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Introduction to Homological algebra

Homological Algebra

An introduction.

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This text is a first introduction to *homological algebra*, assuming only very basic prerequisites. For instance we do recall in some detail basic definitions and constructions in the theory of abelian groups and modules, though of course a prior familiarity with these ingredients will be helpful. Also we use very little category theory, if it all. Where universal constructions do appear we spell them out explicitly in components and just mention their category-theoretic names for those readers who want to dig deeper. We do however freely use the words *functor* and *commuting diagram*. The reader unfamiliar with these elementary notions should click on these keywords and follow the hyperlink to the explanation right now.

1. I) Motivation

The subject of *homological algebra* may be motivated by its archetypical application, which is the *singular homology* of a *topological space* X. This example illustrates homological algebra as being concerned with the *abelianization* of what is called the *homotopy theory* of X.

So we begin with some basic concepts in homotopy theory in section *1) Homotopy type of topological spaces*. Then we consider the “abelianization” of this setup in *2) Simplicial and abelian homology*.

Together this serves to motivate many constructions in homological algebra, such as centrally *chain complexes*, *chain maps* and *homology*, but also *chain homotopies*, *mapping cones* etc, which we discuss in detail in chapter II below. In the bulk we develop the general theory of homological algebra in chapter III and chapter IV. Finally we come back to a systematic discussion of the relation to homotopy theory at the end in chapter V. A section VI) Outlook is appended for readers interested in the grand scheme of things.

We do use some basic category theory *language* in the following, but no actual category *theory*. The reader should know what a *category* is, what a *functor* is and what a commuting diagram is. These concepts are more elementary than any genuine concept in homological algebra to appear below and of general use. Where we do encounter universal constructions below we call them by their category-theoretic name but always spell them out in components explicity.

1) Homotopy type of topological spaces

This section reviews some basic notions in topology and homotopy theory. These will all serve as blueprints for corresponding notions in homological algebra.

Definition 1.1. A **topological space** is a set X equipped with a set of subsets $U \subset X$, called **open sets**, which are closed under

1. finite intersections
2. arbitrary unions.

Example 1.2. The Cartesian space $\mathbb{R}^n$ with its standard notion of open subsets given by unions of open balls $D^n \subset \mathbb{R}^n$.

Definition 1.3. For $Y \hookrightarrow X$ an injection of sets and $\{U_i \subset X\}_{i \in I}$ a topology on X, the subspace topology on Y is $\{U_i \cap Y \subset Y\}_{i \in I}$.

Definition 1.4. For $n \in \mathbb{N}$, the **topological n-simplex** is, up to homeomorphism, the topological space whose underlying set is the subset

$$\Delta^n \coloneqq \{\vec{x} \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i = 1 \text{ and } \forall i . x_i \geq 0\} \subset \mathbb{R}^{n+1}$$

of the Cartesian space $\mathbb{R}^{n+1}$, and whose topology is the subspace topology induces from the canonical topology in $\mathbb{R}^{n+1}$.

Example 1.5. For $n = 0$ this is the point, $\Delta^0 = *$.

For $n = 1$ this is the standard interval object $\Delta^1 = [0, 1]$.

For $n = 2$ this is the filled triangle.

For $n = 3$ this is the filled tetrahedron.

Definition 1.6. A homomorphisms between topological spaces $f : X \to Y$ is a **continuous function**:

a function $f : X \to Y$ of the underlying sets such that the preimage of every open set of Y is an open set of X.

Topological spaces with continuous maps between them form the category Top.

Definition 1.7. For $n \in \mathbb{N}$, $\mathrm{n} \geq 1$ and $0 \leq k \leq n$, the k**th** $(n-1)$**-face (inclusion)** of the topological n-simplex, def. 1.4, is the subspace inclusion

$$\delta_k : \Delta^{n-1} \hookrightarrow \Delta^n$$

induced under the coordinate presentation of def. 1.4, by the inclusion

$$\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$$

which "omits" the kth canonical coordinate:

$$(x_0, \cdots, x_{n-1}) \mapsto (x_0, \cdots, x_{k-1}, 0, x_k, \cdots, x_{n-1}) .$$

Example 1.8. The inclusion

$$\delta_0 : \Delta^0 \to \Delta^1$$

is the inclusion

$$\{1\} \hookrightarrow [0, 1]$$

of the "right" end of the standard interval. The other inclusion

$$\delta_1 : \Delta^0 \to \Delta^1$$

is that of the "left" end $\{0\} \hookrightarrow [0, 1]$.

Definition 1.9. For $n \in \mathbb{N}$ and $0 \leq k < n$ the k**th degenerate** (n)**-simplex (projection)** is the surjective map

$$\sigma_k : \Delta^n \to \Delta^{n-1}$$

induced under the barycentric coordinates of def. 1.4 under the surjection

$$\mathbb{R}^{n+1} \to \mathbb{R}^n$$

which sends

$$(x_0, \cdots, x_n) \mapsto (x_0, \cdots, x_k + x_{k+1}, \cdots, x_n) .$$

Definition 1.10. For $X \in$ Top and $n \in \mathbb{N}$, a **singular n-simplex** in X is a continuous map

$$\sigma : \Delta^n \to X$$

from the topological n-simplex, def. 1.4, to X.

Write

$$(\mathrm{Sing}\, X)_n \coloneqq \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X)$$

for the set of singular n-simplices of X.

As n varies, this forms the *singular simplicial complex* of X. This is the topic of the next section, see def. def. 1.33.

Definition 1.11. For $f, g : X \to Y$ two continuous functions between topological spaces, a **left homotopy** $\eta : f \Rightarrow g$ is a commuting diagram in Top of the form

$$\begin{array}{ccc} X & & \\ {}^{(\mathrm{id},\delta_0)}\downarrow & \searrow^{f} & \\ X \times \Delta^1 & \overset{\eta}{\to} & Y \\ {}^{(\mathrm{id},\delta_1)}\uparrow & \nearrow_{g} & \\ X & & \end{array} .$$

Remark 1.12. In words this says that a homotopy between two continuous functions f and g is a continuous 1-parameter deformation of f to g. That deformation parameter is the canonical coordinate along the interval $[0,1]$, hence along the "length" of the cylinder $X \times \Delta^1$.

Proposition 1.13. *Left homotopy is an equivalence relation on* $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$.

The fundamental invariants of a topological space in the context of *homotopy theory* are its *homotopy groups*. We first review the first homotopy group, called the *fundamental group* of X:

Definition 1.14. For X a topological space and $x : * \to X$ a point. A loop in X based at x is a continuous function

$$\gamma : \Delta^1 \to X$$

from the topological 1-simplex, such that $\gamma(0) = \gamma(1) = x$.

A *based homotopy* between two loops is a homotopy

$$\begin{array}{ccc} \Delta^1 & & \\ \downarrow^{(\mathrm{id},\delta_0)} & \searrow^{f} & \\ \Delta^1 \times \Delta^1 & \overset{\eta}{\to} & X \\ \uparrow^{(\mathrm{id},\delta_1)} & \nearrow_{g} & \\ \Delta^1 & & \end{array}$$

such that $\eta(0, -) = \eta(1, -) = x$.

Proposition 1.15. *This notion of based homotopy is an equivalence relation.*

Proof. This is directly checked. It is also a special case of the general discussion at *homotopy*. ▮

Definition 1.16. Given two loops $\gamma_1, \gamma_2 : \Delta^1 \to X$, define their **concatenation** to be the loop

$$\gamma_2 \cdot \gamma_1 : t \mapsto \begin{cases} \gamma_1(2t) & (0 \leq t \leq 1/2) \\ \gamma_2(2(t - 1/2)) & (1/2 \leq t \leq 1) \end{cases} .$$

Proposition 1.17. *Concatenation of loops respects based homotopy classes where it becomes an associative, unital binary pairing with inverses, hence the product in a group.*

Definition 1.18. For X a topological space and $x \in X$ a point, the set of based homotopy equivalence classes of based loops in X equipped with the group structure from prop. 1.17 is the **fundamental group** or **first homotopy group** of (X, x), denoted

$$\pi_1(X, x) \in \mathrm{Grp} .$$

Example 1.19. The fundamental group of the point is trivial: $\pi_1(*) = *$.

Example 1.20. The fundamental group of the circle is the group of integers $\pi_1(S^1) \simeq \mathbb{Z}$.

This construction has a fairly straightforward generalizations to "higher dimensional loops".

Definition 1.21. Let X be a topological space and $x\colon * \to X$ a point. For $(1 \leq n) \in \mathbb{N}$, the **$n$th homotopy group** $\pi_n(X,x)$ of X at x is the group:

- whose elements are left-homotopy equivalence classes of maps $S^n \to (X,x)$ in $\mathrm{Top}^{*/}$;
- composition is given by gluing at the base point (wedge sum) of representatives.

The 0th homotopy group is taken to be the set of connected components.

Example 1.22. For $n = 1$ this reproduces the definition of the fundamental group of def. 1.18.

The homotopy theory of topological spaces is all controled by the following notion. The abelianization of this notion, the notion of *quasi-isomorphism* discussed in def. 2.91 below is central to homological algebra.

Definition 1.23. For $X, Y \in$ Top two topological spaces, a continuous function $f\colon X \to Y$ between them is called a **weak homotopy equivalence** if

1. f induces an isomorphism of connected components

   $$\pi_0(f)\colon \pi_0(X) \overset{\simeq}{\to} \pi_0(Y)$$

   in Set;

2. for all points $x \in X$ and for all $(1 \leq n) \in \mathbb{N}$ f induces an isomorphism on homotopy groups

   $$\pi_n(f,x)\colon \pi_n(X,x) \overset{\simeq}{\to} \pi_n(Y, f(x))$$

   in Grp.

What is called *homotopy theory* is effectively the study of topological spaces not up to isomorphism (here: homeomorphism), but up to weak homotopy equivalence. Similarly, we will see that homological algebra is effectively the study of chain complexes not up to isomorphism, but up to quasi-isomorphism. But this is slightly more subtle than it may seem, in parts due to the following:

Proposition 1.24. *The existence of a weak homotopy equivalence from X to Y is a reflexive and transitive relation on Top, but it is not a symmetric relation.*

Proof. Reflexivity and transitivity are trivially checked. A counterexample to symmetry is the weak homotopy equivalence between the stanard circle and the pseudocircle. ■

But we can consider the genuine equivalence relation *generated* by weak homotopy equivalence:

Definition 1.25. We say two spaces X and Y have the same **(weak) homotopy type** if they are equivalent under the equivalence relation *generated* by weak homotopy equivalence.

Remark 1.26. Equivalently this means that X and Y have the same (weak) homotopy type if there exists a zigzag of weak homotopy equivalences

$$X \leftarrow \; \to \; \leftarrow \dots \to Y\ .$$

One can understand the homotopy type of a topological space just in terms of its homotopy groups and how they act on each other. (This data is called a Postnikov tower of X.) But computing and handling homotopy groups is in general hard, famously so already for the seemingly simple case of the *homotopy groups of spheres*. Therefore we now want to simplify the situation by passing to a "linear/abelian approximation".

2) Simplicial and singular homology

This section discusses how the "abelianization" of a topological space by *singular chains* gives rise to the notion of *chain complexes* and their *homology*.

Above in def. 1.10 we saw that to a topological space X is associated a sequence of sets

$$(\mathrm{Sing}\,X)_n \coloneqq \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X)$$

of singular simplices. Since the topological n-simplices Δ^n from def. 1.4 sit inside each other by the face inclusions of def. 1.7

$$\delta_k\colon \Delta^{n-1} \to \Delta^n$$

and project onto each other by the degeneracy maps, def. 1.9

$$\sigma_k\colon \Delta^{n+1} \to \Delta^n$$

we dually have functions

$$d_k \coloneqq \mathrm{Hom}_{\mathrm{Top}}(\delta_k, X)\colon (\mathrm{Sing}\,X)_n \to (\mathrm{Sing}\,X)_{n-1}$$

that send each singular n-simplex to its k-face and functions

$$s_k \coloneqq \mathrm{Hom}_{\mathrm{Top}}(\sigma_k, X) : (\mathrm{Sing}\, X)_n \to (\mathrm{Sing}\, X)_{n+1}$$

that regard an n-simplex as beign a degenerate ("thin") $(n+1)$-simplex. All these sets of simplices and face and degeneracy maps between them form the following structure.

Definition 1.27. A **simplicial set** $S \in \mathrm{sSet}$ is

- for each $n \in \mathbb{N}$ a set $S_n \in \mathrm{Set}$ – the **set of n-simplices**;
- for each injective map $\delta_i : \overline{n-1} \to \overline{n}$ of totally ordered sets $\bar{n} \coloneqq \{0 < 1 < \cdots < n\}$

  a function $d_i : S_n \to S_{n-1}$ – the ith **face map** on n-simplices;
- for each surjective map $\sigma_i : \overline{n+1} \to \bar{n}$ of totally ordered sets

  a function $\sigma_i : S_n \to S_{n+1}$ – the ith **degeneracy map** on n-simplices;

such that these functions satisfy the *simplicial identities*.

Definition 1.28. The **simplicial identities** satisfied by face and degeneracy maps as above are (whenever these maps are composable as indicated):

1. $d_i \circ d_j = d_{j-1} \circ d_i$ if $i < j$,
2. $s_i \circ s_j = s_j \circ s_{i-1}$ if $i > j$.
3. $d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1 \\ s_j \circ d_{i-1} & \text{if } i > j+1 \end{cases}$

It is straightforward to check by explicit inspection that the evident injection and restriction maps between the sets of singular simplices make $(\mathrm{Sing}\, X)_\bullet$ into a simplicial set. We now briefly indicate a systematic way to see this using basic category theory, but the reader already satisfied with this statement should jump ahead to the abelianization of $(\mathrm{Sing}\, X)_n$ in prop. 1.37 below.

Definition 1.29. The **simplex category** Δ is the full subcategory of Cat on the free categories of the form

$$\begin{aligned} [0] &\coloneqq \{0\} \\ [1] &\coloneqq \{0 \to 1\} \\ [2] &\coloneqq \{0 \to 1 \to 2\} \\ &\vdots \end{aligned}.$$

Remark 1.30. This is called the "simplex category" because we are to think of the object $[n]$ as being the "spine" of the n-simplex. For instance for $n = 2$ we think of $0 \to 1 \to 2$ as the "spine" of the triangle. This becomes clear if we don't just draw the morphisms that *generate* the category $[n]$, but draw also all their composites. For instance for $n = 2$ we have_

$$[2] = \left\{ \begin{matrix} & & 1 & & \\ & \nearrow & & \searrow & \\ 0 & & \to & & 2 \end{matrix} \right\}.$$

Proposition 1.31. *A functor*

$$S : \Delta^{\mathrm{op}} \to \mathrm{Set}$$

from the opposite category of the simplex category to the category Set of sets is canonically identified with a simplicial set, def. 1.27.

Proof. One checks by inspection that the simplicial identities characterize precisely the behaviour of the morphisms in $\Delta^{\mathrm{op}}([n], [n+1])$ and $\Delta^{\mathrm{op}}([n], [n-1])$. ■

This makes the following evident:

Example 1.32. The topological simplices from def. 1.4 arrange into a *cosimplicial object in Top*, namely a functor

$$\Delta^\bullet : \Delta \to \mathrm{Top}\,.$$

With this now the structure of a simplicial set on the singular simplices $(\mathrm{Sing}\, X)_\bullet$, def. 1.10, is manifest: it is just the *nerve* of X with respect to $\Delta^\bullet$, namely:

Definition 1.33. For X a topological space its **simplicial set of singular simplicies** (often called the **singular simplicial complex**)

$$(\mathrm{Sing}\, X)_\bullet : \Delta^{\mathrm{op}} \to \mathrm{Set}$$

is given by composition of the functor from example 1.32 with the hom functor of Top:

$$(\operatorname{Sing} X) : [n] \mapsto \operatorname{Hom}_{\mathrm{Top}}(\Delta^n, X) \ .$$

Remark (aside) 1.34. It turns out that homotopy type of the topological space X is entirely captured by its singular simplicial complex $\operatorname{Sing} X$ (this is the content of the *homotopy hypothesis-theorem*).

Now we abelianize the singular simplicial complex $(\operatorname{Sing} X)_\bullet$ in order to make it *simpler* and hence more tractable.

Definition 1.35. A **formal linear combination** of elements of a set $S \in$ Set is a function

$$a : S \to \mathbb{Z}$$

such that only finitely many of the values $a_s \in \mathbb{Z}$ are non-zero.

Identifying an element $s \in S$ with the function $S \to \mathbb{Z}$, which sends s to $1 \in \mathbb{Z}$ and all other elements to 0, this is written as

$$a = \sum_{s \in S} a_s \cdot s \ .$$

In this expression one calls $a_s \in \mathbb{Z}$ the coefficient of s in the formal linear combination.

Remark 1.36. For $S \in$ Set, the **group of formal linear combinations** $\mathbb{Z}[S]$ is the group whose underlying set is that of formal linear combinations, def. 1.35, and whose group operation is the pointwise addition in $\mathbb{Z}$:

$$(\sum_{s \in S} a_s \cdot s) + (\sum_{s \in S} b_s \cdot s) = \sum_{s \in S} (a_s + b_s) \cdot s \ .$$

For the present purpose the following statement may be regarded as just introducing different terminology for the group of formal linear combinations:

Proposition 1.37. *The group $\mathbb{Z}[S]$ is the **free abelian group** on S.*

Definition 1.38. For $S_\bullet$ a simplicial set, def. 1.27, the free abelian group $\mathbb{Z}[S_n]$ is called the group of (simplicial) **n-chains** on S.

Definition 1.39. For X a topological space, an n-chain on the singular simplicial complex $\operatorname{Sing} X$ is called a **singular n-chain** on X.

This construction makes the sets of simplices into abelian groups. But this allows to *formally add* the different face maps in the simplicial set to one single boundary map:

Definition 1.40. For S a simplicial set, its **alternating face map differential** in degree n is the linear map

$$\partial : \mathbb{Z}[S_n] \to \mathbb{Z}[S_{n-1}]$$

defined on basis elements $\sigma \in S_n$ to be the alternating sum of the simplicial face maps:

$$\partial \sigma \coloneqq \sum_{k=0}^{n} (-1)^k d_k \sigma \ . \qquad (1)$$

Proposition 1.41. *The simplicial identity, def. 1.28 part (1), implies that the alternating sum boundary map of def. 1.40 squares to 0:*

$$\partial \circ \partial = 0 \ .$$

Proof. By linearity, it is sufficient to check this on a basis element $\sigma \in S_n$. There we compute as follows:

$$\begin{aligned} \partial \partial \sigma &= \partial\left(\textstyle\sum_{j=0}^{n} (-1)^j d_j \sigma\right) \\ &= \textstyle\sum_{j=0}^{n} \sum_{i=0}^{n-1} (-1)^{i+j} d_i d_j \sigma \\ &= \textstyle\sum_{0 \leq i < j \leq n} (-1)^{i+j} d_i d_j \sigma + \sum_{0 \leq j \leq i < n} (-1)^{i+j} d_i d_j \sigma \\ &= \textstyle\sum_{0 \leq i < j \leq n} (-1)^{i+j} d_{j-1} d_i \sigma + \sum_{0 \leq j \leq i < n} (-1)^{i+j} d_i d_j \sigma \\ &= -\textstyle\sum_{0 \leq i \leq j < n} (-1)^{i+j} d_j d_i \sigma + \sum_{0 \leq j \leq i < n} (-1)^{i+j} d_i d_j \sigma \\ &= 0 \end{aligned} \ .$$

Here

1. the first equality is (1);
2. the second is (1) together with the linearity of d;

3. the third is obtained by decomposing the sum into two summands;

4. the fourth finally uses the simplicial identity def. 1.28 (1) in the first summand;

5. the fifth relabels the summation index j by $j+1$;

6. the last one observes that the resulting two summands are negatives of each other.

∎

Example 1.42. Let X be a topological space. Let $\sigma^1 : \Delta^1 \to X$ be a singular 1-simplex, regarded as a 1-chain

$$\sigma^1 \in C_1(X) .$$

Then its boundary $\partial\sigma \in H_0(X)$ is

$$\partial\sigma^1 = \sigma(0) - \sigma(1)$$

or graphically (using notation as for orientals)

$$\partial\left(\sigma(0) \stackrel{\sigma}{\to} \sigma(1)\right) = (\sigma(0)) - (\sigma(1)) .$$

In particular σ is a 1-cycle precisely if $\sigma(0) = \sigma(1)$, hence precisely if σ is a loop.

Let $\sigma^2 : \Delta^2 \to X$ be a singular 2-chain. The boundary is

$$\partial \left(\begin{array}{ccc} & \sigma(1) & \\ {}^{\sigma(0,1)}\nearrow & \Downarrow^{\sigma} & \searrow^{\sigma^{1,2}} \\ \sigma(0) & \underset{\sigma(0,2)}{\longrightarrow} & \sigma(2) \end{array} \right) = \left(\begin{array}{ccc} & \sigma(1) & \\ {}^{\sigma(0,1)}\nearrow & & \\ \sigma(0) & & \end{array} \right) - \left(\sigma(0) \underset{\sigma(0,2)}{\longrightarrow} \sigma(2) \right) + \left(\begin{array}{ccc} & \sigma(1) & \\ & & \searrow^{\sigma^{1,2}} \\ & & \sigma(2) \end{array} \right) .$$

Hence the boundary of the boundary is:

$$\begin{aligned} \partial\,\partial\sigma &= \partial\left(\left(\begin{array}{ccc} & \sigma(1) & \\ {}^{\sigma(0,1)}\nearrow & & \\ \sigma(0) & & \end{array} \right) - \left(\sigma(0) \underset{\sigma(0,2)}{\longrightarrow} \sigma(2) \right) + \left(\begin{array}{ccc} & \sigma(1) & \\ & & \searrow^{\sigma^{1,2}} \\ & & \sigma(2) \end{array} \right) \right) \\ &= \left(\sigma(0) \right) - \left(\sigma(1) \right) - \left(\sigma(0) \right) + \left(\sigma(2) \right) + \left(\sigma(1) \right) - \left(\sigma(2) \right) \\ &= 0 \end{aligned}$$

Definition 1.43. For S a simplicial set, we call the collection

1. of abelian groups of chains $C_n(S) \coloneqq \mathbb{Z}[S_n]$, prop. 1.37;

2. and boundary homomorphisms $\partial_n : C_{n+1}(S) \to C_n(X)$, def. 1.40

(for all $n \in \mathbb{N}$) the **alternating face map chain complex** of S:

$$C_\bullet(S) = [\cdots \stackrel{\partial_2}{\to} \mathbb{Z}[S_2] \stackrel{\partial_1}{\to} \mathbb{Z}[S_1] \stackrel{\partial_0}{\to} \mathbb{Z}[S_0]] .$$

Specifically for $S = \mathrm{Sing}\, X$ we call this the **singular chain complex** of X.

This motivates the general definition:

Definition 1.44. A *chain complex of abelian groups* $C_\bullet$ is a collection $\{C_n \in \mathrm{Ab}\}_n$ of abelian groups together with group homomorphisms $\{\partial_n : C_{n+1} \to C_n\}$ such that $\partial \circ \partial = 0$.

We turn to this definition in more detail in the next section. The thrust of this construction lies in the fact that the chain complex $C_\bullet(\mathrm{Sing}\, X)$ remembers the abelianized fundamental group of X, as well as aspects of the higher homotopy groups: in its *chain homology*.

Definition 1.45. For $C_\bullet(S)$ a chain complex as in def. 1.43, and for $n \in \mathbb{N}$ we say

- an n-chain of the form $\partial\sigma \in C(S)_n$ is an n-**boundary**;

- a chain $\sigma \in C_n(S)$ is an n**-cycle** if $\partial\sigma = 0$

  (every 0-chain is a 0-cycle).

By linearity of ∂ the boundaries and cycles form abelian sub-groups of the group of chains, and we write

$$B_n \coloneqq \mathrm{im}(\partial_n) \subset C_n(S)$$

for the group of n-boundaries, and

$$Z_n \coloneqq \ker(\partial_n) \subset C_(S)$$

for the group of n-cycles.

Remark 1.46. This means that a singular chain is a cycle if the formal linear combination of the oriented boundaries of all its constituent singular simplices sums to 0.

Remark 1.47. More generally, for R any unital ring one can form the degreewise free module $R[\mathrm{Sing}\,X]$ over R. The corresponding homology is the *singular homology with coefficients in R, denoted $H_n(X,R)$. This generality we come to below in the next section.*

Definition 1.48. For $C_\bullet(S)$ a chain complex as in def. 1.43 and for $n \in \mathbb{N}$, the **degree-n chain homology group** $H_n(C(S)) \in \mathrm{Ab}$ is the quotient group

$$H_n(C(S)) \coloneqq \frac{\ker(\partial_{n-1})}{\mathrm{im}(\partial_n)} = \frac{Z_n}{B_n}$$

of the n-cycles by the n-boundaries – where for $n = 0$ we declare that $\partial_{-1} \coloneqq 0$ and hence $Z_0 \coloneqq C_0$.

Specifically, the chain homology of $C_\bullet(\mathrm{Sing}\,X)$ is called the **singular homology** of the topological space X.

One usually writes $H_n(X,\mathbb{Z})$ or just $H_n(X)$ for the singular homology of X in degree n.

Remark 1.49. So $H_0(C_\bullet(S)) = C_0(S)/\mathrm{im}(\partial_0)$.

Example 1.50. For X a topological space we have that the degree-0 singular homology

$$H_0(X) \simeq \mathbb{Z}[\pi_0(X)]$$

is the free abelian group on the set of connected components of X.

Example 1.51. For X a connected, orientable manifold of dimension n we have

$$H_n(X) \simeq \mathbb{Z}\ .$$

The precise choice of this isomorphism is a choice of orientation on X. With a choice of orientation, the element $1 \in \mathbb{Z}$ under this identification is called the **fundamental class**

$$[X] \in H_n(X)$$

of the manifold X.

Definition 1.52. Given a continuous map $f: X \to Y$ between topological spaces, and given $n \in \mathbb{N}$, every singular n-simplex $\sigma: \Delta^n \to X$ in X is sent to a singular n-simplex

$$f_* \sigma : \Delta^n \overset{\sigma}{\to} X \overset{f}{\to} Y$$

in Y. This is called the **push-forward** of σ along f. Accordingly there is a push-forward map on groups of singular chains

$$(f_*)_n : C_n(X) \to C_n(Y)\ .$$

Proposition 1.53. *These push-forward maps make all diagrams of the form*

$$\begin{array}{ccc} C_{n+1}(X) & \overset{(f_*)_{n+1}}{\longrightarrow} & C_{n+1}(Y) \\ \downarrow^{\partial^X_n} & & \downarrow^{\partial^Y_n} \\ C_n(X) & \overset{(f_*)_n}{\longrightarrow} & C_n(Y) \end{array}$$

commute.

Proof. It is in fact evident that push-forward yields a functor of singular simplicial complexes

$$f_* : \mathrm{Sing}\,X \to \mathrm{Sing}\,Y\ .$$

From this the statement follows since $\mathbb{Z}[-]: \mathrm{sSet} \to \mathrm{sAb}$ is a functor. ∎

Therefore we have an "abelianized analog" of the notion of topological space:

Definition 1.54. For $C_\bullet, D_\bullet$ two chain complexes, def. 1.44, a homomorphism between them – called a **chain map** $f_\bullet: C_\bullet \to D_\bullet$ – is for each $n \in \mathbb{N}$ a homomorphism $f_n: C_n \to D_n$ of abelian groups, such that $f_n \circ \partial^C_n = \partial^D_n \circ f_{n+1}$:

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow^{\partial^C_{n+1}} & & \downarrow^{\partial^D_{n+1}} \\
C_{n+1} & \xrightarrow{f_{n+1}} & D_{n+1} \\
\downarrow^{\partial^C_n} & & \downarrow^{\partial^D_n} \\
C_n & \overset{f_n}{\to} & D_n \\
\downarrow^{\partial^C_{n-1}} & & \downarrow^{\partial^D_{n-1}} \\
\vdots & & \vdots
\end{array}
.
$$

Composition of such chain maps is given by degreewise composition of their components. Clearly, chain complexes with chain maps between them hence form a category – the *category of chain complexes* in abelian groups, – which we write

$$\mathrm{Ch}_\bullet(\mathrm{Ab})) \in \mathrm{Cat}\,.$$

Accordingly we have:

Proposition 1.55. *Sending a topological space to its singular chain complex $C_\bullet(X)$, def. 1.43, and a continuous map to its push-forward chain map, prop. 1.53, constitutes a functor*

$$C_\bullet(-) : \mathrm{Top} \to \mathrm{Ch}_\bullet(\mathrm{Ab})$$

from the category Top of topological spaces and continuous maps, to the category of chain complexes.

In particular for each $n \in \mathbb{N}$ singular homology extends to a functor

$$H_n(-) : \mathrm{Top} \to \mathrm{Ab}\,.$$

We close this section by stating the basic properties of singular homology, which make precise the sense in which it is an abelian approximation to the homotopy type of X. The proof of these statements requires some of the tools of homological algebra that we develop in the later chapters, as well as some tools in algebraic topology.

Proposition 1.56. *If $f : X \to Y$ is a continuous map between topological spaces which is a weak homotopy equivalence, def, 1.23, then the induced morphism on singular homology groups*

$$H_n(f) : H_n(X) \to H_n(Y)$$

is an isomorphism.

(A proof (via CW approximations) is spelled out for instance in (Hatcher, prop. 4.21)).

We therefore also have an "abelian analog" of weak homotopy equivalences:

Definition 1.57. For $C_\bullet, D_\bullet$ two chain complexes, a chain map $f_\bullet : C_\bullet \to D_\bullet$ is called a **quasi-isomorphism** if it induces isomorphisms on all homology groups:

$$f_n : H_n(C) \overset{\simeq}{\to} H_n(D)\,.$$

In summary: **chain homology sends weak homotopy equivalences to quasi-isomorphisms**. Quasi-isomorphisms of chain complexes are the abelianized analog of weak homotopy equivalences of topological spaces.

In particular we have the analog of prop. 1.24:

Proposition 1.58. *The relation "There exists a quasi-isomorphism from $C_\bullet$ to $D_\bullet$." is a reflexive and transitive relation, but it is not a symmetric relation.*

Proof. Reflexivity and transitivity are evident. An explicit counter-example showing the non-symmetry is the chain map

$$
\begin{array}{ccccccccc}
\cdots & \to & 0 & \to & \mathbb{Z} & \overset{\cdot 2}{\to} & \mathbb{Z} & \to & 0 & \to & \cdots \\
\cdots & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \cdots \\
\cdots & \to & 0 & \to & 0 & \to & \mathbb{Z}/2\mathbb{Z} & \to & 0 & \to & \cdots
\end{array}
$$

from the chain complex concentrated on the morphism of multiplication by 2 on integers, to the chain complex concentrated on the cyclic group of order 2.

This clearly induces an isomorphism on all homology groups. But there is not even a non-zero chain map in the other direction, since there is no non-zero group homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$. ▮

Accordingly, as for homotopy types of topological spaces, in homological algebra one regards two chain complexes $C_\bullet$, $D_\bullet$ as essentially equivalent – "of the same weak homology type" – if there is a zigzag of quasi-isomorphisms

$$C_\bullet \leftarrow \rightarrow \leftarrow \cdots \rightarrow D_\bullet$$

between them. This is made precise by the central notion of the *derived category* of chain complexes. We turn to this below in section *Derived categories and derived functors*.

But quasi-isomorphisms are a little *coarser* than weak homotopy equivalences. The singular chain functor $C_\bullet(-)$ forgets some of the information in the homotopy types of topological spaces. The following series of statements characterizes to some extent what exactly is lost when passing to singular homology, and which information is in fact retained.

First we need a comparison map:

Definition 1.59. **(Hurewicz homomorphism)**

For (X,x) a pointed topological space, the **Hurewicz homomorphism** is the function

$$\Phi : \pi_k(X,x) \to H_k(X)$$

from the kth homotopy group of (X,x) to the kth singular homology group defined by sending

$$\Phi : (f : S^k \to X)_{\sim} \mapsto f_*[S_k]$$

a representative singular k-sphere f in X to the push-forward along f of the fundamental class $[S_k] \in H_k(S^k)$, example 1.51.

Proposition 1.60. *For X a topological space the Hurewicz homomorphism in degree 0 exhibits an isomorphism between the free abelian group $\mathbb{Z}[\pi_0(X)]$ on the set of path connected components of X and the degree-0 singular homlogy:*

$$\mathbb{Z}[\pi_0(X)] \simeq H_0(X) \,.$$

Since a homotopy group in positive degree depends on the homotopy type of the connected component of the base point, while the singular homology does not depend on a basepoint, it is interesting to compare these groups only for the case that X is connected.

Proposition 1.61. *For X a path-connected topological space the Hurewicz homomorphism in degree 1*

$$\Phi : \pi_1(X,x) \to H_1(X)$$

is surjective. Its kernel is the commutator subgroup of $\pi_1(X,x)$. Therefore it induces an isomorphism from the abelianization $\pi_1(X,x)^{\mathrm{ab}} \coloneqq \pi_1(X,x)/[\pi_1,\pi_1]$:

$$\pi_1(X,x)^{\mathrm{ab}} \xrightarrow{\simeq} H_1(X) \,.$$

For higher connected X we have the

Theorem 1.62. *If X is (n-1)-connected for $n \geq 2$ then*

$$\Phi : \pi_n(X,x) \to H_n(X)$$

is an isomorphism.

This is known as the *Hurewicz theorem*.

This gives plenty of motivation for studying

1. chain complexes

2. chain homology

3. quasi-isomorphism

of chain complexes. This is essentially what homological algebra is about. In the next section we start to develop these notions more systematically.

2. II) Chain complexes

Chain complexes of modules with chain maps between them form a category, the *category of chain complexes*, which is where all of homological algebra takes place. We first construct this category and discuss its most fundamental properties in *3) Categories of chain complexes* . Then we consider more interesting properties of this category: the most elementary and still already profoundly useful is the phenomenon of exact sequences and specifically of homology exact sequences, discussed in *4) Homology exact sequences*. In 5) Homotopy fiber sequences and mapping cones we explain how these are the shadow under the homology functor of homotopy fiber sequences of chain complexes constructed using *mapping cones*. The construction of the *connecting homomorphism* obtained this way may be understood as a special case of the *basic diagram chasing lemmas* in double complexes, such as the *snake lemma*, which we discuss in *6) Double complexes and the diagram chasing lemmas*.

This serves to provide a rich set of tools that is needed when in the next chaper *III) Abelian homotopy theory* we turn to the actual category of interest, which is not quite that of chain complexes and chain maps, but the localization of this at the quasi-isomorphisms: the *derived category*.

3) Categories of chain complexes

In def. 1.43 we had encountered complexes of *singular chains*, of formal linear combinations of simplices in a topological space. Here we discuss such chain complexes in their own right in a bit more depth.

Also, above a singular chain was taken to be a formal sum of singular simplices with coefficients in the abelian group of integers $\mathbb{Z}$. It is just as straightforward, natural and useful to allow the coefficients to be an arbitrary abelian group A, or in fact to be a module over a ring. We have to postpone proper discussion of motivating examples for this step below in chapter III and chapter IV, but the reader eager to see a deeper motivation right now might look at *Modules – As generalized vector bundles*. See also the archetypical example 2.32 below.

So we start by developing a bit of the theory of abelian groups, rings and modules.

Definition 2.1. Write Ab $\in$ Cat for the category of abelian groups and group homomorphisms between them:

- an object is a group A such that for all elements $a_1, a_2 \in A$ we have that the group product of a_1 with a_2 is the same as that of a_2 with a_1, which we write $a_1 + a_2 \in A$ (and the neutral element is denoted by $0 \in A$);
- a morphism $\phi : A_1 \to A_2$ is a group homomorphism, hence a function of the underlying sets, such that for all elements as above $\phi(a_1 + a_2) = \phi(a_1) + \phi(a_2)$.

Among the basic constructions that produce new abelian groups from given ones are the *tensor product of abelian groups* and the *direct sum* of abelian groups. These we discuss now.

Definition 2.2. For A, B and C abelian groups and $A \times B$ the cartesian product group, a **bilinear map**

$$f : A \times B \to C$$

is a function of the underlying sets which is linear – hence is a group homomorphism – in each argument separately.

Remark 2.3. In terms of elements this means that a bilinear map $f : A \times B \to C$ is a function of sets that satisfies for all elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ the two relations

$$f(a_1 + a_2, b_1) = f(a_1, b_1) + f(a_2, b_1)$$

and

$$f(a_1, b_1 + b_2) = f(a_1, b_1) + f(a_1, b_2) \; .$$

Notice that this is *not* a group homomorphism out of the product group. The product group $A \times B$ is the group whose elements are pairs (a, b) with $a \in A$ and $b \in B$, and whose group operation is

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2 \; , \; b_1 + b_2) \; .$$

A *group homomorphism*

$$\phi : A \times B \to C$$

hence satisfies

$$\phi(a_1 + a_2, b_1 + b_2) = \phi(a_1, b_1) + \phi(a_2, b_2)$$

and hence in particular

$$\begin{aligned} \phi(a_1 + a_2, b_1) &= \phi(a_1, b_1) + \phi(a_2, 0) \\ \phi(a_1, b_1 + b_2) &= \phi(a_1, b_1) + \phi(0, b_2) \end{aligned}$$

which is (in general) different from the behaviour of a bilinear map.

Definition 2.4. For A, B two abelian groups, their **tensor product of abelian groups** is the abelian group $A \otimes B$ which is the quotient group of the free group on the product (direct sum) $A \times B$ by the relations

- $(a_1, b) + (a_2, b) \sim (a_1 + a_2, b)$
- $(a, b_1) + (a, b_2) \sim (a, b_1 + b_2)$

for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$.

