THE GROUPOID PICTURE OF QUANTUM MECHANICS

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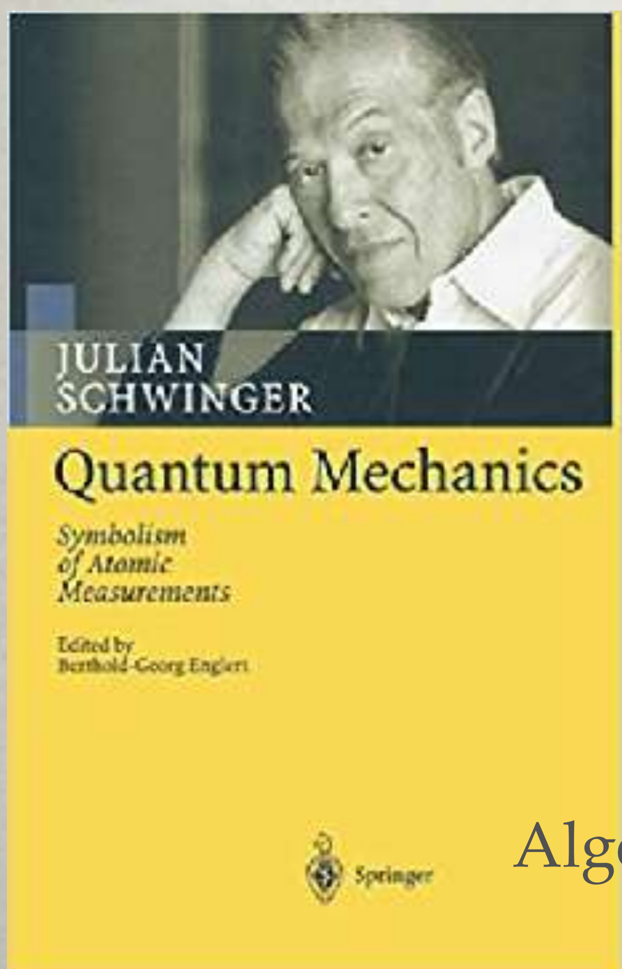
1. Introduction

The Foundations of Quantum Mechanics, J. Schwinger and Groupoids

Symbolic language

Julian S. Schwinger (1918-1994)

"The laws of atomic physics must be expressed in a non-classical mathematical language that constitutes a symbolic expression of the properties of microscopic measurements" In "Quantum Kinematics and Dynamics" (1991).



Physics in the earlier 20th century

The puzzle of Quantum Mechanics

Deep impact in Mathematics (1930-)

J. von Neumann's Hilbert space formalism and the theory of rings of operators

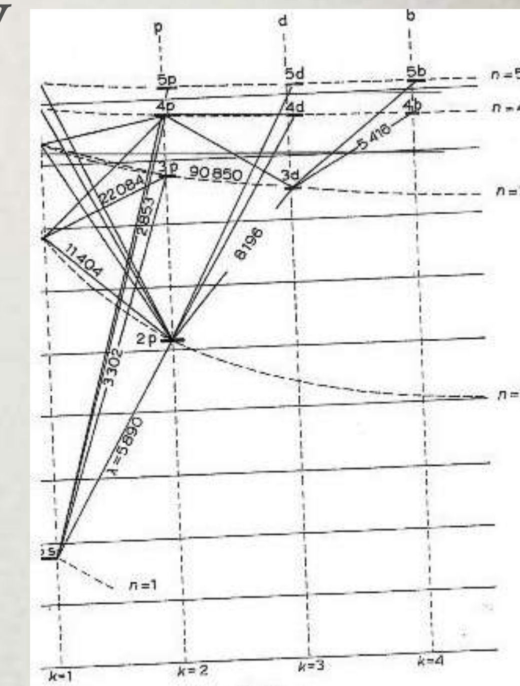


Fig. 2. Sodium.

R. Ladenburg.Zs.f. Physik, 4 (1921) 451-468

Transitions ("quantum jumps")

N. Bohr, W. Heisenberg (1925)

The bold leap was taken by Heisenberg in identifying observable effects associated with atoms being associated not with single stationary orbits but with jumps or transitions between two orbits, i.e.,with emission or absorption frequencies Composition of frequencies which describes the groupoidal composition law was clearly written by Heisenberg in section 1 of his paper in Zs. f. Phys.33 (1925) 879-893.

