t(\gamma) = 0$ and there is nothing to prove. So assume $Q(\gamma) \geq 0$.

Then choose a generic small triangle Δ such that all $Z_0(\delta) \in \Delta$ with $Q(\delta) \geq 0$ are contained in the ray ℓ_0 through $Z_0(\gamma) \in \Delta$. For t close to 0 we may assume that $M := \{\delta \in \Gamma \mid Q(\delta) \geq 0, Z_t(\delta) \in \Delta\}$ is independent of t. The set of rays $I_t := \{\mathbb{R}_{>0} Z_t(\delta) \mid \delta \in M\}$ may depend on t, but it will always be finite. For $t = 0$ it reduces to ℓ_0 or, in other words, $Z_0(\delta) \in \ell_0$ for all $\delta \in M$.

By continuity, the γ-component $\log(A_{t\Delta})_\gamma$ of $\log(A_{t\Delta})$ will be constant. For $t = 0$ it is $a_0(\gamma)$. For $t \neq 0$ one writes $A_{t\Delta} = \overrightarrow{\prod}_{\ell \in I_t} A_{t\ell\cap\Delta}$. Then $A_{t\ell\cap\Delta} = \exp\left(\sum a_t(\delta)\right)$, where the sum runs over all $\delta \in M$ with $Z_t(\delta) \in \ell$. Applying the Campbell–Hausdorff formula to the product $\overrightarrow{\prod}_{\ell \in I_t} \exp\left(\sum a_t(\delta)\right)$, one computes its γ-component as a sum of Lie expressions $[\ldots[\ldots[\ ,\ ]\ldots]$ in the $a_t(\delta)$ with $\delta \in M$.

i) Consider first the case that to all Lie expressions entering the formula for $\log(A_{t\Delta})_\gamma$ only one of the rays in I_t contributes. Whenever a Lie expression in the formula for $\log(A_{t\Delta})_\gamma$ involves only one ray, say $\ell \in I_t$, then $Z_t(\gamma) \in \ell$. Since $Z_t(\gamma)$ can only be contained in one of the rays in I_t, all Lie expressions contributing to $\log(A_{t\Delta})_\gamma$ involve the same single ray. In the Campbell–Hausdorff formula for $X * Y$ there is no monomial of degree > 1 involving only X (or Y). Thus, $a_t(\gamma)$ appears in $\log(A_{t\ell\cap\Delta}) = \sum a_t(\delta)$ and there can be no other contribution of higher degree to $\log(A_{t\Delta})_\gamma$ involving only ℓ. Thus, $a_t(\gamma) = \log(A_{t\Delta})_\gamma$ which is locally constant.

ii) Now consider the case that one of the Lie expressions in the Campbell–Hausdorff formula for $\log(A_{t\Delta})_\gamma$ involves more than one ray. Then $\gamma = \delta_1 + \ldots + \delta_k$, $k > 1$, with $Z_t(\delta_i)$ spanning different rays. If the $Z_t(\delta_i)$ are written in clockwise order, then for $\gamma_1 := \delta_1$ and $\gamma_2 := \delta_2 + \ldots + \delta_k$ the images $Z_t(\gamma_1)$ and $Z_t(\gamma_2)$ are linearly independent in $\mathbb{R}^2$. Hence γ_1 and γ_2 are linearly independent in $\Gamma_\mathbb{R}$. On the other hand, $Z_0(\gamma_i) \in \mathbb{R}_{>0} Z_0(\gamma)$ which then contradicts the assumption that Z_0 is not contained in any γ-wall. $\square$

Consider a path $(Z_t, a_t) = (Z_t, A_t) \in \mathrm{Stab}_Q(\mathfrak{g})$, $t \in [0,1]$, such that each wall W_{γ_1,γ_2} is intersected only finitely often. A value $t \in [0,1]$ is a *discontinuity point* for $\gamma \in \Gamma$ if $\gamma = \gamma_1 + \gamma_2$ with $Q(\gamma_i) \geq 0$ and $Z_t \in W_{\gamma_1,\gamma_2}$.