In words: it is the group whose elements are presented by pairs of elements in A and B and such that the group operation for one argument fixed is that of the other group in the other argument.

Remark 2.5. There is a canonical function of the underlying sets

$$A \times B \overset{\otimes}{\to} A \otimes B \,.$$

On elements this sends (a, b) to the equivalence class that it represents under the above equivalence relations.

Proposition 2.6. *A function of underlying sets $f : A \times B \to C$ is a bilinear function precisely if it factors by the morphism of 2.5 through a group homomorphism $\phi : A \otimes B \to C$ out of the tensor product:*

$$f : A \times B \overset{\otimes}{\to} A \otimes B \overset{\phi}{\to} C \,.$$

Proposition 2.7. *Equipped with the tensor product $\otimes$ of def. 2.4 Ab becomes a monoidal category.*

The unit object in $(\mathrm{Ab}, \otimes)$ is the additive group of integers $\mathbb{Z}$.

This means:

1. forming the tensor product is a functor in each argument

$$A \otimes (-) : \mathrm{Ab} \to \mathrm{Ab} \,,$$

2. there is an associativity natural isomorphism $(A \otimes B) \otimes C \overset{\simeq}{\to} A \otimes (B \otimes C)$ which is "coherent" in the sense that all possible ways of using it to rebracket a given expression are equal.

3. There is a unit natural isomorphism $A \otimes \mathbb{Z} \overset{\simeq}{\to} A$ which is compatible with the asscociativity isomorphism in the evident sense.

Proof. To see that $\mathbb{Z}$ is the unit object, consider for any abelian group A the map

$$A \otimes \mathbb{Z} \to A$$

which sends for $n \in \mathbb{N} \subset \mathbb{Z}$

$$(a, n) \mapsto n \cdot a \coloneqq \underbrace{a + a + \cdots + a}_{n \text{ summands}} \,.$$

Due to the quotient relation defining the tensor product, the element on the left is also equal to

$$(a, n) = (a, \underbrace{1 + 1 \cdots + 1}_{n \text{ summands}}) = \underbrace{(a,1) + (a,1) + \cdots + (a,1)}_{n \text{ summands}} \,.$$

This shows that $A \otimes \mathbb{Z} \to A$ is in fact an isomorphism.

The other properties are similarly direct to check. ▮

We see simple but useful examples of tensor products of abelian groups put to work below in the context of example 2.60 and then in many of the applications to follow. An elementary but not entirely trivial example that may help to illustrate the nature of the tensor product is the following.

Example 2.8. For $a, b \in \mathbb{N}$ and positive, we have

$$\mathbb{Z}_a \otimes \mathbb{Z}_b \simeq \mathbb{Z}_{\mathrm{LCM}(a,b)} \,,$$

where $\mathrm{LCM}(-,-)$ denotes the least common multiple.

Definition 2.9. Let $I \in$ Set be a set and $\{A_i\}_{i \in I}$ an I-indexed family of abelian groups. The **direct sum** $\oplus_{i \in I} \in \mathrm{Ab}$ is the coproduct of these objects in Ab.

This means: the direct sum is an abelian group equipped with a collection of homomorphisms

$$\begin{array}{ccccc} A_j && \cdots && A_k \\ & {}_{\iota_j}\searrow & \cdots & \swarrow_{\iota_k} \\ && \oplus_{i \in I} A_i \end{array} \,,$$

which is characterized (up to unique isomorphism) by the following universal property: for every other abelian group K equipped with maps

$$\begin{array}{ccccc} A_j && \cdots && A_k \\ & {}_{f_j}\searrow & \cdots & \swarrow_{f_k} \\ && K \end{array}$$

there is a unique homomorphism $\phi \colon \oplus_{i \in I} A_i \to K$ such that $f_i = \phi \circ \iota_i$ for all $i \in I$.

Explicitly in terms of elements we have:

Proposition 2.10. *The direct sum $\oplus_{i \in I} A_i$ is the abelian group whose ements are formal sums*

$$a_1 + a_2 + \cdots + a_k$$

of finitely many elements of the $\{A_i\}$, with addition given by componentwise addition in the corresponding A_i.

Example 2.11. If each $A_i = \mathbb{Z}$, then the direct sum is again the free abelian group on I

$$\oplus_{i \in I} \mathbb{Z} \simeq \mathbb{Z}[I] \;.$$

Proposition 2.12. *The tensor product of abelian groups distributes over arbitrary direct sums:*

$$A \otimes (\oplus_{i \in I} B_i) \simeq \; \oplus_{i \in I} A \otimes B_o \;.$$

Example 2.13. For $I \in \mathrm{Set}$ and $A \in \mathrm{Ab}$, the direct sum of $|I|$ copies of A with itself is equivalently the tensor product of abelian groups of the free abelian group on I with A:

$$\oplus_{i \in I} A \simeq (\oplus_{i \in I} \mathbb{Z}) \otimes A \simeq (\mathbb{Z}[I]) \otimes A \;.$$

Remark 2.14. Together, tensor product and direct sum of abelian groups make Ab into what is called a *bimonoidal category*.

This now gives us enough structure to define rings and consider basic examples of their modules.

Definition 2.15. A ring (unital and not-necessarily commutative) is an abelian group R equipped with

1. an element $1 \in R$

2. a bilinear operation, hence a group homomorphism

   $$\cdot : R \otimes R \to R$$

   out of the tensor product of abelian groups,

such that this is associative and unital with respect to 1.

Remark 2.16. The fact that the product is a bilinear map is the **distributivity law**: for all $r, r_1, r_2 \in R$ we have

$$r \cdot (r_1 + r_2) = r \cdot r_1 + r \cdot r_2$$

and

$$(r_1 + r_2) \cdot r = (r_1 + r_2) \cdot r \;.$$

Example 2.17.

- The integers $\mathbb{Z}$ are a ring under the standard addition and multiplication operation.
- For each n, this induces a ring structure on the cyclic group $\mathbb{Z}_n$, given by operations in $\mathbb{Z}$ modulo n.
- The rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$ and complex numbers are rings under their standard operations (in fact these are even *fields*).

Example 2.18. For R a ring, the polynomials

$$r_0 + r_1 x + r_2 x^2 + \cdots + r_n x^n$$

(for arbitrary $n \in \mathbb{N}$) in a variable x with coefficients in R form another ring, the *polynomial ring* denoted $R[x]$. This is the free R-associative algebra on a single generator x.

Example 2.19. For R a ring and $n \in \mathbb{N}$, the set $M(n, R)$ of $n \times n$-matrices with coefficients in R is a ring under elementwise addition and matrix multiplication.

Example 2.20. For X a topological space, the set of continuous functions $C(X, \mathbb{R})$ or $C(X, \mathbb{C})$ with values in the real numbers or complex numbers is a ring under pointwise (points in X) addition and multiplication.

Just as an outlook and a suggestion for how to think geometrically of the objects appearing here, we mention the following.

Remark 2.21. The *Gelfand duality theorem* says that if one remembers certain extra structure on the rings of functions $C(X, \mathbb{C})$ in example 2.20 – called the structure of a *C-star algebra*, then this construction

$$C(-, \mathbb{C}) : \mathrm{Top} \overset{\simeq}{\to} C^\ast \mathrm{Alg}^{\mathrm{op}} \overset{\mathrm{forget}}{\longrightarrow} \mathrm{Ring}^{\mathrm{op}}$$

is an equivalence of categories between that of topological spaces, and the opposite category of $C^\ast$-algebras. Together with remark 2.33 further below this provides a useful dual geometric way of thinking about the theory of modules.

From now on and throughout, we take R to be a commutative ring.

Definition 2.22. A **module** N over a ring R is

1. an object $N \in$ Ab, hence an abelian group;
2. equipped with a morphism

   $$\alpha : R \otimes N \to N$$

   in Ab; hence a function of the underlying sets that sends elements

   $$(r,n) \mapsto rn \coloneqq \alpha(r,n)$$

   and which is a bilinear function in that it satisfies

   $$(r, n_1 + n_2) \mapsto rn_1 + rn_2$$

   and

   $$(r_1 + r_2, n) \mapsto r_1 n + r_2 n$$

   for all $r, r_1, r_2 \in R$ and $n, n_1, n_2 \in N$;
3. such that the diagram

   $$\begin{array}{ccc} R \otimes R \otimes N & \overset{\cdot_R \otimes \mathrm{Id}_N}{\longrightarrow} & R \otimes N \\ {}^{\mathrm{Id}_R \otimes \alpha}\downarrow && \downarrow^{\alpha} \\ R \otimes N & \to & N \end{array}$$

   commutes in Ab, which means that for all elements as before we have

   $$(r_1 \cdot r_2) n = r_1 (r_2 n) \; .$$
4. such that the diagram

   $$\begin{array}{ccccc} 1 \otimes N && \overset{1 \otimes \mathrm{id}_N}{\longrightarrow} && R \otimes N \\ & \searrow && \swarrow_{\alpha} \\ && N \end{array}$$

   commutes, which means that on elements as above

   $$1 \cdot n = n \; .$$

Example 2.23. The ring R is naturally a module over itself, by regarding its multiplication map $R \otimes R \to R$ as a module action $R \otimes N \to N$ with $N \coloneqq R$.

Example 2.24. More generally, for $n \in \mathbb{N}$ the n-fold direct sum of the abelian group underlying R is naturally a module over R

$$R^n \coloneqq R^{\oplus n} \coloneqq \underbrace{R \oplus R \oplus \cdots \oplus R}_{n \text{ summands}} \; .$$

The module action is componentwise:

$$r \cdot (r_1, r_2, \cdots, r_n) = (r \cdot r_1, r \cdot r_2, \cdot r \cdot r_n) \; .$$

Example 2.25. Even more generally, for $I \in$ Set any set, the direct sum $\oplus_{i \in I} R$ is an R-module.

This is the **free module** (over R) on the set S.

The set I serves as the basis of a free module: a general element $v \in \oplus_i R$ is a formal linear combination of elements of I with coefficients in R.

For special cases of the ring R, the notion of R-module is equivalent to other notions:

Example 2.26. For $R = \mathbb{Z}$ the integers, an R-module is equivalently just an abelian group.

Example 2.27. For $R = k$ a field, an R-module is equivalently a vector space over k.

Every finitely-generated free k-module is a free module, hence every finite dimensional vector space has a basis. For infinite dimensions this is true if the axiom of choice holds.

Example 2.28. For N a module and $\{n_i\}_{i \in I}$ a set of elements, the linear span

$$\langle n_i \rangle_{i \in I} \hookrightarrow N \,,$$

(hence the completion of this set under addition in N and multiplication by R) is a submodule of N.

Example 2.29. Consider example 2.28 for the case that the module is $N = R$, the ring itself, as in example

2.23. Then a submodule is equivalently (called) an *ideal* of R.

Definition 2.30. Write RMod for the category or R-modules and R-linear maps between them.

Example 2.31. For $R = \mathbb{Z}$ we have $\mathbb{Z}\,\mathrm{Mod} \simeq \mathrm{Ab}$.

Example 2.32. Let X be a topological space and let

$$R \coloneqq C(X, \mathbb{C})$$

be the ring of continuous functions on X with values in the complex numbers.

Given a complex vector bundle $E \to X$ on X, write $\Gamma(E)$ for its set of continuous sections. Since for each point $x \in X$ the fiber E_x of E over x is a $\mathbb{C}$-module (by example 2.27), $\Gamma(X)$ is a $C(X, \mathbb{C})$-module.

Just as an outlook and a suggestion for how to think of modules geometrically, we mention the following.

Remark 2.33. The *Serre-Swan theorem* says that if X is Hausdorff and compact with ring of functions $C(X, \mathbb{C})$ – as in remark 2.21 above – then $\Gamma(X)$ is a *projective* $C(X, \mathbb{C})$-module and indeed there is an equivalence of categories between projective $C(X, \mathbb{C})$-modules and complex vector bundles over X. (We introduce the notion of *projective modules* below in Derived categories and derived functors.)

We now discuss a bunch of properties of the category RMod which together will show that there is a reasonable concept of chain complexes of R-modules, in generalization of how there is a good concept of chain complexes of abelian groups. In a more abstract category theoretical context than we invoke here, all of the following properties are summarized in the following statement.

Theorem 2.34. *Let R be a commutative ring. Then $R\,\mathrm{Mod}$ is an abelian category.*

But for the moment we ignore this further abstraction and just consider the following list of properties.

Definition 2.35. An object in a category which is both an initial object and a terminal object is called a **zero object**.

Remark 2.36. This means that $0 \in \mathcal{C}$ is a zero object precisely if for every other object A there is a unique morphism $A \to 0$ to the zero object as well as a unique morphism $0 \to A$ from the zero object.

Proposition 2.37. *The trivial group is a zero object in Ab.*

The trivial module is a zero object in RMod.

Proof. Clearly the 0-module 0 is a terminal object, since every morphism $N \to 0$ has to send all elements of N to the unique element of 0, and every such morphism is a homomorphism. Also, 0 is an initial object because a morphism $0 \to N$ always exists and is unique, as it has to send the unique element of 0, which is the neutral element, to the neutral element of N. ∎

Definition 2.38. In a category with an initial object 0 and pullbacks, the **kernel** $\ker(f)$ of a morphism $f : A \to B$ is the pullback $\ker(f) \to A$ along f of the unique morphism $0 \to B$

$$\begin{array}{ccc} \ker(f) & \to & 0 \\ {}^{p}\downarrow && \downarrow \\ A & \overset{f}{\to} & B \end{array}.$$

Remark 2.39. More explicitly, this characterizes the object $\ker(f)$ as the object (unique up to unique isomorphism) that satisfies the following universal property:

for every object C and every morphism $h : C \to A$ such that $f \circ h = 0$ is the zero morphism, there is a unique morphism $\phi : C \to \ker(f)$ such that $h = p \circ \phi$.

Example 2.40. In the category Ab of abelian groups, the kernel of a group homomorphism $f : A \to B$ is the subgroup of A on the set $f^{-1}(0)$ of elements of A that are sent to the zero-element of B.

Example 2.41. More generally, for R any ring, this is true in RMod: the kernel of a morphism of modules is the preimage of the zero-element at the level of the underlying sets, equipped with the unique sub-module structure on that set.

Definition 2.42. In a category with zero object, the **cokernel** of a morphism $f : A \to B$ is the pushout $\mathrm{coker}(f)$ in

$$\begin{array}{ccc} A & \overset{f}{\to} & B \\ \downarrow && \downarrow^{i} \\ 0 & \to & \mathrm{coker}(f) \end{array}.$$

Remark 2.43. More explicitly, this characterizes the object $\mathrm{coker}(f)$ as the object (unique up to unique

isomorphism) that satisfies the following universal property:

for every object C and every morphism $h: B \to C$ such that $h \circ f = 0$ is the zero morphism, there is a unique morphism $\phi: \mathrm{coker}(f) \to C$ such that $h = \phi \circ i$.

Example 2.44. In the category Ab of abelian groups the cokernel of a morphism $f: A \to B$ is the quotient group of B by the image (of the underlying morphism of sets) of f.

Proposition 2.45. *$R\,\mathrm{Mod}$ has all kernels. The kernel of a homomorphism $f: N_1 \to N_2$ is the set-theoretic preimage $U(f)^{-1}(0)$ equipped with the induced R-module structure.*

$R\,\mathrm{Mod}$ has all cokernels. The cokernel of a homomorphism $f: N_1 \to N_2$ is the quotient abelian group

$$\mathrm{coker}\, f = \frac{N_2}{\mathrm{im}(f)}$$

of N_2 by the image of f.

The reader unfamiliar with the general concept of monomorphism and epimorphism may take the following to *define* these in Ab to be simply the injections and surjections.

Proposition 2.46. *$U: R\,\mathrm{Mod} \to \mathrm{Set}$ preserves and reflects monomorphisms and epimorphisms:*

A homomorphism $f: N_1 \to N_2$ in $R\,\mathrm{Mod}$ is a monomorphism / epimorphism precisely if $U(f)$ is an injection / surjection.

Proof. Suppose that f is a monomorphism, hence that $f: N_1 \to N_2$ is such that for all morphisms $g_1, g_2: K \to N_1$ such that $f \circ g_1 = f \circ g_2$ already $g_1 = g_2$. Let then g_1 and g_2 be the inclusion of submodules generated by a single element $k_1 \in K$ and $k_2 \in K$, respectively. It follows that if $f(k_1) = f(k_2)$ then already $k_1 = k_2$ and so f is an injection. Conversely, if f is an injection then its image is a submodule and it follows directly that f is a monomorphism.

Suppose now that f is an epimorphism and hence that $f: N_1 \to N_2$ is such that for all morphisms $g_1, g_2: N_2 \to K$ such that $f \circ g_1 = f \circ g_2$ already $g_1 = g_2$. Let then $g_1: N_2 \to \frac{N_2}{\mathrm{im}(f)}$ be the natural projection. and let $g_2: N_2 \to 0$ be the zero morphism. Since by construction $f \circ g_1 = 0$ and $f \circ g_2 = 0$ we have that $g_1 = 0$, which means that $\frac{N}{\mathrm{im}(f)} = 0$ and hence that $N = \mathrm{im}(f)$ and so that f is surjective. The other direction is evident on elements. ■

Definition 2.47. For $N_1, N_2 \in R\,\mathrm{Mod}$ two modules, define on the hom set $\mathrm{Hom}_{R\,\mathrm{Mod}}(N_1, N_2)$ the structure of an abelian group whose addition is given by argumentwise addition in N_2: $(f_1 + f_2): n \mapsto f_1(n) + f_2(n)$.

Proposition 2.48. *With def. 2.47 $R\,\mathrm{Mod}$ composition of morphisms*

$$\circ: \mathrm{Hom}(N_1, N_2) \times \mathrm{Hom}(N_2, N_3) \to \mathrm{Hom}(N_1, N_3)$$

is a bilinear map, hence is equivalently a morphism

$$\mathrm{Hom}(N_1, N_2) \otimes \mathrm{Hom}(N_2, N_3) \to \mathrm{Hom}(N_1, N_3)$$

out of the tensor product of abelian groups.

This makes $R\,\mathrm{Mod}$ into an Ab-enriched category.

Proof. Linearity of composition in the second argument is immediate from the pointwise definition of the abelian group structure on morphisms. Linearity of the composition in the first argument comes down to linearity of the second module homomorphism. ■

Remark 2.49. In fact $R\,\mathrm{Mod}$ is even a closed category, but this we do not need for showing that it is abelian.

Prop. 2.37 and prop. 2.48 together say that:

Corollary 2.50. *$R\,\mathrm{Mod}$ is an pre-additive category.*

Proposition 2.51. *$R\,\mathrm{Mod}$ has all products and coproducts, being direct products and direct sums.*

The products are given by cartesian product of the underlying sets with componentwise addition and R-action.

The direct sum is the subobject of the product consisting of tuples of elements such that only finitely many are non-zero.

Proof. The defining universal properties are directly checked. Notice that the direct product $\prod_{i \in I} N_i$ consists of arbitrary tuples because it needs to have a projection map

$$p_j: \prod_{i \in I} N_i \to N_j$$

to each of the modules in the product, reproducing all of a possibly infinite number of non-trivial maps $\{K \to N_j\}$. On the other hand, the direct sum just needs to contain all the modules in the sum

$$\iota_j : N_j \to \oplus_{i \in I} N_i$$

and since, being a module, it needs to be closed only under addition of *finitely* many elements, so it consists only of linear combinations of the elements in the N_j, hence of finite formal sums of these. ▮

Together cor. 2.50 and prop. 2.51 say that:

Corollary 2.52. *$R\,\mathrm{Mod}$ is an additive category.*

Proposition 2.53. *In $R\,\mathrm{Mod}$*

- *every monomorphism is the kernel of its cokernel;*
- *every epimorphism is the cokernel of its kernel.*

Proof. Using prop. 2.45 this is directly checked on the underlying sets: given a monomorphism $K \hookrightarrow N$, its cokernel is $N \to \frac{N}{K}$, The kernel of that morphism is evidently $K \hookrightarrow N$. ▮

Now cor. 2.50 and prop. 2.53 imply theorem 2.34, by definition.

Now we finally have all the ingredients to talk about chain complexes of R-modules. The following definitions are the direct analogs of the definitions of chain complexes of abelian groups in *Simplicial and singular homology* above.

Definition 2.54. A ($\mathbb{Z}$-graded) **chain complex** in RMod is

- a collection of objects $\{C_n\}_{n \in \mathbb{Z}}$,
- and of morphisms $\partial_n : C_n \to C_{n-1}$

$$\cdots \overset{\partial_3}{\to} C_2 \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0 \overset{\partial_0}{\to} C_{-1} \overset{\partial_{-1}}{\longrightarrow} \cdots$$

such that

$$\partial_n \circ \partial_{n+1} = 0$$

(the zero morphism) for all $n \in \mathbb{N}$.

Definition 2.55. For $C_\bullet$ a chain complex and $n \in \mathbb{N}$

- the morphisms ∂_n are called the **differentials** or **boundary maps**;
- the elements of C_n are called the n**-chains**;
- for $n \geq 1$ the elements in the kernel

  $$Z_n \coloneqq \ker(\partial_{n-1})$$

  of $\partial_{n-1} : C_n \to C_{n-1}$ are called the n**-cycles**

  and for $n = 0$ we say that every 0-chain is a 0-cycle

  $$Z_0 \coloneqq C_0$$

  (equivalently we declare that $\partial_{-1} = 0$).

- the elements in the image

  $$B_n \coloneqq \mathrm{im}(\partial_n)$$

  of $\partial_n : C_{n+1} \to C_n$ are called the n**-boundaries**;

Notice that due to $\partial\partial = 0$ we have canonical inclusions

$$0 \hookrightarrow B_n \hookrightarrow Z_n \hookrightarrow C_n \; .$$

- the cokernel

  $$H_n \coloneqq Z_n / B_n$$

  is called the degree-n **chain homology** of $C_\bullet$.

  $$0 \to B_n \to Z_n \to H_n \to 0 \; .$$

Definition 2.56. A **chain map** $f : V_\bullet \to W_\bullet$ is a collection of morphism $\{f_n : V_n \to W_n\}_{n \in \mathbb{Z}}$ in $\mathcal{A}$ such that all the diagrams

$$\begin{array}{ccc} V_{n+1} & \overset{d_n^V}{\to} & V_n \\ \downarrow^{f_{n+1}} & & \downarrow^{f_n} \\ W_{n+1} & \overset{d_n^W}{\longrightarrow} & W_n \end{array}$$

commute, hence such that all the equations

$$f_n \circ d_n^V = d_{n+1}^W \circ f_{n+1}$$

hold.

Proposition 2.57. *For $f : C_\bullet \to D_\bullet$ a chain map, it respects boundaries and cycles, so that for all $n \in \mathbb{Z}$ it restricts to a morphism*

$$B_n(f) : B_n(C_\bullet) \to B_n(D_\bullet)$$

and

$$Z_n(f) : Z_n(C_\bullet) \to Z_n(D_\bullet) \; .$$

In particular it also respects chain homology

$$H_n(f) : H_n(C_\bullet) \to H_n(D_\bullet) \; .$$

Corollary 2.58. *Conversely this means that taking chain homology is a functor*

$$H_n(-) : \mathrm{Ch}_\bullet(\mathcal{A}) \to \mathcal{A}$$

from the category of chain complexes in $\mathcal{A}$ to $\mathcal{A}$ itself.

This establishes the basic objects that we are concerned with in the following. But as before, we are not so much interested in chain complexes up to chain map isomorphism, rather, we are interested in them up to a notion of homotopy equivalence. This we begin to study in the next section *Homology exact sequences and homotopy fiber sequences*. But in order to formulate that neatly, it is useful to have the *tensor product of chain complexes*. We close this section with introducing that notion.

Definition 2.59. For $X, Y \in \mathrm{Ch}_\bullet(\mathcal{A})$ write $X \otimes Y \in \mathrm{Ch}_\bullet(\mathcal{A})$ for the chain complex whose component in degree n is given by the direct sum

$$(X \otimes Y)_n := \oplus_{i+j=n} X_i \otimes_R Y_j$$

over all tensor products of components whose degrees sum to n, and whose differential is given on elements (x, y) of homogeneous degree by

$$\partial^{X \otimes Y}(x, y) = (\partial^X x, y) + (-1)^{\deg(x)}(x, \partial^Y y) \; .$$

Example 2.60. **(square as tensor product of interval with itself)**

For R some ring, let $I_\bullet \in \mathrm{Ch}_\bullet(R\,\mathrm{Mod})$ be the chain complex given by

$$I_\bullet = \left[\cdots \to 0 \to 0 \to R \overset{\partial_0^I}{\to} R \oplus R \right],$$

where $\partial_0^I = (-\mathrm{id}, \mathrm{id})$.

This is the normalized chain complex of the simplicial chain complex of the standard simplicial interval, the 1-simplex Δ_1, which means: we may think of

$$I_0 = R \oplus R \simeq R[\{(0), (1)\}]$$

as the R-linear span of two basis elements labelled "(0)" and "(1)", to be thought of as the two 0-chains on the endpoints of the interval. Similarly we may think of

$$I_1 = R \simeq R[\{(0 \to 1)\}]$$

as the free R-module on the single basis element which is the unique non-degenerate 1-simplex $(0 \to 1)$ in Δ^1.

Accordingly, the differential ∂_0^I is the oriented boundary map of the interval, taking this basis element to

$$\partial_0^I : (0 \to 1) \mapsto (1) - (0)$$

and hence a general element $r \cdot (0 \to 1)$ for some $r \in R$ to

$$\partial_0^I : r \cdot (0 \to 1) \mapsto r \cdot (1) - r \cdot (0) \; .$$

We now write out in full details the tensor product of chain complexes of $I_\bullet$ with itself, according to def.

2.59:

$$S_\bullet \coloneqq I_\bullet \otimes I_\bullet \,.$$

By definition and using the above choice of basis element, this is in low degree given as follows:

$$\begin{aligned} S_0 & = I_0 \oplus I_0 \\ & = (R \oplus R) \otimes (R \oplus R) \\ & \simeq R \oplus R \oplus R \oplus R \\ & = \left\{ r_{00} \cdot ((0),(0)') + r_{01} \cdot ((0),(1)') + r_{10} \cdot ((1),(0)') + r_{11} \cdot ((1),(1)') \,\vert\, r_{\cdot,\cdot} \in R \right\} \end{aligned} \,,$$

where in the last line we express a general element as a linear combination of the canonical basis elements which are obtained as tensor products $(a,b) \in R \otimes R$ of the previous basis elements. Notice that by the definition of tensor product of modules we have relations like

$$r((0),(1)') = (r(0),(1)') = ((0),r(1)')$$

etc.

Similarly then, in degree-1 the tensor product chain complex is

$$\begin{aligned} (I \otimes I)_1 & = (I_0 \otimes I_1) \oplus (I_1 \otimes I_0) \\ & \simeq R \otimes (R \oplus R) \oplus (R \oplus R) \otimes R \\ & \simeq R \oplus R \oplus R \oplus R \\ & \simeq \{ r_0 \cdot ((0),(0 \to 1)') + r_1 \cdot ((1),(0 \to 1)') + \tilde r_0 \cdot ((0 \to 1),(0)') + \tilde r_1 \cdot ((0 \to 1),(1)') \,\vert\, r_\cdot, \tilde r_\cdot \in R \} \end{aligned} \,.$$

And finally in degree 2 it is

$$\begin{aligned} (I \otimes I)_2 & \simeq I_1 \otimes I_1 \\ & \simeq R \otimes R \\ & \simeq R \\ & \simeq \{ r \cdot ((0 \to 1),(0 \to 1)') \,\vert\, r \in R \} \end{aligned} \,.$$

All other contributions that are potentially present in $(I \otimes I)_\bullet$ vanish (are the 0-module) because all higher terms in $I_\bullet$ are.

The tensor product basis elements appearing in the above expressions have a clear geometric interpretation: we can label a square with them as follows

$$\begin{array}{ccc} ((0),(1)') & \underset{((0 \to 1),(0))}{\longrightarrow} & ((1),(1)') \\ {}^{((0),(0 \to 1)')}\uparrow & \circlearrowleft^{((0 \to 1),(0 \to 1)')} & \uparrow^{((1),(0 \to 1)')} \\ ((0),(0)') & \underset{((0 \to 1),(0)')}{\longrightarrow} & ((1),(0)') \end{array} \,.$$

This diagram indicates a cellular square and identifies its canonical singular chains with the elements of $(I \otimes I)_\bullet$. The arrows indicate the orientation. For instance the fact that

$$\begin{aligned} \partial^{I \otimes I}((0 \to 1),(0)') & = (\partial^I (0 \to 1),(0)') + (-1)^1 ((0 \to 1), \partial^I(0)) \\ & = ((1) - (0),\, (0)') - 0 \\ & = ((1),(0)') - ((0),(0)') \end{aligned}$$

says that the oriented boundary of the bottom morphism is the bottom right element (its target) minus the bottom left element (its source), as indicated. Here we used that the differential of a degree-0 element in $I_\bullet$ is 0, and hence so is any tensor product with it.

Similarly the oriented boundary of the square itself is computed to

$$\begin{aligned} \partial^{I \otimes I}((0 \to 1),(0 \to 1)') & = (\partial^I(0 \to 1),(0 \to 1)') - ((0 \to 1), \partial^I(0 \to 1)) \\ & = ((1) - (0),(0 \to 1)') - ((0 \to 1),(1)' - (0)') \\ & = ((1),(0 \to 1)') - ((0),(0 \to 1)') - ((0 \to 1),(1)') + ((0 \to 1),(0)') \end{aligned} \,,$$

which can be read as saying that the boundary is the evident boundary thought of as oriented by drawing it *counterclockwise* into the plane, so that the right arrow (which points up) contributes with a +1 prefactor, while the left arrow (which also points up) contributes with a -1 prefactor.

Proposition 2.61. *Equipped with the standard tensor product of chain complexes $\otimes$, def. 2.59 the category of chain complexes is a monoidal category $(\mathrm{Ch}_\bullet(R\,\mathrm{Mod}), \otimes)$. The unit object is the chain complex concentrated in degree 0 on the tensor unit R of $R\,\mathrm{Mod}$.*

Definition 2.62. We write $\mathrm{Ch}_\bullet^{\mathrm{ub}}$ for the category of *unbounded* chain complexes.

Definition 2.63. For $X, Y \in \mathrm{Ch}_\bullet^{\mathrm{ub}}(\mathcal{A})$ any two objects, define a chain complex $[X, Y] \in \mathrm{Ch}_\bullet^{\mathrm{ub}}(\mathcal{A})$ to have components

$$[X, Y]_n := \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{R\,\mathrm{Mod}}(X_i, Y_{i+n})$$

(the collection of degree-n maps between the underlying graded modules) and whose differential is defined on homogeneously graded elements $f \in [X, Y]_n$ by

$$df := d_Y \circ f - (-1)^n f \circ d_X \,.$$

This defines a functor

$$[-,-] : \mathrm{Ch}_\bullet^{\mathrm{ub}}(\mathcal{A})^{\mathrm{op}} \times \mathrm{Ch}_\bullet^{\mathrm{ub}}(\mathcal{A}) \to \mathrm{Ch}_\bullet^{\mathrm{ub}}(\mathcal{A}) \,.$$

Proposition 2.64. *This functor*

$[-,-] : \mathrm{Ch}_\bullet^{\mathrm{ub}} \times \mathrm{Ch}_\bullet^{\mathrm{ub}} \to \mathrm{Ch}_\bullet^{\mathrm{ub}}$

is the internal hom of the category of chain complexes.

Proposition 2.65. *The collection of cycles of the internal hom $[X, Y]_\bullet$ in degree 0 coincides with the external hom functor*

$$Z_0([X, Y]) \simeq \mathrm{Hom}_{\mathrm{Ch}_\bullet^{\mathrm{ub}}}(X, Y) \,.$$

The chain homology of the internal hom $[X, Y]$ in degree 0 coincides with the homotopy classes of chain maps.

Proof. By Definition 2.63 the 0-cycles in $[X, Y]$ are collections of morphisms $\{f_k : X_k \to Y_k\}$ such that

$$f_{k+1} \circ d_X = d_Y \circ f_k \,.$$

This is precisely the condition for f to be a chain map.

Similarly, the boundaries in degree 0 are precisely the collections of morphisms of the form

$$\lambda_{k+1} \circ d_X + d_Y \circ \lambda_k$$

for a collection of maps $\{\lambda_k : X_k \to Y_{k+1}\}$. This are precisely the null homotopies. ∎

Proposition 2.66. *The monoidal category $(\mathrm{Ch}_\bullet, \otimes)$ is a closed monoidal category, the internal hom is the standard internal hom of chain complexes.*

4) Homology exact sequences

With the basic definition of the category of chain complexes in hand, we now consider the first application, which is as simple as it is of ubiquitous use in mathematics: *long exact sequences in homology*. This is the “abelianization”, in the sense of the discussion in 2) above, of what in homotopy theory are *long exact sequences of homotopy groups*. But both concepts, in turn, are just the shadow on homology groups/homotopy groups, respectively of homotopy fiber sequences of the underlying chain complexes/topological spaces themselves. Since these are even more useful, in particular in chapter III) below, we discuss below in 5) how to construct these using *chain homotopy* and *mapping cones*.

First we need the fundamental notion of *exact sequences*. As before, we fix some commutative ring R throughout and consider the category of modules over R, which we will abbreviate

$$\mathcal{A} \coloneqq R\,\mathrm{Mod} \,.$$

Definition 2.67. An **exact sequence** in $\mathcal{A}$ is a chain complex $C_\bullet$ in $\mathcal{A}$ with vanishing chain homology in each degree:

$$\forall n \in \mathbb{N} \,.\, H_n(C) = 0 \,.$$

Definition 2.68. A **short exact sequence** is an exact sequence, def. 2.67 of the form

$$\cdots \to 0 \to 0 \to A \to B \to C \to 0 \to 0 \to \cdots \,.$$

One usually writes this just “$0 \to A \to B \to C \to 0$” or even just “$A \to B \to C$”.

Remark 2.69. A general exact sequence is sometimes called a **long exact sequence**, to distinguish from the special case of a short exact sequence.

Beware that there is a difference between $A \to B \to C$ being exact (at B) and $A \to B \to C$ being a “short exact sequence” in that $0 \to A \to B \to C \to 0$ is exact at A, B and C. This is illustrated by the following proposition.

Proposition 2.70. *Explicitly, a sequence of morphisms*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

in $\mathcal{A}$ is short exact, def. 2.68, precisely if

1. *i is a monomorphism,*
2. *p is an epimorphism,*
3. *and the image of i equals the kernel of p (equivalently, the coimage of p equals the cokernel of i).*

Proof. The third condition is the definition of exactness at B. So we need to show that the first two conditions are equivalent to exactness at A and at C.

This is easy to see by looking at elements when $\mathcal{A} \simeq R\mathrm{Mod}$, for some ring R (and the general case can be reduced to this one using one of the embedding theorems):

The sequence being exact at

$$0 \to A \to B$$

means, since the image of $0 \to A$ is just the element $0 \in A$, that the kernel of $A \to B$ consists of just this element. But since $A \to B$ is a group homomorphism, this means equivalently that $A \to B$ is an injection.

Dually, the sequence being exact at

$$B \to C \to 0$$

means, since the kernel of $C \to 0$ is all of C, that also the image of $B \to C$ is all of C, hence equivalently that $B \to C$ is a surjection. ▮

Example 2.71. Let $\mathcal{A} = \mathbb{Z}\mathrm{Mod} \simeq \mathrm{Ab}$. For $n \in \mathbb{N}$ with $n \geq 1$ let $\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$ be the linear map/homomorphism of abelian groups which acts by the ordinary multiplication of integers by n. This is clearly an injection. The cokernel of this morphism is the projection to the quotient group, which is the cyclic group $\mathbb{Z}_n \coloneqq \mathbb{Z}/n\mathbb{Z}$. Hence we have a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}_n \;.$$

A typical use of a long exact sequence, notably of the *homology long exact sequence* to be discussed, is that it allows to determine some of its entries in terms of others.

The characterization of short exact sequences in prop. 2.70 is one example for this. Another is this:

Proposition 2.72. *If part of an exact sequence looks like*

$$\cdots \to 0 \to C_{n+1} \xrightarrow{\partial_n} C_n \to 0 \to \cdots ,$$

then ∂_n is an isomorphism and hence

$$C_{n+1} \simeq C_n \;.$$

Often it is useful to make the following strengthening of short exactness explicit.

Definition 2.73. A short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ in $\mathcal{A}$ is called **split** if either of the following equivalent conditions hold

1. There exists a section of p, hence a homomorphism $s : B \to C$ such that $p \circ s = \mathrm{id}_C$.
2. There exists a retract of i, hence a homomorphism $r : B \to A$ such that $r \circ i = \mathrm{id}_A$.
3. There exists an isomorphism of sequences with the sequence

   $$0 \to A \to A \oplus C \to C \to 0$$

   given by the direct sum and its canonical injection/projection morphisms.

Proposition 2.74. ***(splitting lemma)***

The three conditions in def. 2.73 are indeed equivalent.

Proof. It is clear that the third condition implies the first two: take the section/retract to be given by the canonical injection/projection maps that come with a direct sum.

Conversely, suppose we have a retract $r : B \to A$ of $i : A \to B$. Write $P : B \xrightarrow{r} A \xrightarrow{i} B$ for the composite. Notice that by $r \circ i = \mathrm{id}$ this is an idempotent: $P \circ P = P$, hence a projector.

