Universal property of Kasparov bivariant
K-theory

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Abstract

It will be proved that Kasparov’s bivariant $K$-theory is the theory
of satellites of the Grothendieck functor of homotopy classes of homo-
morphisms with respect to pre(co)sheaves of semi-split extensions of
separable $C^*$-algebras.

To this end the theory of satellites of arbitrary functors with respect to
(co)presheaves of categories (constructed in [2]) will be used.

In what follows we will work in the category $\mathcal{A}_{C^*}$ of separable $C^*$-algebras.
So all considered $C^*$-algebras will be separable. The basic notion we shall need
is the notion of semi-split extension of $C^*$-algebras.

Recall some definitions and results concerning extensions of $C^*$-algebras
([1,3,4]) needed to expose the main theorem.

Let

$$0 \longrightarrow B \xrightarrow{\varphi} X \xrightarrow{\psi} A \longrightarrow 0$$

(1)

be an extension of $A$ by $B$, i.e. the sequence (1) is an exact sequence of $C^*$-
algebras. It will be said that (1) is a split extension if there is a commutative
diagram of $C^*$-algebras.

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where $0 : B \to B$ is the trivial map. We will investigate only extensions of the form $E : 0 \to K \otimes B \overset{\varphi}{\to} X \overset{\psi}{\to} A \to 0$ where $K$ is the $C^*$-algebra of compact operators on the infinite dimensional Hilbert space and $K \otimes B$ is the spatial tensor product of $K$ and $B$.

Two extensions $E$ and $E'$ of $A$ by $K \otimes B$ will be called isomorphic if there is a commutative diagram

\[
\begin{array}{c}
0 \to B \overset{\varphi}{\to} X \overset{\psi}{\to} A \to 0 \\
\downarrow 0 \quad \downarrow \alpha \quad \| \quad \|
\end{array}
\]

\[
0 \to B \overset{\varphi}{\to} X \overset{\psi}{\to} A \to 0
\]

Let $E^1(A, B)$ be the set of equivalence classes of isomorphic extensions of $A$ by $K \otimes B$. If $f : A' \to A$ is a homomorphism of $C^*$-algebras the map

\[
E^1(f, B) : E^1(A, B) \to E^1(A', B)
\]

is defined in the usual way. Namely for $E : 0 \to K \otimes B \overset{\varphi}{\to} X \overset{\psi}{\to} A \to 0$ take the fiber product $X'$ of $X \overset{\psi}{\to} A \leftarrow f : A'$. Then $E^1(f, B)([E]) = [E']$ where $E' : 0 \to K \otimes B \overset{\varphi'}{\to} X' \overset{\psi'}{\to} A' \to 0$ with $\varphi'$ and $\psi'$ natural maps. $E^1(-, B)$ becomes a contravariant functor from $\mathcal{A}_C^*$ to the category $\text{Sets}$.

For any extension (1) of $C^*$-algebras there is a uniquely defined commutative diagram

\[
\begin{array}{c}
E : 0 \to B \overset{\varphi}{\to} X \overset{\psi}{\to} A \to 0 \\
\downarrow \quad \| \quad \downarrow \tau_E
\end{array}
\]

\[
E : 0 \to B \overset{\sigma}{\to} M(B) \overset{\eta}{\to} O(B) \to 0
\]

where $M(B)$ is the multiplier algebra of $B$, $\sigma$ is the natural injection and $\eta : M(B) \to O(B) = M(B)\sigma(B)$ is the canonical surjection. The homomorphism $\tau_E$ is called the Busby invariant associated to the given extension $E$ of $A$ by $B$. 
$E^1(A,B)$ can be defined also as a covariant functor in the second variable. In effect let $g : B \to B'$ be a homomorphism of $C^*$-algebras. Take by Lemma 1.2 [4] the homomorphism 

$$(K \otimes g)\neq : M(K \otimes B) \to M(K \otimes B').$$

For $E : 0 \to K \otimes B \xrightarrow{\psi} X \xrightarrow{\psi} A \to 0$ one gets a commutative diagram

\[
\begin{array}{cccccc}
E : 0 & \to & K \otimes B & \xrightarrow{\psi} & X & \xrightarrow{\psi} & A & \to 0 \\
0 & \to & K \otimes B & \to & M(K \otimes B) & \to & O(K \otimes B) & \to 0 \\
0 & \to & K \otimes B' & \to & M(K \otimes B') & \to & O(K \otimes B') & \to 0 \\
\end{array}
\]

with $(K \otimes g)\neq : K \otimes B \to K \otimes B'$ and let $E'$ be the extension of $A$ by $K \otimes B'$ whose Busby invariant is $\lambda g\tau_E$. Then define