Lemma 4.6. *A fixed γ has only finitely many discontinuity points.*

Proof. First note that $Z_t(\gamma)$ is bounded for $t \in [0,1]$. Then if Z_t is contained in a γ-wall W_{γ_1,γ_2}, one has $|Z_t(\gamma_i)| \leq |Z_t(\gamma)|$, because $Z_t(\gamma_1)$ and $Z_t(\gamma_2)$ are both contained in $\mathbb{R}_{>0}Z_t(\gamma)$. Since $Q(\gamma_i) \geq 0$, the inequality in the support property holds and hence the $|\gamma_i|$ are bounded universally by $C\cdot\max\{|Z_t(\gamma) \mid t \in [0,1]\}$. Thus the set of potential (γ_1,γ_2) for which $Z_t \in W_{\gamma_1,\gamma_2}$ with $\gamma_1+\gamma_2=\gamma$ is bounded and discrete and hence finite. (See the arguments in the proof of Lemma 3.3 and Remark 3.4. One may have to decompose $[0,1]$ in finitely many intervals and so to work with finitely many constants C and norms $\|\ \|$, but this does not affect the argument.) □

So for fixed γ and any $t \in [0,1]$ the value of $a_t(\gamma)$ will be constant for $t+\varepsilon$ and for $t-\varepsilon$ for small $\varepsilon > 0$. They are denoted $a_t^+(\gamma)$ respectively $a_t^-(\gamma)$. Of course, if t is not a discontinuity point for γ then $a_t^{\pm}(\gamma) = a_t(\gamma)$. In general the relation between $a_t^{\pm}(\gamma)$ and $a_t(\gamma)$ is described by the following

Proposition 4.7. Wall-crossing formula: *Let $\ell = \mathbb{R}_{>0}Z_t(\gamma)$. Then the following formula holds in $G_{\ell,Z_t,Q}$:*

$$\prod_{\mu}^{\rightarrow} \exp\left(\sum_{n\geq 1} a_t^{\pm}(n\mu)\right) = \exp\left(\sum_{\mu,n\geq 1} a_t(n\mu)\right), \tag{4.2}$$

where on both sides μ runs through all primitive elements in Γ with $Z_t(\mu) \in \ell := \mathbb{R}_{>0}Z_t(\gamma)$. The product is taken in clockwise order with respect to $Z_{t\pm\varepsilon}(\mu)$.

Proof. As we will see, the proof actually uses that $Z_{t\pm\varepsilon}$ is not contained in any wall W_{δ_1,δ_2} (where $\delta_1+\delta_2$ may or may not be γ). In fact, it is enough to exclude walls W_{δ_1,δ_2} with $Z_t(\delta_i) \in \mathbb{R}_{>0}Z_t(\gamma)$. But it does not matter on how many walls W_{γ_1,γ_2} (with $\gamma_1+\gamma_2=\gamma$ or not) Z_t happens to lie.

First observe that the right hand side of the equation is just $A_{t,\ell}$. Next, consider a sequence of triangles Δ_i satisfying the following conditions:
i) $\Delta_i \cap \ell = (0,i]\cdot Z_t(\gamma)$,
ii) The sector corresponding to Δ_i is generic (in the sense of Proposition 4.1, iii))
iii) If ε is small and $\delta \in \Gamma$ with $Q(\delta) \geq 0$ and $Z_{t-\varepsilon}(\delta) \in \Delta_i$, then $Z_t(\delta) \in \ell$.

Then $A_{t,\ell\cap\Delta_i} = A_{t,\Delta_i}$ and $\log(A_{t,\Delta_i}) = \log(A_{t\pm\varepsilon,\Delta_i})$ for small ε. (Use continuity and condition ii).) Now $A_{t\pm\varepsilon,\Delta_i}$ can be written as the finite clockwise product $\prod A_{t\pm\varepsilon,\Delta_i\cap\ell_j}$, where the ℓ_j run through the rays spanned by $Z_{t\pm\varepsilon}(\delta_j) \in \Delta_i$ and $Q(\delta_j) \geq 0$ (see Remark 3.8).

Use $A_{t\pm\varepsilon,\Delta_i\cap\ell_j} = \exp\left(\sum a_{t\pm\varepsilon}(n\mu)\right)_{\Delta_i}$ with the sum over all primitive $\mu \in \Gamma$ with $Z_{t\pm\varepsilon}(\mu) \in \ell_j$ and all $n \geq 1$.

As $Z_{t\pm\varepsilon}$ is not contained in any wall (of type W_{δ_1,δ_2} with $Z_t(\delta_i) \in \ell$), there is at most one primitive μ for each ℓ_j actually contributing. Hence $A_{t\pm\varepsilon,\Delta_i} = \prod A_{t\pm\varepsilon,\Delta_i\cap\ell_j} = \prod_\mu \exp\left(\sum_n a_{t\pm\varepsilon}(n\mu)\right)_{\Delta_i}$ with the product over all primitive μ with $Z_{t\pm\varepsilon}(\mu) \in \Delta_i$.