A. Connes (1995)

Ritz-Rydberg composition principle for emission or absorption frequencies of radiation for atoms interacting with electromagnetic fields is properly understood as a groupoid

J. Schwinger's
Algebra of selective measurements (1960)

Groupoids

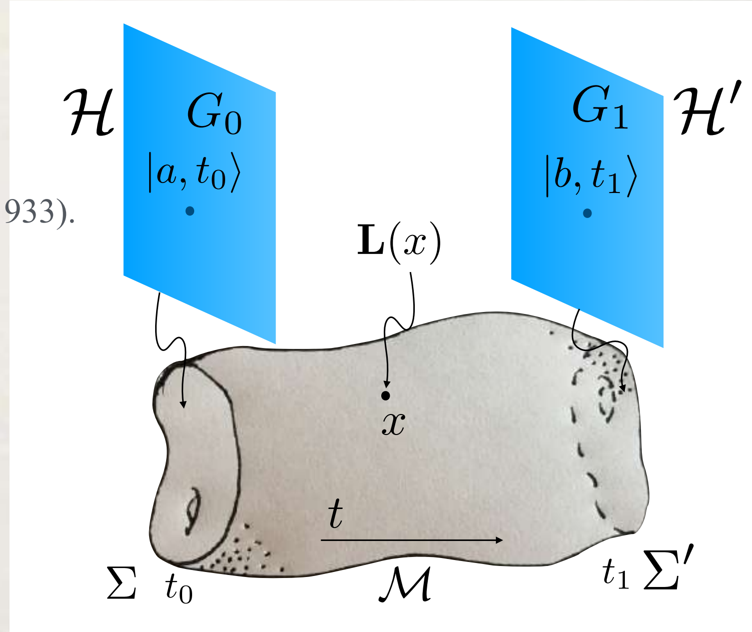
Categorical thinking in Physics

J. V. Neumann. Mathematische Begründung der Quantenmechanik. Springer, Berlin (1931).
F.J. Murray, J. von Neumann, On rings of operators, Ann. Math. 37 (1936), 116--22

1. Groupoids and Transitions I: The Algebra of Selective Measurements

Dirac's question: What is the role of the Lagrangian in Quantum Mechanics?

P.A.M. Dirac. *The Lagrangian in Quantum Mechanics*. Physikalische Zeitschrift der Sovietunion, Band 3, Heft 1 (1933).



R. Feynman's dynamical principle

$$Z_\varphi(J) = \int_{\partial\phi=\varphi} \mathcal{D}\phi \; e^{\frac{i}{\hbar}\int_M \mathcal{L}(\phi)d^4x + \langle J,\phi\rangle}$$

Schwinger's quantum variational principle

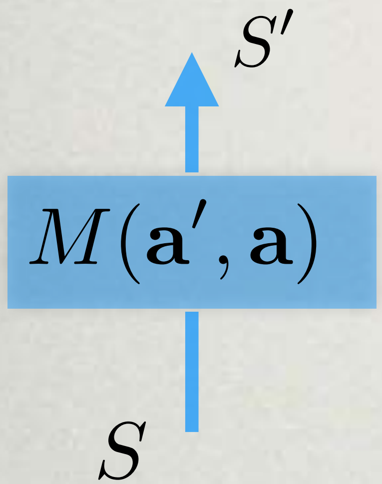
$$\delta\varphi_{b,t_1;a,t_0} = i\langle b, t_1|\delta\int_{t_0}^{t_1}\mathbf{L}(s)ds|a, t_0\rangle = \langle b, t_1|G_1 - G_0|a, t_0\rangle$$