$$E^1(A,g) : E^1(A,B) \to E^1(A,B')$$

by $[E] \mapsto [E']$. So $E^1(A,-)$ becomes a covariant functor from $A^1_{C^*}$ to $Sets$. A sum $\oplus$ is defined on the set $E^1(A,B)$ as follows. Let $\tau_{E_1}$ and $\tau_{E_2}$ be the Busby invariant of $E_1$ and $E_2$ respectively where $[E_1], [E_2] \in E^1(A,B)$. Consider the homomorphism $\tau : A \to O(K \otimes B)$ given by

$$\tau(a) = \begin{pmatrix} \tau_{E_1}(a) & 0 \\ 0 & \tau_{E_2}(a) \end{pmatrix} \in M_2 \otimes O(K \otimes B) \approx O(K \otimes B)$$

and take the extension $E$ denoted by $E_1 \oplus E_2$ with Busby invariant $\tau$. Define

$$[E_1] \oplus [E_2] = [E].$$

We arrive to the definition of a semi-split extension of $A$ by $K \otimes B$. Let $A$ and $B$ be $C^*$-algebras. An extension $E$ of $A$ by $K \otimes B$ is called a semi-split extension if there is an extension $E_-$ of $A$ by $K \otimes B$ such that $E \oplus E_-$ is a split extension.

It will be said that two semi-split extensions $E_1$ and $E_2$ of $A$ by $K \otimes B$ are unitary equivalent up to splitting if there exists split extensions $F_1$, $F_2$ of $A$ by $K \otimes B$ and a unitary element $u \in M(K \otimes B)$ such that there is a commutative diagram
where \( \text{ad } u \) is a derivation given by \( x \mapsto \sigma^{-1}(u \sigma(x) u^*) \) with \( x \in K \otimes B \) and \( \sigma : K \otimes B \to M(K \otimes B) \).

Let \( \text{ext}^1(A, B) \) be the set of semi-split extensions of \( A \) by \( K \otimes B \). Then \( \text{ext}^1(\_ , \_ ) \) is a subbifunctor of \( E^1(\_ , \_ ) \). Moreover \( \text{ext}^1(A, B) \) is a commutative monoid under the sum \( \oplus \) and its quotient set \( \text{Ext}^1(A, B) \) by the unitary equivalence up to splitting becomes an abelian group with sum induced by \( \oplus \) and it is a bifunctor from \( \mathbb{A}^*_C \) to the category \( \mathbb{A}^*_C \) of abelian groups. It was proved by Kasparov [3] that in fact \( \text{Ext}^1(A, B) \) is isomorphic to \( KK^1(A, B) \) and so is a homotopy functor under both variables.

Define a presheaf \( \mathcal{G} \) of categories over the category \( \mathbb{A}^*_C \) of separable \( C^* \)-algebras as follows. For any \( A \in \text{Ob} \mathbb{A}^*_C \), the objects of the category \( \mathcal{G}(A) \) are semi-split extensions \( E \) of the \( C^* \)-algebra \( A \)

\[
E : 0 \to K \otimes X \xrightarrow{\varphi} Y \xrightarrow{\psi} A \to 0.
\]

A morphism of \( \mathcal{G}(A) \) is a triple \((\alpha, \beta, 1_A) : E \to E'\) such that the diagram

\[
\begin{array}{ccc}
E : 0 & \to & K \otimes X \xrightarrow{\varphi} Y \xrightarrow{\psi} A \to 0 \\
& \downarrow \alpha & \downarrow \beta & \| \\
E' : 0 & \to & K \otimes X' \xrightarrow{\varphi'} Y' \xrightarrow{\psi'} A \to 0
\end{array}
\]

is commutative. If \( f : A' \to A \) is a homomorphism of \( C^* \)-algebras then the covariant functor \( \mathcal{G}(f) : \mathcal{G}(A) \to \mathcal{G}(A') \) is given by

\[
\mathcal{G}(f)(E) = \text{ext}^1(f, K \otimes X)(E)
\]

for \( E : 0 \to K \otimes X \xrightarrow{\varphi} Y \xrightarrow{\psi} A \to 0 \in \text{Ob} \mathcal{G}(A) \) and for a morphism \( E \to E' \) of \( \mathcal{G}(A) \) the morphism \( \mathcal{G}(f)(E) \to \mathcal{G}(f)(E') \) is defined in a natural way. The trace \((S, s)\) in the category \( \mathbb{A}^*_C \) of the presheaf \( \mathcal{G} \) is given by \( S_A(E) = K \otimes X \) for \( E : 0 \to K \otimes X \xrightarrow{\varphi} Y \xrightarrow{\psi} A \to 0 \) and for any \( C^* \)-algebra \( A \), and \( S_A(\alpha, \beta, 1_A) = \alpha \) for \((\alpha, \beta, 1_A) : E \to E'\). If \( f : A' \to A \) is a homomorphism of \( C^* \)-algebras then for \( E : 0 \to K \otimes X \xrightarrow{\varphi} Y \xrightarrow{\psi} A \to 0 \)
and $A \in Ob \mathcal{A}_C$, the homomorphism $s_E(f) : S_A(G(f)(E)) \rightarrow S_A(E)$ is the identity map $1_K : K \otimes X \rightarrow K \otimes X$.