Then every element $b \in B$ can be decomposed as $b = (b - P(b)) + P(b)$ hence with $b - P(b) \in \ker(r)$ and $P(b) \in \mathrm{im}(i)$. Moreover this decomposition is unique since if $b = i(a)$ while at the same time $r(b) = 0$ then $0 = r(i(a)) = a$. This shows that $B \simeq \mathrm{im}(i) \oplus \ker(r)$ is a direct sum and that $i : A \to B$ is the canonical inclusion of

$\mathrm{im}(i)$. By exactness it then follows that $\ker(r) \simeq \ker(p)$ and hence that $B \simeq A \oplus C$ with the canonical inclusion and projection.

The implication that the second condition also implies the third is formally dual to this argument. ∎

Moreover, of particular interest are exact sequences *of chain complexes*. We consider this concept in full beauty below in section 5). In order to motivate the discussion there we here content ourselves with the following quick definition, which already admits discussion of some of its rich consequences.

Definition 2.75. A sequence of chain maps of chain complexes

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is a **short exact sequence of chain complexes** in $\mathcal{A}$ if for each n the component

$$0 \to A_n \to B_n \to C_n \to 0$$

is a short exact sequence in $\mathcal{A}$, according to def. 2.68.

Definition 2.76. Consider a short exact sequence of chain complexes as in def. 2.75. For $n \in \mathbb{Z}$, define a group homomorphism

$$\delta_n : H_n(C) \to H_{n-1}(A)\,,$$

called the nth **connecting homomorphism** of the short exact sequence, by sending

$$\delta_n : [c] \mapsto [\partial^B \hat{c}]_A\,,$$

where

1. $c \in Z_n(C)$ is a cycle representing the given homology group $[c]$;
2. $\hat{c} \in C_n(B)$ is any lift of that cycle to an element in B_n, which exists because p is a surjection (but which no longer needs to be a cycle itself);
3. $[\partial^B \hat{c}]_A$ is the A-homology class of $\partial^B \hat{c}$ which is indeed in $A_{n-1} \hookrightarrow B_{n-1}$ by exactness (since $p(\partial^B \hat{c}) = \partial^C p(\hat{c}) = \partial^C c = 0$) and indeed in $Z_{n-1}(A) \hookrightarrow A_{n-1}$ since $\partial^A \partial^B \hat{c} = \partial^B \partial^B \hat{c} = 0$.

Proposition 2.77. *Def. 2.76 is indeed well defined in that the given map is independent of the choice of lift $\hat{c}$ involved and in that the group structure is respected.*

Proof. To see that the construction is well-defined, let $\tilde{c} \in B_n$ be another lift. Then $p(\hat{c} - \tilde{c}) = 0$ and hence $\hat{c} - \tilde{c} \in A_n \hookrightarrow B_n$. This exhibits a homology-equivalence $[\partial^B \hat{c}]_A \simeq [\partial^B \tilde{c}]_A$ since $\partial^A(\hat{c} - \tilde{c}) = \partial^B \hat{c} - \partial^B \tilde{c}$.

To see that δ_n is a group homomorphism, let $[c] = [c_1] + [c_2]$ be a sum. Then $\hat{c} \coloneqq \hat{c}_1 + \hat{c}_2$ is a lift and by linearity of ∂ we have $[\partial^B \hat{c}]_A = [\partial^B \hat{c}_1] + [\partial^B \hat{c}_2]$. ∎

Proposition 2.78. *Under chain homology $H_\bullet(-)$ the morphisms in the short exact sequence together with the connecting homomorphisms yield the **homology long exact sequence***

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \stackrel{\delta_n}{\to} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to \cdots\,.$$

Proof. Consider first the exactness of $H_n(A) \xrightarrow{H_n(i)} H_n(B) \xrightarrow{H_n(p)} H_n(C)$.

It is clear that if $a \in Z_n(A) \hookrightarrow Z_n(B)$ then the image of $[a] \in H_n(B)$ is $[p(a)] = 0 \in H_n(C)$. Conversely, an element $[b] \in H_n(B)$ is in the kernel of $H_n(p)$ if there is $c \in C_{n+1}$ with $\partial^C c = p(b)$. Since p is surjective let $\hat{c} \in B_{n+1}$ be any lift, then $[b] = [b - \partial^B \hat{c}]$ but $p(b - \partial^B c) = 0$ hence by exactness $b - \partial^B \hat{c} \in Z_n(A) \hookrightarrow Z_n(B)$ and so $[b]$ is in the image of $H_n(A) \to H_n(B)$.

It remains to see that

1. the image of $H_n(B) \to H_n(C)$ is the kernel of δ_n;
2. the kernel of $H_{n-1}(A) \to H_{n-1}(B)$ is the image of δ_n.

This follows by inspection of the formula in def. 2.76. We spell out the first one:

If $[c]$ is in the image of $H_n(B) \to H_n(C)$ we have a lift $\hat{c}$ with $\partial^B \hat{c} = 0$ and so $\delta_n[c] = [\partial^B \hat{c}]_A = 0$. Conversely, if for a given lift $\hat{c}$ we have that $[\partial^B \hat{c}]_A = 0$ this means there is $a \in A_n$ such that $\partial^A a \coloneqq \partial^B a = \partial^B \hat{c}$. But then $\tilde{c} \coloneqq \hat{c} - a$ is another possible lift of c for which $\partial^B \tilde{c} = 0$ and so $[c]$ is in the image of $H_n(B) \to H_n(C)$. ∎

Example 2.79. The connecting homomorphism of the long exact sequence in homology induced from short exact sequences of the form in example 2.71 is called a *Bockstein homomorphism*.

We now discuss a deeper, more conceptual way of understanding the origin of long exact sequences in homology and the nature of connecting homomorphisms. This will give first occasion to see some actual

homotopy theory of chain complexes at work, and hence serves also as a motivating example for the discussions to follow in chapter III).

For this we need the notion of *chain homotopy*, which is the abelianized analog of the notion of *homotopy* of continuous maps above in def. 1.11. We now first introduce this concept by straightforwardly mimicking the construction in def. 1.11 with topological spaces replaced by chain complexes. Then we use chain homotopies to construct mapping cones of chain maps. Finally we explain how these refine the above long exact sequences in homology groups to *homotopy cofiber sequences* of the chain complexes themselves.

A *chain homotopy* is a homotopy in $\mathrm{Ch}_\bullet(\mathcal{A})$. We first give the explicit definition, the more abstract characterization is below in prop. 2.84.

Definition 2.80. A *chain homotopy* $\psi: f \Rightarrow g$ between two chain maps $f, g: C_\bullet \to D_\bullet$ in $\mathrm{Ch}_\bullet(\mathcal{A})$ is a sequence of morphisms

$$\{(\psi_n : C_n \to D_{n+1}) \in \mathcal{A} \,|\, n \in \mathbb{N}\}$$

in $\mathcal{A}$ such that

$$f_n - g_n = \partial^D \circ \psi_n + \psi_{n-1} \partial^C .$$

Remark 2.81. It may be useful to illustrate this with the following graphics, which however is *not* a commuting diagram:

$$\begin{array}{ccc}
\vdots && \vdots \\
\downarrow && \downarrow \\
C_{n+1} & \overset{f_{n+1} - g_{n+1}}{\longrightarrow} & D_{n+1} \\
\downarrow^{\partial^C_n} & \nearrow_{\psi_n} & \downarrow^{\partial^D_n} \\
C_n & \overset{f_n - g_n}{\longrightarrow} & D_n \\
\downarrow^{\partial^C_{n-1}} & \nearrow_{\psi_{n-1}} & \downarrow^{\partial^D_{n-1}} \\
C_{n-1} & \overset{f_{n-1} - g_{n-1}}{\longrightarrow} & D_{n-1} \\
\downarrow && \downarrow \\
\vdots && \vdots
\end{array} \,.$$

Instead, a way to encode chain homotopies by genuine diagrammatics is below in prop. 2.84, for which we introduce the *interval object* for chain complexes:

Definition 2.82. Let

$$I_\bullet \coloneqq N_\bullet(C(\Delta[1]))$$

be the normalized chain complex in $\mathcal{A}$ of the simplicial chains on the simplicial 1-simplex:

$$I_\bullet = [\cdots \to 0 \to 0 \to R \overset{(-\mathrm{id}, \mathrm{id})}{\longrightarrow} R \oplus R] \,.$$

Remark 2.83. This is the standard *interval in chain complexes*. Indeed it is manifestly the "abelianization" of the standard interval object Δ^1 in sSet/Top: the 1-simplex.

Proposition 2.84. *A chain homotopy $\psi: f \Rightarrow g$ is equivalently a commuting diagram*

$$\begin{array}{ccc}
C_\bullet && \\
\downarrow & \searrow^{f} & \\
I_\bullet \otimes C_\bullet & \overset{(f,g,\psi)}{\longrightarrow} & D_\bullet \\
\uparrow & \nearrow_{g} & \\
C_\bullet &&
\end{array}$$

in $\mathrm{Ch}_\bullet(\mathcal{A})$, hence a genuine left homotopy with respect to the interval object in chain complexes.

Proof. For notational simplicity we discuss this in $\mathcal{A} =$ Ab.

Observe that $N_\bullet(\mathbb{Z}(\Delta[1]))$ is the chain complex

$$(\cdots \to 0 \to 0 \to \mathbb{Z} \overset{(-\mathrm{id}, \mathrm{id})}{\longrightarrow} \mathbb{Z} \oplus \mathbb{Z} \to 0 \to 0 \to \cdots)$$

where the term $\mathbb{Z} \oplus \mathbb{Z}$ is in degree 0: this is the free abelian group on the set $\{(0), (1)\}$ of 0-simplices in $\Delta[1]$. The other copy of $\mathbb{Z}$ is the free abelian group on the single non-degenerate edge $(0 \to 1)$ in $\Delta[1]$. (All other simplices of $\Delta[1]$ are degenerate and hence do not contribute to the normalized chain complex which we are discussing here.) The single nontrivial differential sends $1 \in \mathbb{Z}$ to $(-1, 1) \in \mathbb{Z} \oplus \mathbb{Z}$, reflecting the fact that one of the vertices is the 0-boundary the other the 1-boundary of the single nontrivial edge.

It follows that the tensor product of chain complexes $I_\bullet \otimes C_\bullet$ is

$$\begin{array}{ccccccccc} & & (I \otimes C)_2 & \to & (I \otimes C)_1 & \to & (I \otimes C)_0 & \to & \cdots \\ \cdots & \to & C_1 \oplus C_2 \oplus C_2 & \to & C_0 \oplus C_1 \oplus C_1 & \to & C_{-1} \oplus C_0 \oplus C_0 & \to & \cdots \end{array} .$$

Therefore a chain map $(f, g, \psi) : I_\bullet \otimes C_\bullet \to D_\bullet$ that restricted to the two copies of $C_\bullet$ is f and g, respectively, is characterized by a collection of commuting diagrams

$$\begin{array}{ccc} C_{n+1} \oplus C_{n+1} \oplus C_n & \xrightarrow{(f_{n+1}, g_{n+1}, \psi_n)} & D_n \\ {}^{\partial^{I \otimes C}}\downarrow & & \downarrow^{\partial^D} \\ C_n \oplus C_n \oplus C_{n-1} & \xrightarrow{(f_n, g_n, \psi_{n-1})} & D_{n-1} \end{array} .$$

On the elements $(1,0,0)$ and $(0,1,0)$ in the top left this reduces to the chain map condition for f and g, respectively. On the element $(0,0,1)$ this is the equation for the chain homotopy

$$f_n - g_n - \psi_{n-1} d_C = d_D \psi_n .$$

∎

Let $C_\bullet, D_\bullet \in \mathrm{Ch}_\bullet(\mathcal{A})$ be two chain complexes.

Definition 2.85. Define the relation *chain homotopic* on $\mathrm{Hom}(C_\bullet, D_\bullet)$ by

$$(f \sim g) \Leftrightarrow \exists (\psi : f \Rightarrow g) .$$

Proposition 2.86. *Chain homotopy is an equivalence relation on* $\mathrm{Hom}(C_\bullet, D_\bullet)$.

Definition 2.87. Write $\mathrm{Hom}(C_\bullet, D_\bullet)_\sim$ for the quotient of the hom set $\mathrm{Hom}(C_\bullet, D_\bullet)$ by chain homotopy.

Proposition 2.88. *This quotient is compatible with composition of chain maps.*

Accordingly the following category exists:

Definition 2.89. Write $\mathcal{K}_\bullet(\mathcal{A})$ for the category whose objects are those of $\mathrm{Ch}_\bullet(\mathcal{A})$, and whose morphisms are chain homotopy classes of chain maps:

$$\mathrm{Hom}_{\mathcal{K}_\bullet(\mathcal{A})}(C_\bullet, D_\bullet) \coloneqq \mathrm{Hom}_{\mathrm{Ch}_\bullet(\mathcal{A})}(C_\bullet, D_\bullet)_\sim .$$

This is usually called the **(strong) homotopy category of chain complexes** in $\mathcal{A}$.

Remark 2.90. Beware, as we will discuss in detail below in 8), that another category that would deserve to carry this name instead is called the *derived category* of $\mathcal{A}$. In the derived category one also quotients out chain homotopy, but one allows that first the domain of the two chain maps f and g is refined along a quasi-isomorphism.

Definition 2.91. A chain map $f_\bullet : C_\bullet \to D_\bullet$ in $\mathrm{Ch}_\bullet(\mathcal{A})$ is called a **quasi-isomorphism** if for each $n \in \mathbb{N}$ the induced morphisms on chain homology groups

$$H_n(f) : H_n(C) \to H_n(D)$$

is an isomorphism.

Remark 2.92. Quasi-isomorphisms are also called, more descriptively, **homology isomorphisms** or **$H_\bullet$-isomorphisms**. See at *homology localization* for more on this.

With the homotopy theoretic notions of chain homotopy and quasi-isomorphism in hand, we can now give a deeper explanation of long exact sequences in homology. We first give now a heuristic discussion that means to serve as a guide through the constructions to follow. The reader wishing to skip this may directly jump ahead to definition 2.95.

While the notion of a short exact sequence of chain complexes is very useful for computations, it does not have invariant meaning if one considers chain complexes as objects in (abelian) homotopy theory, where one takes into account chain homotopies between chain maps and takes equivalence of chain complexes not to be given by isomorphism, but by quasi-isomorphism.

For if a chain map $A_\bullet \to B_\bullet$ is the degreewise kernel of a chain map $B_\bullet \to C_\bullet$, then if $\hat{A}_\bullet \xrightarrow{\simeq} A_\bullet$ is a quasi-isomorphism (for instance a projective resolution of $A_\bullet$) then of course the composite chain map $\hat{A}_\bullet \to B_\bullet$ is in general far from being the degreewise kernel of $C_\bullet$. Hence the notion of degreewise kernels of chain maps and hence that of short exact sequences is not meaningful in the homotopy theory of chain complexes in $\mathcal{A}$ (for instance: not in the derived category of $\mathcal{A}$).

That short exact sequences of chain complexes nevertheless play an important role in homological algebra is due to what might be called a "technical coincidence":

Proposition 2.93. *If $A_\bullet \to B_\bullet \to C_\bullet$ is a short exact sequence of chain complexes, then the commuting square*

$$\begin{tikzcd} A_\bullet \arrow[r] \arrow[d] & 0 \arrow[d] \\ B_\bullet \arrow[r] & C_\bullet \end{tikzcd}$$

is not only a pullback square in $\mathrm{Ch}_\bullet(\mathcal{A})$, exhibiting $A_\bullet$ as the fiber of $B_\bullet \to C_\bullet$ over $0 \in C_\bullet$, it is in fact also a homotopy pullback.

This means it is universal not just among commuting such squares, but also among such squares which commute possibly only up to a chain homotopy ϕ:

$$\begin{tikzcd} Q_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow_\phi"] & 0 \arrow[d] \\ B_\bullet \arrow[r] & C_\bullet \end{tikzcd}$$

and with morphisms between such squares being maps $A_\bullet \to A'_\bullet$ correspondingly with further chain homotopies filling all diagrams in sight.

Equivalently, we have the formally dual result

Proposition 2.94. *If $A_\bullet \to B_\bullet \to C_\bullet$ is a short exact sequence of chain complexes, then the commuting square*

$$\begin{tikzcd} A_\bullet \arrow[r] \arrow[d] & 0 \arrow[d] \\ B_\bullet \arrow[r] & C_\bullet \end{tikzcd}$$

is not only a pushout square in $\mathrm{Ch}_\bullet(\mathcal{A})$, exhibiting $C_\bullet$ as the cofiber of $A_\bullet \to B_\bullet$ over $0 \in C_\bullet$, it is in fact also a homotopy pushout.

But a central difference between fibers/cofibers on the one hand and homotopy fibers/homotopy cofibers on the other is that while the (co)fiber of a (co)fiber is necessarily trivial, the homotopy (co)fiber of a homotopy (co)fiber is in general far from trivial: it is instead the looping $\Omega(-)$ or suspension $\Sigma(-)$ of the codomain/domain of the original morphism: by the pasting law for homotopy pullbacks the pasting composite of successive homotopy cofibers of a given morphism $f : A_\bullet \to B_\bullet$ looks like this:

$$\begin{tikzcd} A_\bullet \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Swarrow_\phi"] & B_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & 0 \arrow[d] \\ 0 \arrow[r] & \mathrm{cone}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & A[1]_\bullet \arrow[r] \arrow[d, "f[1]"] \arrow[dr, phantom, "\Swarrow"] & 0 \arrow[d] \\ & 0 \arrow[r] & B[1] \arrow[r] \arrow[d] & \mathrm{cone}(f)[1]_\bullet \arrow[r] \arrow[d] & \cdots \arrow[d, phantom, "\ddots"] \\ & & \vdots & {} & {} \end{tikzcd}$$

here

- $\mathrm{cone}(f)$ is a specific representative of the homotopy cofiber of f called the *mapping cone* of f, whose construction comes with an explicit chain homotopy ϕ as indicated, hence $\mathrm{cone}(f)$ is homology-equivalence to $C_\bullet$ above, but is in general a "bigger" model of the homotopy cofiber;
- $A[1]$ etc. is the suspension of a chain complex of A, hence the same chain complex but pushed up in degree by one.

In conclusion we get from every morphim of chain complexes a long **homotopy cofiber sequence**

$$\cdots \to A_\bullet \xrightarrow{f} B_\bullet \to \mathrm{cone}(f) \to A[1]_\bullet \xrightarrow{f[1]} B[1]_\bullet \to \mathrm{cone}(f)[1]_\bullet \to \cdots .$$

And applying the chain homology functor to this yields the long exact sequence in chain homology which is traditionally said to be associated to the short exact sequence $A_\bullet \to B_\bullet \to C_\bullet$.

In conclusion this means that it is not really the passage to homology groups which "makes a short exact sequence become long". It's rather that passing to homology groups is a shadow of passing to chain complexes regarded up to quasi-isomorphism, and *this* is what makes every short exact sequence be realized as but a special presentation of a stage in a long homotopy fiber sequence.

We give a precise account of this story in the next section.

5) Homotopy fiber sequences and mapping cones

We have seen in 4) the long exact sequence in homology implied by a short exact sequence of chain complexes, constructed by an elementary if somewhat un-illuminating formula for the connecting

homomorphism. We ended 4) by sketching how this formula arises as the shadow under the homology functor of a *homotopy fiber sequence* of chain complexes, constructed using *mapping cones*. This we now discuss in precise detail.

In the following we repeatedly mention that certain chain complexes are colimits of certain diagrams of chain complexes. The reader unfamiliar with colimits may simply ignore them and regard the given chain complex as arising by definition. However, even a vague intuitive understanding of the indicated colimits as formalizations of "gluing" of chain complexes along certain maps should help to motivate why these definitions are what they are. The reader unhappy even with this can jump ahead to prop. 2.105 and take this and the following propositions up to and including prop. 2.112 as definitions.

The notion of a mapping cone that we introduce now is something that makes sense whenever

1. there is a notion of cylinder object, such as the topological cylinder $[0,1] \times X$ over a topological space, or the chain complex cylinder $I_\bullet \otimes X_\bullet$ of a chain complex from def. 2.82.

2. there is a way to *glue* objects along maps between them, a notion of colimit.

Definition 2.95. *For $f : X \to Y$ a morphism in a category with cylinder objects $\mathrm{cyl}(-)$, the* ***mapping cone*** *or* ***homotopy cofiber*** *of f is the colimit in the following diagram*

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "i_1"] & Y \arrow[d] \\ X \arrow[r, "i_0"] \arrow[d] & \mathrm{cyl}(X) \arrow[dr] & \\ * \arrow[rr] & & \mathrm{cone}(f) \end{tikzcd}$$

in C using any cylinder object $\mathrm{cyl}(X)$ for X.

Remark 2.96. Heuristically this says that $\mathrm{cone}(f)$ is the object obtained by

1. forming the cylinder over X;

2. gluing to one end of that the object Y as specified by the map f.

3. shrinking the other end of the cylinder to the point.

Heuristically it is clear that this way every cycle in Y that happens to be in the image of X can be "continuously" translated in the cylinder-direction, keeping it constant in Y, to the other end of the cylinder, where it becomes the point. This means that every homotopy group of Y in the image of f vanishes in the mapping cone. Hence in the mapping cone **the image of X under f in Y is removed up to homotopy**. This makes it clear how $\mathrm{cone}(f)$ is a homotopy-version of the cokernel of f. And therefore the name "mapping cone".

Another interpretation of the mapping cone is just as important:

Remark 2.97. A morphism $\eta : \mathrm{cyl}(X) \to Y$ out of a cylinder object is a left homotopy $\eta : g \Rightarrow h$ between its restrictions $g := \eta(0)$ and $h := \eta(1)$ to the cylinder boundaries

$$\begin{tikzcd} X \arrow[d, "i_0"] \arrow[dr, "g"] & \\ \mathrm{cyl}(X) \arrow[r, "\eta"] & Y \\ X \arrow[u, "i_1"] \arrow[ur, "h"'] & \end{tikzcd} .$$

Therefore prop. 2.95 says that the mapping cone is the universal object with a morphism i from Y and a left homotopy from $i \circ f$ to the zero morphism.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\swArrow_\eta"] & Y \arrow[d] \\ * \arrow[r] & \mathrm{cone}(f) \end{tikzcd}$$

The interested reader can find more on the conceptual background of this construction at *factorization lemma* and at *homotopy pullback*.

Proposition 2.98. *This colimit, in turn, may be computed in two stages by two consecutive pushouts in C, and in two ways by the following pasting diagram:*

$$\begin{array}{ccccc} & & X & \xrightarrow{f} & Y \\ & & \downarrow^{i_1} & & \downarrow \\ X & \xrightarrow{i_0} & \mathrm{cyl}(X) & \to & \mathrm{cyl}(f) \\ \downarrow & & \downarrow & & \downarrow \\ * & \to & \mathrm{cone}(X) & \to & \mathrm{cone}(f) \end{array} .$$

Here every square is a pushout, (and so by the pasting law is every rectangular pasting composite).

This now is a basic fact in ordinary category theory. The pushouts appearing here go by the following names:

Definition 2.99. The pushout

$$\begin{array}{ccc} X & \xrightarrow{i_0} & \mathrm{cyl}(X) \\ \downarrow & & \downarrow \\ * & \to & \mathrm{cone}(X) \end{array}$$

defines the **cone** $\mathrm{cone}(X)$ over X (with respect to the chosen cylinder object): the result of taking the cylinder over X and identifying one X-shaped end with the point.

The pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \mathrm{cyl}(X) & \to & \mathrm{cyl}(f) \end{array}$$

defines the **mapping cylinder** $\mathrm{cyl}(f)$ of f, the result of identifying one end of the cylinder over X with Y, using f as the gluing map.

The pushout

$$\begin{array}{ccc} \mathrm{cyl}(x) & \to & \mathrm{cyl}(f) \\ \downarrow & & \downarrow \\ \mathrm{cone}(X) & \to & \mathrm{cone}(f) \end{array}$$

defines the **mapping cone** $\mathrm{cone}(f)$ of f: the result of forming the cyclinder over X and then identifying one end with the point and the other with Y, via f.

Remark 2.100. As in remark 2.96 all these step have evident heuristic geometric interpretations:

1. $\mathrm{cone}(X)$ is obtained from the cylinder over X by contracting one end of the cylinder to the point;
2. $\mathrm{cyl}(f)$ is obtained from the cylinder over X by gluing Y to one end of the cylinder, as specified by the map f;

We discuss now this general construction of the mapping cone $\mathrm{cone}(f)$ for a chain map f between chain complexes. The end result is prop. 2.112 below, reproducing the classical formula for the mapping cone.

Definition 2.101. Write $*_\bullet \in \mathrm{Ch}_\bullet(\mathcal{A})$ for the chain complex concentrated on R in degree 0

$$*_\bullet 0 = [\cdots \to 0 \to 0 \to R] .$$

Remark 2.102. This may be understood as the normalized chain complex of chains of simplices on the terminal simplicial set Δ^0, the 0-simplex.

Definition 2.103. Let $I_\bullet \in \mathrm{Ch}_\bullet(\mathcal{A})$ be given by

$$I_\bullet = (\cdots 0 \to 0 \to R \xrightarrow{(-\mathrm{id},\mathrm{id})} R \oplus R) .$$

Denote by

$$i_0 : *_\bullet \to I_\bullet$$

the chain map which in degree 0 is the canonical inclusion into the second summand of a direct sum and by

$$i_1 : *_\bullet \to I_\bullet$$

correspondingly the canonical inclusion into the first summand.

Remark 2.104. This is the standard interval object in chain complexes.

It is in fact the normalized chain complex of chains on a simplicial set for the canonical simplicial interval, the 1-simplex:

$$I_\bullet = C_\bullet(\Delta[1]) \,.$$

The differential $\partial^I = (-\mathrm{id}, \mathrm{id})$ here expresses the alternating face map complex boundary operator, which in terms of the three non-degenerate basis elements is given by

$$\partial(0 \to 1) = (1) - (0) \,.$$

We decompose the proof of this statement is a sequence of substatements.

Proposition 2.105. *For $X_\bullet \in \mathrm{Ch}_\bullet$ the tensor product of chain complexes*

$$(I \otimes X)_\bullet \in \mathrm{Ch}_\bullet$$

is a cylinder object of $X_\bullet$ for the structure of a category of cofibrant objects on $\mathrm{Ch}_\bullet$ whose cofibrations are the monomorphisms and whose weak equivalences are the quasi-isomorphisms (the substructure of the standard injective model structure on chain complexes).

Example 2.106. In example 2.60 above we saw the cyclinder over the interval itself: the square.

Proposition 2.107. *The complex $(I \otimes X)_\bullet$ has components*

$$(I \otimes X)_n = X_n \oplus X_n \oplus X_{n-1}$$

and the differential is given by

$$\begin{array}{ccc} X_{n+1} \oplus X_{n+1} & \overset{\partial^X \oplus \partial^X}{\longrightarrow} & X_n \oplus X_n \\ \oplus & \nearrow_{(-\mathrm{id}, \mathrm{id})} & \oplus \\ X_n & \underset{-\partial^X}{\longrightarrow} & X_{n-1} \end{array} \,,$$

hence in matrix calculus by

$$\partial^{I \otimes X} = \left(\begin{array}{cc} \partial^X \oplus \partial^X & (-\mathrm{id}, \mathrm{id}) \\ 0 & -\partial^X \end{array} \right) : (X_{n+1} \oplus X_{n+1}) \oplus X_n \to (X_n \oplus X_n) \oplus X_{n-1} \,.$$

Proof. By the formula discussed at tensor product of chain complexes the components arise as the direct sum

$$(I \otimes X)_n = (R_{(0)} \otimes X_n) \oplus (R_{(1)} \otimes X_n) \oplus (R_{(0 \to 1)} \otimes X_{(n-1)})$$

and the differential picks up a sign when passed past the degree-1 term $R_{(0 \to 1)}$:

$$\begin{aligned} \partial^{I \otimes X}((0 \to 1), x) &= ((\partial^I (0 \to 1)), x) - ((0 \to 1), \partial^X x) \\ &= (-(0) + (1), x) - ((0 \to 1), \partial^X x) \\ &= -((0), x) + ((1), x) - ((0 \to 1), \partial^X x) \end{aligned} \,.$$

∎

Remark 2.108. The two boundary inclusions of $X_\bullet$ into the cylinder are given in terms of def. 2.103 by

$$i_0^X : X_\bullet \simeq *_\bullet \otimes X_\bullet \overset{i_0 \otimes \mathrm{id}_X}{\longrightarrow} (I \otimes X)_\bullet$$

and

$$i_1^X : X_\bullet \simeq *_\bullet \otimes X_\bullet \overset{i_1 \otimes \mathrm{id}_X}{\longrightarrow} (I \otimes X)_\bullet$$

which in components is the inclusion of the second or first direct summand, respectively

$$X_n \hookrightarrow X_n \oplus X_n \oplus X_{n-1} \,.$$

One part of definition 2.99 now reads:

Definition 2.109. For $f_\bullet : X_\bullet \to Y_\bullet$ a chain map, the mapping cylinder $\mathrm{cyl}(f)$ is the pushout

$$\begin{array}{ccc} \mathrm{cyl}(f)_\bullet & \leftarrow & Y_\bullet \\ \uparrow && \uparrow^{f} \\ I_\bullet \otimes X_\bullet & \overset{i_0}{\leftarrow} & X_\bullet \end{array} \,.$$

Proposition 2.110. *The components of $\mathrm{cyl}(f)$ are*

$$\mathrm{cyl}(f)_n = X_n \oplus Y_n \oplus X_{n-1}$$

and the differential is given by

$$\begin{array}{ccc} X_{n+1} \oplus Y_{n+1} & \overset{\partial^X \oplus \partial^Y}{\longrightarrow} & X_n \oplus Y_n \\ \oplus & \nearrow_{(-\mathrm{id}, f)} & \oplus \\ X_n & \underset{-\partial^X}{\longrightarrow} & X_{n-1} \end{array},$$

hence in matrix calculus by

$$\partial^{\mathrm{cyl}(f)} = \begin{pmatrix} \partial^X \oplus \partial^Y & (-\mathrm{id}, f_n) \\ 0 & -\partial^X \end{pmatrix} : (X_{n+1} \oplus Y_{n+1}) \oplus X_n \to (X_n \oplus Y_n) \oplus X_{n-1}\,.$$

Proof. The colimits in a category of chain complexes $\mathrm{Ch}_\bullet(\mathcal{A})$ are computed in the underlying presheaf category of towers in $\mathcal{A}$. There they are computed degreewise in $\mathcal{A}$ (see at limits in presheaf categories). Here the statement is evident:

the pushout identifies one direct summand X_n with Y_n along f_n and so where previously a id_{X_n} appeared on the diagonl, there is now f_n. ▮

The last part of definition 2.99 now reads:

Definition 2.111. For $f_\bullet : X_\bullet \to Y_\bullet$ a chain map, the **mapping cone** $\mathrm{cone}(f)$ is the pushout

$$\begin{array}{ccc} \mathrm{cone}(f) & \leftarrow & \mathrm{cyl}(f) \\ \uparrow && \uparrow \\ \mathrm{cone}(X) & \leftarrow & X \otimes I \\ \uparrow && \uparrow^{i_1} \\ 0 & \leftarrow & X \end{array}$$

Proposition 2.112. *The components of the mapping cone $\mathrm{cone}(f)$ are*

$$\mathrm{cone}(f)_n = Y_n \oplus X_{n-1}$$

with differential given by

$$\begin{array}{ccc} Y_{n+1} & \overset{\partial^Y}{\to} & Y_n \\ \oplus & \nearrow_{f_n} & \oplus \\ X_n & \underset{-\partial^X}{\longrightarrow} & X_{n-1} \end{array},$$

and hence in matrix calculus by

$$\partial^{\mathrm{cone}(f)} = \begin{pmatrix} \partial^Y_n & f_n \\ 0 & -\partial^X_n \end{pmatrix} : Y_{n+1} \oplus X_n \to Y_n \oplus X_{n-1}\,.$$

Proof. As before the pushout is computed degreewise. This identifies the remaining unshifted copy of X with 0. ▮

Proposition 2.113. *For $f : X_\bullet \to Y_\bullet$ a chain map, the canonical inclusion $i : Y_\bullet \to \mathrm{cone}(f)_\bullet$ of $Y_\bullet$ into the mapping cone of f is given in components*

$$i_n : Y_n \to \mathrm{cone}(f)_n = Y_n \oplus X_{n-1}$$

by the canonical inclusion of a summand into a direct sum.

Proof. This follows by starting with remark 2.108 and then following these inclusions through the formation of the two colimits as discussed above. ▮

Using these mapping cones of chain maps, we now explain how the long exact sequences of homology groups, prop. 2.78, are a shadow under homology of genuine homotopy cofiber sequences of the chain complexes themselves.

Let $f : X_\bullet \to Y_\bullet$ be a chain map and write $\mathrm{cone}(f) \in \mathrm{Ch}_\bullet(\mathcal{A})$ for its mapping cone as explicitly given in prop. 2.112.

Definition 2.114. Write $X[1]_\bullet \in \mathrm{Ch}_\bullet(\mathcal{A})$ for the suspension of a chain complex of X. Write

$$p : \mathrm{cone}(f) \to X[1]_\bullet$$

for the chain map which in components

$$p_n : \mathrm{cone}(f)_n \to X[1]_n$$

is given, via prop. 2.112, by the canonical projection out of a direct sum

$$p_n : Y_n \oplus X_{n-1} \to X_{n-1}\ .$$

This defines the mapping cone construction on chain complex. Its definition as a universal left homotopy should make the following proposition at least plausible, which we cannot prove yet at this point, but which we state nevertheless to highlight the meaning of the mapping cone construction. The tools for the proof of propositions like this are discussed further below in 7) Derived categories and derived functors.

Proposition 2.115. *The chain map $p : \mathrm{cone}(f)_\bullet \to X[1]_\bullet$ represents the homotopy cofiber of the canonical map $i : Y_\bullet \to \mathrm{cone}(f)_\bullet$.*

Proof. By prop. 2.113 and def. 2.114 the sequence

$$Y_\bullet \xrightarrow{i} \mathrm{cone}(f)_\bullet \xrightarrow{p} X[1]_\bullet$$

is a short exact sequence of chain complexes (since it is so degreewise, in fact degreewise it is even a split exact sequence, def. 2.73). In particular we have a cofiber pushout diagram

$$\begin{array}{ccc} Y_\bullet & \overset{i}{\hookrightarrow} & \mathrm{cone}(f)_\bullet \\ \downarrow && \downarrow \\ 0 & \to & X[1]_\bullet \end{array}\ .$$

Now, in the injective model structure on chain complexes all chain complexes are cofibrant objects and an inclusion such as $i : Y_\bullet \hookrightarrow \mathrm{cone}(f)_\bullet$ is a cofibration. By the detailed discussion at homotopy limit this means that the ordinary colimit here is in fact a homotopy colimit, hence exhibits p as the homotopy cofiber of i. ■

Accordingly one says:

Corollary 2.116. *For $f_\bullet : X_\bullet \to Y_\bullet$ a chain map, there is a **homotopy cofiber sequence** of the form*

$$X_\bullet \xrightarrow{f_\bullet} Y_\bullet \xrightarrow{i_\bullet} \mathrm{cone}(f)_\bullet \xrightarrow{p_\bullet} X[1]_\bullet \xrightarrow{f[1]_\bullet} Y_\bullet \xrightarrow{i[1]_\bullet} \mathrm{cone}(f)_\bullet \xrightarrow{p[1]_\bullet} X[2]_\bullet \to \cdots$$

In order to compare this to the discussion of connecting homomorphisms, we now turn attention to the case that $f_\bullet$ happens to be a monomorphism. Notice that this we can always assume, up to quasi-isomorphism, for instance by prolonging f by the map into its mapping cylinder

$$X_\bullet \to Y_\bullet \xrightarrow{\simeq} \mathrm{cyl}(f)\ .$$

By the axioms on an abelian category in this case we have a short exact sequence

$$0 \to X_\bullet \xrightarrow{f_\bullet} Y_\bullet \xrightarrow{p_\bullet} Z_\bullet \to 0$$

of chain complexes. The following discussion revolves around the fact that now $\mathrm{cone}(f)_\bullet$ as well as $Z_\bullet$ are both models for the homotopy cofiber of f.

Lemma 2.117. *Let*

$$X_\bullet \xrightarrow{f_\bullet} Y_\bullet \xrightarrow{p_\bullet} Z_\bullet$$

be a short exact sequence of chain complexes.

The collection of linear maps

$$h_n : Y_n \oplus X_{n-1} \to Y_n \to Z_n$$

constitutes a chain map

$$h_\bullet : \mathrm{cone}(f)_\bullet \to Z_\bullet\ .$$

This is a quasi-isomorphism. The inverse of $H_n(h_\bullet)$ is given by sending a representing cycle $z \in Z_n$ to

$$(\hat{z}_n, \partial^Y \hat{z}_n) \in Y_n \oplus X_{n+1}\,,$$

where $\hat{z}_n$ is any choice of lift through p_n and where $\partial^Y \hat{z}_n$ is the formula expressing the connecting homomorphism in terms of elements, as discussed at Connecting homomorphism – In terms of elements.