J. Schwinger's Algebra of selective measurements (1960)

Groupoids

1. Groupoids and Transitions I: The Algebra of Selective Measurements

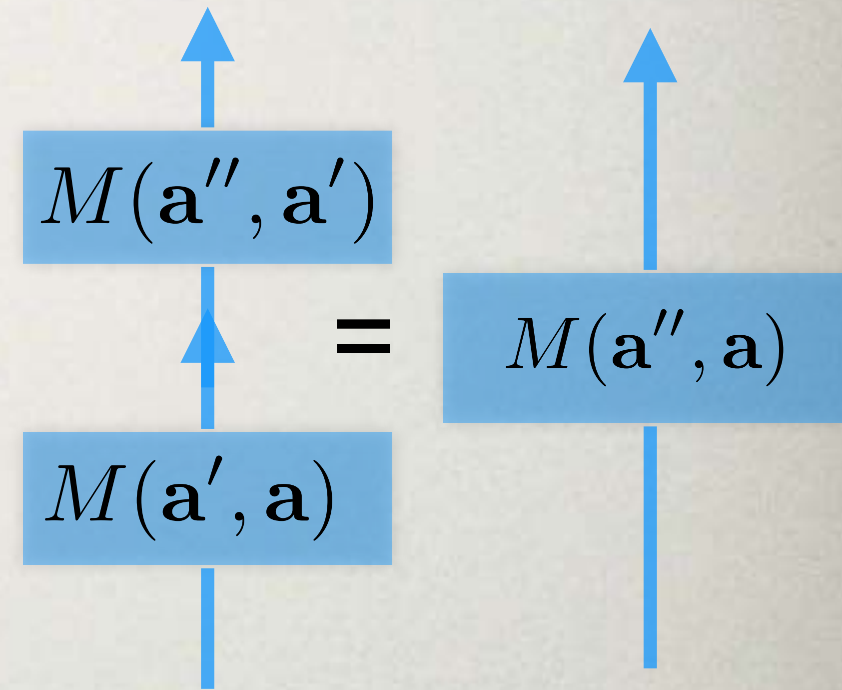
J. Schwinger's Algebra of selective measurements (1960)



$$\mathcal{E}(\mathbf{a}') = \{S' \in \mathcal{E} \mid \mathbf{a}' = \langle \mathbf{A} : S' \rangle\}$$

$$\mathcal{E}$$

$$\mathcal{E}(\mathbf{a}) = \{S \in \mathcal{E} \mid \mathbf{a} = \langle \mathbf{A} : S \rangle\}$$



Symbolic mathematical language

$$M(\mathbf{a}'', \mathbf{a}) := M(\mathbf{a}'', \mathbf{a}') \circ M(\mathbf{a}', \mathbf{a}) \qquad M(\mathbf{a}) := M(\mathbf{a}, \mathbf{a})$$

Units: $M(\mathbf{a}', \mathbf{a}) \circ M(\mathbf{a}) = M(\mathbf{a}', \mathbf{a}) \quad M(\mathbf{a}') \circ M(\mathbf{a}', \mathbf{a}) = M(\mathbf{a}', \mathbf{a})$

Associativity: $M(\mathbf{a}''', \mathbf{a}'') \circ (M(\mathbf{a}'', \mathbf{a}') \circ M(\mathbf{a}', \mathbf{a})) = (M(\mathbf{a}''', \mathbf{a}'') \circ (M(\mathbf{a}'', \mathbf{a}')) \circ M(\mathbf{a}', \mathbf{a})$

Invertibility: $M(\mathbf{a}') = M(\mathbf{a}', \mathbf{a}) \circ M(\mathbf{a}, \mathbf{a}') \qquad M(\mathbf{a}) = M(\mathbf{a}, \mathbf{a}') \circ M(\mathbf{a}', \mathbf{a})$

Groupoid

1.2. Groupoids and Transitions II

Axioms

(derived from a few simple physical principles)

Outcomes: $a, b, x, y, .. \in \Omega$

Transitions: $\alpha\colon a \to b, \beta\colon x \to y \in \Gamma$

$$s, t\colon \Gamma \to \Omega \qquad s(\alpha) = a, t(\alpha) = b$$

$$\Gamma^{(2)} = \{(\alpha, \beta) \in \Gamma \times \Gamma \mid s(\alpha) = t(\beta)\}$$