Therefore $A_{t,\ell\cap\Delta_i} = A_{t,\Delta_i} = A_{t\pm\varepsilon,\Delta_i} = \prod_\mu \exp\left(\sum_n a_{t\pm\varepsilon}(n\mu)\right)_{\Delta_i}$ with $\mu \in \Gamma$ primitive. In other words, the Δ_i-truncations of the two sides of the asserted equation coincide. □

Example 4.8. Suppose $\Gamma = \mathbb{Z}^2$ and $Q(x,y) = xy$. We shall write $a(\gamma) = a(m,n)$ for $\gamma = (m,n)$. Consider a path in $\mathrm{Stab}_Q(\mathfrak{g})$ with $Z_t(\Gamma)$ of rank one and such that $Z_{t-\varepsilon}$ is injective and orientation preserving and $Z_{t+\varepsilon}$ is injective and orientation reversing. Then the wall crossing formula says:

$$\prod\nolimits_{(m,n)=1}^{\rightarrow} \exp\left(\sum_{k\geq 1} a^{-}(m,n)\right) = \prod\nolimits_{(m,n)=1}^{\leftarrow} \exp\left(\sum_{k\geq 1} a^{+}(m,n)\right).$$

So only the ordering of the product with respect to the natural ordering of $m/n \in \mathbb{Q}$ has changed while crossing the wall.

Proposition 4.7 can in particular be used to express for fixed γ the value $a_{t+\varepsilon}(\gamma)$ after going through a discontinuity points for γ in terms of $a_{t-\varepsilon}(\gamma)$ and a finite collection of $a_{t-\varepsilon}(\gamma_i)$. Since there are only finitely many discontinuity points $t \in [0,1]$ for a fixed γ, one would hope that this eventually leads to expressing $a_1(\gamma)$ in terms of $a_0(\gamma)$ and finitely many $a_0(\gamma_i)$. As after each wall crossing, there are more elements one has to keep track of, the finiteness needs to be proved carefully.

Proposition 4.9. *Suppose $(Z_t, a_t) \in \mathrm{Stab}_Q(\mathfrak{g})$, $t \in [0,1]$, is a continuous path. Then $a_1(\gamma)$ can be expressed (using the Lie algebra structure of $\mathfrak{g}$) in terms of finitely many $a_0(\gamma_i)$.*

Proof. TBC □

The arguments used to prove the Proposition 4.7 also show that any continuous path $Z_t \in \Gamma_{\mathbb{C}}^*$, $t \in [0,1]$, with Q negative definite on all $\ker(Z_t)$ can be lifted to a unique continuous path in $\mathrm{Stab}_Q(\mathfrak{g})$ once $(Z_0, a_0) \in \mathrm{Stab}_Q(\mathfrak{g})$ is chosen. If a path connecting Z_0 and Z_1 is chosen generically in the sense that only one wall at a time is crossed, then the small perturbations of the path will not change the lift (Z_1, a_1). So if it can be shown that the monodromy of a loop around an intersection point of two or more walls will have trivial monodromy, then the lift (Z_1, a_1) will only depend on (Z_0, a_0) and not on the path connecting Z_0 and Z_1.

This is the rough idea for the following

Proposition 4.10. (Theorem 3) *Fix $(Z_0, a_0) \in \mathrm{Stab}_Q(\mathfrak{g})$. Then over the connected component of U of the open set $\{Z \in \Gamma_{\mathbb{C}}^* \mid Q|_{\ker(Z)} < 0\}$ that contains Z_0 the natural projection $\mathrm{Stab}_Q(\mathfrak{g}) \longrightarrow \Gamma_{\mathbb{C}}^*$, $(Z, a) \longmapsto Z$ admits a unique continuous section $U \longrightarrow \mathrm{Stab}_Q(\mathfrak{g})$ through (Z_0, a_0).*

Proof. 1. Let us first look at the 'global monodromy' of the connected component U of the open set $\{Z \in \Gamma_{\mathbb{C}}^* \mid Q|_{\ker(Z)} < 0\}$. Claim: $\pi_1(U) \simeq \mathbb{Z}$ with a generator given by $e^{2\pi i t} \cdot Z$, $t \in [0,1]$. Indeed, TBC

If Z is not contained in any of the walls, then neither is any of the $Z_t := e^{2\pi i t} \cdot Z$. Hence, the wall crossing formula in this case says that the constant path $a_t \equiv a_0$ yields the unique lift of the path Z_t.

2. The difficult part is to show that going from Z_0 to Z_1 does not depend on the choice of the path connecting Z_0 and Z_1. The path can be chosen generic, i.e. crossing walls for at most finitely many $t \in [0,1]$. (In principle a path could also spend time within a wall, but it would jump only when meeting another wall.)