Finally, the morphism $i_\bullet : Y_\bullet \to \mathrm{cone}(f)_\bullet$ is eqivalent in the homotopy category (the derived category) to the zigzag

$$\begin{array}{ccc} && \mathrm{cone}(f)_\bullet \\ && \downarrow^{h}_{\simeq} \\ Y_\bullet & \to & Z_\bullet \end{array}\ .$$

Proof. To see that $h_\bullet$ defines a chain map recall the differential $\partial^{\mathrm{cone}(f)}$ from prop. 2.112, which acts by

$$\partial^{\mathrm{cone}(f)}(x_{n-1}, \hat{z}_n) = (-\partial^X x_{n-1}, \partial^Y \hat{z}_n + x_{n-1})$$

and use that x_{n-1} is in the kernel of p_n by exactness, hence

$$\begin{aligned} h_{n-1}\partial^{\mathrm{cone}(f)}(x_{n-1}, \hat{z}_n) &= h_{n-1}(-\partial^X x_{n-1}, \partial^Y \hat{z}_n + x_{n-1}) \\ &= p_{n-1}(\partial^Y \hat{z}_n + x_{n-1}) \\ &= p_{n-1}(\partial^Y \hat{z}_n) \\ &= \partial^Z p_n \hat{z}_n \\ &= \partial^Z h_n(x_{n-1}, \hat{z}_n) \end{aligned} .$$

It is immediate to see that we have a commuting diagram of the form

$$\begin{array}{ccc} & & \mathrm{cone}(f)_\bullet \\ & {}^{\mathllap{i_\bullet}}\nearrow & \downarrow^{\mathrlap{h}}_{\simeq} \\ Y_\bullet & \to & Z_\bullet \end{array}$$

since the composite morphism is the inclusion of Y followed by the bottom morphism on Y.

Abstractly, this already implies that $\mathrm{cone}(f)_\bullet \to Z_\bullet$ is a quasi-isomorphism, for this diagram gives a morphism of cocones under the diagram defining $\mathrm{cone}(f)$ in prop. 2.95 and by the above both of these cocones are homotopy-colimiting.

But in checking the claimed inverse of the induced map on homology groups, we verify this also explicity:

We first determine those cycles $(x_{n-1}, y_n) \in \mathrm{cone}(f)_n$ which lift a cycle z_n. By lemma 2.95 a lift of chains is any pair of the form $(x_{n-1}, \hat{z}_n)$ where $\hat{z}_n$ is a lift of z_n through $Y_n \to X_n$. So x_{n-1} has to be found such that this pair is a cycle. By prop. 2.112 the differential acts on it by

$$\partial^{\mathrm{cone}(f)}(x_{n-1}, \hat{z}_n) = (-\partial^X x_{n-1}, \partial^Y \hat{z}_n + x_{n-1})$$

and so the condition is that

$x_{n-1} \coloneqq -\partial^Y \hat{z}_n$ (which implies $\partial^X x_{n-1} = -\partial^X \partial^Y \hat{z}_n = -\partial^Y \partial^Y \hat{z}_n = 0$ due to the fact that f_n is assumed to be an inclusion, hence that ∂^X is the restriction of ∂^Y to elements in X_n).

This condition clearly has a unique solution for every lift $\hat{z}_n$ and a lift $\hat{z}_n$ always exists since $p_n : Y_n \to Z_n$ is surjective, by assumption that we have a short exact sequence of chain complexes. This shows that $H_n(h_\bullet)$ is surjective.

To see that it is also injective we need to show that if a cycle $(-\partial^Y \hat{z}_n, \hat{z}_n) \in \mathrm{cone}(f)_n$ maps to a cycle $z_n = p_n(\hat{z}_n)$ that is trivial in $H_n(Z)$ in that there is c_{n+1} with $\partial^Z c_{n+1} = z_n$, then also the original cycle was trivial in homology, in that there is (x_n, y_{n+1}) with

$$\partial^{\mathrm{cone}(f)}(x_n, y_{n+1}) \coloneqq (-\partial^X x_n, \partial^Y y_{n+1} + x_n) = (-\partial^Y \hat{z}_n, \hat{z}_n) \; .$$

For that let $\hat{c}_{n+1} \in Y_{n+1}$ be a lift of c_{n+1} through p_n, which exists again by surjectivity of p_{n+1}. Observe that

$$p_n(\hat{z}_n - \partial^Y \hat{c}_{n+1}) = z_n - \partial^Z(p_n \hat{c}_{n+1}) = z_n - \partial^Z(c_{n+1}) = 0$$

by assumption on z_n and c_{n+1}, and hence that $\hat{z}_n - \partial^Y \hat{c}_{n+1}$ is in X_n by exactness.

Hence $(z_n - \partial^Y \hat{c}_{n+1}, \hat{c}_{n+1}) \in \mathrm{cone}(f)_n$ trivializes the given cocycle:

$$\begin{aligned} \partial^{\mathrm{cone}(f)}(\hat{z}_n - \partial^Y \hat{c}_{n+1}, \hat{c}_{n+1}) &= (-\partial^X(\hat{z}_n - \partial^Y \hat{c}_{n+1}), \partial^Y \hat{c}_{n+1} + (\hat{z}_n - \partial^Y \hat{c}_{n+1})) \\ &= (-\partial^Y(\hat{z}_n - \partial^Y \hat{c}_{n+1}), \hat{z}_n) \\ &= (-\partial^Y \hat{z}_n, \hat{z}_n) \end{aligned} .$$

∎

Theorem 2.118. *Let*

$$X_\bullet \stackrel{f_\bullet}{\to} Y_\bullet \to Z_\bullet$$

be a short exact sequence of chain complexes.

Then the chain homology functor

$$H_n(-) : \mathrm{Ch}_\bullet(\mathcal{A}) \to \mathcal{A}$$

sends the homotopy cofiber sequence of f, cor. 2.116, to the long exact sequence in homology induced by

the given short exact sequence, hence to

$$H_n(X_\bullet) \to H_n(Y_\bullet) \to H_n(Z_\bullet) \overset{\delta}{\to} H_{n-1}(X_\bullet) \to H_{n-1}(Y_\bullet) \to H_{n-1}(Z_\bullet) \overset{\delta}{\to} H_{n-2}(X_\bullet) \to \cdots ,$$

where δ_n is the nth connecting homomorphism.

Proof. By lemma 2.117 the homotopy cofiber sequence is equivalen to the zigzag

$$\begin{array}{ccccccccc} &&&&&&& \mathrm{cone}(f)[1]_\bullet & \to & \cdots \\ &&&&&&& \downarrow^{h[1]_\bullet}_{\simeq} \\ &&&& \mathrm{cone}(f)_\bullet & \to & X[1]_\bullet \overset{f[1]_\bullet}{\longrightarrow} Y[1]_\bullet & \to & Z[1]_\bullet \\ &&&& \downarrow^{h_\bullet}_{\simeq} \\ X_\bullet & \overset{f_\bullet}{\to} & Y_\bullet & \to & Z_\bullet \end{array} \,.$$

Observe that

$$H_n(X[k]_\bullet) \simeq H_{n-k}(X_\bullet) \,.$$

It is therefore sufficient to check that

$$H_n\left(\begin{array}{ccc} \mathrm{cone}(f)_\bullet & \to & X[1]_\bullet \\ \downarrow^{\simeq} \\ Z_\bullet \end{array} \right) \;:\; H_n(Z_\bullet) \to H_n(\mathrm{cone}(f)_\bullet) \to H_{n-1}(X_\bullet)$$

equals the connecting homomorphism δ_n induced by the short exact sequence.

By prop. 2.117 the inverse of the vertical map is given by choosing lifts and forming the corresponding element given by the connecting homomorphism. By prop. 2.115 the horizontal map is just the projection, and hence the assignment is of the form

$$[z_n] \mapsto [x_{n-1}, y_n] \mapsto [x_{n-1}] \,.$$

So in total the image of the zig-zag under homology sends

$$[z_n]_Z \mapsto -[\partial^Y \hat{z}_n]_X \,.$$

By the discussion there, this is indeed the action of the connecting homomorphism. ∎

In summary, the above says that for every chain map $f_\bullet : X_\bullet \to Y_\bullet$ we obtain maps

$$X_\bullet \overset{f}{\to} Y_\bullet \xrightarrow{\left(\begin{smallmatrix} 0 \\ \mathrm{id}_{Y_\bullet} \end{smallmatrix}\right)} \mathrm{cone}(f)_\bullet \xrightarrow{\left(\begin{smallmatrix} \mathrm{id}_{X[1]_\bullet} & 0 \end{smallmatrix}\right)} X[1]_\bullet$$

which form a homotopy fiber sequence and such that this sequence continues by forming suspensions, hence for all $n \in \mathbb{Z}$ we have

$$X[n]_\bullet \overset{f}{\to} Y[n]_\bullet \xrightarrow{\left(\begin{smallmatrix} 0 \\ \mathrm{id}_{Y[n]_\bullet} \end{smallmatrix}\right)} \mathrm{cone}(f)[n]_\bullet \xrightarrow{\left(\begin{smallmatrix} \mathrm{id}_{X[n+11]_\bullet} & 0 \end{smallmatrix}\right)} X[n+1]_\bullet$$

To amplify this quasi-cyclic behaviour one sometimes depicts the situation as follows:

$$\begin{array}{ccccc} X_\bullet && \overset{f}{\to} && Y_\bullet \\ & {}_{[1]}\nwarrow && \swarrow \\ && \mathrm{cone}(f)_\bullet \end{array}$$

and hence speaks of a "triangle", or **distinguished triangle** or **mapping cone triangle** of f.

- distinguished triangle = period of homotopy fiber sequence .

Due to these "triangles" one calls the homotopy category of chain complexes localized at the quasi-isomorphisms, hence the derived category which we discuss below in 8), a **triangulated category**.

6) Double complexes and the diagram chasing lemmas

We have seen in the discussion of the connecting homomorphism in the homology long exact sequence in 4) above that given an *exact sequence of chain complexes* – hence in particular a chain complex of chain complexes – there are interesting ways to relate elements on the far right to elements on the far left in lower degree. In 5) we had given the conceptual explanation of this phenomenon in terms of long homotopy fiber sequences. But often it is just computationally useful to be able to efficiently establish and compute these "long diagram chase"-relations, independently of a homotopy-theoretic interpretation. Such

computational tools we discuss here.

A *chain complex of chain complex* is called a *double complex* and so we first introduce this elementary notion and the corresponding notion notion of total complex. (Total complexes are similarly elementary to define but will turn out to play a deeper role as models for homotopy colimits, this we indicate further below in chapter V)).

There is a host of classical diagram-chasing lemmas that relate far-away entries in double complexes that enjoy suitable exactness properties. These go by names such as the *snake lemma* or the *3x3 lemma*. The underlying mechanism of all these lemmas is made most transparent in the *salamander lemma*. This is fairly trivial to establish, and the notions it induces allow quick transparent proofs of all the other diagram-chasing lemmas.

The discussion to go here is kept at ***salamander lemma***. *See there.*

3. III) Abelian homotopy theory

We have seen in section II) that the most interesting properties of the category of chain complexes is all secretly controled by the phenomenon of chain homotopy and quasi-isomorphism. Strictly speaking these two phenomena point beyond plain category theory to the richer context of general abstract *homotopy theory*. Here we discuss properties of the category of chain complexes from this genuine homotopy-theoretic point of view. The result of passing the category of chain complexes to genuine homotopy theory is called the derived category (of the underlying abelian category $\mathcal{A}$, say of modules) and we start in 7) with a motivation of the phenomenon of this "homotopy derivation" and the discussion of the necessary *resolutions* of chain complexes. This naturally gives rise to the general notion of derived functors which we discuss in 8). Examples of these are ubiquituous in homological algebra, but as in ordinary enriched category theory two stand out as being of more fundamental importance, the derived functor "Ext" of the hom-functor and the derived functor "Tor" of the tensor product functor. Their properties and uses we discuss in 9).

7) Chain homotopy and resolutions

We now come back to the category $\mathcal{K}(\mathcal{A})$ of def. 2.89, the "homotopy category of chain complexes" in which chain-homotopic chain maps are identified. This would seem to be the right context to study the homotopy theory of chain complexes, but one finds that there are still chain maps which ought to be identified in homotopy theory, but which are still not identified in $\mathcal{K}(\mathcal{A})$. This is our motivating example 3.1 below.

We discuss then how this problem is fixed by allowing to first "resolve" chain complexes quasi-isomorphically by "good representatives" called *projective resolutions* or *injective resolutions*. Many of the computations in the following sections – and in homological algebra in general – come down to operating on such resolutions. We end this section by prop. 3.29 below, which shows that the above problem indeed goes away when allowing chain complexes to be resolved.

In the next section, 8), we discuss how this process of forming resolutions functorially extends to the whole category of modules.

So we start here with this simple example that shows the problem with bare chain homotopies and indicates how these have to be *resolved*:

Example 3.1. In $\mathrm{Ch}_\bullet(\mathcal{A})$ for $\mathcal{A} =$ Ab consider the chain map

$$\begin{array}{ccccccccc} \cdots & \to & 0 & \to & 0 & \to & 0 & \to & \mathbb{Z}_2 \\ && \downarrow && \downarrow && \downarrow && \downarrow^{\mathrm{id}} \\ \cdots & \to & 0 & \to & \mathbb{Z} & \stackrel{\cdot 2}{\to} & \mathbb{Z} & \stackrel{\mathrm{mod}\, 2}{\longrightarrow} & \mathbb{Z}_2 \end{array}.$$

The codomain of this map is an exact sequence, hence is quasi-isomorphic to the 0-chain complex. Thereofore in homotopy theory it should behave entirely as the 0-complex itself. In particular, every chain map to it should be chain homotopic to the zero morphism (have a *null homotopy*).

But the above chain map is chain homotopic precisely only to itself. This is because the degree-0 component of any chain homotopy out of this has to be a homomorphism of abelian groups $\mathbb{Z}_2 \to \mathbb{Z}$, and this must be the 0-morphism, because $\mathbb{Z}$ is a free group, but $\mathbb{Z}_2$ is not.

This points to the problem: the components of the domain chain complex are not *free enough* to admit sufficiently many maps out of it.

Consider therefore a free resolution of the above domain complex by the quasi-isomorphism

$$\begin{array}{ccccccccc} \cdots & \to & 0 & \to & 0 & \to & \mathbb{Z} & \stackrel{\cdot 2}{\to} & \mathbb{Z} \\ && \downarrow && \downarrow && \downarrow && \downarrow^{\mathrm{mod}\, 2} \\ \cdots & \to & 0 & \to & 0 & \to & 0 & \to & \mathbb{Z}_2 \end{array},$$

where now the domain complex consists entirely of free groups. The composite of this with the original chain map is now

$$\begin{array}{ccccccccc} \cdots & \to & 0 & \to & 0 & \to & \mathbb{Z} & \overset{\cdot 2}{\to} & \mathbb{Z} \\ && \downarrow && \downarrow && \downarrow^{0} && \downarrow^{\mathrm{mod}\,2} \\ \cdots & \to & 0 & \to & \mathbb{Z} & \overset{\cdot 2}{\to} & \mathbb{Z} & \overset{\mathrm{mod}\,2}{\longrightarrow} & \mathbb{Z}_2 \end{array}.$$

This is the corresponding resolution of the original chain map. And *this* indeed has a null homotopy:

$$\begin{array}{ccccccccc} \cdots & \to & 0 & \to & 0 & \to & \mathbb{Z} & \overset{\cdot 2}{\to} & \mathbb{Z} \\ && \downarrow & \swarrow & \downarrow & \swarrow_{-\mathrm{id}} & \downarrow^{0} & \swarrow_{\mathrm{id}} & \downarrow^{\mathrm{mod}\,2} \\ \cdots & \to & 0 & \to & \mathbb{Z} & \overset{\cdot 2}{\to} & \mathbb{Z} & \overset{\mathrm{mod}\,2}{\longrightarrow} & \mathbb{Z}_2 \end{array}.$$

So *resolving the domain by a sufficiently free complex* makes otherwise missing chain homotopies exist. Below in lemma 3.30 we discuss the general theory behind the kind of situation of this example. But to get there we first need some basic notions and facts.

Notably, in general it is awkward to insist on actual *free resolutions*. But it is easy to see, and this we discuss now, that essentially just as well is a resolution by modules which are direct summands of free modules.

Definition 3.2. An object P of a category C is a **projective object** if it has the left lifting property against epimorphisms.

This means that P is projective if for any morphism $f : P \to B$ and any epimorphism $q : A \to B$, f factors through q by some morphism $P \to A$.

$$\begin{array}{ccc} && A \\ & {}^{\exists}\nearrow & \downarrow^{q} \\ P & \overset{f}{\to} & B \end{array}.$$

An equivalent way to say this is that:

Definition 3.3. An object P is projective precisely if the hom-functor $\mathrm{Hom}(P,-)$ preserves epimorphisms.

Remark 3.4. The point of this lifting property will become clear when we discuss the construction of projective resolutions a bit further below: they are built by applying this property degreewise to obtain suitable chain maps.

We will be interested in projective objects in the category RMod: *projective modules*. Before we come to that, notice the following example (which the reader may on first sight feel is pedantic and irrelevant, but for the following it is actually good to make this explicit).

Example 3.5. In the category Set of sets the following are equivalent

- every object is projective;
- the axiom of choice holds.

Remark 3.6. We will assume here throughout the axiom of choice in Set, as usual. The point of the above example, however, is that one could just as well replace Set by another "base topos" which will behave essentially precisely like Set, but in general will not validate the axiom of choice. Homological algebra in such a more general context is the theory of complexes of abelian sheaves/sheaves of abelian groups and ultimately the theory of *abelian sheaf cohomology*.

This is a major aspect of homological algebra. While we will not discuss this further here in this introduction, the reader might enjoy keeping in mind that all of the following discussion of resolutions of R-modules goes through in this wider context of sheaves of modules except for subtleties related to the (partial) failure of example 3.5 for the category of sheaves.

We now characterize projective modules.

Lemma 3.7. *Assuming the axiom of choice, a free module $N \simeq R^{(S)}$ is projective.*

Proof. Explicitly: if $S \in \mathrm{Set}$ and $F(S) = R^{(S)}$ is the free module on S, then a module homomorphism $F(S) \to N$ is specified equivalently by a function $f : S \to U(N)$ from S to the underlying set of N, which can be thought of as specifying the images of the unit elements in $R^{(S)} \simeq \oplus_{s \in S} R$ of the $|S|$ copies of R.

Accordingly then for $\tilde{N} \to N$ an epimorphism, the underlying function $U(\tilde{N}) \to U(N)$ is an epimorphism, and the axiom of choice in Set says that we have all lifts $\tilde{f}$ in

$$\begin{array}{ccc} && U(\tilde{N}) \\ & {}^{\tilde{f}}\nearrow & \downarrow \\ S & \overset{f}{\to} & U(N) \end{array}.$$

By adjunction these are equivalently lifts of module homomorphisms

$$\begin{array}{ccc} & & \tilde{N} \\ & \nearrow & \downarrow \\ R^{(S)} & \to & N \end{array} .$$

∎

Lemma 3.8. *If $N \in R\,\mathrm{Mod}$ is a direct summand of a free module, hence if there is $N' \in R\,\mathrm{Mod}$ and $S \in \mathrm{Set}$ such that*

$$R^{(S)} \simeq N \oplus N' ,$$

then N is a projective module.

Proof. Let $\tilde{K} \to K$ be a surjective homomorphism of modules and $f : N \to K$ a homomorphism. We need to show that there is a lift $\tilde{f}$ in

$$\begin{array}{ccc} & & \tilde{K} \\ & {}^{\tilde{f}}\nearrow & \downarrow \\ N & \stackrel{f}{\to} & K \end{array} .$$

By definition of direct sum we can factor the identity on N as

$$\mathrm{id}_N : N \to N \oplus N' \to N .$$

Since $N \oplus N'$ is free by assumption, and hence projective by lemma 3.7, there is a lift $\hat{f}$ in

$$\begin{array}{ccccc} & & & & \tilde{K} \\ & & & {}^{\hat{f}}\nearrow & \downarrow \\ N & \to & N \oplus N' & \to & K \end{array} .$$

Hence $\tilde{f} : N \to N \oplus N' \stackrel{\hat{f}}{\to} \tilde{K}$ is a lift of f. ∎

Proposition 3.9. *An R-module N is projective precisely if it is the direct summand of a free module.*

Proof. By lemma 3.8 if N is a direct summand then it is projective. So we need to show the converse.

Let $F(U(N))$ be the free module on the set $U(N)$ underlying N, hence the direct sum

$$F(U(n)) = \oplus_{n \in U(n)} R .$$

There is a canonical module homomorphism

$$\oplus_{n \in U(N)} R \to N$$

given by sending the unit $1 \in R_n$ of the copy of R in the direct sum labeled by $n \in U(n)$ to $n \in N$.

(Abstractly this is the counit $\epsilon : F(U(N)) \to N$ of the free/forgetful-adjunction $(F \dashv U)$.)

This is clearly an epimorphism. Thefore if N is projective, there is a section s of ϵ. This exhibits N as a direct summand of $F(U(N))$. ∎

We discuss next how to build resolutions of chain complexes by projective modules. But before we come to that it is useful to also introduce the *dual* notion. So far we have concentrated on chain complexes with degrees in the natural numbers: non-negative degrees. For a discussion of resolutions we need a more degree-symmetric perspective, which of course is straightforward to obtain.

Definition 3.10. A cochain complex $C^\bullet$ in $\mathcal{A} = R\,\mathrm{Mod}$ is a sequence of morphism

$$C^0 \stackrel{d^0}{\to} C^1 \stackrel{d^1}{\to} C^2 \stackrel{d^2}{\to} \cdots$$

in $\mathcal{A}$ such that $d \circ d = 0$. A homomorphism of cochain complexes $f^\bullet : C^\bullet \to D^\bullet$ is a collection of morphisms $\{f^n : C^n \to D^n\}$ such that $d^n_D \circ f^n = f^n \circ d^n_C$ for all $n \in \mathbb{N}$.

We write $\mathrm{Ch}^\bullet(\mathcal{A})$ for the category of cochain complexes.

Example 3.11. Let $N \in \mathcal{A}$ be a fixed module and $C_\bullet \in \mathrm{Ch}_\bullet(\mathcal{A})$ a chain complex. Then applying degreewise the hom-functor out of the components of $C_\bullet$ into N yields a cochain complex in $\mathbb{Z}\,\mathrm{Mod} \simeq \mathrm{Ab}$:

$$\mathrm{Hom}_{\mathcal{A}}(C_\bullet, N) = \left[\mathrm{Hom}_{\mathcal{A}}(C_0, N) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(\partial_0, N)} \mathrm{Hom}_{\mathcal{A}}(C_1, N) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(\partial_1, N)} \mathrm{Hom}_{\mathcal{A}}(C_2, N) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(\partial_2, N)} \cdots \right] .$$

Example 3.12. In example 3.11 let $\mathcal{A} = \mathbb{Z}\mathrm{Mod} = \mathrm{Ab}$, let $N = \mathbb{Z}$ and let $C_\bullet = \mathbb{Z}[\mathrm{Sing}(X)]$ be the singular

simplicial complex of a topological space X. Write

$$C^\bullet(X) \coloneqq \mathrm{Hom}_{\mathbb{Z}[\mathrm{Sing}\, X],\mathbb{Z}} \,.$$

Then $H^\bullet(C(X))$ is called the *singular cohomology* of X.

Remark 3.13. Example 3.11 is just a special case of the internal hom of def. 2.63: we may regard cochain complexes in non-negative degree equivalently as chain complexes in positive degree.

Accordingly we say for $C^\bullet$ a cochain complex that

- an element in C^n is an n-cochain
- an element in $\mathrm{im}(d^{n-1})$ is an n-coboundary
- al element in $\ker(d^n)$ is an n-cocycle.

But equivalently we may regard a cochain in degree n as a chain in degree $(-n)$ and so forth. And this is the perspective used in all of the following.

The role of projective objects, def. 3.2, for chain complexes is played, dually, by *injective objects* for cochain complexes:

Definition 3.14. An object I a category is **injective** if all diagrams of the form

$$\begin{array}{ccc} X & \to & I \\ \downarrow & & \\ Z & & \end{array}$$

with $X \to Z$ a monomorphism admit an extension

$$\begin{array}{ccc} X & \to & I \\ \downarrow & \nearrow_{\exists} & \\ Z & & \end{array} \,.$$

Since we are interested in refining modules by projective or injective modules, we have the following terminology.

Definition 3.15. A category

- **has enough projectives** if for every object X there is a projective object Q equipped with an epimorphism $Q \to X$;
- **has enough injectives** if for every object X there is an injective object P equipped with a monomorphism $X \to P$.

We have essentially already seen the following statement.

Proposition 3.16. *Assuming the axiom of choice, the category RMod has enough projectives.*

Proof. Let $F(U(N))$ be the free module on the set $U(N)$ underlying N. By lemma 3.7 this is a projective module.

The canonical morphism

$$F(U(n)) = \bigoplus_{n \in U(n)} R \to N$$

is clearly a surjection, hence an epimorphism in RMod. ∎

We now show that similarly $R\,\mathrm{Mod}$ has enough injectives. This is a little bit more work and hence we proceed with a few preparatory statements.

The following basic statement of algebra we cite here without proof (but see at *injective object* for details).

Proposition 3.17. *Assuming the axiom of choice, an abelian group A is injective as a $\mathbb{Z}$-module precisely if it is a divisible group, in that for all integers $n \in \mathbb{N}$ we have $nG = G$.*

Example 3.18. By prop. 3.17 the following abelian groups are injective in Ab.

The group of rational numbers $\mathbb{Q}$ is injective in Ab, as is the additive group of real numbers $\mathbb{R}$ and generally that underlying any field. The additive group underlying any vector space is injective. The quotient of any injective group by any other group is injective.

Example 3.19. **Not** injective in Ab are the cyclic groups $\mathbb{Z}/n\mathbb{Z}$.

Proposition 3.20. *Assuming the axiom of choice, the category $\mathbb{Z}$Mod $\simeq$ Ab has enough injectives.*

Proof. By prop. 3.17 an abelian group is an injective $\mathbb{Z}$-module precisely if it is a divisible group. So we need to show that every abelian group is a subgroup of a divisible group.

To start with, notice that the group $\mathbb{Q}$ of rational numbers is divisible and hence the canonical embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$ shows that the additive group of integers embeds into an injective $\mathbb{Z}$-module.

Now by the discussion at *projective module* every abelian group A receives an epimorphism $(\oplus_{s \in S} \mathbb{Z}) \to A$ from a free abelian group, hence is the quotient group of a direct sum of copies of $\mathbb{Z}$. Accordingly it embeds into a quotient $\tilde{A}$ of a direct sum of copies of $\mathbb{Q}$.

$$\begin{array}{ccc} \ker & \stackrel{=}{\to} & \ker \\ \downarrow && \downarrow \\ (\oplus_{s \in S} \mathbb{Z}) & \hookrightarrow & (\oplus_{s \in S} \mathbb{Q}) \\ \downarrow && \downarrow \\ A & \hookrightarrow & \tilde{A} \end{array}$$

Here $\tilde{A}$ is divisible because the direct sum of divisible groups is again divisible, and also the quotient group of a divisible groups is again divisble. So this exhibits an embedding of any A into a divisible abelian group, hence into an injective $\mathbb{Z}$-module. ∎

Proposition 3.21. *Assuming the axiom of choice, for R a ring, the category RMod has enough injectives.*

The proof uses the following lemma.

Write $U : R\,\mathrm{Mod} \to \mathrm{Ab}$ for the forgetful functor that forgets the R-module structure on a module N and just remembers the underlying abelian group $U(N)$.

Lemma 3.22. *The functor $U : R\,\mathrm{Mod} \to \mathrm{Ab}$ has a right adjoint*

$$R_* : \mathrm{Ab} \to R\,\mathrm{Mod}$$

given by sending an abelian group A to the abelian group

$$U(R_*(A)) \coloneqq \mathrm{Ab}(U(R), A)$$

equipped with the R-module struture by which for $r \in R$ an element $(U(R) \stackrel{f}{\to} A) \in U(R_*(A))$ is sent to the element $r f$ given by

$$r f : r' \mapsto f(r' \cdot r) \,.$$

This is called the ***coextension of scalars*** *along the ring homomorphism $\mathbb{Z} \to R$.*

The unit of the $(U \dashv R_*)$ adjunction

$$\epsilon_N : N \to R_*(U(N))$$

is the R-module homomorphism

$$\epsilon_N : N \to \mathrm{Hom}_{\mathrm{Ab}}(U(R), U(N))$$

given on $n \in N$ by

$$j(n) : r \mapsto r n \,.$$

***Proof*. of prop. 3.21**

Let $N \in R\,\mathrm{Mod}$. We need to find a monomorphism $N \to \tilde{N}$ such that $\tilde{N}$ is an injective R-module.

By prop. 3.20 there exists a monomorphism

$$i : U(N) \hookrightarrow D$$

of the underlying abelian group into an injective abelian group D.

Now consider the $(U \dashv R_*)$-adjunct

$$N \to R_*(D)$$

of i, hence the composite

$$N \stackrel{\eta_N}{\to} R_*(U(N)) \stackrel{R_*(i)}{\longrightarrow} R_*(D)$$

with R_* and η_N from lemma 3.22. On the underlying abelian groups this is

$$U(N) \stackrel{U(\eta_N)}{\longrightarrow} \mathrm{Hom}_{\mathrm{Ab}}(U(R), U(N)) \stackrel{\mathrm{Hom}_{\mathrm{Ab}}(U(R), i)}{\longrightarrow} \mathrm{Hom}_{\mathrm{Ab}}(U(R), U(D)) \,.$$

Hence this is monomorphism. Therefore it is now sufficient to see that $\mathrm{Hom}_{\mathrm{Ab}}(U(R), U(D))$ is an injective R-module.

This follows from the existence of the adjunction isomorphism given by lemma 3.22

$$\mathrm{Hom}_{\mathrm{Ab}}(U(K), U(D)) \simeq \mathrm{Hom}_{R\,\mathrm{Mod}}(K, \mathrm{Hom}_{\mathrm{Ab}}(U(R), U(D)))$$

natural in $K \in R\,\mathrm{Mod}$ and from the injectivity of $D \in \mathrm{Ab}$.

$$\begin{array}{ccc} U(K) & \to & D \\ \downarrow & \nearrow & \\ U(L) & & \end{array} \quad \leftrightarrow \quad \begin{array}{ccc} K & \to & R_* D \\ \downarrow & \nearrow & \\ L & & \end{array} .$$

∎

Now we can state the main definition of this section and discuss its central properties.

Definition 3.23. For $X \in \mathcal{A}$ an object, an **injective resolution** of X is a cochain complex $J^\bullet \in \mathrm{Ch}^\bullet(\mathcal{A})$ (in non-negative degree) equipped with a quasi-isomorphism

$$i : X \overset{\simeq}{\to} J^\bullet$$

such that $J^n \in \mathcal{A}$ is an injective object for all $n \in \mathbb{N}$.

Remark 3.24. In components the quasi-isomorphism of def. 3.23 is a chain map of the form

$$\begin{array}{ccccccccc} X & \to & 0 & \to & \cdots & \to & 0 & \to & \cdots \\ \downarrow^{i^0} & & \downarrow & & & & \downarrow & & \\ J^0 & \overset{d^0}{\to} & J^1 & \overset{d^1}{\to} & \cdots & \to & J^n & \overset{d^n}{\to} & \cdots \end{array} .$$

Since the top complex is concentrated in degree 0, this being a quasi-isomorphism happens to be equivalent to the sequence

$$0 \to X \overset{i^0}{\to} J^0 \overset{d^0}{\to} J^1 \overset{d^1}{\to} J^2 \overset{d^2}{\to} \cdots$$

being an exact sequence. In this form one often finds the definition of injective resolution in the literature.

Definition 3.25. For $X \in \mathcal{A}$ an object, a **projective resolution** of X is a chain complex $J_\bullet \in \mathrm{Ch}_\bullet(\mathcal{A})$ (in non-negative degree) equipped with a quasi-isomorphism

$$p : J_\bullet \overset{\simeq}{\to} X$$

such that $J_n \in \mathcal{A}$ is a projective object for all $n \in \mathbb{N}$.

Remark 3.26. In components the quasi-isomorphism of def. 3.25 is a chain map of the form

$$\begin{array}{ccccccccc} \cdots & \overset{\partial_n}{\to} & J_n & \overset{\partial_{n-1}}{\longrightarrow} & \cdots & \to & J_1 & \overset{\partial_0}{\to} & J_0 \\ & & \downarrow & & & & \downarrow & & \downarrow^{p_0} \\ \cdots & \to & 0 & \to & \cdots & \to & 0 & \to & X \end{array}$$

Since the bottom complex is concentrated in degree 0, this being a quasi-isomorphism happens to be equivalent to the sequence

$$\cdots J_2 \overset{\partial_1}{\to} J_1 \overset{\partial_0}{\to} J_0 \overset{p_0}{\to} X \to 0$$

being an exact sequence. In this form one often finds the definition of projective resolution in the literature.

We first discuss the existence of injective/projective resolutions, and then the functoriality of their constructions.

Proposition 3.27. *Let $\mathcal{A}$ be an abelian category with enough injectives, such as our $R\mathrm{Mod}$ for some ring R.*

Then every object $X \in \mathcal{A}$ has an injective resolution, def. 3.23.

Proof. Let $X \in \mathcal{A}$ be the given object. By remark 3.24 we need to construct an exact sequence of the form

$$0 \to X \to J^0 \overset{d^0}{\to} J^1 \overset{d^1}{\to} J^2 \overset{d^2}{\to} \cdots \to J^n \to \cdots$$

such that all the $J^\cdot$ are injective objects.

This we now construct by induction on the degree $n \in \mathbb{N}$.

In the first step, by the assumption of enough injectives we find an injective object J^0 and a monomorphism

$$X \hookrightarrow J^0$$

hence an exact sequence

$$0 \to X \to J^0 \, .$$

Assume then by induction hypothesis that for $n \in \mathbb{N}$ an exact sequence

$$X \to J^0 \overset{d^0}{\to} \cdots \to J^{n-1} \xrightarrow{d^{n-1}} J^n$$

has been constructed, where all the $J^\cdot$ are injective objects. Forming the cokernel of d^{n-1} yields the short exact sequence

$$0 \to J^{n-1} \xrightarrow{d^{n-1}} J^n \overset{p}{\to} J^n / J^{n-1} \to 0 \, .$$

By the assumption that there are enough injectives in $\mathcal{A}$ we may now again find a monomorphism $J^n / J^{n-1} \overset{i}{\hookrightarrow} J^{n+1}$ into an injective object J^{n+1}. This being a monomorphism means that

$$J^{n-1} \xrightarrow{d^{n-1}} J^n \xrightarrow{d^n \coloneqq i \circ p} J^{n+1}$$

is exact in the middle term. Therefore we now have an exact sequence

$$0 \to X \to J^0 \to \cdots \to J^{n-1} \xrightarrow{d^{n-1}} J^n \overset{d^n}{\to} J^{n+1}$$

which completes the induction step. ▮

The following proposition is formally dual to prop. 3.27.

Proposition 3.28. *Let $\mathcal{A}$ be an abelian category with enough projectives (such as RMod for some ring R).*

Then every object $X \in \mathcal{A}$ has a projective resolution, def. 3.25.

Proof. Let $X \in \mathcal{A}$ be the given object. By remark 3.26 we need to construct an exact sequence of the form

$$\cdots \overset{\partial_2}{\to} J_2 \overset{\partial_1}{\to} J_1 \overset{\partial_0}{\to} J_0 \to X \to 0$$

such that all the $J_\cdot$ are projective objects.

This we we now construct by induction on the degree $n \in \mathbb{N}$.

In the first step, by the assumption of enough projectives we find a projective object J_0 and an epimorphism

$$J_0 \to X$$

hence an exact sequence

$$J_0 \to X \to 0 \, .$$

Assume then by induction hypothesis that for $n \in \mathbb{N}$ an exact sequence

$$J_n \xrightarrow{\partial_{n-1}} J_{n-1} \to \cdots \overset{\partial_0}{\to} J_0 \to X \to 0$$

has been constructed, where all the $J_\cdot$ are projective objects. Forming the kernel of ∂_{n-1} yields the short exact sequence

$$0 \to \ker(\partial_{n-1}) \overset{i}{\to} J_n \xrightarrow{\partial_{n-1}} J_{n-1} \to 0 \, .$$

By the assumption that there are enough projectives in $\mathcal{A}$ we may now again find an epimorphism $p : J_{n+1} \to \ker(\partial_{n-1})$ out of a projective object J_{n+1}. This being an epimorphism means that

$$J_{n+1} \xrightarrow{\partial_n \coloneqq i \circ p} J_n \xrightarrow{\partial_{n-1}}$$

is exact in the middle term. Therefore we now have an exact sequence

$$J_{n+1} \overset{\partial_n}{\to} J_n \xrightarrow{\partial_{n-1}} \cdots \overset{\partial_0}{\to} J_0 \to X \to 0 \, ,$$

which completes the induction step. ▮

To conclude this section we now show that all this work indeed serves to solve the problem indicated above in example 3.1.

Proposition 3.29. *Let $f^\bullet : X^\bullet \to J^\bullet$ be a chain map of cochain complexes in non-negative degree, out of an exact complex $0 \simeq_{\mathrm{qi}} X^\bullet$ to a degreewise injective complex $J^\bullet$. Then there is a null homotopy*

$$\eta : 0 \Rightarrow f^\bullet$$

Proof. By definition of chain homotopy we need to construct a sequence of morphisms $(\eta^{n+1} : X^{n+1} \to J^n)_{n \in \mathbb{N}}$ such that

$$f^n = \eta^{n+1} \circ d_X^n + d_J^{n-1} \circ \eta^n \,.$$

for all n. We now construct this by induction over n.

It is convenient to start at $n = -1$, take $\eta^{\leq 0} \coloneqq 0$ and $f^{<0} \coloneqq 0$. Then the above condition holds for $n = -1$.