Composition law: $\circ\colon \Gamma^{(2)} \to \Gamma$

Groupoid

$$\Gamma \rightrightarrows \Omega$$

Associative

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma$$

Units $1_a\colon a \to a$

$$\alpha \circ 1_a = 1_b \circ \alpha = \alpha$$

Inverse

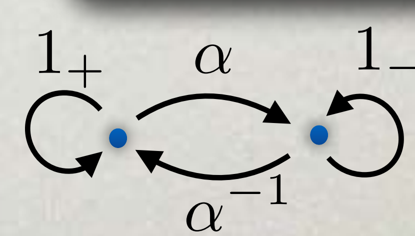
$$\exists \alpha^{-1}\colon b \to a$$

$$\alpha^{-1} \circ \alpha = 1_a \quad \alpha \circ \alpha^{-1} = 1_b$$

Some simple examples

Discrete output systems

(Abstract) qubit $A_2 \rightrightarrows \Omega_2 = \{+, -\}$



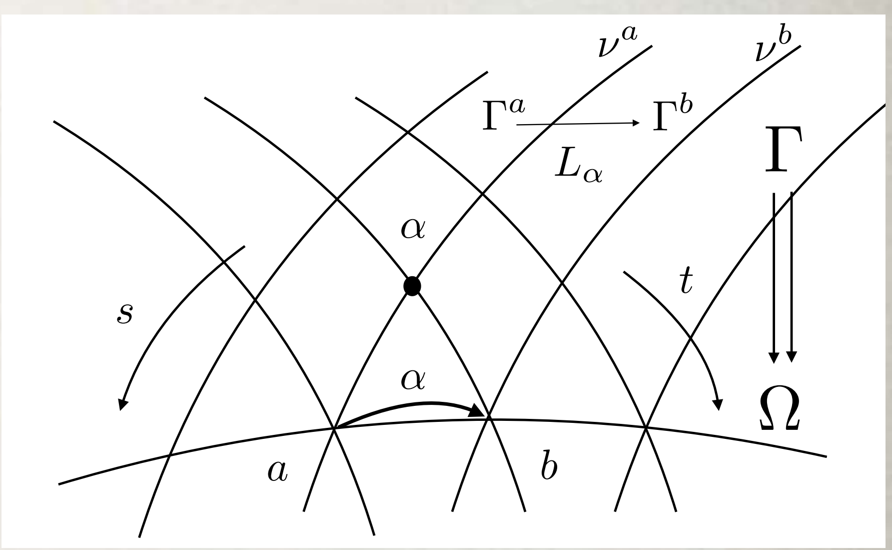
"Harmonic oscillator" $A_\infty \rightrightarrows \Omega_\infty$

$\alpha_n\colon n \to n+1 \quad \alpha_n^{-1}\colon n+1 \to n$

Ciaglia, F.M.; AI.; Marmo, G. Schwinger's Picture of Quantum Mechanics I: Groupoids. Int. J. of Geom. Meth. in Mod. Phys. (2019}, 16, 1950119 (31).

1. Groupoids and Transitions III

The statistical interpretation



The job of the "theoretical physicist" is to construct a mathematical model of the random phenomena observed when studying a physical system:

Kolmogorov's spaces (1931): $(\Omega, \mathcal{B}, P)$

Mackey-Hahn-Connes-Renault measure groupoids:

Transverse function $a \mapsto \nu^a \, , \operatorname{supp} \nu^a \subset t^{-1}(a) = \Gamma^a$

$\alpha \circ \nu^a = \nu^b \qquad \alpha \colon a \to b \qquad \nu = \int_\Omega \nu^a P\{da\}$

Measure groupoid $(\Gamma \rightrightarrows \Omega, [\nu])$

Action groupoids (von Neumann's group-measure construction)

G group acting on Ω, $(g, x) \mapsto gx$, $g \in G$, $x \in \Omega$

$\Gamma(G, \Omega) = G \times \Omega \rightrightarrows \Omega \, , \; s(g,x) = x, \; t(g,x) = gx \quad (g', gx) \circ (g, x) = (g'g, x)$

(Ω, P) Kolmogorov's space $\quad$ G locally compact group

Measurable action, the measure P is G-cuasi-invariant

N.N. Cencov. "Statistical Decision Rules and Optimal Inference" AMS (1982).
A. Connes. "Sur la théorie non commutative de l'intégration." Algébres d'opérateurs. Springer, Berlin, Heidelberg, 1979. 19-143
J. Renault: A Groupoid Approach to C*-Algebras, Lect. Notes in Math. 793, Springer-Verlag, Berlin 1980.

1. Groupoids and Transitions IV

Kinematical groupoids (Lie groupoids)

$\mathscr{M}$ spacetime, metric η

Riemann groupoid

$$\mathcal{R}(\mathscr{M}) = \{T_{yx} \colon T_x\mathscr{M} \to T_y\mathscr{M} \mid T_{yx}^* \eta_y = \eta_x\} \rightrightarrows \mathscr{M}$$

Poincaré groupoid

$$\mathcal{P}(\mathscr{M}) = \{(p_y, T_{yx}, p_x) \in T^*\mathscr{M} \times \mathcal{R}(\mathscr{M}) \times T^*\mathscr{M} \mid T_{yx}^* p_y = p_x\} \rightrightarrows T^*\mathscr{M}$$

Wigner's program: Elementary Quantum Systems

Irreducible unitary representations of the Poincaré group
Irreducible unitary representations of Poincaré's groupoid

Theorem. There is a one-to-one correspondence between (irreducible) unitary representations of Poincaré's grouped and (irreducible) unitary representations of the isotropy group along the orbits $p_\mu p^\mu = m^2$.