First observe, that the collection of walls is locally finite. Indeed, TBC

3. Suppose $(Z_t, a_t) \in \mathrm{Stab}_Q(\mathfrak{g})$, $t \in [0,1)$ is a continuous path with Z_t not contained in any wall for $t < 1$ and such that Z_t, $t \in [0,1)$, can be extended to a continuous path Z_t, $t \in [0,1]$ in $\Gamma_{\mathbb{C}}^*$ with Q still negative definite on $\ker(Z_1)$. The wall crossing formula determines a_1 uniquely. With this choice, (Z_1, a_1) is a stability condition and (Z_t, a_t), $t \in [0,1]$, is a continuous path in $\mathrm{Stab}_Q(\mathfrak{g})$.

If Z_1 is contained in exactly one wall, then the path can be perturbed a little without changing a_1. If, however, Z_1 is contained in two walls $W = W_{\gamma_1,\gamma_2}$ and $W' = W_{\gamma_1',\gamma_2'}$, then the value of a_1 will in general change. So, in this case one has to show that crossing both walls is independent of the chosen path. Or, in other words, one has to show that also for $Z \in W \cap W'$ the lift locally around Z is unique.

In [1] only the case of the intersection of two walls (and not more) is sketched and we will follow them. Set $\Lambda := \langle \gamma_1, \gamma_2 \rangle$ and $\Lambda' := \langle \gamma_1', \gamma_2' \rangle$. Then $L_1 := \mathbb{R}Z(\Lambda)$ and $L_2 := \mathbb{R}Z(\Lambda')$ are both of dimension one.

There are two cases, $L_1 \neq L_2$ or $L_1 = L_2$. In the first case, the wall crossing formula related to Λ_1 and Λ_2 happen in different slices of the Lie algebra and therefore commute. Thus only the case $L_1 = L_2$ needs to be discussed. Then $\Gamma' := \Lambda + \Lambda'$ is of rank at least 3 and $\mathbb{R}Z(\Gamma') = L_1 = L_2 =: L$.

One may assume $\Gamma' = \Gamma$, as only for elements γ in Γ' crossing the walls W and W' will affect $a(\gamma)$. Now pick a decomposition in strict sectors $\mathbb{R}^2 \setminus \{0\} = V_1 \bigsqcup V_2 \bigsqcup V_3 \bigsqcup V_4$ with $L \setminus \{0\}$ contained in the interior of $V_1 \bigsqcup V_3$. Then condition iii) of Proposition 4.1 holds for all four sectors V_i and therefore A_{V_i} is locally constant around Z.

Thus for all Z' close to Z any lift contained in a small neighbourhood of (Z, A) will have constant A_{V_i} and will, in particular, be unique. Now use Remark 4.11. □

Remark 4.11. Suppose a decomposition $\mathbb{R}^2 \setminus \bigsqcup V_i$ into strict sectors is given. Then for a given Z a stability condition (Z, A) uniquely determined by the A_{V_i}. One has to show that the $a(\gamma)$ are determined. Consider the sector $V = V_i$ that contains $Z(\gamma)$. Then choose a decreasing sequence of triangles $\Delta_1 \subset \Delta_2 \subset \ldots$ in V such that $\Delta_{i+1} \setminus \Delta_i$ contains exactly one $Z(\gamma)$ with $a(\gamma) \neq 0$ (or $Q(\gamma) \geq 0$), it shall be called γ_{i+1}. Clearly, A_{Δ_1} determines $a(\gamma_1)$. In fact, $\exp(a(\gamma_1)) = A_{\Delta_1}$. Then proceed by induction. Suppose $a(\gamma_1), \ldots a(\gamma_i)$ are already determined. Use FP to write $A_{\Delta_{i+1}}$ as a product of $A_{\ell_k \cap \Delta_{i+1}}$, where the rays ℓ_k are spanned by $Z(\gamma_1), \ldots, Z(\gamma_i)$. Then $a(\gamma_{i+1})$ will occur in the logarithm of one of the factors. All the other ones will involve only $a(\gamma_1), \ldots, a(\gamma_i)$. The Campbell–Hausdorff formula then allows one to express $a(\gamma_{i+1})$ in terms of $\log(A_{\Delta_{i+1}}$ and $a(\gamma_1), \ldots, a(\gamma_i)$. This proves the claim.

Clearly, one consequence of the theorem is that $\mathrm{Stab}(\mathfrak{g}) \longrightarrow \Gamma_{\mathbb{C}}^*$ is a local (in $\mathrm{Stab}(\mathfrak{g})$!) homeomorphism, which is reminiscent of Bridgeland's result.

References

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