Then in the induction step assume that for given $n \in \mathbb{N}$ we have constructed $\eta^{\bullet \leq n}$ satisfying the above condition for $f^{<n}$

First define now

$$g^n \coloneqq f^n - d_J^{n-1} \circ \eta^n$$

and observe that by induction hypothesis

$$\begin{aligned} g^n \circ d_X^{n-1} &= f^n \circ d_X^{n-1} - d_J^{n-1} \circ \eta^n \circ d_X^{n-1} \\ &= f^n \circ d_X^{n-1} - d_J^{n-1} \circ f^{n-1} + d_J^{n-1} \circ d_J^{n-2} \circ \eta^{n-1} \\ &= 0 + 0 \\ &= 0 \end{aligned} \,.$$

This means that g^n factors as

$$X^n \to X^n / \mathrm{im}(d_X^{n-1}) \overset{g^n}{\to} J^n \,,$$

where the first map is the projection to the quotient.

Observe then that by exactness of $X^\bullet$ the morphism $X^n / \mathrm{im}(d_X^{n-1}) \overset{d_X^n}{\to} X^{n+1}$ is a monomorphism. Together this gives us a diagram of the form

$$\begin{array}{ccc} X^n / \mathrm{im}(d_X^{n-1}) & \overset{d_X^n}{\to} & X^{n+1} \\ \downarrow^{g^n} & \swarrow_{\eta^{n+1}} & \\ J^n & & \end{array} \,,$$

where the morphism η^{n+1} may be found due to the defining right lifting property of the injective object J^n against the top monomorphism.

Observing that the commutativity of this diagram is the chain homotopy condition involving η^n and η^{n+1}, this completes the induction step. ▮

The formally dual statement of prop 3.29 is the following.

Lemma 3.30. *Let $f_\bullet : P_\bullet \to Y_\bullet$ be a chain map of chain complexes in non-negative degree, into an exact complex $0 \simeq_{\mathrm{qi}} Y_\bullet$ from a degreewise projective complex $X^\bullet$. Then there is a null homotopy*

$$\eta : 0 \Rightarrow f_\bullet$$

Proof. This is formally dual to the proof of prop 3.29. ▮

Hence we have seen now that injective and projective resolutions of chain complexes serve to make chain homotopy interact well with quasi-isomorphism. In the next section we show that this construction lifts from single chain complexes to chain maps between chain complexes and in fact to the whole category of chain complexes. The resulting "resolved" category of chain complexes is the *derived category*, the true home of the abelian homotopy theory of chain complexes.

8) The derived category

In the previous section we have seen that every object $A \in \mathcal{A}$ admits an injective resolution and a projective resolution. Here we lift this construction to morphisms and then to the whole category of chain complexes, up to chain homotopy.

The following proposition says that, when injectively resolving objects, the morphisms between these objects lift to the resolutions, and the following one, prop. 3.32, says that this lift is unique up to chain homotopy.

Proposition 3.31. *Let $f : X \to Y$ be a morphism in $\mathcal{A}$. Let*

$$i_Y : Y \overset{\simeq}{\to} Y^\bullet$$

be an injective resolution of Y and

$$i_X : X \overset{\simeq}{\to} X^\bullet$$

any monomorphism that is a quasi-isomorphism (possibly but not necessarily an injective resolution). Then there is a chain map $f^\bullet : X^\bullet \to Y^\bullet$ giving a commuting diagram

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d, "f"] & X^\bullet \arrow[d, "f^\bullet"] \\ Y \arrow[r, "\simeq"] & Y^\bullet \end{tikzcd} .$$

Proof. By definition of chain map we need to construct morphisms $(f^n : X^n \to Y^n)_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$ the diagrams

$$\begin{tikzcd} X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[d, "f^{n+1}"] \\ Y^n \arrow[r, "d_Y^n"] & Y^{n+1} \end{tikzcd}$$

commute (the defining condition on a chain map) and such that the diagram

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f"] & X^0 \arrow[d, "f^0"] \\ Y \arrow[r, "i_Y"] & Y^0 \end{tikzcd}$$

commutes in $\mathcal{A}$ (which makes the full diagram in $\mathrm{Ch}^\bullet(\mathcal{A})$ commute).

We construct these $f^\bullet = (f^n)_{n \in \mathbb{N}}$ by induction.

To start the induction, the morphism f^0 in the last diagram above can be found by the defining right lifting property of the injective object Y^0 against the monomorphism i_X.

Assume then that for some $n \in \mathbb{N}$ component maps $f^{\bullet \leq n}$ have been obtained such that $d_Y^k \circ f^k = f^{k+1} \circ d_X^k$ for all $0 \leq k < n$. In order to construct f^{n+1} consider the following diagram, which we will describe/construct stepwise from left to right:

$$\begin{tikzcd} X^n \arrow[r] \arrow[d, "f^n"'] \arrow[dr, "g^n"] & X^n/\mathrm{im}(d_X^{n-1}) \arrow[r, hook, "d_X^n"] \arrow[d, "h^n"] & X^{n+1} \arrow[dl, "f^{n+1}"] \\ Y^n \arrow[r, "d_Y^n"'] & Y^{n+1} & \end{tikzcd} .$$

Here the morphism f^n on the left is given by induction assumption and we define the diagonal morphism to be the composite

$$g^n \coloneqq d_Y^n \circ f^n .$$

Observe then that by the chain map property of the $f^{\bullet \leq n}$ we have

$$d_Y^n \circ f^n \circ d_X^{n-1} = d_Y^n \circ d_Y^{n-1} \circ f^{n-1} = 0$$

and therefore g^n factors through $X^n/\mathrm{im}(d_X^{n-1})$ via some h^n as indicated in the middle of the above diagram. Finally the morphism on the top right is a monomorphism by the fact that $X^\bullet$ is exact in positive degrees (being quasi-isomorphic to a complex concentrated in degree 0) and so a lift f^{n+1} as shown on the far right of the diagram exists by the defining lifting property of the injective object Y^{n+1}.

The total outer diagram now commutes, being built from commuting sub-diagrams, and this is the required chain map property of $f^{\bullet \leq n+1}$ This completes the induction step. ▮

Proposition 3.32. *The morphism $f_\bullet$ in prop. 3.31 is the unique one up to chain homotopy making the given diagram commute.*

Proof. Given two cochain maps $g_1^\bullet, g_2^\bullet$ making the diagram commute, a chain homotopy $g_1^\bullet \Rightarrow g_2^\bullet$ is equivalently a null homotopy $0 \Rightarrow g_2^\bullet - g_1^\bullet$ of the difference, which sits in a square of the form

$$\begin{tikzcd} X \arrow[r, "\simeq", "h^\bullet"'] \arrow[d, "0"] & X^\bullet \arrow[d, "f^\bullet \coloneqq g_2^\bullet - g_1^\bullet"] \\ Y \arrow[r, "\simeq"] & Y^\bullet \end{tikzcd}$$

with the left vertical morphism being the zero morphism (and the bottom an injective resolution). Hence we have to show that in such a diagram $f^\bullet$ is null-homotopic.

This we may reduce to the statement of prop. 3.29 by considering instead of $f^\bullet$ the induced chain map of augmented complexes

$$\begin{array}{ccccccccc} 0 & \to & X & \overset{h^0}{\to} & X^0 & \overset{d_X^0}{\to} & X^1 & \to & \cdots \\ \downarrow^{f^{-2}=0} && \downarrow^{f^{-1}=0} && \downarrow^{f^0} && \downarrow^{f^1} && \\ 0 & \to & Y & \to & Y^0 & \overset{d_J^0}{\to} & Y^1 & \to & \cdots \end{array} \,,$$

where the second square from the left commutes due to the commutativity of the original square of chain complexes in degree 0.

Since $h^\bullet$ is a quasi-isomorphism, the top chain complex is exact, by remark 3.24. Moreover the bottom complex consists of injective objects from the second degree on (the former degree 0). Hence the induction in the proof of prop. 3.29 implies the existence of a null homotopy

$$\begin{array}{ccccccccc} 0 & \to & X & \to & X^0 & \overset{d_X^0}{\to} & X^1 & \to & \cdots \\ \downarrow^{f^{-2}=0} & \swarrow_{\eta^{-1}=0} & \downarrow^{f^{-1}=0} & \swarrow_{\eta^0=0} & \downarrow^{f^0} & \swarrow_{\eta^1} & \downarrow^{f^1} && \\ 0 & \to & Y & \to & Y^0 & \overset{d_Y^0}{\to} & Y^1 & \to & \cdots \end{array}$$

starting with $\eta^{-1} = 0$ and $\eta^0 = 0$ (notice that the proof prop. 3.29 was formulated exactly this way), which works because $f^{-1} = 0$. The de-augmentation $\{f^{\bullet \geq 0}\}$ of this is the desired null homotopy of $f^\bullet$. ▮

We now discuss how the injective/projective resolutions constructed above are functorial if regarded in the homotopy category of chain complexes, def. 2.89. For definiteness, to be able to distinguish chain complexes from cochain complexes, introduce the following notation.

Definition 3.33. **(the derived category)**

Write as before

$$\mathcal{K}_\bullet(\mathcal{A}) \in \mathrm{Cat}$$

for the strong chain homotopy category of chain complexes, from def. 2.89.

Write similarly now

$$\mathcal{K}^\bullet(\mathcal{A}) \in \mathrm{Cat}$$

for the strong chain homotopy category of co-chain complexes.

Write furthermore

$$\mathcal{D}_\bullet(\mathcal{A}) \coloneqq \mathcal{K}_\bullet(\mathcal{P}_{\mathcal{A}}) \hookrightarrow \mathcal{K}_\bullet(\mathcal{A})$$

for the full subcategory on the degreewise projective chain complexes, and

$$\mathcal{D}^\bullet(\mathcal{A}) \coloneqq \mathcal{K}^\bullet(\mathcal{I}_{\mathcal{A}}) \hookrightarrow \mathcal{K}^\bullet(\mathcal{A})$$

for the full subcategory on the degreewise injective cochain complexes.

These subcategories – or any category equivalent to them – are called the (strictly bounded above/below) **derived category** of $\mathcal{A}$.

Remark 3.34. Often one defines the derived category by more general abstract means than we have introduced here, namely as the *localization* of the category of chain complexes at the quasi-isomorphims. If one does this, then the simple definition def. 3.33 is instead a *theorem*. The interested reader can find more details and further pointers *here*.

Theorem 3.35. *If $\mathcal{A}$ has enough injectives, def. 3.15, then there exists a functor*

$$P : \mathcal{A} \to \mathcal{D}^\bullet(\mathcal{A}) = \mathcal{K}^\bullet(\mathcal{I}_{\mathcal{A}})$$

together with natural isomorphisms

$$H^0(-) \circ P \simeq \mathrm{id}_{\mathcal{A}}$$

and

$$H^{n \geq 1}(-) \circ P \simeq 0 \,.$$

Proof. By prop. 3.27 every object $X^\bullet \in \mathrm{Ch}^\bullet(\mathcal{A})$ has an injective resolution. Proposition 3.31 says that for $X \to X^\bullet$ and $X \to \tilde X^\bullet$ two resolutions there is a morphism $X^\bullet \to \tilde X^\bullet$ in $\mathcal{K}^\bullet(\mathcal{A})$ and prop. 3.32 says that this morphism is unique in $\mathcal{K}^\bullet(\mathcal{A})$. In particular it is therefore an isomorphism in $\mathcal{K}^\bullet(\mathcal{A})$ (since the composite with the reverse lifted morphism, also being unique, has to be the identity).

So choose one such injective resolution $P(X)^\bullet$ for each $X^\bullet$.

Then for $f: X \to Y$ any morphism in $\mathcal{A}$, proposition 3.27 again says that it can be lifted to a morphism between $P(X)^\bullet$ and $P(Y)^\bullet$ and proposition 3.31 says that there is an image in $\mathcal{K}^\bullet(\mathcal{A})$, unique for morphism making the given diagram commute.

This implies that this assignment of morphisms is functorial, since then also the composites are unique. ∎

Dually we have:

Theorem 3.36. *If $\mathcal{A}$ has enough projectives, def. 3.15, then there exists a functor*

$$Q : \mathcal{A} \to \mathcal{D}_\bullet(\mathcal{A}) = \mathcal{K}_\bullet(\mathcal{P}_{\mathcal{A}})$$

together with natural isomorphisms

$$H_0(-) \circ P \simeq \mathrm{id}_{\mathcal{A}}$$

and

$$H_{n \geq 1}(-) \circ P \simeq 0 \,.$$

For actually working with the derived category, the following statement is of central importance, which we record here without proof (which requires a bit of localization theory). It says that for computing hom-sets in the derived category, it is in fact sufficient to just resolve the domain or the codomain.

Proposition 3.37. *Let $X_\bullet, Y_\bullet \in \mathrm{Ch}_\bullet(\mathcal{A})$. We have natural isomorphisms*

$$\mathcal{D}_\bullet(Q(X)_\bullet, Q(Y)_\bullet) \simeq \mathcal{K}_\bullet(Q(X)_\bullet, Y_\bullet) \,.$$

Dually, for $X^\bullet, Y^\bullet \in \mathrm{Ch}^\bullet(\mathcal{A})$, we have a natural isomorphism

$$\mathcal{D}^\bullet(P(X)_\bullet, P(Y)^\bullet) \simeq \mathcal{K}^\bullet(X^\bullet, P(Y)^\bullet) \,.$$

In conclusion we have found that there are *resolution functors* that embed $\mathcal{A}$ in the homotopically correct context of resolved chain complexes with chain maps up to chain homotopy between them.

In the next section we discuss the general properties of this "homotopically correct context": the derived category.

9) Derived functors

In the previous section we have seen how the entire category $\mathcal{A}$ (= RMod) embeds into its derived category, the category of degreewise injective cochain complexes

$$P : \mathcal{A} \to \mathcal{D}^\bullet(\mathcal{A}) = \mathcal{K}^\bullet(\mathcal{I}_{\mathcal{A}})$$

or degreewise projective chain complexes

$$Q : \mathcal{A} \to \mathcal{D}_\bullet(\mathcal{A}) = \mathcal{K}_\bullet(\mathcal{P}_{\mathcal{A}})$$

modulo chain homotopy. This construction of the derived category naturally gives rise to the following notion of derived functors.

Definition 3.38. For $\mathcal{A}, \mathcal{B}$ two abelian categories (e.g. RMod and R'Mod), a functor

$$F : \mathcal{A} \to \mathcal{B}$$

is called an **additive functor** if

1. F maps the zero object to the zero object, $F(0) \simeq 0 \in \mathcal{B}$;
2. given any two objects $x, y \in \mathcal{A}$, there is an isomorphism $F(x \oplus y) \cong F(x) \oplus F(y)$, and this respects the inclusion and projection maps of the direct sum:

$$\begin{tikzcd} x \arrow[dr, "i_x"] & & y \arrow[dl, "i_y"'] & & F(x) \arrow[dr, "i_{F(x)}"] & & F(y) \arrow[dl, "i_{F(y)}"'] \\ & x \oplus y \arrow[dl, "p_x"'] \arrow[dr, "p_y"] & \overset{F}{\mapsto} & & & F(x \oplus y) \cong F(x) \oplus F(y) \arrow[dl, "p_{F(X)}"'] \arrow[dr, "p_{F(y)}"] & \\ x & & y & & F(x) & & F(y) \end{tikzcd}$$

Definition 3.39. Given an additive functor $F : \mathcal{A} \to \mathcal{A}'$, it canonically induces a functor

$$\mathrm{Ch}_\bullet(F) : \mathrm{Ch}_\bullet(\mathcal{A}) \to \mathrm{Ch}_\bullet(\mathcal{A}')$$

between categories of chain complexes (its "prolongation") by applying it to each chain complex and to all the diagrams in the definition of a chain map. Similarly it preserves chain homotopies and hence it passes to

the quotient given by the strong homotopy category of chain complexes

$$\mathcal{K}(F) : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{A}') \,.$$

Remark 3.40. If $\mathcal{A}$ and $\mathcal{A}'$ have enough projectives, then their derived categories are

$$\mathcal{D}_\bullet(\mathcal{A}) \simeq \mathcal{K}_\bullet(\mathcal{P}_{\mathcal{A}})$$

and

$$\mathcal{D}^\bullet(\mathcal{A}) \simeq \mathcal{K}^\bullet(\mathcal{I}_{\mathcal{A}})$$

etc. One wants to accordingly *derive* from F a functor $\mathcal{D}_\bullet(\mathcal{A}) \to \mathcal{D}_\bullet(\mathcal{A})$ between these derived categories. It is immediate to achieve this on the domain category, there we can simply precompose and form

$$\mathcal{A} \to \mathcal{D}_\bullet(\mathcal{A}) \simeq \mathcal{K}_\bullet(\mathcal{P}_{\mathcal{A}}) \hookrightarrow \mathcal{K}_\bullet(\mathcal{A}) \xrightarrow{\mathcal{K}_\bullet(F)} \mathcal{K}_\bullet(\mathcal{A}') \,.$$

But the resulting composite lands in $\mathcal{K}_\bullet(\mathcal{A}')$ and in general does not factor through the inclusion $\mathcal{D}_\bullet(\mathcal{A}') = \mathcal{K}_\bullet(\mathcal{P}_{\mathcal{A}'}) \hookrightarrow \mathcal{K}_\bullet(\mathcal{A}')$.

In a more general abstract discussion than we present here, one finds that by applying a projective resolution functor *on chain complexes*, one can enforce this factorization. However, by definition of resolution, the resulting chain complex is quasi-isomorphic to the one obtained by the above composite.

This means that if one is only interested in the "weak chain homology type" of the chain complex in the image of a derived functor, then forming chain homology groups of the chain complexes in the images of the above composite gives the desired information. This is what def. 3.44 and def. 3.45 below do.

Definition 3.41. Let $\mathcal{A}, \mathcal{A}'$ be two abelian categories, for instance $\mathcal{A} = R$Mod and $\mathcal{A}' = R'$Mod. Then a functor $F : \mathcal{A} \to \mathcal{A}'$ which preserves direct sums (and hence in particular the zero object) is called

- a **left exact functor** if it preserves kernels;
- a **right exact functor** if it preserves cokernels;
- an **exact functor** if it is both left and right exact.

Here to "preserve kernels" means that for every morphism $X \xrightarrow{f} Y$ in $\mathcal{A}$ we have an isomorphism on the left of the following commuting diagram

$$\begin{array}{ccccc} F(\ker(f)) & \to & F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow^{\simeq} && \downarrow^{=} && \downarrow^{=} \\ \ker(F(f)) & \to & F(X) & \xrightarrow{F(f)} & F(Y) \end{array} ,$$

hence that both rows are exact. And dually for right exact functors.

We record the following immediate consequence of this definition (which in the literature is often taken to be the definition).

Proposition 3.42. *If F is a left exact functor, then for every exact sequence of the form*

$$0 \to A \to B \to C$$

also

$$0 \to F(A) \to F(B) \to F(C)$$

is an exact sequence. Dually, if F is a right exact functor, then for every exact sequence of the form

$$A \to B \to C \to 0$$

also

$$F(A) \to F(B) \to F(C) \to 0$$

is an exact sequence.

Proof. If $0 \to A \to B \to C$ is exact then $A \hookrightarrow B$ is a monomorphism by prop. 2.70. But then the statement that $A \to B \to C$ is exact at B says precisely that A is the kernel of $B \to C$. So if F is left exact then by definition also $F(A) \to F(B)$ is the kernel of $F(B) \to F(C)$ and so is in particular also a monomorphism. Dually for right exact functors. ■

Remark 3.43. Proposition 3.42 is clearly the motivation for the terminology in def. 3.41: a functor is left exact if it preserves short exact sequences to the left, and right exact if it preserves them to the right.

Now we can state the main two definitions of this section.

Definition 3.44. Let

$$F : \mathcal{A} \to \mathcal{A}'$$

be a left exact functor between abelian categories such that $\mathcal{A}$ has enough injectives. For $n \in \mathbb{N}$ the **nth right derived functor** of F is the composite

$$R^n F : \mathcal{A} \xrightarrow{P} \mathcal{K}^\bullet(\mathcal{I}_{\mathcal{A}}) \xrightarrow{\mathcal{K}^\bullet(F)} \mathcal{K}^\bullet(\mathcal{A}') \xrightarrow{H^n(-)} \mathcal{A}' ,$$

where

- P is the injective resolution functor of theorem 3.35;
- $\mathcal{K}(F)$ is the prolongation of F according to def. 3.39;
- $H^n(-)$ is the n-chain homology functor. Hence

$$(R^n F)(X^\bullet) \coloneqq H^n(F(P(X)^\bullet)) .$$

Dually:

Definition 3.45. Let

$$F : \mathcal{A} \to \mathcal{A}'$$

be a right exact functor between abelian categories such that $\mathcal{A}$ has enough projectives. For $n \in \mathbb{N}$ the **nth left derived functor** of F is the composite

$$L_n F : \mathcal{A} \xrightarrow{Q} K_\bullet(\mathcal{P}_{\mathcal{A}}) \xrightarrow{\mathcal{K}_\bullet(F)} \mathcal{K}_\bullet(\mathcal{A}') \xrightarrow{H_n(-)} \mathcal{A}' ,$$

where

- Q is the projective resolution functor of theorem 3.36;
- $\mathcal{K}(F)$ is the prolongation of F according to def. 3.39;
- $H_n(-)$ is the n-chain homology functor. Hence

$$(L_n F)(X_\bullet) \coloneqq H_n(F(Q(X)_\bullet)) .$$

The following proposition says that in degree 0 these derived functors coincide with the original functors.

Proposition 3.46. *Let $F : \mathcal{A} \to \mathcal{B}$ a left exact functor, def. 3.41 in the presence of enough injectives. Then for all $X \in \mathcal{A}$ there is a natural isomorphism*

$$R^0 F(X) \simeq F(X) .$$

Dually, if F is a right exact functor in the presence of enough projectives, then

$$L_0 F(X) \simeq F(X) .$$

Proof. We discuss the first statement, the second is formally dual.

By remark 3.24 an injective resolution $X \overset{\simeq_{\mathrm{qi}}}{\longrightarrow} X^\bullet$ is equivalently an exact sequence of the form

$$0 \to X \hookrightarrow X^0 \to X^1 \to \cdots .$$

If F is left exact then it preserves this excact sequence by definition of left exactness, and hence

$$0 \to F(X) \hookrightarrow F(X^0) \to F(X^1) \to \cdots$$

is an exact sequence. But this means that

$$R^0 F(X) \coloneqq \ker(F(X^0) \to F(X^1)) \simeq F(X) .$$

∎

The following immediate consequence of the definition is worth recording:

Proposition 3.47. *Let F be an additive functor.*

- *If F is right exact and $N \in \mathcal{A}$ is a projective object, then*

$$L_n F(N) = 0 \quad \forall n \geq 1 .$$

- *If F is left exact and $N \in \mathcal{A}$ is a injective object, then*

$$R^n F(N) = 0 \quad \forall n \geq 1 .$$

Proof. If N is projective then the chain complex $[\cdots \to 0 \to 0 \to N]$ is already a projective resolution and hence by definition $L_n F(N) \simeq H_n(0)$ for $n \geq 1$. Dually if N is an injective object. ▮

For proving the basic property of derived functors below in prop. 3.50 which continues these basis statements to higher degree, in a certain way, we need the following technical lemma.

Lemma 3.48. *For $0 \to A \stackrel{i}{\to} B \stackrel{p}{\to} C \to 0$ a short exact sequence in an abelian category with enough projectives, there exists a commuting diagram of chain complexes*

$$\begin{array}{ccccccccc} 0 & \to & A_\bullet & \to & B_\bullet & \to & C_\bullet & \to & 0 \\ & & \downarrow^{f_\bullet} & & \downarrow^{g_\bullet} & & \downarrow^{h_\bullet} \\ 0 & \to & A & \stackrel{i}{\to} & B & \stackrel{p}{\to} & C & \to & 0 \end{array}$$

where

- *each vertical morphism is a projective resolution;*

and in addition

- *the top row is again a short exact sequence of chain complexes.*

Proof. By prop. 3.27 we can choose $f_\bullet$ and $h_\bullet$. The task is now to construct the third resolution $g_\bullet$ such as to obtain a short exact sequence of chain complexes, hence degreewise a short exact sequence, in the two row.

To construct this, let for each $n \in \mathbb{N}$

$$B_n \coloneqq A_n \oplus C_n$$

be the direct sum and let the top horizontal morphisms be the canonical inclusion and projection maps of the direct sum.

Let then furthermore (in matrix calculus notation)

$$g_0 = \left((j_0)_A \quad (j_0)_B \right) : A_0 \oplus C_0 \to B$$

be given in the first component by the given composite

$$(g_0)_A : A_0 \oplus C_0 \to A_0 \stackrel{f_0}{\to} A \stackrel{i}{\hookrightarrow} B$$

and in the second component we take

$$(j_0)_C : A_0 \oplus C_0 \to C_0 \stackrel{\zeta}{\to} B$$

to be given by a lift in

$$\begin{array}{ccc} & & B \\ & {}^{\zeta}\nearrow & \downarrow^{p} \\ C_0 & \stackrel{h_0}{\to} & C \end{array},$$

which exists by the left lifting property of the projective object C_0 (since $C_\bullet$ is a projective resolution) against the epimorphism $p : B \to C$ of the short exact sequence.

In total this gives in degree 0

$$\begin{array}{ccccc} A_0 & \hookrightarrow & A_0 \oplus C_0 & \to & C_0 \\ {}^{f_0}\downarrow & & \downarrow^{((g_0)_A, (g_0)_C)} & \swarrow_{\zeta} & \downarrow^{h_0} \\ A & \stackrel{i}{\hookrightarrow} & B & \stackrel{p}{\to} & C \end{array}.$$

Let then the differentials of $B_\bullet$ be given by

$$d^{B_\bullet}_k = \begin{pmatrix} d^{A_\bullet}_k & (-1)^k e_k \\ 0 & d^{C_\bullet}_k \end{pmatrix} : A_{k+1} \oplus C_{k+1} \to A_k \oplus C_k \,,$$

where the $\{e_k\}$ are constructed by induction as follows. Let e_0 be a lift in

$$\begin{array}{ccccc} & & & & A_0 \\ & & & {}^{e_0}\nearrow & \downarrow^{f_0} \\ \zeta \circ d^{C_\bullet}_0 : & C_1 & \to & A & \hookrightarrow B \end{array}$$

which exists since C_1 is a projective object and $A_0 \to A$ is an epimorphism by $A_\bullet$ being a projective resolution. Here we are using that by exactness the bottom morphism indeed factors through A as indicated, because

the definition of ζ and the chain complex property of $C_\bullet$ gives

$$\begin{aligned} p \circ \zeta \circ d_0^{C_\bullet} &= h_0 \circ d_0^{C_\bullet} \\ &= 0 \circ h_1 \\ &= 0 \end{aligned} .$$

Now in the induction step, assuming that e_{n-1} has been been found satisfying the chain complex property, let e_n be a lift in

$$\begin{array}{ccccccc} & & & & A_n & & \\ & & & {}^{e_n}\nearrow & \downarrow^{d^{A_\bullet}_{n-1}} & & \\ e_{n-1} \circ d_n^{C_\bullet} \colon & C_{n+1} & \hookrightarrow & \ker(d^{A_\bullet}_{n-1}) = \operatorname{im}(d^{A_\bullet}_{n-1}) & \to & A_{n-1} \end{array} ,$$

which again exists since C_{n+1} is projective. That the bottom morphism factors as indicated is the chain complex property of e_{n-1} inside $d_{n-1}^{B_\bullet}$.

To see that the $d^{B_\bullet}$ defines this way indeed squares to 0 notice that

$$d_n^{B_\bullet} \circ d_{n+1}^{B_\bullet} = \begin{pmatrix} 0 & (-1)^n \left(e_n \circ d^{C_\bullet}_{n+1} - d_n^{A_\bullet} \circ e_{n+1} \right) \\ 0 & 0 \end{pmatrix} .$$

This vanishes by the very commutativity of the above diagram.

This establishes $g_\bullet$ such that the above diagram commutes and the bottom row is degreewise a short exact sequence, in fact a split exact sequence, by construction.

To see that $g_\bullet$ is indeed a quasi-isomorphism, consider the homology long exact sequence associated to the short exact sequence of cochain complexes $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$. In positive degrees it implies that the chain homology of $B_\bullet$ indeed vanishes. In degree 0 it gives the short sequence $0 \to A \to H_0(B_\bullet) \to B \to 0$ sitting in a commuting diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \hookrightarrow & H_0(B_\bullet) & \to & C & \to & 0 \\ \downarrow & & \downarrow^{=} & & \downarrow & & \downarrow^{=} & & \downarrow \\ 0 & \to & A & \hookrightarrow & B & \to & C & \to & 0 \end{array} ,$$

where both rows are exact. That the middle vertical morphism is an isomorphism then follows by the five lemma. ∎

The formally dual statement to lemma 3.48 is the following.

Lemma 3.49. *For $0 \to A \to B \to C \to 0$ a short exact sequence in an abelian category with enough injectives, there exists a commuting diagram of cochain complexes*

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A^\bullet & \to & B^\bullet & \to & C^\bullet & \to & 0 \end{array}$$

where

- *each vertical morphism is an injective resolution;*

and in addition

- *the bottom row is again a short exact sequence of cochain complexes.*

The central general fact about derived functors to be discussed here is now the following.

Proposition 3.50. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and assume that $\mathcal{A}$ has enough injectives.*

Let $F \colon \mathcal{A} \to \mathcal{B}$ be a left exact functor and let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence in $\mathcal{A}$.

Then there is a long exact sequence of images of these objects under the right derived functors $R^\bullet F(-)$ of def. 3.44

$$\begin{array}{ccccccccccccccccc} 0 & \to & R^0F(A) & \to & R^0F(B) & \to & R^0F(C) & \stackrel{\delta_0}{\to} & R^1F(A) & \to & R^1F(B) & \to & R^1F(C) & \stackrel{\delta_1}{\to} & R^2F(A) & \to & \cdots \\ & & \downarrow^{\simeq} & & \downarrow^{\simeq} & & \downarrow^{\simeq} & & & & & & & & & & \\ 0 & \to & F(A) & \to & F(B) & \to & F(C) & & & & & & & & & & \end{array}$$

in $\mathcal{B}$.

Proof. By lemma 3.49 we can find an injective resolution

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

of the given exact sequence which is itself again an exact sequence of cochain complexes.

Since A^n is an injective object for all n, its component sequences $0 \to A^n \to B^n \to C^n \to 0$ are indeed split exact sequences (see the discussion there). Splitness is preserved by any functor F (and also since F is additive it even preserves the direct sum structure that is chosen in the proof of lemma 3.48) and so it follows that

$$0 \to F(\tilde{A}^\bullet) \to F(\tilde{B}^\bullet) \to F(\tilde{C}^\bullet) \to 0$$

is a again short exact sequence of cochain complexes, now in $\mathcal{B}$. Hence we have the corresponding homology long exact sequence from prop. 2.78:

$$\cdots \to H^{n-1}(F(A^\bullet)) \to H^{n-1}(F(B^\bullet)) \to H^{n-1}(F(C^\bullet)) \overset{\delta}{\to} H^n(F(A^\bullet)) \to H^n(F(B^\bullet)) \to H^n(F(C^\bullet)) \overset{\delta}{\to} H^{n+1}(F(A^\bullet)) \to H^{n+1}(F(B^\bullet)) \to H^{n+1}(F(C^\bullet)$$

By construction of the resolutions and by def. 3.44, this is equal to

$$\cdots \to R^{n-1}F(A) \to R^{n-1}F(B) \to R^{n-1}F(C) \overset{\delta}{\to} R^nF(A) \to R^nF(B) \to R^nF(C) \overset{\delta}{\to} R^{n+1}F(A) \to R^{n+1}F(B) \to R^{n+1}F(C) \to \cdots .$$

Finally the equivalence of the first three terms with $F(A) \to F(B) \to F(C)$ is given by prop. 3.46. ∎

Remark 3.51. Prop. 3.50 implies that one way to interpret $R^1F(A)$ is as a "measure for how a left exact functor F fails to be an exact functor". For, with $A \to B \to C$ any short exact sequence, this proposition gives the exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A)$$

and hence $0 \to F(A) \to F(B) \to F(C) \to 0$ is a short exact sequence itself precisely if $R^1F(A) \simeq 0$.

Dually, if F is right exact functor, then $L_1F(C)$ "measures how F fails to be exact" for then

$$L_1F(C) \to F(A) \to F(B) \to F(C) \to 0$$

is an exact sequence and hence is a short exact sequence precisely if $L_1F(C) \simeq 0$.

Notice that in fact we even have the following statement (following directly from the definition).

Proposition 3.52. *Let F be an additive functor which is an exact functor. Then*

$$R^{\geq 1}F = 0$$

and

$$L_{\geq 1}F = 0 \,.$$

Proof. Because an exact functor preserves all exact sequences. If $Y_\bullet \to A$ is a projective resolution then also $F(Y)_\bullet$ is exact in all positive degrees, and hence $L_{n \geq 1}F(A))H_{n\geq}(F(Y)) = 0$. Dually for R^nF. ∎

Conversely:

Definition 3.53. Let $F : \mathcal{A} \to \mathcal{B}$ be a left or right exact additive functor. An object $A \in \mathcal{A}$ is called an F-**acyclic object** if all positive-degree right/left derived functors of F are zero on A.

Acyclic objects are useful for computing derived functors on non-acyclic objects. More generally, we now discuss how the derived functor of an additive functor F may also be computed not necessarily with genuine injective/projective resolutions, but with (just) "F-injective"/"F-projective resolutions".

While projective resolutions in $\mathcal{A}$ are *sufficient* for computing *every* left derived functor on $\mathrm{Ch}_\bullet(\mathcal{A})$ and injective resolutions are sufficient for computing *every* right derived functor on $\mathrm{Ch}^\bullet(\mathcal{A})$, if one is interested just in a single functor F then such resolutions may be more than *necessary*. A weaker kind of resolution which is still sufficient is then often more convenient for applications. These *F-projective resolutions* and *F-injective resolutions*, respectively, we discuss now. A special case of both are *F-acyclic resolutions*.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor.

Definition 3.54. Assume that F is left exact. An additive full subcategory $\mathcal{I} \subset \mathcal{A}$ is called F**-injective** (or: consisting of F-injective objects) if

1. for every object $A \in \mathcal{A}$ there is a monomorphism $A \to \tilde{A}$ into an object $\tilde{A} \in \mathcal{I} \subset \mathcal{A}$;

2. for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ with $A, B \in \mathcal{I} \subset \mathcal{A}$ also $C \in \mathcal{I} \subset \mathcal{A}$;

3. for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ with $A \in \mathcal{I} \subset \mathcal{A}$ also $0 \to F(A) \to F(B) \to F(C) \to 0$ is a short exact sequence in $\mathcal{B}$.

And dually:

Definition 3.55. Assume that F is right exact. An additive full subcategory $\mathcal{P} \subset \mathcal{A}$ is called **F-projective** (or: consisting of F-projective objects) if

1. for every object $A \in \mathcal{A}$ there is an epimorphism $\tilde{A} \to A$ from an object $\tilde{A} \in \mathcal{P} \subset \mathcal{A}$;

2. for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ with $B, C \in \mathcal{P} \subset \mathcal{A}$ also $A \in \mathcal{P} \subset \mathcal{A}$;

3. for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ with $C \in \mathcal{I} \subset \mathcal{A}$ also $0 \to F(A) \to F(B) \to F(C) \to 0$ is a short exact sequence in $\mathcal{B}$.

With the $\mathcal{I}, \mathcal{P} \subset \mathcal{A}$ as above, we say:

Definition 3.56. For $A \in \mathcal{A}$,

- an **F-injective resolution** of A is a cochain complex $I^\bullet \in \mathrm{Ch}^\bullet(\mathcal{I}) \subset \mathrm{Ch}^\bullet(\mathcal{A})$ and a quasi-isomorphism

$$A \overset{\simeq_{\mathrm{qi}}}{\longrightarrow} I^\bullet$$

- an **F-projective resolution** of A is a chain complex $Q_\bullet \in \mathrm{Ch}_\bullet(\mathcal{P}) \subset \mathrm{Ch}^\bullet(\mathcal{A})$ and a quasi-isomorphism

$$Q_\bullet \overset{\simeq_{\mathrm{qi}}}{\longrightarrow} A \,.$$

Let now $\mathcal{A}$ have enough projectives / enough injectives, respectively, def. 3.15.

Example 3.57. For $F : \mathcal{A} \to \mathcal{B}$ an additive functor, let $\mathrm{Ac} \subset \mathcal{A}$ be the full subcategory on the F-acyclic objects, def. 3.53. Then

- if F is left exact, then $\mathcal{I} \coloneqq \mathrm{Ac}$ is a subcategory of F-injective objects;

- if F is right exact, then $\mathcal{P} \coloneqq \mathrm{Ac}$ is a subcategory of F-projective objects.

Proof. Consider the case that F is right exact. The other case works dually. Then the first condition of def. 3.54 is satisfied because every injective object is an F-acyclic object and by assumption there are enough of these.

For the second and third condition of def. 3.54 use that there is the long exact sequence of derived functors prop. 3.50

$$0 \to A \to B \to C \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to R^2F(B) \to R^2F(C) \to \cdot\ .$$

For the second condition, by assumption on A and B and definition of F-acyclic object we have $R^nF(A) \simeq 0$ and $R^nF(B) \simeq 0$ for $n \geq 1$ and hence short exact sequences

$$0 \to 0 \to R^nF(C) \to 0$$

which imply that $R^nF(C) \simeq 0$ for all $n \geq 1$, hence that C is acyclic.