Wigner, E. "On unitary representations of the inhomogeneous Lorentz group." Annals of Mathematics (1939): 149-204
Landsmann, N.P.. "Lie groupoids and Lie algebroids in Physics and Noncommutative Geometry." J. Geom. Phys., 56 (2006): 24-54

2. Algebras of Observables I
From von Neumann to Connes

The algebra of observables of a quantum system

Hilbert space $\mathcal{H}$

Von Neumann algebras $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ $\mathcal{M} = \mathcal{M}^\dagger$

$\mathcal{M} = \mathcal{M}''$, $\mathcal{M}$ closed in the WOT (SOT)

The von Neumann algebra of a measure groupoid

$\Gamma \rightrightarrows \Omega$ countable

$$\lambda \colon \mathbb{C}[\Gamma] \to \mathcal{B}(L^2(\Gamma)),\ (\lambda(\alpha)\Psi)(\beta) = \Psi(\alpha^{-1} \circ \beta),\ t(\alpha) = t(\beta))$$

$$\nu(\Gamma) = \lambda(\mathbb{C}[\Gamma])'' = \overline{\lambda(\mathbb{C}[\Gamma])}^{\mathrm{WOT}}$$

Convolution algebra

$$f, g \in C_c(\Gamma),\ f \star g(\alpha) = \int f(\beta) g(\beta^{-1} \circ \alpha) d\nu^{t(\alpha)}(\beta)$$

$$\lambda(f)\Psi = f \star \Psi \qquad \nu(\Gamma) = \overline{\lambda(C_c(\Gamma))}$$

2. Algebras of Observables II

Simple examples

$\mathcal{M}$ such that $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}\mathbf{1}$ is called a factor

Discrete output systems

Qubit $\nu(A_2) = M_2(\mathbb{C})$ Factor Type I_n

Harmonic oscillator $\nu(A_\infty) = \mathcal{B}(l^2)$ Factor Type I_∞

Action groupoids $\Gamma(G, \Omega) \rightrightarrows \Omega$

$$\nu(\Gamma(G, \Omega)) = L^\infty(\Omega, \mu) \rtimes G$$ Crossed product algebra

The type of a quantum system

States: $\rho \colon \nu(\Gamma) \to \mathbb{C}\,;\, \rho \geq 0\,;\, \rho(\mathbf{1}) = 1$

GNS representation $\pi_\rho \colon \nu(\Gamma) \to \mathcal{B}(\mathcal{H}_\rho)$ $\quad \mathcal{H}_\rho = \overline{\nu(\Gamma)/\mathcal{J}_\rho}$

$$\pi_\rho(a)[b] = [ab]$$

$$\mathcal{J}_\rho = \{a \mid \rho(a^* a) = 0\}$$

Non-degenerate representation: $\varphi \colon \Gamma \to \mathbb{C}$ positive-definite function

$$\sum \bar{\zeta}_i \zeta_j \varphi(\alpha_i^{-1} \circ \alpha_j) \geq 0,\ i, j = 1, \ldots, N,\ \forall N \in \mathbb{N},\ \alpha_i \in \Gamma,\ \zeta_i \in \mathbb{C}.$$

3. Classical systems and the cuasi-classical approximation

Von Neumann (and Birkhoff) again

$\mathcal{M}$ von Neumann algebra of observables

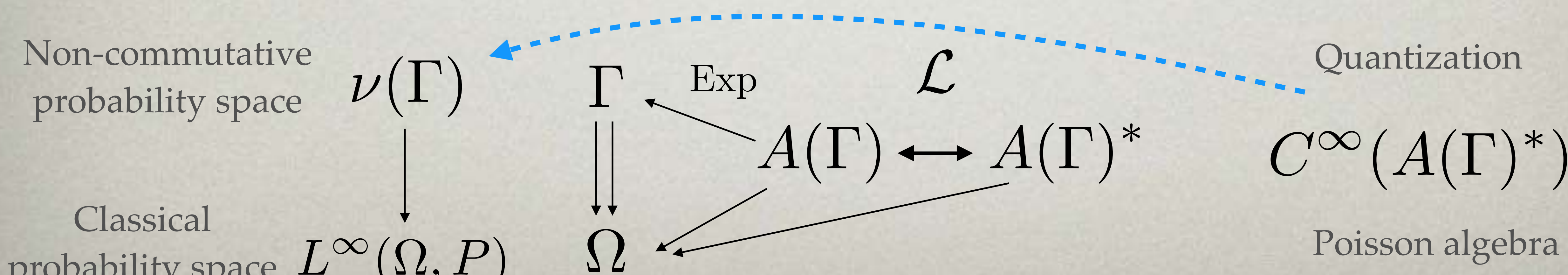
$\mathcal{P}(\mathcal{M}) = \{p = p^\dagger = p^2\}$ lattice of projections