We see that the presheaf $G(S, s)$ of semi-split extensions over $\mathcal{A}_C^s$ is completely analogous to the presheaf of short exact sequences of modules with its trace over the category of modules [2].

Let $A$ and $B$ be two $C^*$-algebras and let $\text{hom}(A, K \otimes B)$ be the set of all $C^*$-homomorphisms from $A$ into $K \otimes B$. Let $\text{hom}^*(A, K \otimes B)$ be the set of equivalence classes of homotopic $C^*$-homomorphisms from $A$ into $K \otimes B$. Then one can define on $\text{hom}(A, K \otimes B)$ a sum $\oplus$ by $f \oplus g = h$ where

$$h(a) = \begin{pmatrix} f(a) & 0 \\ 0 & g(a) \end{pmatrix} \in M_2 \otimes (K \otimes B) \approx K \otimes B$$

for $a \in A$ and $f, g \in \text{hom}(A, K \otimes B)$. The sum $\oplus$ induces on $\text{hom}^*(A, K \otimes B)$ a structure of commutative monoid and let $K \text{hom}^*(A, K \otimes B)$ be its Grothendieck group. One gets a bifunctor $K \text{hom}^*(\mathcal{A}_C^s, \mathbf{Ab})$ from $\mathcal{A}_C^s$ to $\mathbf{Ab}$.

**Definition 1.** It will be said that a connected pair $(T^0, \vartheta, T^1)$ of contravariant functors from $\mathcal{A}_C^s$ to $\mathbf{Ab}$ with respect to the presheaf $G(S, s)$ of semi-split extensions satisfies condition (i) if for any unitary element $u \in M(K \otimes B)$ the equality

$$\delta_E T^0(\text{ad } u) = \delta_E$$

holds for any $E : 0 \rightarrow K \otimes B \xrightarrow{\varphi} X \xrightarrow{\psi} A \rightarrow 0 \in \text{Ob } G(A)$, $A \in Ob \mathcal{A}_C^s$.

Denote by $\mathbf{L}$ be the class of all connected pairs of functors satisfying condition (i). Let $E : 0 \rightarrow K \otimes X \xrightarrow{\varphi} Y \xrightarrow{\psi} A \rightarrow 0 \in G(A)$. Define a homomorphism

$$\vartheta_E : \text{hom}^*(K \otimes X, K \otimes B) \rightarrow \text{Ext}^1(A, B)$$

by $\vartheta_E([g]) = \text{ext}^1(A, g)(E)$ for $g : K \otimes X \rightarrow K \otimes B$ and extend $\vartheta_E$ to a homomorphism

$$\vartheta_E : K \text{hom}^*(K \otimes X, K \otimes B) \rightarrow \text{Ext}^1(A, B).$$

**Theorem 2.** The pair $(K \text{hom}^*(\mathcal{A}_C^s, \mathbf{Ab}), \vartheta), (\text{Ext}^1(\mathcal{A}_C^s, \mathbf{Ab}))$ is a right universal pair of contravariant functors with respect to the class $\mathbf{L}$.  

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Let $H$ be the copresheaf of categories of semi-split extensions over the category $\mathcal{A}_C^s$ of stable separable $C^*$-algebras with its (dually defined) natural trace $(S,s)$ in the category $\mathcal{A}_C^s$.

**Definition 3.** It will be said that a connected pair $(T_0, \kappa, T_1)$ of functors $T_0 : \mathcal{A}_C^s \to \mathbf{Ab}$, $T_1 : \mathcal{A}_C^s \to \mathbf{Ab}$ with respect to $H(S,s)$ satisfies condition (j) if for any unitary element $u \in M(K \otimes B)$ the equality

$$T_1(ad\ u)\kappa_E = \kappa_E$$

holds for any $E : K \otimes B \to X \to Y \in \text{Ob} H(K \otimes B)$, $K \otimes B \in \text{Ob} \mathcal{A}_C^s$.

For any $E : K \otimes B \to X \to Y$ define a connecting homomorphism

$$\eta_E : K \text{hom}^s(A,Y) \to \text{Ext}^1(A,B)$$

given by $[g] \mapsto [\text{ext}^1(g,K \otimes B)]$ where $A$ is a separable $C^*$-algebra.

**Theorem 4.** The pair $(K \text{hom}^s(A,-), \eta, \text{Ext}^1(A,-))$ is a universal pair of functors with respect to the class of all connected pairs of functors satisfying condition (j).

Note that Theorems 2 and 4 can be extended in a natural way to the functors $\text{Ext}^n$.

References