Similarly, the third condition is equivalent to $R^1F(A) \simeq 0$. ■

Example 3.58. The F-projective/injective resolutions by acyclic objects as in example 3.57 are called **F-acyclic resolutions**.

Let $\mathcal{A}$ be an abelian category with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive left exact functor with right derived functor $R_\bullet F$, def. 3.44. Finally let $\mathcal{I} \subset \mathcal{A}$ be a subcategory of F-injective objects, def. 3.54.

Lemma 3.59. *If a cochain complex $A^\bullet \in \mathrm{Ch}^\bullet(\mathcal{I}) \subset \mathrm{Ch}^\bullet(\mathcal{A})$ is quasi-isomorphic to 0,*

$$X^\bullet \overset{\simeq_{\mathrm{qi}}}{\longrightarrow} 0$$

then also $F(X^\bullet) \in \mathrm{Ch}^\bullet(\mathcal{B})$ is quasi-isomorphic to 0

$$F(X^\bullet) \overset{\simeq_{\mathrm{qi}}}{\longrightarrow} 0 \,.$$

Proof. Consider the following collection of short exact sequences obtained from the long exact sequence $X^\bullet$:

$$0 \to X^0 \overset{d^0}{\to} X^1 \overset{d^1}{\to} \mathrm{im}(d^1) \to 0$$
$$0 \to \mathrm{im}(d^1) \to X^2 \overset{d^2}{\to} \mathrm{im}(d^2) \to 0$$
$$0 \to \mathrm{im}(d^2) \to X^3 \overset{d^3}{\to} \mathrm{im}(d^3) \to 0$$

and so on. Going by induction through this list and using the second condition in def. 3.54 we have that all the $\mathrm{im}(d^n)$ are in $\mathcal{I}$. Then the third condition in def. 3.54 says that all the sequences

$$0 \to F(\mathrm{im}(d^n)) \to F(X^n + 1) \to F(\mathrm{im}(d^{n+1})) \to 0$$

are exact. But this means that

$$0 \to F(X^0) \to F(X^1) \to F(X^2) \to \cdots$$

is exact, hence that $F(X^\bullet)$ is quasi-isomorphic to 0. ▮

Theorem 3.60. *For $A \in \mathcal{A}$ an object with F-injective resolution $A \overset{\simeq_{qi}}{\longrightarrow} I_F^\bullet$, def. 3.56, we have for each $n \in \mathbb{N}$ an isomorphism*

$$R^n F(A) \simeq H^n(F(I_F^\bullet))$$

between the nth right derived functor, def. 3.44 of F evaluated on A and the cochain cohomology of F applied to the F-injective resolution $I_F^\bullet$.

Proof. By prop. 3.27 we can also find an injective resolution $A \overset{\simeq_{qi}}{\longrightarrow} I^\bullet$. By prop. 3.31 there is a lift of the identity on A to a chain map $I_F^\bullet \to I^\bullet$ such that the diagram

$$\begin{array}{ccc} A & \overset{\simeq_{qi}}{\longrightarrow} & I_F^\bullet \\ \downarrow^{\mathrm{id}} & & \downarrow^{f} \\ A & \overset{\simeq_{qi}}{\longrightarrow} & I^\bullet \end{array}$$

commutes in $\mathrm{Ch}^\bullet(\mathcal{A})$. Therefore by the 2-out-of-3 property of quasi-isomorphisms it follows that f is a quasi-isomorphism

Let $\mathrm{Cone}(f) \in \mathrm{Ch}^\bullet(\mathcal{A})$ be the mapping cone of f and let $I^\bullet \to \mathrm{Cone}(f)$ be the canonical chain map into it. By the explicit formulas for mapping cones, we have that

1. there is an isomorphism $F(\mathrm{Cone}(f)) \simeq \mathrm{Cone}(F(f))$;

2. $\mathrm{Cone}(f) \in \mathrm{Ch}^\bullet(\mathcal{I}) \subset \mathrm{Ch}^\bullet(\mathcal{A})$ (because F-injective objects are closed under direct sum).

The first implies that we have a homology exact sequence

$$\cdots \to H^n(I^\bullet) \to H^n(I_F^\bullet) \to H^n(\mathrm{Cone}(f)^\bullet) \to H^{n+1}(I^\bullet) \to H^{n+1}(I_F^\bullet) \to H^{n+1}(\mathrm{Cone}(f)^\bullet) \to \cdots .$$

Observe that with $f^\bullet$ a quasi-isomorphism $\mathrm{Cone}(f^\bullet)$ is quasi-isomorphic to 0. Therefore the second item above implies with lemma 3.59 that also $F(\mathrm{Cone}(f))$ is quasi-isomorphic to 0. This finally means that the above homology exact sequences consists of exact pieces of the form

$$0 \to (R^n F(A) \coloneqq H^n(I^\bullet) \overset{\simeq}{\to} H^n(I_F^\bullet) \to 0 .$$

▮

This concludes the discussion of the general definition and the general properties of derived functors that we will consider here. In the next section we discuss the two archetypical examples.

10) Fundamental examples of derived functors

We introduce here the two archetypical examples of derived functors and discuss their basic properties. In the next chapter *IV) The fundamental theorems* we discuss how to use these derived functors for obtaining deeper statements.

Above we have seen the definition and basic general properties of *derived functors* obtained from left/right exact functors between abelian categories.

Of all functors, a most fundamental one is the hom-functor of a given category. For categories such as RMod considered here, it comes with its *left adjoint*, the tensor product functor, which is hence equally fundamentally important. Here we discuss the derived functors of these two basic functors in detail.

For simplicity – this here being an introduction – we will discuss various statements only over $R = \mathbb{Z}$, hence for abelian groups. The main simplification that this leads to is the following.

Proposition 3.61. *Every subgroup of a free abelian group is itself a free group.*

This is a classical fact going back to Dedekind, now known (in its generalization to not-necessarily abelian groups) as the Nielsen-Schreier theorem. For us it is interesting due to the following consequence

Proposition 3.62. *Assuming the axiom of choice, every abelian group A admits a projective resolution, def. 3.25, concentrated in degree 0 and degree 1, hence a resolution which under remark 3.26 corresponds to a short exact sequence*

$$0 \to F_1 \to F_0 \to A \to 0$$

where F_0 and F_1 are projective, indeed free.

Proof. By the proof of prop. 3.16 there is an epimorphism $F_0 \to A$ out of a free abelian group (take for instance $F_0 = F(U(A))$, the free abelian group in the underlying set of A). By prop. 3.61 the kernel of this epimorphism is itself a free group, and hence by prop. 3.9 is itself projective. Take this kernel to be $F_1 \hookrightarrow F_0$. ▮

This fact drastically constrains the complexity of right derived functors on abelian groups:

Proposition 3.63. *Let $F : \mathrm{Ab} \to \mathrm{Ab}$ be an additive functor which is left exact functor. Then its right derived functors $R^n F$ vanish for all $n \geq 2$.*

Proof. By prop. 3.62 there is a projective resolution of any $A \in \mathrm{Ab}$ of the form $F_\bullet = [\cdots \to 0 \to 0 \to F_1 \to F_0]$. This implies the claim by def. 3.44. ▮

Remark 3.64. The conclusion of prop. 3.62 holds more generally over every ring which is a principal ideal domain. This includes in particular $R = k$ a field, in which case $R\,\mathrm{Mod} \simeq k\mathrm{Vect}$. On the other hand, every k-vector space is already projective itself, so that in this case the whole theory of right derived functors trivializes.

a) The derived Hom functor and group extensions

For $\mathcal{A}$ an abelian category, such as RMod, the hom-sets naturally have the structure of an abelian group themselves. This means that the hom-functor of $\mathcal{A}$ is

$$\mathrm{Hom}_{\mathcal{A}}(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}\,,$$

where $\mathcal{A}^{\mathrm{op}}$ is the opposite category of $\mathcal{A}$. This functor sends a morphism

$$\begin{array}{c} (X_1, A_1) \\ (\uparrow, \downarrow) \\ (X_2, A_2) \end{array} \in \mathcal{A}^{\mathrm{op}} \times \mathcal{A}$$

to the linear map which sends a homomorphism $(X_1 \stackrel{f}{\to} A_1) \in \mathrm{Hom}(X_1, A_1)$ to the composite homomorphism

$$\begin{array}{ccc} X_1 & \stackrel{f}{\to} & A_2 \\ \uparrow && \downarrow \\ X_2 && A_2 \end{array} \in \mathrm{Hom}(X_2, A_2)\,.$$

In particular if we hold the first argument fixed on an object $X \in \mathcal{A}$, then this yields a functor

$$\mathrm{Hom}(X, -) : \mathcal{A} \to \mathrm{Ab}$$

and if we keep the second argument fixed on an object $A \in \mathcal{A}$, then this yields a functor

$$\mathrm{Hom}(-, A) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}\,.$$

This functor we have already seen above in example 3.11.

A very basic fact is the following.

Proposition 3.65. *The functor $\mathrm{Hom}(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}$ is a left exact functor, def. 3.41. In particular for every $X \in \mathcal{A}$ the functor $\mathrm{Hom}(X, -) : \mathcal{A} \to \mathrm{Ab}$ is left exact, and for every $A \in \mathcal{A}$ the functor $\mathrm{Hom}(-, A) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ is left exact.*

Remark 3.66. A kernel in the opposite category $\mathcal{A}^{\mathrm{op}}$ is equivalently a cokernel in $\mathcal{A}$. Hence if we regard $\mathrm{Hom}(-, A)$ instead as a contravariant functor from $\mathcal{A}$ to Ab, then the statement that it is left exact means that (on top of preserving direct sums) it sends cokernels in $\mathcal{A}$ to kernels in Ab.

We therefore have the corresponding right derived functor:

Definition 3.67. For given $A \in \mathcal{A}$, write

$$\mathrm{Ext}^\bullet(-, A) \coloneqq R^\bullet \mathrm{Hom}(-, A) : \mathcal{A} \to \mathrm{Ab}$$

for the right derived functor, def. 3.44, of the hom-functor in the first argument, according to prop. 3.65.

This is called the **Ext**-functor.

The basic property of the derived Hom-functor/Ext-functor is that it classifies *group extensions* by (suspensions of) A. This we now discuss in detail, starting from a basic discussion of group extensions themselves.

The following definition essentially just repeats that of a short exact sequence above in def. 2.68, but now we consider it for G a possibly nonabelian group and think of it slightly differently regarding these sequences

up to homomorphisms as in def. 3.70 below. Equivalently we may think of the following as a discussion of the *classification* of short exact sequences when the leftmost and rightmost component are held fixed.

Definition 3.68. Two consecutive homomorphisms of groups

$$A \overset{i}{\hookrightarrow} \hat{G} \overset{p}{\to} G \tag{2}$$

are a **short exact sequence** if

1. i is monomorphism,

2. p an epimorphism

3. the image of i is all of the kernel of p: $\ker(p) \simeq \mathrm{im}(i)$.

We say that such a short exact sequence exhibits $\hat{G}$ as a **group extension** of G by A.

If $A \hookrightarrow \hat{G}$ factors through the center of $\hat{G}$ we say that this is a **central extension**.

Remark 3.69. Sometimes in the literature one sees $\hat{G}$ called an extension "of A by G". This is however in conflict with terms such as *central extension*, *extension of principal bundles*, etc, where the extension is always regarded *of the base, by the fiber*.

Definition 3.70. A homomorphism of extensions $f : \hat{G}_1 \to \hat{G}_2$ of a given G by a given A is a group homomorphism of this form which fits into a commuting diagram

$$\begin{tikzcd} & \hat{G}_1 \arrow[dr] \arrow[dd, "f"] & \\ A \arrow[ur] \arrow[dr] & & G . \\ & \hat{G}_2 \arrow[ur] & \end{tikzcd}$$

Proposition 3.71. *A morphism of extensions as in def. 3.70 is necessarily an isomorphism.*

$$\begin{array}{ccccccccc} 1 \to & A & \overset{i}{\to} & \hat{G}_1 & \overset{p}{\to} & G & \to 1 \\ & \downarrow = & & \downarrow \epsilon & & \downarrow = & \\ 1 \to & A & \overset{i'}{\to} & \hat{G}_2 & \overset{p'}{\to} & G & \to 1 \end{array} . \tag{3}$$

Proof. By the short five lemma. ∎

Definition 3.72. For G and A groups, write $\mathrm{Ext}(G, A)$ for the set of equivalence classes of extensions of G by A, as above and $\mathrm{CentrExt}(G, A) \hookrightarrow \mathrm{Ext}(G, A)$ for for the central extensions. If G and A are both abelian, write

$$\mathrm{AbExt}(G, A) \hookrightarrow \mathrm{CentrExt}(G, A)$$

for the subset of *abelian groups* $\hat{G}$ that are (necessarily central) extensions of G by A.

We discuss now the following two ways that the Ext^1 knows about such group extensions.

1. Central extensions of a possibly non-abelian group G are classified by the degree-2 group cohomology $H^2_{\mathrm{Grp}}(G, A)$ of G with coefficients in A, and this in turn is equivalently computed by $\mathrm{Ext}^1_{\mathbb{Z}[G]\mathrm{Mod}}(\mathbb{Z}, A)$, where $\mathbb{Z}[G]$ is the group ring of G.

   This is theorem 3.89 below.

2. Abelian extensions of an abelian gorup G are classified by $\mathrm{Ext}^1_{\mathrm{Ab}}(G, A)$. In fact, generally, in an abelian category $\mathcal{A}$ extensions of $G \in \mathcal{A}$ by $A \in \mathcal{A}$ (in the sense of short exact sequences $A \to \hat{G} \to G$) are classified by $\mathrm{Ext}^1_{\mathcal{A}}(G, A)$.

This is prop 3.96 below.

We first discuss now *group cohomology*:

Definition 3.73. Let G be group and A an abelian group (regarded as being equipped with the trivial G-action).

Then a **group 2-cocycle** on G with coefficients in A is a function

$$c : G \times G \to A$$

such that for all $(g_1, g_2) \in G \times G$ it satisfies the equation

$$c(g_1, g_2) - c(g_1, g_2 \cdot g_3) + c(g_1 \cdot g_2, g_3) - c(g_2, g_3) = 0 \quad \in A \tag{4}$$

(called the **2-cocycle condition**).

For $c, \tilde c$ two such cocycles, a **coboundary** $h : c \to \tilde c$ between them is a function

$$h : G \to A$$

such that for all $(g_1, g_2) \in G \times G$ the equation

$$\tilde c(g_1, g_2) = (c + dh)(g_1, g_2)\,, \qquad (5)$$

holds in A, where

$$(dh)(g_1, g_2) \coloneqq h(g_1 g_2) - h(g_1) - h(g_2)$$

is a **2-coboundary**.

The degree-2 **group cohomology** is the set

$$H^2_{\mathrm{Grp}}(G, A) = 2\mathrm{Cocycles}(G, A)/\mathrm{Coboundaries}(G, A)$$

of equivalence classes of group 2-cocycles modulo group coboundaries. This is itself naturally an abelian group under pointwise addition of cocycles in A

$$[c_1] + [c_2] = [c_1 + c_2]$$

where

$$c_1 + c_2 : (g_1, g_2) \mapsto c_1(g_1, g_2) + c_2(g_1, g_2)\,.$$

The following says that in the computation of $H^2_{\mathrm{Grp}}(G, A)$ one may concentrate on nice representatives that are called *normalized* cocycles:

Definition 3.74. A group 2-cocycle $c : G \times G \to A$, def. 3.73 is called **normalized** if

$$\forall_{g_0, g_1 \in G}\ \left(g_0 = e \text{ or } g_1 = e\right) \Rightarrow \left(c(g_0, g_1) = 0\right)\,.$$

Lemma 3.75. *For $c : G \times G \to A$ a group 2-cocycle, we have for all $g \in G$ that*

$$c(e, g) = c(e, e) = c(g, e)\,.$$

Proof. The cocycle condition (4) evaluated on

$$(g^{-1}, g, e) \in G^3$$

says that

$$c(g^{-1}, g) + c(e, e) = c(g, e) + c(g^{-1}, g)$$

hence that

$$c(e, e) = c(g, e)\,.$$

Similarly the 2-cocycle condition applied to

$$(e, g, g^{-1}) \in G^3$$

says that

$$c(e, g) + c(g, g^{-1}) = c(g, g^{-1}) + c(e, e)$$

hence that

$$c(e, g) = c(e, e)\,.$$

∎

Proposition 3.76. *Every group 2-cocycle $c : G \times G \to A$ is cohomologous to a normalized one, def. 3.74.*

Proof. By lemma 3.75 it is sufficient to show that c is cohomologous to a cocycle $\tilde c$ satisfying $\tilde c(e, e) = e$. Now given c, let $h : G \to A$ be given by

$$h(g) \coloneqq c(g, g)\,.$$

Then $\tilde c \coloneqq c + dc$ has the desired property, with (5):

$$\begin{aligned} \tilde c(e, e) &\coloneqq (c + dh)(e, e) \\ &= c(e, e) + c(e \cdot e, e \cdot e) - c(e, e) - c(e, e)\,. \\ &= 0 \end{aligned}$$

The fundamental classification theorem is now the following. This does not yet involve the Ext-functor explicitly.

Theorem 3.77. *There is a natural equivalence*

$$\mathrm{CentrExt}(G,A) \simeq H^2_{\mathrm{Grp}}(G,A)\;.$$

We prove this below as prop. 3.82. Here we first introduce stepwise the ingredients that go into the proof.

Definition 3.78. **(central extension associated to group 2-cocycle)**

Let $[c] \in H^2_{\mathrm{Grp}}(G,A)$ be a group 2-cocycle. Choose $c : G \times G \to A$ to be a representative of the cohomology class by a normalized cocycle, def. 3.74, which can always be done by prop. 3.76.

Define a group

$$G \times_c A \in \mathrm{Grpd}$$

as follows.

Let the underlying set of $G \times_c A$ be the cartesian product $U(G) \times U(A)$ of the underlying sets of G and A. The group operation on this is given by

$$(g_1, a_1) \cdot (g_2, a_2) \coloneqq (g_1 \cdot g_2,\; a_1 + a_2 + c(g_1, g_2))\;.$$

Proposition 3.79. *This defines indeed a group: the cocycle condition on c gives precisely the associativity of the product on $G \times_c A$. Moreover, the construction extends to a homomorphism*

$$\mathrm{Rec} : H^2_{\mathrm{Grp}}(G,A) \to \mathrm{Ext}(G,A)\;.$$

Proof. Forming the product of three elements of $G \times_c A$ bracketed to the left is, according to def. 3.78,

$$\big((g_1,a_1)\cdot(g_2,a_2)\big)\cdot(g_3,a_3) = \big(g_1 g_2 g_3\;,\; a_1 + a_2 + a_3 + c(g_1,g_2) + c(g_1 g_2, g_3)\big)\;.$$

Bracketing the same three elements to the right yields

$$(g_1,a_1)\cdot\big((g_2,a_2)\cdot(g_3,a_3)\big) = \big(g_1 g_2 g_3\;,\; a_1 + a_2 + a_3 + c(g_2,g_3) + c(g_1, g_2 g_3)\big)\;.$$

The difference between the two expressions is read off to be precisely

$$(1, (dc)(g_1, g_2, g_3))\,,$$

where dc denotes the group cohomology differential of c. Hence this vanishes precisely if c is a group 2-cocycle, hence we have an associative product.

To see that it has inverses, notice that for all (g,a) we have

$$(g,a)\cdot(g^{-1}, -a - c(g,g^{-1})) = (e, a - a - c(g,g^{-1}) + c(g,g^{-1}))$$

and hence inverses in $G \times_c A$ are given by

$$(g,a)^{-1} = (g^{-1}, -a - c(g,g^{-1}))\;.$$

Therefore $G \times_c A$ is indeed a group.

Using that c is a normalized cocycle by assumption, we find that the inclusion

$$i : A \to G \times_c A$$

given by $a \mapsto (e,a)$ is a group homomorphism. Moreover, the projection on the underlying sets evidently yields a group homomorphism $p : G \times_c A \to G$ given by $(g,a) \mapsto g$. The kernel of this is A, and hence

$$A \overset{i}{\hookrightarrow} G \times_c A \overset{p}{\to} G$$

is indeed a group extension. It is a central extension again using the assumption that c is normalized $c(g,e) = c(e,g) = 0$:

$$(g,a)\cdot(e,\tilde a) = (g, a + \tilde a + 0) = (e,\tilde a)\cdot(g,a)\;.$$

To see that the construction is independent of the choice of coycle c representing $[c]$, let $\tilde c$ be another representative which differs by a coboundary $h : G \to A$ with

$$\tilde c(g_1, g_2) \coloneqq c(g_1,g_2) - h(g_1) - h(g_2) + h(g_1 g_2)\;.$$

We claim that then we have a homomorphism of central extensions (hence an isomorphism) of the form

$$
\begin{array}{ccccc}
A & \to & G \times_c A & \to & G \\
\downarrow^{=} & & \downarrow^{(\mathrm{id}_G, p_2 - h \circ p_1)} & & \downarrow \\
A & \to & G \times_{\tilde c} A & \to & G
\end{array}
.
$$

To see this we check for all elements that

$$
\begin{aligned}
(g_1, a_1 - h(g_1)) \cdot (g_2, a_2 - h(g_2)) &= (g_1 g_2, a_1 + a_2 - h(g_1) - h(g_2) + c(g_1, g_2)) \\
&= (g_1 g_2, a_1 + a_2 + \tilde c(g_1, g_2) - h(g_1 g_2))
\end{aligned}
.
$$

Hence the construction of $G \times_c A$ indeed defines a function $H^2_{\mathrm{Grp}}(G, A) \to \mathrm{CentrExt}(G, A)$. ■

Assume the axiom of choice in the ambient foundations.

Definition 3.80. **(2-cocycle extracted from central extension)**

Given a central extension $A \to \hat G \to B$ define a group 2-cocycle $c : G \times G \to A$ as follows.

Choose a section $\sigma : U(G) \to U(\hat G)$ of the underlying sets (which exists by the axiom of choice and the fact that $p : \hat G \to G$ is by definition an epimorphism). Then define c by

$$
c : (g_1, g_2) \mapsto -\sigma(g_1)^{-1} \sigma(g_2)^{-1} \sigma(g_1 g_2) \in A \,,
$$

where on the right we are using that by the section-property of σ and the group homomorphism property of p

$$
p(\sigma(g_1)^{-1} \sigma(g_2)^{-1} \sigma(g_1 g_2)) = 1
$$

and hence by the exactness of the extension the argument is in $A \hookrightarrow \hat G$.

Proposition 3.81. *The construction of prop. 3.80 indeed yields a 2-cocycle in group cohomology. It extends to a morphism*

$$
\mathrm{Extr} : \mathrm{Ext}(G, A) \to H^2_{\mathrm{Grp}}(G, A) \,.
$$

Proof. The cocycle condition to be checked is that

$$
c(g_1, g_2) - c(g_0 g_1, g_2) + c(g_0, g_1 g_2) - c(g_0, g_1) = 1
$$

for all $g_0, g_1, g_2 \in G$. Writing this out with def. 3.80 yields

$$
\sigma(g_1)^{-1} \sigma(g_2)^{-1} \sigma(g_1 g_2) \big(\sigma(g_0 g_1)^{-1} \sigma(g_2)^{-1} \sigma(g_0 g_1 g_2)\big)^{-1} \sigma(g_0)^{-1} \sigma(g_1 g_2)^{-1} \sigma(g_0 g_1 g_2) \big(\sigma(g_0)^{-1} \sigma(g_1)^{-1} \sigma(g_0 g_1)\big)^{-1} \,.
$$

Here it is sufficient to observe that for every term also the inverse term appears.

To see that this is a well-defined map to $H^2_{\mathrm{grp}}(G, A)$ we need to check that for $\tilde\sigma : G \to \hat G$ a different choice of section, the corresponding cocycles differ by a group coboundary $\tilde c - c = dh$. Clearly this is obtained by setting

$$
h : g \mapsto \tilde\sigma(g) \sigma(g)^{-1} \,,
$$

where we use that the right hand side is in $A \hookrightarrow \hat G$ since because both σ and $\tilde\sigma$ are sections of p, the image of the right hand under p is the neutral element in G. ■

Proposition 3.82. *The two morphisms of def. 3.78 and def. 3.80 exhibit an bijection*

$$
H^2_{\mathrm{Grp}}(G, A) \underset{\mathrm{Rec}}{\overset{\mathrm{Extr}}{\underset{\to}{\overset{\longleftarrow}{\simeq}}}} \mathrm{CentrExt}(G, A) \,.
$$

Proof. Let $[c] \in H^2_{\mathrm{Grp}}(G, A)$. Then by construction of $\hat G \coloneqq G \times_c A$ there is a canonical section of the underlying function of sets $U(G \times_c A) \to U(G)$ given by $(\mathrm{id}_{U(G)}, 0) U(G) \to U(G) \times U(A)$. The cocycle induced by this section sends

$$
\begin{aligned}
(g_1, g_2) \mapsto & (g_1, 0)(g_2, 0)(g_1 g_2, 0)^{-1} \\
& = (g_1, 0)(g_1, 0)((g_1 g_2)^{-1}, -c(g_1 g_2, (g_1 g_2)^{-1})) \\
& = (g_1 g_2, c(g_1, g_2))((g_1 g_2)^{-1}, -c(g_1 g_2, (g_1 g_2)^{-1})) \\
& = (e, c(g_1, g_2) - c(g_1 g_2, (g_1 g_2)^{-1}) + c(g_1 g_2, (g_1 g_2)^{-1})) \\
& = (e, c(g_1, g_2))
\end{aligned}
,
$$

which is $c(g_1, g_2) \in A \hookrightarrow G \times_c A$, and hence this recovers the 2-cocycle that we started with.

This shows that $\mathrm{Extr} \circ \mathrm{Rec} = \mathrm{id}$ and in particular that Rec is a surjection. It is readily seen that the kernel of

Rec is trivial, and so it is an equivalence. ▮

Remark 3.83. The central extension of an abelian group G by an abelian group A need not itself be abelian.

But from the above classification we can read off the condition for the extension to be central.

Proposition 3.84. *The central extension of an abelian group G is itself abelian if the coresponding cocycle $c : G \times G \to A$ is symmetric, in that*

$$c(g_1, g_2) = c(g_2, g_1)$$

for all $g_1, g_2 \in G$.

With the general classification of group extensions in hand, we now turn back to the Ext-functor. First we discuss a choice of projective resolution that yields group cocycles.

Definition 3.85. For G a group, the **group ring** $\mathbb{Z}[G]$ is the ring

1. whose underlying abelian group is the free abelian group on the underlying set of G;
2. whose multiplication is given on basis elements by the group operation.

Definition 3.86. Write

$$\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$$

for the homomorphism of abelian groups which forms the sum of R-coefficients of the formal linear combinations that constitute the group ring

$$\epsilon : r \mapsto \sum_{g \in G} r_g \; .$$

This is called the augmentation map.

Definition 3.87. For $n \in \mathbb{N}$ let

$$Q_n^u \coloneqq F(U(G)^{\times^n})$$

be the free module over the group ring $\mathbb{Z}[G]$ on n-tuples of elements of G (hence $Q_0^u \simeq \mathbb{Z}[G]$ is the free module on a single generator).

For $n \geq 1$ let $\partial_{n-1} : Q_n^u \to Q_{n-1}^u$ be given on basis elements by

$$\partial_{n-1}(g_1, \cdots, g_n) \coloneqq g_1[g_2, \cdots, g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1, \cdots, g_i g_{i+1}, g_{i+2}, \cdots, g_n] + (-1)^n [g_1, \cdots, g_{n-1}] ,$$

where in the first summand we have the coefficient $g_1 \in G \hookrightarrow \mathbb{Z}[G]$ times the basis element $[g_2, \cdots, g_n]$ in $F(U(G)^{n-1})$.

In particular

$$\partial_0 : [g] \mapsto g[*] - [*] = g - e \in \mathbb{Z}[G] \; . \; .$$

Write furthermore Q_n for the quotient module $Q_n^u \to Q^n$ which is the cokernel of the inclusion of those elements for which one of the g_i is the unit element.

Proposition 3.88. *The construction in def. 3.87 defines chain complexes $Q_\bullet^u$ and $Q_\bullet$ of $\mathbb{Z}[G]$-modules. Moreover, with the augmentation map of def. 3.86 these are projective resolutions*

$$\epsilon : Q_\bullet^u \xrightarrow{\simeq_{\mathrm{qi}}} \mathbb{Z}$$
$$\epsilon : Q_\bullet \xrightarrow{\simeq_{\mathrm{qi}}} \mathbb{Z}$$

of $\mathbb{Z}$ equipped with the trivial $\mathbb{Z}[G]$-module structure in $\mathbb{Z}[G]$Mod.

Proof. The proof that we have indeed a chain complex is much like the proof of the existence of the alternating face map complex of a simplicial group, because writing

$$\partial_n^0[g_1, \cdots, g_n] \coloneqq g_1[g_2, \cdots, g_n]$$
$$\partial_n^i[g_1, \cdots, g_n] \coloneqq [g_1, \cdots, g_i g_{i+1}, g_{i+2}, \cdots, g_n] \;\; \text{for} 1 \leq i \leq n-1$$
$$\partial_n[g_1, \cdots, g_n] \coloneqq [g_1, \cdots, g_{n-1}]$$

one finds that these satisfy the simplicial identities and that $\partial_n = \sum_{i=0}^n (-1)^i \partial_n^i$.

That the augmentation map is a quasi-isomorphism is equivalent, by remark 3.26, to the augmentation

$$\cdots \xrightarrow{\partial_2} \mathbb{Z}[G]^2 \xrightarrow{\partial_1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

being an exact sequence. In fact we show that it is a split exact sequence by constructing for the canonical chain map to the 0-complex a null homotopy $s_\bullet$. To that end, let

$$s_{-1} : \mathbb{Z} \to Q_0^u$$

be given by sending $1 \in \mathbb{Z}$ to the single basis element in $Q_0^u \coloneqq \mathbb{Z}[G][*] \simeq \mathbb{Z}[G]$, and let for $n \in \mathbb{N}$

$$s_n : Q_n^u \to Q_{n+1}^u$$

be given on basis elements by

$$s_n(g[g_1, \cdots, g_n]) \coloneqq [g, g_1, \cdots, g_n] \;.$$

In the lowest degrees we have

$$\epsilon \circ s_{-1} = \mathrm{id}_{\mathbb{Z}}$$

because

$$\epsilon(s_{-1}(1)) = \epsilon([*]) = \epsilon(e) = 1$$

and

$$\partial_0 \circ s_0 + s_{-1} \circ \epsilon = \mathrm{id}_{Q_0^u}$$

because for all $g \in G$ we have

$$\begin{aligned} \partial_0(s_0(g[*])) + s_{-1}(\epsilon(g[*])) &= \partial_0([g]) + s_{-1}(1) \\ &= g[*] - [*] + [*] \;. \\ &= g[*] \end{aligned}$$

For all remaining $n \geq 1$ we find

$$\partial_n \circ s_n + s_{n-1} \circ \partial_{n-1} = \mathrm{id}_{Q_n^u}$$

by a lengthy but straightforward computation. This shows that every cycle is a boundary, hence that we have a resolution.

Finally, since the chain complex $Q_\bullet^u$ consists by construction degreewise of free modules hence of a projective modules, it is a projective resolution. ∎

Theorem 3.89. *For A an abelian group equipped with a linear G-action and for $n \in \mathbb{N}$, the degree-n group cohomology $H^n_{\mathrm{grp}}(G, A)$ of G with coefficients in A is equivalently given by*

$$\begin{aligned} H^n_{\mathrm{Grp}}(G, A) &\simeq \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A) \\ &\simeq H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(Q_n^u, A)) \quad , \\ &\simeq H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(Q_n, A)) \;. \end{aligned}$$

where on the right we canonically regard $A \in \mathbb{Z}[G]$Mod.

Proof. By the free functor adjunction we have that

$$\mathrm{Hom}_{\mathbb{Z}[G]}(F_n^u, A) \simeq \mathrm{Hom}_{\mathrm{Set}}(U(G)^{\times n}, U(A))$$

is the set of functions from n-tuples of elements of G to elements of A. It is immediate to check that these are in the kernel of $\mathrm{Hom}_{\mathbb{Z}[G]}(\partial_n, A)$ precisely if they are cocycles in the group cohomology (by comparison with the explicit formulas there) and that they are gorup cohomology coboundaries precisely if they are in the image of $\mathrm{Hom}_{\mathbb{Z}[G]}(\partial_{n-1}, A)$. This establishes the first equivalences.

Similarly one finds that $H^n(\mathrm{Hom}(F_n, A)))$ is the sub-group of *normalized* cocycles. By the discussion at *group cohomology* these already support the entire group cohomology (every cocycle is comologous to a normalized one). ∎

This finishes the discussion of the classification of central extensions of groups by $\mathrm{Ext}^1_{\mathbb{Z}[G]}$.

Now we discuss the general statement that $\mathrm{Ext}^1_{\mathcal{A}}$ classifies extensions in $\mathcal{A}$, hence in particular *abelian extension* of abelian groups if $\mathcal{A} =$ Ab.

Definition 3.90. Given $A, G \in \mathcal{A}$, an **extension** of G by A is a short exact sequence of the form

$$0 \to A \to \hat{G} \to G \to 0 \;.$$

Two extensions $\hat{G}_1$ and $\hat{G}_2$ are called *equivalent* if there is a morphism $f : \hat{G}_1 \to \hat{G}_2$ in $\mathcal{A}$ such that we have a

commuting diagram

$$
\begin{tikzcd}
 & \hat G_1 \arrow[dr] \arrow[dd, "f"] & \\
A \arrow[ur] \arrow[dr] & & G \,. \\
 & \hat G_2 \arrow[ur] &
\end{tikzcd}
$$

Write $\mathrm{Ext}(G,A)$ for the set of equivalence classes of extensions of G by A.

Remark 3.91. By the short five lemma a morphism f as above is necessarily an isomorphism and hence we indeed have an equivalence relation.

Definition 3.92. If $\mathcal{A}$ has enough projectives, define a function

$$\mathrm{Extr} : \mathrm{Ext}(G,A) \to \mathrm{Ext}^1(G,A)$$

from the group of extensions, def. 3.90, to the first Ext functor group as follows. Choose any projective resolution $Y_\bullet \overset{\simeq_{\mathrm{qi}}}{\longrightarrow} G$, which exists by prop. 3.27. Regard then $A \to \hat G \to G \to 0$ as a resolution

$$
\begin{array}{ccccccccc}
\cdots & \to & 0 & \to & 0 & \to & A & \to & \hat G \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & 0 & \to & 0 & \to & 0 & \to & G
\end{array}
$$

of G, by remark 3.26. By prop. 3.31 there exists then a commuting diagram of the form

$$
\begin{array}{ccc}
Y_2 & \to & 0 \\
\downarrow^{\partial_1^Y} & & \downarrow \\
Y_1 & \overset{c}{\to} & A \\
\downarrow^{\partial_0^Y} & & \downarrow \\
Y_0 & \to & \hat G \\
\downarrow & & \downarrow \\
G & \overset{\mathrm{id}}{\to} & G
\end{array}
$$

lifting the identity map on G two a chain map between the two resolutions.

By the commutativity of the top square, the morphism c is 1-cocycle in $\mathrm{Hom}(Y_\bullet, N)$, hence defines an element in $\mathrm{Ext}^1(G,A) \coloneqq H^1(\mathrm{Hom}(Y_\bullet, N))$.

Proposition 3.93. *The construction of def. 3.92 is indeed well defined in that it is independent of the choice of projective resolution as well as of the choice of chain map between the projective resolutions.*

Proof. First consider the same projective resolution but another lift $\tilde c$ of the identity. By prop. 3.32 any other choice $\tilde c$ fitting into a commuting diagram as above is related by a chain homotopy to c.

$$
\begin{array}{ccc}
Y_2 & \to & 0 \\
\downarrow^{\partial_1^Y} & \nearrow_{\eta_1 = 0} & \downarrow \\
Y_1 & \overset{c - \tilde c}{\longrightarrow} & A \\
\downarrow^{\partial_0^Y} & \nearrow_{\eta_0} & \downarrow \\
Y_0 & \to & \hat G \\
\downarrow & \nearrow & \downarrow \\
G & \to & G
\end{array}
\,.
$$

The chain homotopy condition here says that

$$c - \tilde c = \eta_0 \circ \partial_0^Y$$

and hence that in $\mathrm{Hom}(Y_\bullet, N)$ we have that $d\eta_0 = c - \tilde c$ is a coboundary. Therefore for the given choice of resolution $Y_\bullet$ we have obtained a well-defined map

$$\mathrm{Ext}(G,A) \to \mathrm{Ext}^1(G,A) \,.$$

If moreover $Y'_\bullet \overset{\simeq_{\mathrm{qi}}}{\longrightarrow} G$ is another projective resolution, with respect to which we define such a map as above, then lifting the identity map on G to a chain map between these resolutions in both directions, by prop. 3.31, establishes an isomorphism between the resulting maps, and hence the construction is independent

also of the choice of resolution. ▮

Definition 3.94. *Define a function*

$$\mathrm{Rec} : \mathrm{Ext}^1(G, A) \to \mathrm{Ext}(G, A)$$

as follows. For $Y_\bullet \to G$ a projective resolution of G and $[c] \in \mathrm{Ext}^1(G, A) \simeq H^1(\mathrm{Hom}_{\mathcal{A}}(F_\bullet, A))$ an element of the Ext-group, let

$$\begin{array}{ccc} Y_2 & \to & 0 \\ \downarrow & & \downarrow \\ Y_1 & \stackrel{c}{\to} & A \\ \downarrow \\ Y_0 \\ \downarrow \\ G \end{array}$$

be a representative. By the commutativity of the top square this restricts to a morphism

$$\begin{array}{ccc} Y_1/Y_2 & \stackrel{c}{\to} & A \\ \downarrow \\ Y_0 & & , \\ \downarrow \\ G \end{array}$$

where now the left column is itself an extension of G by the cokernel Y_1/Y_2 (because by exactness the kernel of $Y_1 \to Y_0$ is the image of Y_2 so that the kernel of $Y_1/Y_2 \to Y_0$ is zero). Form then the pushout of the horizontal map along the two vertical maps. This yields

$$\begin{array}{ccc} Y_1/Y_2 & \stackrel{c}{\to} & A \\ \downarrow & & \downarrow \\ Y_0 & \to & Y_0 \coprod_{Y_1/Y_2} A \\ \downarrow & & \downarrow \\ G & \stackrel{\mathrm{id}}{\to} & G \end{array} .$$

Here the top right is indeed G, by the pasting law for pushouts and using that the left vertical composite is the zero morphism. Moreover, the top right morphism is indeed a monomorphism as it is the pushout of a map of modules along an injection. Similarly the top right morphism is an epimorphism.