Classical system $(\mathcal{P}(\mathcal{M}), \vee, \wedge, ')$ is Boolean

Birkhoff, G.; J., Von Neumann. *The Logic of Quantum Mechanics.* Annals of Mathematics (1936) **37**, 823–843

Theorem. The system described by the measure groupoid $(\Gamma \rightrightarrows \Omega, \nu)$ is classical iff it is totally disconnected and its isotropy groups are Abelian, in which case the von Neumann algebra $\nu(\Gamma)$ is Abelian. $\nu(\Gamma) \cong L^\infty(\widetilde{\Omega}, \mu)$

$\Gamma \rightrightarrows \Omega$ Lie groupoid, $(A(\Gamma), \mu)$ its Lie algebroid.



N.P. Landsman, *Deformation quantization and the Baum-Connes conjecture*, Comm. Math. Phys. **237** (2003) 87–103.

F.M. Ciaglia, F. Di Cosmo, AI, G. Marmo. *Evolution of Classical and Quantum States in the Groupoid Picture of Quantum Mechanics*, Entropy, **22** (11), 1292 (2020)

4. Dynamics

Groupoid $(\Gamma \rightrightarrows \Omega, [\nu])$ (Quantum mechanical system)

One-parameter groups of automorphisms $\varphi_t \colon \nu(\Gamma) \to \nu(\Gamma)$

von Neumann algebras are dynamical

State ρ, GNS representation π_ρ supported on $\mathcal{H}_\rho$.

Fundamental vector (cyclic) $|0\rangle = [\mathbf{1}]$

Separating

$\nu(\Gamma) \cong \pi_\rho(\nu(\Gamma))|0\rangle \subset \mathcal{H}_\rho$

Tomita-Takesaki theory:

$J \colon \mathcal{H}_\rho \to \mathcal{H}_\rho$ antilinear isometry

$\Delta = \Delta^\dagger \geq 0$ non-negative self-adjoint

$$J\nu(\Gamma)J = \nu(\Gamma)' \qquad \Delta^{it}\nu(\Gamma)\Delta^{-it} = \nu(\Gamma)$$

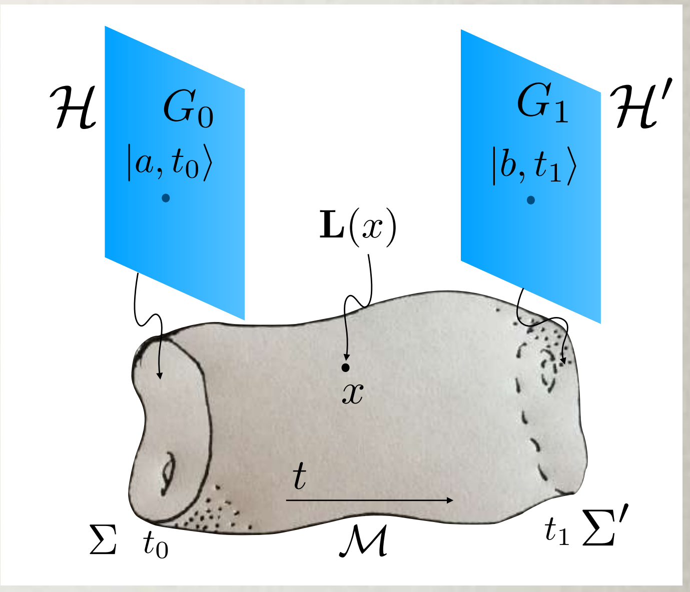
Modular flow: $\varphi_t(a) = \Delta^{it} a \Delta^{-it}$

Is this sensible?