Hence $A \to Y_0 \coprod_{Y_1/Y_2} Y_0 \to G$ is an element in $\mathrm{Ext}(G, A)$ which we assign to c.

Proposition 3.95. *The construction of def. 3.94 is indeed well defined in that it is independent of the choice of projective resolution as well as of the choice of representative of the Ext-element.*

Proof. The coproduct $Y_0 \coprod_{Y_1/Y_2} A$ is equivalently

$$\mathrm{coker}(Y_1/Y_2 \xrightarrow{(\mathrm{incl}, -c)} Y_0 \oplus A) .$$

For a different representative $\tilde{c}$ of $[c]$ there is by construction a

$$\begin{array}{ccc} Y_1 & \stackrel{\tilde{c} - c}{\longrightarrow} & A \\ {}^{\partial_0}\downarrow & \nearrow_{\lambda} & . \\ Y_0 \end{array}$$

Define from this a map between the two cokernels induced by the commuting diagram

$$\begin{array}{ccc} Y_1/Y_2 & \stackrel{\mathrm{id}}{\to} & Y_1/Y_2 \\ \downarrow^{(\mathrm{id}, -c)} & & \downarrow^{(\mathrm{id}, -\tilde{c})} \\ Y_0 \oplus A & \xrightarrow{\left(\begin{array}{cc} \mathrm{id} & 0 \\ \lambda & \mathrm{id} \end{array}\right)} & Y_0 \oplus A \end{array} .$$

By construction this respects the inclusion of $A \stackrel{(0, \mathrm{id})}{\hookrightarrow} Y_0 \oplus A \to Y_0 \coprod_{Y_1/Y_2} A$. It also manifestly respects the projection to G. Therefore this defines a morphism and hence by remark 3.91 even an isomorphism of extensions. ▮

Proposition 3.96. *The functions*

$$\mathrm{Extr} : \mathrm{Ext}(G, A) \leftrightarrow \mathrm{Ext}^1(G, A) : \mathrm{Rec}$$

from def. 3.92 to def. 3.94 are inverses of each other and hence exhibit a bijection between extensions of G by A and $\mathrm{Ext}^1(G, A)$.

Proof. By straightforward unwinding of the definitions.

In one direction, starting with a $c \in \mathrm{Ext}^1(G, A)$ and constructing the extension by pushout, the resulting pushout diagram

$$\begin{array}{ccc} Y_1 & \stackrel{c}{\to} & A \\ \downarrow && \downarrow \\ Y_0 & \to & Y_0 \coprod^c_{Y_1/Y_2} A \\ \downarrow && \downarrow \\ G & \stackrel{\mathrm{id}}{\to} & G \end{array}$$

at the same time exhibits c as the cocycle extracted from the extension $A \to Y_0 \coprod^c_{Y_1/Y_2} A \to G$.

Conversely, when starting with an extension $A \to \hat G \to G$ then extracting a c by a choice of projective resolution and constructing from that another extension by pushout, the universal property of the pushout yields a morphism of exensions, which by remark 3.91 is an isomorphism of extensions, hence an equality in $\mathrm{Ext}(G, A)$. ▮

This concludes our discussion of the derived Hom-functor and its relation to extensions and group extensions in low degree. Of course also the higher Ext-groups classify higher extensions, but this we will not discuss here. Instead we turn now to the left adjoint of Hom-functor, the functor that forms tensor product of modules.

b) The derived tensor product functor and torsion subgroups

We discuss now the construction and the basic properties of the derived functors of the following tensor product functors.

Let R be a commutative ring. Above in def. 2.4 we considered the tensor product of abelian groups, hence of $\mathbb{Z}$-modules. This directly generalizes to a tensor product of R-modules as follows.

Definition 3.97. For $N, N' \in R\mathrm{Mod}$ two R-modules, their tensor product of modules over R

$$N \otimes_R N' \in R\,\mathrm{Mod}$$

is defined to be the R-module

* whose underlying abelian group is the quotient of the free abelian group on $U(N) \times U(N')$, hence on the set of pairs $\{(n, n') | n \in N, n' \in N'\}$, by the bilinearity relations (for all tuples of elements for which these expressions makes sense)

  $$(n_1 + n_2, n') = (n_1, n') + (n_2, n')$$

  and

  $$(n, n'_1 + n'_2) = (n, n'_1) + (n, n'_2)$$

  (as for tensor products of abelian groups)

  and

  $$(rn, n') = (n, rn') \;.$$

* whose R-action is given by

  $$r(n, n') = (rn, n') \sim (n, rn') \;.$$

We then have statements analog to those for tensor products of abelian groups. or instance as in prop. 2.7 we have:

Example 3.98. For $N \in R\mathrm{Mod}$ any module and for R regarded as a module over itself, example 2.23, there is an isomorphism

$$R \otimes_R N \stackrel{\simeq}{\to} N$$

given by sending

$$(r, n) \sim (1r, n) \sim (1, n) \mapsto n \;.$$

Definition 3.99. Let $R \in R\mathrm{Mod}$ be an R-module. The operation of forming the tensor product of modules

with N extends to a functor

$$(-) \otimes_R N : R\,\mathrm{Mod} \to R\,\mathrm{Mod}$$

by sending a homomorphism $f : N_1 \to N_2$ of R-modules to the homomorphism

$$f \otimes_R N : N_1 \otimes_R N \to N_2 \otimes_R N$$

given by

$$(n_1, n) \mapsto (f(n_1), n)\ .$$

This is well-defined precisely by the fact that f is a homomorphism of R-modules by assumption.

Proposition 3.100. *For every $N \in R$Mod, the functor $(-) \otimes_R N$ from def. 3.99*

- *is an additive functor;*
- *is a right exact functor.*

Therefore we may consider its left derived functor, according to def. 3.45.

Definition 3.101. For $N \in R\,\mathrm{Mod}$ and $n \in \mathbb{N}$, write

$$\mathrm{Tor}_n^R(-, N) \coloneqq L_n((-) \otimes_R N) : R\,\mathrm{Mod} \to R\,\mathrm{Mod}$$

for the left derived functor of the tensor product of modules-functor – the **Tor-functor**.

Remark 3.102. We could just as well consider deriving the tensor product functor in the second variable. Indeed both choices give the same result. We postpone the proof of this until we have developed the tool of spectral sequences below in 12). See prop. 4.68 below.

The name "Tor" derives from the basic relation of this functor to torsion subgroups. This we discuss now.

Definition 3.103. *An abelian group is called torsion if its elements are "nilpotent", hence if all its elements have finite order.*

Definition 3.104. For $A \in$ Ab and $p \in \mathbb{N}$, write

$${}_pA \coloneqq \{a \in A \,|\, p \cdot a = 0\}$$

for the p**-torsion subgroup** consisting of all those elements whose p-fold sum with themselves gives 0.

Proposition 3.105. *For $p \in \mathbb{N}$, $p \geq 1$, for $\mathbb{Z}_p \coloneqq \mathbb{Z}/p\mathbb{Z}$ the cyclic group and for $A \in$ Ab $\simeq \mathbb{Z}$Mod any abelian group, we have an isomorphism*

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_p, A) \simeq {}_pA\ .$$

For $p = 0$ we have

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, A) \simeq 0\ .$$

Proof. For the first statement, the short exact sequence

$$0 \to \mathbb{Z} \overset{\cdot p}{\to} \mathbb{Z} \overset{\mathrm{mod}\, p}{\longrightarrow} \mathbb{Z}/p\mathbb{Z} \to 0$$

constitutes a projective resolution (even a free resolution) of $\mathbb{Z}/p\mathbb{Z}$. Accordingly we have

$$\begin{aligned} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, A) &\simeq H_1([\cdots \to 0 \to \mathbb{Z} \otimes A \overset{(\cdot p)\otimes A}{\longrightarrow} \mathbb{Z} \otimes A) \\ &\simeq \ker((\cdot p) \otimes A) \\ &\simeq \{a \in A \,|\, p \cdot a = 0\} \end{aligned} .$$

Here in the last step we use that $(\cdot p) \otimes A$ acts as

$$\begin{aligned} (1, a) &\mapsto (p, a) \\ &= p \cdot (1, a) \\ &= (1, p \cdot a) \end{aligned} .$$

The second statement follows since $\mathbb{Z}$ is already free so that $[\cdots \to 0 \to 0 \to \mathbb{Z}]$ is a projective resolution. ∎

Proposition 3.106. *For $N \in R$Mod, the functor $\mathrm{Tor}_n^R(-, N)$ respects direct sums.*

Proof. Let $S \in$ Set and let $\{N_s\}_{s \in S}$ be an S-family of R-modules. Observe that

1. if $\{(F_s)_\bullet\}_{s \in S}$ is a family of projective resolutions, then their degreewise direct sum $(\oplus_{s \in S} F)_\bullet$ is a projective resolution of $\oplus_{s \in S} N_s$.

2. the tensor product functor distributes over direct sums, by prop. 2.12;

3. the chain homology functor preserves direct sums.

Using this we have

$$\begin{aligned} \mathrm{Tor}_n^R(\oplus_{s\in S} N_n, N) &\simeq H_n(((\oplus_{s\in S} F)\otimes N)_\bullet) \\ &\simeq H_n(\oplus_{s\in S}(F_s\otimes N)_\bullet) \\ &\simeq \oplus_{s\in S} H_n((F_s\otimes N)_\bullet) \\ &\simeq \oplus_{s\in S}\mathrm{Tor}_n(N_s, N) \end{aligned} .$$

∎

Proposition 3.107. *Let A be a finite abelian group and B any abelian group. Then $\mathrm{Tor}_1(A,B)$ is a torsion group. Specifically, $\mathrm{Tor}_1(A,B)$ is a direct sum of torsion subgroups of A.*

Proof. By a fundamental fact about finite abelian groups (see *this theorem*), A is a direct sum of cyclic groups $A \simeq \oplus_k \mathbb{Z}_{p_k}$. By prop. 3.106 Tor_1 respects this direct sum, so that

$$\mathrm{Tor}_1(A,B) \simeq \oplus_k \mathrm{Tor}_1(\mathbb{Z}_{p_k}, B) .$$

By prop. 3.105 every direct summand on the right is a torsion group and hence so is the whole direct sum. ∎

In fact this statement is true without assuming finiteness. The full statement is theorem 3.115 below, which we come to after preparing a few more properties of Tor.

One aspect here is that the order of A and B does not matter:

Proposition 3.108. *For $N_1, N_2 \in Ab$ and $n \in \mathbb{N}$ there is a natural isomorphism*

$$\mathrm{Tor}_n(A,B) \simeq \mathrm{Tor}_n(B,A) .$$

Proof. By prop. 3.50 there is always a short exact sequence

$$0 \to F_1 \to F_0 \to N \to 0$$

exhibiting a projective resolution of any module N. It follows that $\mathrm{Tor}_{n\geq 2}(-,-) = 0$.

Let then $0 \to F_1 \to F_2 \to N_2 \to 0$ be such a short resolution for N_2. Then by the long exact sequence of a derived functor, prop. 3.50, this induces an exact sequence of the form

$$0 \to \mathrm{Tor}_1(N_1,F_1) \to \mathrm{Tor}_1(N_1,F_0) \to \mathrm{Tor}_1(N_1,N_2) \to N_1\otimes F_1 \to N_1\otimes F_0 \to N_1\otimes N_2 \to 0 .$$

By prop. 3.47, since by construction F_0 and F_1 are already projective modules themselves this collapses to an exact sequence

$$0 \to \mathrm{Tor}_1(N_1,N_2) \hookrightarrow N_1\otimes F_1 \to N_1\otimes F_0 \to N_1\otimes N_2 \to 0 .$$

To the last three terms we apply the natural symmetric braiding in $(R\,\mathrm{Mod}, \otimes_R)$ isomorphism to get

$$\begin{array}{ccccccccccc} 0 & \to & \mathrm{Tor}_1(N_1,N_2) & \hookrightarrow & N_1\otimes F_1 & \to & N_1\otimes F_0 & \to & N_1\otimes N_2 & \to & 0 \\ && \downarrow && \downarrow^{\simeq} && \downarrow^{\simeq} && \downarrow^{\simeq} && \\ 0 & \to & \mathrm{Tor}_1(N_2,N_1) & \hookrightarrow & F_1\otimes N_1 & \to & F_0\otimes N_1 & \to & N_2\otimes N_1 & \to & 0 \end{array} .$$

This exhibits a morphism $\mathrm{Tor}_1(N_1,N_2) \to \mathrm{Tor}_1(N_2,N_1)$ as the morphism induced on kernels from an isomorphism between two morphisms. Hence this is itself an isomorphism. (This is just by the universal property of the kernel, but one may also think of it as a simple application of the four lemma/five lemma.) ∎

In order to understand more of theorem 3.115 we need to understand the acyclic objects of the tensor product functor, def. 3.53. These are called the *flat modules*.

Definition 3.109. An R-module N is **flat** if tensoring with N over R as a functor from RMod to itself

$$(-)\otimes_R N : R\,\mathrm{Mod} \to R\,\mathrm{Mod}$$

is an exact functor, def. 3.41.

Remark 3.110. The condition in def. 3.109 has the following immediate equivalent reformulations:

1. N is flat precisely if $(-)\otimes_R N$ is a left exact functor,

   because tensoring with any module is generally already a right exact functor;

2. N is flat precisely if $(-)\otimes_R N$ sends monomorphisms (injections) to monomorphisms,

because for a right exact functor to also be left exact the only remaining condition is that it preserves the monomorphisms on the left of a short exact sequence;

3. N is flat precisely if the degree-1 Tor-functor $\mathrm{Tor}_1^{R\,\mathrm{Mod}}(-,N)$ is zero,

   because by remark 3.51 $L_1 F$ is the obstruction to a right exact functor F being left exact;

4. N is flat precisely if all higher Tor functors $\mathrm{Tor}_{\geq 1}(R\,\mathrm{Mod})(-,N)$ are zero,

   by prop. 3.52;

5. N is flat precisely if N is an acyclic object with respect to the tensor product functor;

   because the Tor functor is symmetric in both arguments by prop. 3.108 and by definition of acyclic object, def. 3.53.

A particularly simple kind of injection of R-modules are the injections of finitely generated ideals $I \hookrightarrow R$ into the ring R, regarded as a module over itself, by example 2.23. According to remark 3.110 N being flat implies that also $I \otimes_R N \to R \otimes_R N \simeq N$ is a monomorphism. The following theorem says that this is indeed already sufficient to imply that $(-) \otimes_R N$ preserves also all other monomorphisms.

Theorem 3.111. *An R-module N is flat already if for all inclusions $I \hookrightarrow R$ of a finitely generated ideal into R, regarded as a module over itself, the induced morphism*

$$I \otimes_R N \to R \otimes_R N \simeq N$$

is an injection.

We will not prove this here. But this does imply the following explicit element-wise characterization of flat modules.

Proposition 3.112. *A module N is flat precisely if for every finite linear combination of zero, $\sum_i r_i n_i = 0 \in N$ with $\{r_i \in R\}_i$, $\{n_i \in N\}$ there are elements $\{\tilde{n}_j \in N\}_j$ and linear combinations*

$$n_i = \sum_j b_{ij} \tilde{n}_j \quad \in N$$

with $\{b_{ij} \in R\}_{i,j}$ such that for all j we have linear combinations of 0 in R

$$\sum_i r_i b_{ij} = 0 \quad \in R\;.$$

Proof. A finite set $\{r_i \in R\}_i$ corresponds to the inclusion of a finitely generated ideal $I \hookrightarrow R$.

By theorem 3.111 N is flat precisely if $I \otimes_R N \to N$ is an injecton. This in turn is the case precisely if the only element of the tensor product $I \otimes_R R$ that is 0 in $R \otimes_R N = N$ is already 0 on $I \otimes_R N$.

Now by definition of tensor product of modules an element of $I \otimes_R N$ is of the form $\sum_i (r_i, n_i)$ for some $\{n_i \in N\}$. Under the inclusion $I \otimes_R N \to N$ this maps to the actual linear combination $\sum_i r_i n_i$. This map is injective if whenever this linear combination is 0, already $\sum_i (r_i, n_i)$ is 0.

But the latter is the case precisely if this is equal to a combination $\sum_j (\tilde{r}_j, \tilde{n}_j)$ where all the $\tilde{r}_j$ are 0. This implies the claim. ∎

By the same kind of reasoning as in the proof of prop. 3.112 one finds:

Proposition 3.113. ***(Lazard's criterion)***

A module is flat if and only if it is a filtered colimit of free modules.

Using this we can now show the following.

Proposition 3.114. *For $N \in R\,\mathrm{Mod}$ a module and $n \in \mathbb{N}$, the functor*

$$\mathrm{Tor}_n^R(-,N) : R\,\mathrm{Mod} \to R\,\mathrm{Mod}$$

respects filtered colimits.

Proof. Let hence $A : I \to R\,\mathrm{Mod}$ be a filtered diagram of modules. For each A_i, $i \in I$ we may find a projective resolution and in fact a free resolution $(Y_i)_\bullet \overset{\simeq_{\mathrm{qi}}}{\longrightarrow} A$. Since chain homology commutes with filtered colimits (this is discussed at *chain homology - respect for filtered colimits*), this means that

$$(\underset{\longrightarrow_i}{\lim} Y_i)_\bullet \to A$$

is still a quasi-isomorphism. Moreover, by Lazard's criterion, def. 3.113 the degreewise filtered colimits of

projective modules $\underset{\longrightarrow_i}{\lim}(Y_i)_n$ for each $n \in \mathbb{N}$ are flat modules. This means that $\underset{\longrightarrow_i}{\lim}(Y_i)_\bullet \to A$ is flat resolution of A. By remark 3.110 this means that it is a $(-) \otimes N$-acyclic resolution. Then by example 3.57 and theorem 3.60 it follows that

$$\mathrm{Tor}_n^{\mathbb{Z}}(A,N) \simeq H_n((\underset{\longrightarrow_i}{\lim} Y_i) \otimes N) \,.$$

Now the tensor product of modules is a left adjoint functor (the right adjoint being the internal hom of modules) and so it commutes over the filtered colimit to yield, using again that chain homology commutes with filtered colimits,

$$\begin{aligned} \cdots &\simeq H_n(\underset{\longrightarrow_i}{\lim}(Y_i \otimes N)) \\ &\simeq \underset{\longrightarrow_i}{\lim} H_n(Y_i \otimes N) \\ &\simeq \underset{\longrightarrow_i}{\lim} \mathrm{Tor}_n(A_i, N) \end{aligned} \,.$$

∎

Using this we now can now proof the generalization of prop. 3.107.

Theorem 3.115. *For $A, B \in$ Ab, $\mathrm{Tor}_1(A,B)$ is a torsion group which is a filtered colimit of direct sums of torsion subgroups of either A or B.*

Proof. The group A may be expressed as a filtered colimit

$$A \simeq \underset{\longrightarrow_i}{\lim} A_i$$

of all its finitely generated subgroups (this is discussed at *Mod - Limits and colimits*). Each of these is a direct sum of cyclic groups.

By prop. 3.114 $\mathrm{Tor}_1^{\mathbb{Z}}(-,B)$ preserves these colimits. By prop. 3.105 every summand is sent to a torsion subgroup (of either A or B). Therefore by prop. 3.105 $\mathrm{Tor}_1(A,B)$ is a filtered colimit of direct sums of torsion groups. This is itself a torsion group. ∎

This concludes our discussion of the basic properties of the Tor-functor. In the next chapter *The fundamental theorems* we see Ext and Tor put to work to yield deeper statements.

4. IV) The fundamental theorems

We have tried to indicate in the motivation chapter I) that homological algebra arises from homotopy theory by "abelianization", a strict form of *stabilization*. Accordingly a central question is, how and to which extent this process respects basic universal constructions. This is what the "fundamental theorems" of homological algebra are about:

The Künneth theorem, discussed in 10) below, says how passing to singular homology commutes with taking products of spaces. The spectral sequence of a double complex describes how passing to homology commutes with taking homotopy colimits producing either total simplicial sets or total complexes. This we discuss in 12). The constructions and computation going into this involve the fundamentals of iterative *relative homology*, which is expressed by the spectral sequence of a filtered complex which we discuss in detail in 11).

There are more and tighter relation between homotopy theory and homological algebra, which however require a bit more background in simplicial homotopy theory. This we finally turn to in chapter V) below.

11) Universal coefficient theorem and Künneth theorem

We discuss the following three theorems which put the Ext- and Tor-construction of the previous section 10) to use. All three are closely related, the first two are roughly dual to each other, the third is a generalization of the first:

1. The **universal coefficient theorem in homology**, theorem 4.6;
2. the **universal coefficient theorem in cohomology**, theorem 4.9;
3. the **Künneth theorem**, theorem 4.11.

We state these here first, as is traditional, in a version that is not the most general possible, but which is still convenient to use and as general as the standard applications require. (The fully general version requires the technology of spectral sequences, which we turn to below in the next section 12).) In this version these theorems all require an assumption on the base ring R: that it is the ring of integers, or, more generally, that it is a principal ideal domain, for instance also a field or the polynomial ring with coefficients in a field. Or rather, they rely on the following consequence of this assumption on R:

Proposition 4.1. *For R a principal ideal domain, every submodule of a free module over R is itself a free module.*

A detailed proof of this fact can be found at *Principal ideal domain - structure theory of modules*.

Remark 4.2. Over R a field an R-module is a vector space and (assuming the axiom of choice) *every* vector space has a basis, hence is free on that basis.

Indeed, for all of the following three theorems the situation where R is a field is special in that in this case an Ext- or Tor-correction term vanishes and instead of just a short exact sequence with such a term the theorems produce an isomorphism.

The following theorems relate homology/cohomology with *basic coefficients* to those *with coefficients*. To make this notion fully explicit:

Definition 4.3. Let $C_\bullet \in \mathrm{Ch}_\bullet(R\,\mathrm{Mod})$ be a chain complex which is degreewise a free module. Let $A \in R\,\mathrm{Mod}$ be any module. Then we say that

- the chain homology of C **with coefficients** in A is the chain homology

$$H_\bullet(C_\bullet \otimes A)$$

  of the chain complex obtained degreewise by the tensor product of modules with R;

- the cochain cohomology of C **with coefficients** in A in the cochain cohomology

$$H^\bullet(\mathrm{Hom}_R(C_\bullet, A))$$

  of the cochain complex, example 3.11, obtained forming degreewise the hom-object into A.

Remark 4.4. Since for each k the module C_k is free by assumption, hence a direct sum $C_k = \oplus_{s \in S_k} R$, since the tensor product of modules distributes over direct sums, prop. 2.12, and since R is the tensor unit for the tensor product over R, it follows that

$$C_k \otimes_R A \simeq \oplus_{s \in S_k} A \,.$$

Our archetypical and motivating example, introduced in section 2), is still the following:

Example 4.5. For X a topological space and $C_\bullet \coloneqq C_\bullet(R[\mathrm{Sing}\,X])$ the singular chain complex over R, hence for $C_k \coloneqq R[(\mathrm{Sing}\,X)]_k$ the free module on the set of sincular k-simplices for each $k \in \mathbb{N}$ we have that

- $H_\bullet(C_\bullet(R[\mathrm{Sing}\,X]) \otimes_R A)$ is the *singular homology* of X with coefficients in A;
- $H^\bullet(\mathrm{Hom}_R(C_\bullet(R[\mathrm{Sing}\,X]), A))$ as in example 3.11, is the *singular cohomology* of X with coefficients in A.

Now the universal coefficient theorem below says, roughly, that the basic coefficient ring R is already "universal" in that

1. homology with any other coefficients is determined by homology with basic coefficients corrected by a Tor-module;
2. cohomology with any other coefficient is determined by cohomology with basic coefficients corrected by an Ext-module.

After these preliminaries, we finally state and prove the theorems. So

- let R be a ring which is a principal ideal domain,
- let $C_\bullet \in \mathrm{Ch}_\bullet(R\,\mathrm{Mod})$ be a chain complex of free modules over R,
- let $A \in R\,\mathrm{Mod}$ be any R-module,
- write $C_k \otimes_R A$ for the tensor product of modules over R.

Theorem 4.6. ***(universal coefficient theorem in ordinary homology)***

For each $n \in \mathbb{N}$ there is a short exact sequence

$$0 \to H_n(C_\bullet) \otimes_R A \to H_n(C_\bullet \otimes_R A) \to \mathrm{Tor}_1^R(H_{n-1}(C_\bullet), A) \to 0$$

where on the right we have the first Tor-module, def. 3.101, of the chain homology $H_{n-1}(C_\bullet)$ with A.

Remark 4.7. This means in particular that when the Tor-module $\mathrm{Tor}_1^R(H_{n-1}(C_\bullet), A)$ vanishes, then there is an isomorphism

$$H_n(C_\bullet \otimes_R A) \simeq H_n(C_\bullet) \otimes_R A$$

which identifies the homology of $C_\bullet$ with coefficients in A, def. 4.3, with the bare homology of $C_\bullet$ tensored with A.

Before we give the proof we state the following lemma.

Lemma 4.8. *For $C_\bullet$ a chain complex of free modules and $A \in R\mathrm{Mod}$ any module, there is a long exact sequence of the form*

$$\cdots \to B_n \otimes_R A \xrightarrow{i_n \otimes A} Z_n \otimes_R A \to H_n(C_\bullet \otimes_R A) \to B_{n-1} \otimes_R A \xrightarrow{i_{n-1} \otimes A} Z_{n-1} \otimes_R A \to \cdots,$$

where B_n are the boundaries and Z_n the cycles of $C_\bullet$ in degree n and where $i_n : B_n \hookrightarrow Z_n$ is the canonical inclusion.

Proof. Since, by prop. 4.1, every submodule of a free module over our ring R is itself free, such as the submodule of cycles $Z_n \hookrightarrow C_n$, it follows that for each $n \in \mathbb{N}$ we have a splitting, def. 2.73, of the short exact sequence $0 \to Z_n \to C_n \to C_n/Z_n$ and hence, by prop. 2.74, a direct sum decomposition

$$C_n \simeq Z_n \oplus B_{n-1} \,.$$

Here the second direct summand on the right is identified, as indicated, under the differential ∂^C with the boundaries in one degree lower, since by construction ∂^C is injective on C_n/Z_n.

Accordingly, if we regard the graded modules $B_\bullet$ and $Z_\bullet$ of boundaries and cycles as chain complexes with vanishing differential, then we have a sequence of chain maps

$$0 \to Z_\bullet \hookrightarrow C_\bullet \to B_{\bullet-1} \to 0$$

which is degreewise a short exact sequence, hence is a short exact sequence of chain complexes. Now since the tensor product of modules distributes over direct sum, the image of this sequence under $(-) \otimes A$

$$0 \to Z_\bullet \otimes A \hookrightarrow C_\bullet \otimes_R A \to B_{\bullet-1} \otimes_R A \to 0$$

is still a split exact sequence hence in particular still a short exact sequence. The induced homology long exact sequence, as discussed there, is the long exact sequence to be shown: one reads off that it has the right terms and it is straightforward to check that the connecting homomorphisms are indeed given by i_n as stated. ∎

***Proof*. of theorem 4.6**

By lemma 4.8 we have short exact sequences

$$0 \to \mathrm{coker}(i_n \otimes_R A) \to H_n(C_\bullet \otimes_R A) \to \ker(i_n \otimes_R A) \to 0 \,.$$

Since the tensor product of modules is a right exact functor it preserves cokernels and hence

$$\mathrm{coker}(i_n \otimes A) \simeq \mathrm{coker}(i_n) \otimes A = H_n(C) \otimes A \,,$$

which is what we needed to show on the left.

The dual statement were true if $(-) \otimes A$ were also a left exact functor. In general it is not, and the failure is measured by the Tor-group:

Notice that with prop. 4.1 the defining short exact sequence

$$0 \to B_n \stackrel{i_n}{\to} Z_n \to H_n(C_\bullet) \to 0$$

exhibits $[\cdots \to 0 \to B_n \to Z_n] \stackrel{\simeq_{\mathrm{qi}}}{\to} H_n(C)$ as a projective resolution of $H_n(C_\bullet)$, by remark 3.26. Therefore by definition of Tor the group $\mathrm{Tor}_1^R(H_n(C_\bullet), A)$ is the chain homology in degree 1 of

$$[\cdots \to 0 \to B_n \otimes G \xrightarrow{i_n \otimes A} Z_n \otimes_R A] \,,$$

which is

$$\mathrm{Tor}_1(H_n(C_\bullet), A) \simeq \ker(i_n \otimes_R A)$$

and this is indeed what we have to show on the right hand side. ∎

The following statement is a kind of dualization of the previous one. Instead of tensor products of modules it involves the Hom of modules, and instead of Tor-modules it involves Ext-modules as corrections.

Theorem 4.9. ***(universal coefficient theorem in ordinary cohomology)***

Let $C_\bullet \in \mathrm{Ch}_\bullet(R\,\mathrm{Mod})$ be a chain complex of modules over a principal ideal domain R, which is degreewise a free module. Let $A \in R\,\mathrm{Mod}$ be an module. Then there is a short exact sequence

$$0 \to \mathrm{Ext}^1(H_{n-1}(C), A) \to H^n(\mathrm{Hom}_{R\,\mathrm{Mod}}(C_\bullet, A)) \to \mathrm{Hom}_{R\,\mathrm{Mod}}(H_n(C), A) \to 0$$

with the Ext-module on the left.

Lemma 4.10. *Given a homomorphism $A_1 \stackrel{f}{\to} A_2 \stackrel{g}{\to} A_3$ of modules together with a retract $s : A_3 \to A_2$ of g, there is a short exact sequence of cokernels*

$$0 \to \mathrm{coker}\, f \xrightarrow{g'} \mathrm{coker}(g \circ f) \to \mathrm{coker}(g) \to 0 \;.$$

Proof. Since we work in RMod, all the cokernels appearing here (as discussed there) may be expressed as quotients, e.g $\mathrm{coker}(f) \simeq A_2/\mathrm{im}(f)$.

The sequence of inclusions $\mathrm{im}(g \circ f) \hookrightarrow \mathrm{im}(g) \hookrightarrow A_3$ induces the canonical short exact sequence

$$0 \to \frac{\mathrm{im}(g)}{\mathrm{im}(g \circ f)} \to \frac{A_3}{\mathrm{im}(g \circ f)} \to \frac{A_3}{\mathrm{im}(g)} \to 0$$

and we claim that this is already isomorphic to the one stated in the lemma. This is manifestly true for the two terms on the right. For the term on the left observe that g induces a morphism $g' : A_2/\mathrm{im}(f) \to A_3/\mathrm{im}(g \circ f)$. By the existence of the retract s this has itself a retract. Moreover it factors as

$$g' : A_1/\mathrm{im}(f) \to \mathrm{im}(g)/\mathrm{im}(g \circ f) \hookrightarrow A_3/\mathrm{im}(g \circ f) \;.$$

Therefore the first morphism here on the left has to be an isomorphism, too. ∎

Proof (of theorem 12). Write

$$0 \to B_n \to Z_n \to H_n \to 0$$

for the short exact sequence of boundaries, cycles, and homology groups of $C_\bullet$ in degree n. Since C_n is assumed to be a free module and since B_n and Z_n are submodules, it follows that these are also free, by prop. 4.1. Therefore this sequence exhibits a projective resolution of the group H_n. It follows that the Ext-group $\mathrm{Ext}^1(H_n, A)$ is characterized by the short exact sequence

$$\mathrm{Hom}(Z_n, A) \to \mathrm{Hom}(B_n, A) \to \mathrm{Ext}^1(H_n, A) \to 0 \;. \tag{6}$$

Notice also that the short exact sequence

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0 \tag{7}$$

is split because, as before, B_{n-1} is free abelian. Using these two exact sequences on the left and right of the short exact sequence

$$0 \to Z_n/B_n \to C_n/B_n \to C_n/Z_n \to 0$$

shows that this is equivalent to

$$0 \to H_n \to C_n/B_n \xrightarrow{\partial} B_{n-1} \;. \tag{8}$$

Again this splits as B_{n-1} is free abelian.

In addition to these exact sequence consider the decomposition

$$\partial : C_n \to C_n/B_n \to C_n/Z_n \xrightarrow{\simeq} B_{n-1} \hookrightarrow Z_{n-1} \hookrightarrow C_{n-1}$$

and apply $\mathrm{Hom}(-, A)$ to obtain the diagram

$$\begin{tikzcd}
 & & 0 & & \\
 & & \mathrm{Hom}(H_n, A) \arrow[u] & & \\
\mathrm{Hom}(B_n, A) & \mathrm{Hom}(C_n, A) \arrow[l] & \mathrm{Hom}(C_n/B_n, A) \arrow[l, "i"'] \arrow[u] & 0 \arrow[l] & 0 \\
0 & \mathrm{Ext}^1(H_n, A) \arrow[l] & \mathrm{Hom}(B_{n-1}, A) \arrow[l] \arrow[u, "\mathrm{Hom}(\bar\partial, A)"'] & & \mathrm{Hom}(Z_{n-1}, A) \arrow[ll] \arrow[u] \\
 & & 0 \arrow[u] & & \mathrm{Hom}(C_{n-1}, A) \arrow[u] \arrow[ull]
\end{tikzcd}$$

Here the right vertical sequence is exact, because (7) splits, and the left vertical sequence is exact because (8) splits. The upper horizontal sequence is exact because the hom functor takes cokernels to kernels and finally the lower horizontal sequence is the exact sequence (6).

Since therefore i and $\mathrm{Hom}(\bar\partial, A)$ are monomorphisms, it follows that the degree n-cocycles are

$$Z^{n-1} \coloneqq \ker(\mathrm{Hom}(C_{n-1}, A) \to \mathrm{Hom}(C_n, a)) \simeq \ker(\mathrm{Hom}(C_{n-1}, A) \to \mathrm{Hom}(B_{n-1}, A)) \;.$$

Using this for $n-1$ replaced by n shows by the upper horizontal exact sequence that

$$Z^n = \mathrm{Hom}(C_n/B_n, A) \;.$$

Similarly the coboundaries are seen to be

$$B^n \coloneqq \mathrm{im}\mathrm{Hom}(\partial, A) \simeq \mathrm{im}(\mathrm{Hom}(Z_{n-1}, A) \to \mathrm{Hom}(C_n/B_n), A)\ .$$

Together this gives the cochain cohomology as

$$H^n(C, A) \coloneqq Z^n/B^n \simeq \mathrm{coker}(\mathrm{Hom}(Z_n, A) \to \mathrm{Hom}(C_n/B_n, A))\ .$$

Now the universal coefficient theorem follows by going into lemma 4.10 with the identifications $A_1 = \mathrm{Hom}(Z_{n-1}, A)$, $A_2 = \mathrm{Hom}(B_{n-1}, A)$, $A_3 = \mathrm{Hom}(C_n/B_n, A)$. ■

The universal coefficient theorem in homology, theorem, 4.6 involves the tensor product of modules. The following generalizes this to the tensor product of chain complexes, def. 2.59.

Theorem 4.11. *For R a principal ideal domain, given a chain complex $C_\bullet \in \mathrm{Ch}_\bullet(R\,\mathrm{Mod})$ of free modules over R and given any other chain complex $C'_\bullet \in \mathrm{Ch}_\bullet(R\,\mathrm{Mod})$, then for each $n \in \mathbb{N}$ there is a short exact sequence of the form*

$$0 \to \oplus_k \left(H_k(C_\bullet) \otimes_R H_{n-k}(C'_\bullet)\right) \to H_n\left(C_\bullet \otimes_R C'_\bullet\right) \to \oplus_k \mathrm{Tor}_1^R(H_k(C_\bullet), H_{n-k-1}(C'_\bullet)) \to 0\ .$$

Remark 4.12. In the special case that C' is concentrated in degree 0, this is the universal coefficient theorem in ordinary homology, theorem 4.6.

Remark 4.13. In particular if all the Tor-groups on the right vanish, then the theorem asserts an isomorphism

$$H_n\left(C_\bullet \otimes_R C'_\bullet\right) \simeq \oplus_k \left(H_k(C_\bullet) \otimes_R H_{n-k}(C'_\bullet)\right),$$

which identifies the homology of a tensor product with the tensor product of the separate homologies.

This is the case (assuming the axiom of choice) notably if R is a field (since every module over a field is a free module – every vector space has a basis – and every free module is a flat module).