4. Schwinger's variational Principle II

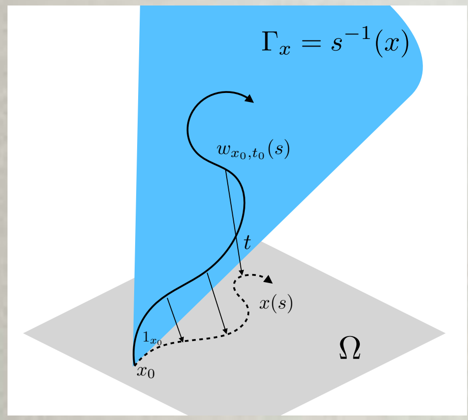
Dynamics via variational principles

Schwinger's quantum variational principle



$$\delta\varphi_{b,t_1;a,t_0} = i\langle b, t_1|\delta \int_{t_0}^{t_1} \mathbf{L}(s)ds|a, t_0\rangle$$
$$= \langle b, t_1|G_1 - G_0|a, t_0\rangle$$

Histories on groupoids $\quad \Gamma - \text{paths}$



$$w_{x_0,t_0} \colon [t_0, t_1] \to \Gamma \ , \ w_{x_0,t_0}(t_0) = 1_{x_0} \ , \ \gamma(s) = t(w_{x_0,t_0}(s))$$

$$s(w_{x_0,t_0}) = (x_0, t_0) \ , \ t(w_{x_0,t_0}) = (t(w_{x_0,t_0})(t_1)), t_1)$$

$$w'_{x_1,t_1} \circ w_{x_0,t_0}(s) = \begin{cases} w_{x_0,t_0}(s) & t_0 \leq s \leq t_1 \\ w'_{x_1,t_1}(s) \circ w_{x_0,t_0}(t_1) & t_1 \leq s \leq t_2 \end{cases}$$

The class of Γ-paths form a measurable category $\mathcal{C}(\Gamma) \rightrightarrows \Omega \times \mathbb{R}$, whose groupoidification $\mathcal{G}(\Gamma) \rightrightarrows \Omega \times \mathbb{R}$ is the groupoid of histories of $\Gamma \rightrightarrows \Omega$.

4. Schwinger's variational Principle III

Universal histories

Homomorphisms of groupoids: $w\colon P(\mathbb{R}) = \mathbb{R}\times\mathbb{R} \to \Gamma$, $w(s,t)\colon x(t) \to x(s)$

Γ-path: $w_{x_0,t_0}(s) = w(s,t_0)$, $x_0 = x(t_0)$.

Histories as Lie algebroid-paths

$w_*\colon A(P(\mathbb{R})) = T\mathbb{R} \to A(\Gamma)$, $A(\Gamma)$-path.

Dirac-Feynman-Schwinger states

$\mathcal{G}(\Gamma)$ groupoid of histories

$$\varphi(w) = \sqrt{p(s(w))p(t(w))}\,e^{i\mathscr{S}(w)}$$

p probability distribution on $\Omega\times\mathbb{R}$

$\mathscr{S}$ is an action, i.e., $\mathscr{S}(w_2\circ w_1) = \mathscr{S}(w_2) + \mathscr{S}(w_1)$, $\mathscr{S}(w^{-1}) = -\mathscr{S}(w)$

Theorem. The function $\varphi\colon \mathcal{G}(\Gamma) \to \mathbb{C}$ is positive-definite, hence it defines a state called a DFS state.

F.M. Ciaglia, AI, G. Marmo. *Schwinger's Picture of Quantum Mechanics III: The Statistical Interpretation.* Int. J. of Geom. Meth. in Mod. Phys. (2019), **16**.

4. Schwinger's variational Principle IV

Main example: $\mathscr{S}(w) = \int_{t_0}^{t_1} \mathcal{L}(\dot{w}(s) \circ w^{-1}(s))ds$

GNS representation associated to a DFS state

$\mathcal{H}_\varphi = L^2(\Omega \times \mathbb{R}, p)$ Space of wave-functions

$\Psi \colon \nu(\mathcal{G}(\Gamma)) \to \mathcal{H}_\varphi$ $\quad a \colon \mathcal{G}(\Gamma) \to \mathbb{C}$

$$\Psi_a(x,t) = \int_{\mathcal{G}(\Gamma)^{(x,t)}} a(w) e^{\frac{i}{\hbar}\mathscr{S}(w)} d\nu(w)$$

Feynman's representation of wave functions

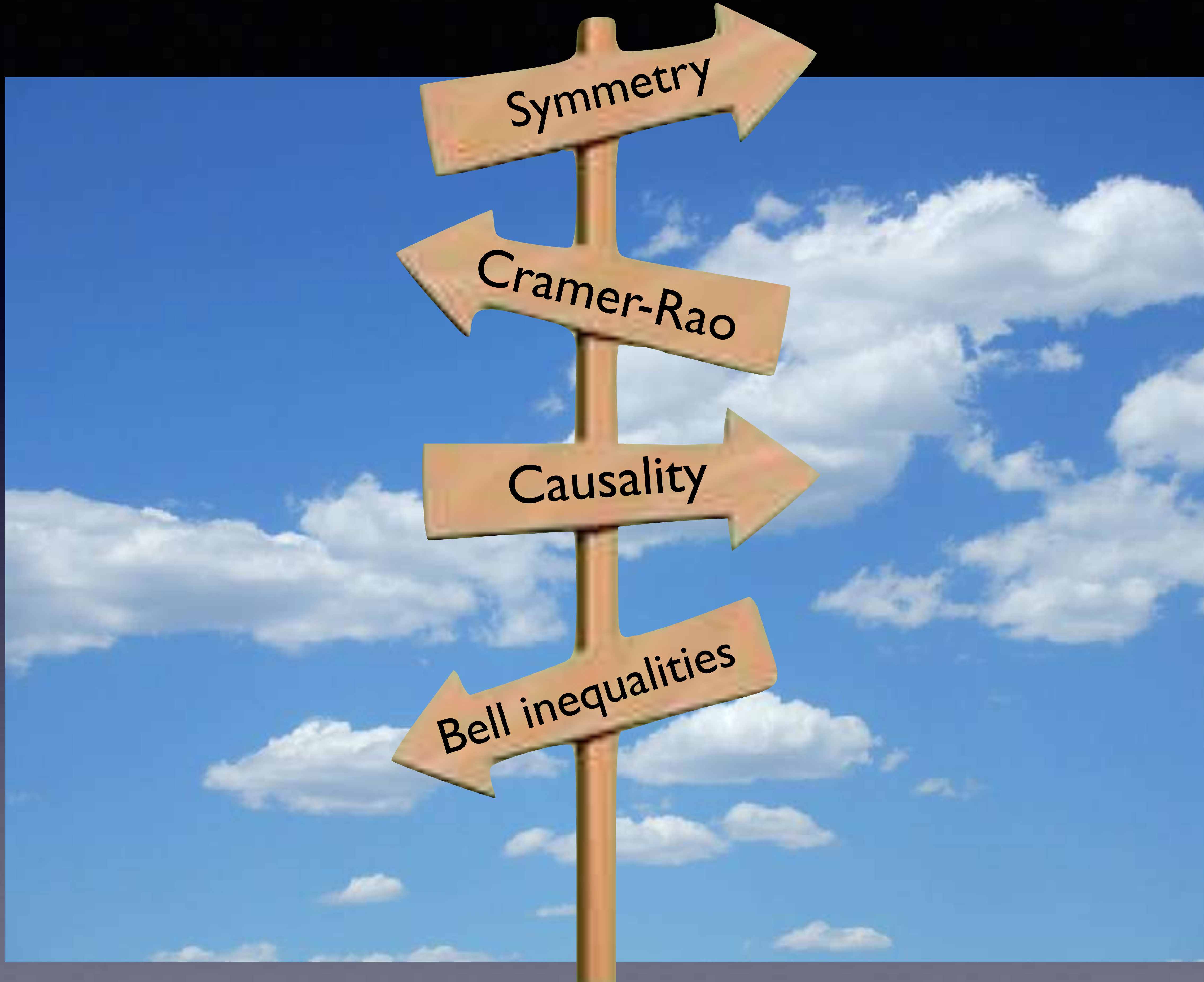
$\ker \Psi = \mathcal{J}_\varphi$ $\quad \mathcal{H}_{\mathrm{DFS}} = \overline{\nu(\mathcal{G}(\Gamma))/\mathcal{J}_\varphi} = L^2(\Omega \times \mathbb{R}, p)$

$\pi_\varphi(a)\Psi_b = \Psi_{a\star b}$

Theorem. A DFS function φ defines a dynamics in the sense of Tomita-Takesaki (generalised) theory on the space of wave functions $\mathcal{H}_{\mathrm{DFS}}$.

F.M. Ciaglia, F. Di Cosmo, AI., G. Marmo, L. Schiavone, A. Zampini. (2021). *A quantum route to the classical Lagrangian formalism.* Modern Physics Letters A, **36**(15), 2150091

F.M. Ciaglia, F. di Cosmo, A. Ibort, G. Marmo. *The groupoidal picture of Quantum Mechanics.* J. Geom. Phys., **197**, 105095 (2024).


Symmetry
Cramer-Rao
Causality
Bell inequalities

Thanks for your attention!