***Proof*. of theorem 4.11**

Notice that since C_k is assumed to be free, hence a direct sum of R with itself, since the tensor product of modules distributes over direct sums, and since chain homology respects direct sums, we have

$$H_n(C_k \otimes_R C') \simeq C_k \otimes_R H_{n-k}(C'_\bullet)\ . \tag{9}$$

First consider now the special case that all the differentials of $C_\bullet$ are zero, so that $H_k(C_\bullet) = C_k$. In this case (9) yields $H_n(C_k \otimes_R C') \simeq H_k(C_\bullet) \otimes_R H_{n-k}(C'_\bullet)$ and therefore

$$\begin{aligned} H_n(C \otimes_R C') &\simeq H_n(\oplus_k C_k \otimes_R C')) \\ &\simeq \oplus_k H_n(C_k \otimes_R C') \\ &\simeq \oplus_k H_k(C_\bullet) \otimes_R H_{n-k}(C'_\bullet) \end{aligned}\ .$$

Since $H_k(C) = C_k$ is a free module by assumption, it has no Tor-terms (by the discussion there) and hence this is the statement to be shown.

Now let $C_\bullet$ be a general chain complex of free modules. Notice that for each n the cycle-chain-boundary-short exact sequence

$$0 \to Z_{n-k} \hookrightarrow C_{n-k} \xrightarrow{\partial_{n-k-1}} B_{n-1} \to 0$$

splits due to the assumption that C_n is a free module, and hence (as discussed at *split exact sequence*) that it exhibits a direct sum decomposition $C_n \simeq Z_n \oplus B_{n-1}$. Since the tensor product of modules distributes over direct sum, it follows that tensoring with any C'_k yields another short exact sequence

$$0 \to Z_{n-k} \otimes_R C'_k \to C_{n-k} \otimes_R C'_k \to B_{n-k-1} \otimes_R C'_k \to 0\ .$$

This means that if we regard the graded modules $Z_\bullet$ and $B_\bullet$ of chains and of boundaries as chain complexes with zero-differentials, then we have a short exact sequence of chain complexes

$$0 \to Z_\bullet \otimes_R C'_\bullet \to C_\bullet \otimes_R C'_\bullet \to B_{\bullet-1} \otimes_R C'_\bullet \to 0\ .$$

This induces its homology long exact sequence, prop. 2.78, of the form

$$\cdots \to H_n(Z \otimes_R C') \to H_n(C \times_R C') \to H_{n-1}(B \otimes_R C') \to H_{n-1}(Z \otimes_R C') \to \cdots\ .$$

Here the terms involving the complexes B and Z of boundaries and cycles may be evalutated, since these have zero differentials, via the special case discussed at the beginning of this proof to yield the long exact sequence

$$\cdots \xrightarrow{i_n \otimes_R C'} \oplus_k (Z_k \otimes_R H_{n-k}(C')) \to H_n(C \otimes_R C') \to \oplus_k (B_k \otimes_R H_{n-k-1}(C')) \xrightarrow{i_{n-1}} \oplus_k (Z_k \otimes_R H_{n-k-1}(C')) \to \cdots,$$

where $i_n \coloneqq H_n(i \otimes C')$ is the morphism induced from the inclusion $i: B_\bullet \hookrightarrow Z_\bullet$ of boundaries into cycles.

This means that by quotienting out an image on the left and a kernel on the right, we obtain a short exact sequence

$$0 \to \mathrm{coker}(i_n) \to H_n(C \otimes_R C') \to \ker(i_{n-1}) \to 0 \; .$$

Since the tensor product of modules is a right exact functor it commutes with the cokernel on the left, as does the formation of direct sums, and so we have

$$\mathrm{coker}\Big(\oplus_k \Big(B_k \otimes_R H_{n-k}(C') \xrightarrow{i_k \otimes H_{n-k}(C')} Z_k \otimes_R H_{n-k}(C') \Big)\Big) \simeq \oplus_k \Big(\Big(\mathrm{coker}\Big(B_k \overset{i_k}{\to} Z_k\Big) \otimes_R H_{n-k}(C')\Big)\Big) \simeq \oplus_k \left(H_k(C) \otimes_R H_{n-k}(C')\right) \; .$$

This is the left term in the short exact sequence to be shown. For the right term the analogous argument does not quite go through, because tensoring is not in addition a left exact functor, in general. The failure to be so is precisely measured by the Tor-module:

Notice that by the assumption that C_n is free and using prop. 4.1 that over our R the submodules $B_n, Z_n \hookrightarrow C_n$ are themselves free modules, the defining short exact sequence $0 \to B_n \overset{i_n}{\to} Z_n \to H_n(C) \to 0$ exhibits a projective resolution of $H_n(C)$. Therefore by definition of Tor we have

$$\mathrm{Tor}_1(H_k(C), H_{n-k}(C')) \simeq \ker(i_k \otimes H_{n-k}(C')) \; .$$

This identifies the term on the right of the exact sequence to be shown. ∎

These theorems are of particular use in the computation of singular cohomology, due to the following fact.

Proposition 4.14. *Let $X, Y \in$ Top two topological spaces. The singular cohomology of their product topological space $X \times Y$ is isomorphic to that of the tensor product of chain complexes of their singular chain complexes separately*

$$H_n(\mathbb{Z}[\mathrm{Sing}(X \times Y)]_\bullet) \simeq H_n(\mathbb{Z}[\mathrm{Sing}\, X]_\bullet \otimes \mathbb{Z}[\mathrm{Sing}\, Y]_\bullet) \; .$$

This is a consequence of the *Eilenberg-Zilber theorem*, which we discuss below in section 14).

Corollary 4.15. *Let $X, Y \in \mathrm{Top}$ be two topological spaces. Let $R = k$ be a field. Then the singular homology of their product topological space in some degree n is the direct sum of the tensor products of the singular homologies of the spaces separately, whose degrees add up to n.*

$$H_n(X \times Y, k) \simeq \oplus_{n_1 + n_2 = n} H_{n_1}(X, k) \otimes_k H_{n_2}(X, k) \; .$$

This finishes the statements and proofs of the universal coefficient theorem and the Künneth theorem. We turn now to a tool that allows to produce more refined theorems of this kind.

12) Relative homology and Spectral sequences

We have motivated – in chapter 2) – *chain complexes* and their *homological algebra* from the singular chain complexes of topological spaces. While these do enjoy many nice formal properties, as we have discussed, they are not well-adapted to *explicit* computations of homology groups: there are in general "too many singular chains" in a topological space to say anything useful about them without further information.

The canonical piece of extra information needed to do explicit computations of homology groups for concrete topological spaces X is a *filtering* of X – a decomposition of X into "layers" – in the form of a sequence of subspaces $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X$ such that in each step there is some information on the structure that is added. In such a situation we can refine the notion of homology to *relative homology*, where one studies *relative cycles* in X_{n+1} whose boundary does not necessarily vanish, but is constrained to be one step lower in filtering degree $X_{n-1} \hookrightarrow X_n$. Such relative homology of filtered topological spaces often allows to compute genuine singular homology by induction over the filtering degree. A particularly explicit realization of this idea is applicable when X_{n+1} is obtained from X_n by specifically attaching sets of $(n+1)$-disks – the basic cells in homotopy theory. We begin this section below by explaining the resulting *cellular homology* of such topological spaces, which are called *CW-complexes*. The central fact about this cellular homology defined in terms of boundaries *relative* to filtering degree ± 1 is that it does coincide with the genuine singular homology and hence provides an efficient means for computing the latter, when available.

But the argument that shows this directly generalizes to homology *relative to higher shifts in filtering degree*: one finds immediately – and we discuss this in detail below – that for $r \in \mathbb{N}$ the $(r+1)$-relative cycles are themselves the homology of r-relative cycles in the filtered complex in a natural sense. The resulting tower of relative cycles of arbitrary relative degree is called (for no good reason, unfortunately, but ever since the notion was conceived) the *spectral sequence of the filtered complex.*

Via the motivating example of cellular homology we introduce this general notion of *spectral sequences*, see what it has to say about cellular homology and indicate in an outlook how with the same kind of simple argument a plethora of questions in homological algebra can be answered. In particular, given a double complex (as we discussed in section 6)) its total complex is naturally filtered either by row- or by column-degree and hence there is a spectral sequence of a double complex which helps with computing its total homology.

Such total homologies of double complexes are of interest notably whenever one computes the value of a

derived functor not on a single object, but on a chain complex of objects. A plethora of applications of spectral sequences arises this way. At the end of this section we provide some pointers to further reading on this.

We now begin with introducing basics of *relative homology* and then eventually and hopefully seamlessly derive the notion of *spectral sequences* from that.

Let X be a topological space and $A \hookrightarrow X$ a topological subspace. Write $C_\bullet(X)$ for the chain complex of singular homology on X, def. 1.43 and $C_\bullet(A) \hookrightarrow C_\bullet(X)$ for the chain map induced by the subspace inclusion according to def. 1.52.

Definition 4.16. The (degreewise) cokernel of this inclusion, hence the quotient $C_\bullet(X)/C_\bullet(A)$ of $C_\bullet(X)$ by the image of $C_\bullet(A)$ under the inclusion, is the **chain complex of A-relative singular chains**.

- A boundary in this quotient is called an **A-relative singular boundary**,
- a cycle is called an **A-relative singular cycle**.
- The chain homology of the quotient is the **A-relative singular homology of X**

$$H_n(X, A) \coloneqq H_n(C_\bullet(X)/C_\bullet(A)) \;.$$

Remark 4.17. This means that a singular $(n+1)$-chain $c \in C_{n+1}(X)$ is an A-relative cycle precisely if its boundary $\partial c \in C_n(X)$ is, while not necessarily 0, contained in the n-chains of A: $\partial c \in C_n(A) \hookrightarrow C_n(X)$. So the boundary vanishes possibly only "up to contributions coming from A".

We record two evident but important classes of long exact sequences that relative homology groups sit in:

Proposition 4.18. *Let $A \overset{i}{\hookrightarrow} X$ be a topological subspace inclusion. The corresponding relative singular homology, def. 4.16, sits in a long exact sequence of the form*

$$\cdots \to H_n(A) \xrightarrow{H_n(i)} H_n(X) \to H_n(X, A) \xrightarrow{\delta_{n-1}} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} H_{n-1}(X) \to H_{n-1}(X, A) \to \cdots \;.$$

The connecting homomorphism $\delta_n : H_{n+1}(X, A) \to H_n(A)$ sends an element $[c] \in H_{n+1}(X, A)$ represented by an A-relative cycle $c \in C_{n+1}(X)$, to the class represented by the boundary $\partial^X c \in C_n(A) \hookrightarrow C_n(X)$.

Proof. This is the *homology long exact sequence*, prop. 2.78, induced by the defining short exact sequence $0 \to C_\bullet(A) \overset{i}{\hookrightarrow} C_\bullet(X) \to \mathrm{coker}(i) \simeq C_\bullet(X)/C_\bullet(A) \to 0$ of chain complexes. ■

Proposition 4.19. *Let $B \hookrightarrow A \hookrightarrow X$ be a sequence of two topological subspace inclusions. Then there is a long exact sequence of relative singular homology groups of the form*

$$\cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to \cdots \;.$$

Proof. Observe that we have a short exact sequence of chain complexes, def. 2.75

$$0 \to C_\bullet(A)/C_\bullet(B) \to C_\bullet(X)/C_\bullet(B) \to C_\bullet(X)/C_\bullet(A) \to 0 \;.$$

The corresponding homology long exact sequence, prop. 2.78, is the long exact sequence in question. ■

We look at some concrete fundamental examples in a moment. But first it is useful to make explicit the following general sub-notion of relative homology.

Let X still be a given topological space.

Definition 4.20. The **augmentation map** for the singular homology of X is the homomorphism of abelian groups

$$\epsilon : C_0(X) \to \mathbb{Z}$$

which adds up all the coefficients of all 0-chains:

$$\epsilon : : \sum_i n_i \sigma_i \mapsto \sum_i n_i \;.$$

Since the boundary of a 1-chain is in the kernel of this map, by example 1.42, it constitutes a chain map

$$\epsilon : C_\bullet(X) \to \mathbb{Z} \,,$$

where now $\mathbb{Z}$ is regarded as a chain complex concentrated in degree 0.

Definition 4.21. The **reduced singular chain complex** $\tilde{C}_\bullet(X)$ of X is the kernel of the augmentation map, the chain complex sitting in the short exact sequence

$$0 \to \tilde{C}_\bullet(C) \to C_\bullet(X) \overset{\epsilon}{\to} \mathbb{Z} \to 0 \;.$$

The **reduced singular homology** $\tilde{H}_\bullet(X)$ of X is the chain homology of the reduced singular chain complex

$$\tilde{H}_\bullet(X) \coloneqq H_\bullet(\tilde{C}_\bullet(X)) \;.$$

Equivalently:

Definition 4.22. The **reduced singular homology** of X, denoted $\tilde{H}_\bullet(X)$, is the chain homology of the augmented chain complex

$$\cdots \to C_2(X) \overset{\partial_1}{\to} C_1(X) \overset{\partial_0}{\to} C_0(X) \overset{\epsilon}{\to} \mathbb{Z} \to 0 \;.$$

Let X be a topological space, $H_\bullet(X)$ its singular homology and $\tilde{H}_\bullet(X)$ its reduced singular homology, def. 4.21.

Proposition 4.23. *For $n \in \mathbb{N}$ there is an isomorphism*

$$H_n(X) \simeq \begin{cases} \tilde{H}_n(X) & \text{for } n \geq 1 \\ \tilde{H}_0(X) \oplus \mathbb{Z} & \text{for } n = 0 \end{cases}$$

Proof. The homology long exact sequence, prop. 2.78, of the defining short exact sequence $\tilde{C}_\bullet(C) \to C_\bullet(X) \overset{\epsilon}{\to} \mathbb{Z}$ is, since $\mathbb{Z}$ here is concentrated in degree 0, of the form

$$\cdots \to \tilde{H}_n(X) \to H_n(X) \to 0 \to \cdots \to 0 \to \cdots \to \tilde{H}_1(X) \to H_1(X) \to 0 \to \tilde{H}_0(X) \to H_0(X) \overset{\epsilon}{\to} \mathbb{Z} \to 0 \;.$$

Here exactness says that all the morphisms $\tilde{H}_n(X) \to H_n(X)$ for positive n are isomorphisms. Moreover, since $\mathbb{Z}$ is a free abelian group, hence a projective object, the remaining short exact sequence

$$0 \to \tilde{H}_0(X) \to H_0(X) \to \mathbb{Z} \to 0$$

is split, by prop. 2.74, and hence $H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$. ▮

Proposition 4.24. *For $X = *$ the point, the morphism*

$$H_0(\epsilon) : H_0(X) \to \mathbb{Z}$$

is an isomorphism. Accordingly the reduced homology of the point vanishes in every degree:

$$\tilde{H}_\bullet(*) \simeq 0 \;.$$

Proof. By the discussion in section 2) we have that

$$H_n(*) \simeq \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} .$$

Moreover, it is clear that $\epsilon : C_0(*) \to \mathbb{Z}$ is the identity map. ▮

Now we can discuss the relation between reduced homology and relative homology.

Proposition 4.25. *For X an inhabited topological space, its reduced singular homology, def. 4.21, coincides with its relative singular homology relative to any base point $x : * \to X$:*

$$\tilde{H}_\bullet(X) \simeq H_\bullet(X, *) \;.$$

Proof. Consider the sequence of topological subspace inclusions

$$\emptyset \hookrightarrow * \overset{x}{\hookrightarrow} X \;.$$

By prop. 4.19 this induces a long exact sequence of the form

$$\cdots \to H_{n+1}(*) \to H_{n+1}(X) \to H_{n+1}(X, *) \to H_n(*) \to H_n(X) \to H_n(X, *) \to \cdots \to H_1(X) \to H_1(X, *) \to H_0(*) \overset{H_0(x)}{\longrightarrow} H_0(X) \to H_n(X, *) \to 0 \;.$$

Here in positive degrees we have $H_n(*) \simeq 0$ and therefore exactness gives isomorphisms

$$H_n(X) \overset{\simeq}{\to} H_n(X, *) \quad \forall_{n \geq 1}$$

and hence with prop. 4.23 isomorphisms

$$\tilde{H}_n(X) \overset{\simeq}{\to} H_n(X, *) \quad \forall_{n \geq 1} \;.$$

It remains to deal with the case in degree 0. To that end, observe that $H_0(x) : H_0(*) \to H_0(X)$ is a monomorphism: for this notice that we have a commuting diagram

$$\begin{array}{ccc} H_0(*) & \overset{\mathrm{id}}{\to} & H_0(*) \\ {}^{H_0(x)}\downarrow & {}^{H_0(f)}\nearrow & \downarrow^{H_0(\epsilon)}_{\simeq} \\ H_0(X) & \overset{H_0(\epsilon)}{\longrightarrow} & \mathbb{Z} \end{array},$$

where $f : X \to *$ is the terminal map. That the outer square commutes means that $H_0(\epsilon) \circ H_0(x) = H_0(\epsilon)$ and hence the composite on the left is an isomorphism. This implies that $H_0(x)$ is an injection.

Therefore we have a short exact sequence as shown in the top of this diagram

$$\begin{tikzcd} 0 \arrow[r] & H_0(*) \arrow[r, hook, "H_0(x)"] \arrow[dr, "\simeq"'] & H_0(X) \arrow[r] \arrow[d, "H_0(\epsilon)"] & H_0(X,*) \arrow[r] & 0 \\ & & \mathbb{Z} & & \end{tikzcd} .$$

Using this we finally compute

$$\begin{aligned} \tilde{H}_0(X) &\coloneqq \ker H_0(\epsilon) \\ &\simeq \mathrm{coker}(H_0(x)) \; . \\ &\simeq H_0(X,*) \end{aligned}$$

■

With this understanding of homology *relative to a point* in hand, we can now characterize relative homology more generally. From its definition in def. 4.16, it is plausible that the relative homology group $H_n(X,A)$ provides information about the quotient topological space X/A. This is indeed true under mild conditions:

Definition 4.26. A topological subspace inclusion $A \hookrightarrow X$ is called a **good pair** if

1. A is closed inside X;

2. A has an neighbourhood $A \hookrightarrow U \hookrightarrow X$ such that $A \hookrightarrow U$ has a deformation retract.

Proposition 4.27. *If $A \hookrightarrow X$ is a topological subspace inclusion which is good in the sense of def. 4.26, then the A-relative singular homology of X coincides with the reduced singular homology, def. 4.21, of the quotient space X/A:*

$$H_n(X/A) \simeq \tilde{H}_n(X,A) \; .$$

The proof of this is spelled out at *Relative homology – relation to quotient topological spaces*. It needs the proof of the *Excision property* of relative homology. While important, here we will not further dwell on this. The interested reader can find more information behind the above links.

With the general definition of relative homology in hand, we now consider the basic *cells* such that *cell complexes* built from such cells have tractable relative homology groups. Actually, up to weak homotopy equivalence, *every* Hausdorff topological space is given by such a cell complex and hence its relative homology, then called *cellular homology*, is a good tool for computing singular homology rather generally.

Definition 4.28. For $n \in \mathbb{N}$ write

- $D^n \hookrightarrow \mathbb{R}^n \in$ Top for the standard n-disk;

- $S^{n-1} \hookrightarrow \mathbb{R}^n \in$ Top for the standard $(n-1)$-sphere;

  (notice that the 0-sphere is the disjoint union of *two points*, $S^0 = * \coprod *$, and by definition the (-1)-sphere is the empty set)

- $S^{-1} \hookrightarrow D^n$ for the continuous function that includes the $(n-1)$-sphere as the boundary of the n-disk.

Example 4.29. The reduced singular homology of the n-sphere S^n equals the S^{n-1}-relative homology of the n-disk with respect to the canonical boundary inclusion $S^{n-1} \hookrightarrow D^n$: for all $n \in \mathbb{N}$

$$\tilde{H}_\bullet(S^n) \simeq H_\bullet(D^n, S^{n-1}) \; .$$

Proof. The n-sphere is homeomorphic to the n-disk with its entire boundary identified with a point:

$$S^n \simeq D^n / S^{n-1} \; .$$

Moreover the boundary inclusion is a *good pair* in the sense of def. 4.26. Therefore the example follows with prop. 4.27. ■

When forming cell complexes from disks, then each relative dimension will be a wedge sum of disks:

Definition 4.30. For $\{x_i : * \to X_i\}_i$ a set of pointed topological spaces, their *wedge sum* $\vee_i X_i$ is the result of identifying all base points in their disjoint union, hence the quotient

$$\left(\coprod_i X_i\right) / \left(\coprod_i *\right) .$$

Example 4.31. The wedge sum of two pointed circles is the "figure 8"-topological space.

Proposition 4.32. *Let $\{* \to X_i\}_i$ be a set of pointed topological spaces. Write $\vee_i X_i \in \mathrm{Top}$ for their wedge sum and write $\iota_i : X_i \to \vee_i X_i$ for the canonical inclusion functions.*

Then for each $n \in \mathbb{N}$ the homomorphism

$$(\tilde H_n(\iota_i))_i : \oplus_i \tilde H_n(X_i) \to \tilde H_n(\vee_i X_i)$$

is an isomorphism.

Proof. By prop. 4.27 the reduced homology of the wedge sum is equivalently the relative homology of the disjoint union of spaces relative to their disjoint union of basepoints

$$\tilde H_n(\vee_i X_i) \simeq H_n(\coprod_i X_i, \coprod_i *) .$$

The relative homology preserves these coproducts (sends them to direct sums) and so

$$H_n(\coprod_i X_i, \coprod_i *) \simeq \oplus_i H_n(X_i, *) .$$

∎

The following defines topological spaces which are inductively built by gluing disks to each other.

Definition 4.33. A **CW complex of dimension** (-1) is the empty topological space.

By induction, for $n \in \mathbb{N}$ a **CW complex of dimension** n is a topological space X_n obtained from

1. a CW-complex X_{n-1} of dimension $n-1$;
2. an index set $\mathrm{Cell}(X)_n \in \mathrm{Set}$;
3. a set of continuous maps (the **attaching maps**) $\{f_i : S^{n-1} \to X_{n-1}\}_{i \in \mathrm{Cell}(X)_n}$

as the pushout

$$X_n \simeq \Big(\coprod_{j \in \mathrm{Cell}(X)_n} D^n \Big) \coprod_{j \in \mathrm{Cell}(X)_n S^{n-1}} X_n$$

in

$$\begin{array}{ccc} \coprod_{j \in \mathrm{Cell}(X)_n} S^{n-1} & \overset{(f_j)}{\longrightarrow} & X_{n-1} \\ \downarrow && \downarrow \\ \coprod_{j \in \mathrm{Cell}(X)_n} D^n & \to & X_n \end{array} ,$$

hence as the topological space obtained from X_{n-1} by gluing in n-disks D^n for each $j \in \mathrm{Cell}(X)_n$ along the given boundary inclusion $f_j : S^{n-1} \to X_{n-1}$.

By this construction, an n-dimensional CW-complex is canonically a *filtered topological space*, hence a sequence of topological subspace inclusions of the form

$$\emptyset \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_{n-1} \hookrightarrow X_n$$

which are the right vertical morphisms in the above pushout diagrams.

A general **CW complex** X then is a topological space which is the limiting space of a possibly infinite such sequence, hence a topological space given as the sequential colimit over a tower diagram each of whose morphisms is such a filter inclusion

$$\emptyset \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X .$$

The following basic facts about the singular homology of CW complexes are important.

Now we can state a variant of singular homology adapted to CW complexes which admits a more systematic way of computing its homology groups. First we observe the following.

Proposition 4.34. *The relative singular homology, def. 4.16, of the filtering degrees of a CW complex X, def. 4.33, is*

$$H_n(X_k, X_{k-1}) \simeq \begin{cases} \mathbb{Z}[\mathrm{Cells}(X)_n] & \text{if } k = n \\ 0 & \text{otherwise} \end{cases} ,$$

where $\mathbb{Z}[\mathrm{Cells}(X)_n]$ denotes the free abelian group on the set of n-cells.

Proof. The inclusion $X_{k-1} \hookrightarrow X_k$ is a *good pair* in the sense of def. 4.26. The quotient X_k / X_{k-1} is by definition

of CW-complexes a wedge sum, def. 4.30, of k-spheres, one for each element in $\mathrm{Cell}(X)_k$. Therefore by prop. 4.27 we have an isomorphism $H_n(X_k, X_{k-1}) \simeq \tilde H_n(X_k/X_{k-1})$ with the reduced homology of this wedge sum. The statement then follows by the respect of reduced homology for wedge sums, prop. 4.32. ∎

Proposition 4.35. *For X a CW complex with skeletal filtration $\{X_n\}_n$ as above, and with $k,n \in \mathbb{N}$ we have for the singular homology of X that*

$$(k > n) \Rightarrow (H_k(X_n) \simeq 0)\,.$$

In particular if X is a CW-complex of finite dimension $\dim X$ (the maximum degree of cells), then

$$(k > \dim X) \Rightarrow (H_k(X) \simeq 0)\,.$$

Moreover, for $k < n$ the inclusion

$$H_k(X_n) \overset{\simeq}{\to} H_k(X)$$

is an isomorphism and for $k = n$ we have an isomorphism

$$\mathrm{image}(H_n(X_n) \to H_n(X)) \simeq H_n(X)\,.$$

Proof. By the long exact sequence in relative homology, prop. 4.18 we have an exact sequence of the form

$$H_{k+1}(X_n, X_{n-1}) \to H_k(X_{n-1}) \to H_k(X_n) \to H_k(X_n, X_{n-1})\,.$$

Now by prop. 4.34 the leftmost and rightmost homology groups here vanish when $k \neq n$ and $k \neq n-1$ and hence exactness implies that

$$H_k(X_{n-1}) \overset{\simeq}{\to} H_k(X_n)$$

is an isomorphism for $k \neq n, n-1$. This implies the first claims by induction on n.

Finally for the last claim use that the above exact sequence gives

$$H_{n-1+1}(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}) \to H_{n-1}(X_n) \to 0$$

and hence that with the above the map $H_{n-1}(X_{n-1}) \to H_{n-1}(X)$ is surjective. ∎

We can now discuss the *cellular homology* of a CW complex.

Definition 4.36. For X a CW-complex, def. 4.33, its **cellular chain complex** $H_\bullet^{\mathrm{CW}}(X) \in \mathrm{Ch}_\bullet$ is the chain complex such that for $n \in \mathbb{N}$

- the abelian group of chains is the relative singular homology group, def. 4.16, of $X_n \hookrightarrow X$ relative to $X_{n-1} \hookrightarrow X$:

$$H_n^{\mathrm{CW}}(X) \coloneqq H_n(X_n, X_{n-1})\,,$$

- the differential $\partial_{n+1}^{\mathrm{CW}} : H_{n+1}^{\mathrm{CW}}(X) \to H_n^{\mathrm{CW}}(X)$ is the composition

$$\partial_n^{\mathrm{CW}} : H_{n+1}(X_{n+1}, X_n) \overset{\partial_n}{\to} H_n(X_n) \overset{i_n}{\to} H_n(X_n, X_{n-1})\,,$$

  where ∂_n is the boundary map of the singular chain complex and where i_n is the morphism on relative homology induced from the canonical inclusion of pairs $(X_n, \emptyset) \to (X_n, X_{n-1})$.

Proposition 4.37. *The composition $\partial_n^{\mathrm{CW}} \circ \partial_{n+1}^{\mathrm{CW}}$ of two differentials in def. 4.36 is indeed zero, hence $H_\bullet^{\mathrm{CW}}(X)$ is indeed a chain complex.*

Proof. On representative singular chains the morphism i_n acts as the identity and hence $\partial_n^{\mathrm{CW}} \circ \partial_{n+1}^{\mathrm{CW}}$ acts as the double singular boundary, $\partial_n \circ \partial_{n+1} = 0$. ∎

Remark 4.38. This means that

- a **cellular n-chain** is a singular n-chain required to sit in filtering degree n, hence in $X_n \hookrightarrow X$;
- a **cellular n-cycle** is a singular n-chain whose singular boundary is not necessarily 0, but is contained in filtering degree $(n-2)$, hence in $X_{n-2} \hookrightarrow X$.
- a **cellular n-boundary** is a singular n-chain which is the boundary of a singular $(n+1)$-chain coming from filtering degree $(n+1)$.

This kind of situation – chains that are cycles only up to lower filtering degree and boundaries that come from specified higher filtering degree – has an evident generalization to higher relative filtering degrees. And in this greater generality the concept is of great practical relevance. Therefore before discussing cellular homology further now, we consider this more general "higher-order relative homology" that it suggests (namely the formalism of spectral sequences). After establishing a few fundamental facts about that we will come back in prop. 4.61 below to analyse the above cellular situation using this conceptual tool.

First we abstract the structure on chain complexes that in the above example was induced by the CW-complex structure on the singular chain complex.

Definition 4.39. The structure of a **filtered chain complex** in a chain complex $C_\bullet$ is a sequence of chain map inclusions

$$\cdots \hookrightarrow F_{p-1}C_\bullet \hookrightarrow F_p C_\bullet \hookrightarrow \cdots \hookrightarrow C_\bullet \,.$$

The **associated graded complex** of a filtered chain complex, denoted $G_\bullet C_\bullet$, is the collection of quotient chain complexes

$$G_p C_\bullet \coloneqq F_p C_\bullet / F_{p-1} C_\bullet \,.$$

We say that element of $G_p C_\bullet$ are *in filtering degree* p.

Remark 4.40. In more detail this means that

1. $[\cdots \overset{\partial_n}{\to} C_n \xrightarrow{\partial_{n-1}} C_{n-1} \to \cdots]$ is a chain complex, hence $\{C_n\}$ are objects in $\mathcal{A}$ (R-modules) and $\{\partial_n\}$ are morphisms (module homomorphisms) with $\partial_{n+1} \circ \partial_n = 0$;

2. For each $n \in \mathbb{Z}$ there is a filtering $F_\bullet C_n$ on C_n and all these filterings are compatible with the differentials in that

$$\partial(F_p C_n) \subset F_p C_{n-1}$$

3. The grading associated to the filtering is such that the p-graded elements are those in the quotient

$$G_p C_n \coloneqq \frac{F_p C_n}{F_{p-1} C_n} \,.$$

   Since the differentials respect the grading we have chain complexes $G_p C_\bullet$ in each filtering degree p.

Hence elements in a filtered chain complex are **bi-graded**: they carry a degree as elements of $C_\bullet$ as usual, but now they also carry a filtering degree: for $p, q \in \mathbb{Z}$ we therefore also write

$$C_{p,q} \coloneqq F_p C_{p+q}$$

and call this the collection of (p,q)**-chains** in the filtered chain complex.

Accordingly we have (p,q)-cycles and -boundaries. But for these we may furthermore refine to a notion where also the filtering degree of the boundaries is constrained:

Definition 4.41. Let $F_\bullet C_\bullet$ be a filtered chain complex. Its associated graded chain complex is the set of chain complexes

$$G_p C_\bullet \coloneqq F_p C_\bullet / F_{p-1} C_\bullet$$

for all p.

Then for $r, p, q \in \mathbb{Z}$ we say that

1. $G_p C_{p+q}$ is the module of (p,q)***-chains*** or of $(p+q)$***-chains in filtering degree*** p;

2. $$\begin{aligned} Z^r_{p,q} &\coloneqq \{c \in G_p C_{p+q} \mid \partial c = 0 \mod F_{p-r} C_\bullet\} \\ &= \{c \in F_p C_{p+q} \mid \partial(c) \in F_{p-r} C_{p+q-1}\} / F_{p-1} C_{p+q} \end{aligned}$$

   is the module of r***-almost*** (p,q)***-cycles*** (the $(p+q)$-chains whose differential vanishes modulo terms of filtering degree $p-r$);

3. $B^r_{p,q} \coloneqq \partial(F_{p+r-1} C_{p+q+1})$,

   is the module of r**-almost** (p,q)**-boundaries**.

Similarly we set

$$\begin{aligned} Z^\infty_{p,q} &\coloneqq \{c \in F_p C_{p+q} \mid \partial c = 0\} / F_{p-1} C_{p+q} = Z(G_p C_{p+q}) \\ B^\infty_{p,q} &\coloneqq \partial(F_p C_{p+q+1}) \,. \end{aligned}$$

From this definition we immediately have that the differentials $\partial : C_{p+q} \to C_{p+q-1}$ restrict to the r-almost cycles as follows:

Proposition 4.42. *The differentials of $C_\bullet$ restrict on r-almost cycles to homomorphisms of the form*

$$\partial^r : Z^r_{p,q} \to Z^r_{p-r, q+r-1} \,.$$

These are still differentials: $\partial^2 = 0$.

Proof. By the very definition of $Z^r_{p,q}$ it consists of elements in filtering degree p on which ∂ decreases the filtering degree to $p-r$. Also by definition of differential on a chain complex, ∂ decreases the actual degree $p+q$ by one. This explains that ∂ restricted to $Z^r_{p,q}$ lands in $Z^\bullet_{p-r,q+r-1}$. Now the image constists indeed of actual boundaries, not just r-almost boundaries. But since actual boundaries are in particular r-almost boundaries, we may take the codomain to be $Z^r_{p-r,q+r-1}$. ∎

As before, we will in general index these differentials by their codomain and hence write in more detail

$$\partial^r : Z^r_{p,q} \to Z^r_{p-r,q+r-1} \,.$$

Proposition 4.43. *We have a sequence of canonical inclusions*

$$B^0_{p,q} \hookrightarrow B^1_{p,q} \hookrightarrow \cdots B^\infty_{p,q} \hookrightarrow Z^\infty_{p,q} \hookrightarrow \cdots \hookrightarrow Z^1_{p,q} \hookrightarrow Z^0_{p,q} \,.$$

The following observation is elementary, and yet this is what drives the theory of spectral sequences, as it shows that almost cycles may be computed iteratively by homological means themselves.

Proposition 4.44. *The $(r+1)$-almost cycles are the ∂^r-kernel inside the r-almost cycles:*

$$Z^{r+1}_{p,q} \simeq \ker(Z^r_{p,q} \stackrel{\partial^r}{\to} Z^r_{p-r,q+r-1}) \,.$$

Proof. An element $c \in F_p C_{p+q}$ represents

1. an element in $Z^r_{p,q}$ if $\partial c \in F_{p-r} C_{p+q-1}$
2. an element in $Z^{r+1}_{p,q}$ if even $\partial c \in F_{p-r-1} C_{p+q-1} \hookrightarrow F_{p-r} C_{p+q-1}$.

The second condition is equivalent to ∂c representing the 0-element in the quotient $F_{p-r} C_{p+q-1} / F_{p-r-1} C_{p+q-1}$. But this is in turn equivalent to ∂c being 0 in $Z^r_{p-r,q+r-1} \subset F_{p-r} C_{p+q-1} / F_{p-r-1} C_{p+q-1}$. ∎

With a definition of almost-cycles and almost-boundaries, of course we are now interested in the corresponding homology groups:

Definition 4.45. For $r,p,q \in \mathbb{Z}$ define the **r-almost (p,q)-chain homology** of the filtered complex to be the quotient of the r-almost (p,q)-cycles by the r-almost (p,q)-boundaries, def. 4.41:

$$\begin{aligned} E^r_{p,q} &\coloneqq \frac{Z^r_{p,q}}{B^r_{p,q}} \\ &= \frac{\{x \in F_p C_{p+q} \mid \partial x \in F_{p-r} C_{p+q-1}\}}{\partial(F_{p+r-1} C_{p+q+1}) \oplus F_{p-1} C_{p+q}} \end{aligned}$$

By prop. 4.42 the differentials of $C_\bullet$ restrict on the r-almost homology groups to maps

$$\partial^r : E^r_{p,q} \to E^r_{p-r,q+r-1} \,.$$

The central property of these r-almost homology groups now is their following iterative homological characterization.

Proposition 4.46. *With definition 4.45 we have that $E^{r+1}_{\bullet,\bullet}$ is the ∂^r-chain homology of $E^r_{\bullet,\bullet}$:*

$$E^{r+1}_{p,q} = \frac{\ker(\partial^r : E^r_{p,q} \to E^r_{p-r,q+r-1})}{\mathrm{im}(\partial^r : E^r_{p+r,q-r+1} \to E^r_{p,q})} \,.$$

Proof. By prop. 4.44. ∎

This structure on the collection of r-almost cycles of a filtered chain complex thus obtained is called a *spectral sequence*:

Definition 4.47. A **spectral sequence** of R-modules is

1. a set $\{E^r_{p,q}\}_{p,q,r \in \mathbb{Z}}$ of R-modules;
2. a set $\{\partial^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\}_{r,p,q \in \mathbb{Z}}$ of homomorphisms

such that

1. the ∂^rs are differentials: $\forall_{p,q,r} (\partial^r_{p-r,q+r-1} \circ \partial^r_{p,q} = 0)$;
2. the modules $E^{r+1}_{p,q}$ are the ∂^r-homology of the modules in relative degree r:

$$\forall_{r,p,q} \left(E^{r+1}_{p,q} \simeq \frac{\ker(\partial^r_{p-r,q+r-1})}{\mathrm{im}(\partial^r_{p,q})} \right) .$$

One says that $E^r_{\bullet,\bullet}$ is the **r-page** of the spectral sequence.

Since this turns out to be a useful structure to make explicit, as the above motivation should already

indicate, one introduces the following terminology and basic facts to talk about spectral sequences.

Definition 4.48. Let $\{E^r_{p,q}\}_{r,p,q}$ be a spectral sequence, def. 4.47, such that for each p,q there is $r(p,q)$ such that for all $r \geq r(p,q)$ we have

$$E^{r \geq r(p,q)}_{p,q} \simeq E^{r(p,q)}_{p,q} \; .$$

Then one says that

1. the bigraded object

$$E^\infty \coloneqq \{E^\infty_{p,q}\}_{p,q} \coloneqq \{E^{r(p,q)}_{p,q}\}_{p,q}$$

   is the **limit term** of the spectral sequence;

- the spectral sequence **abuts** to E^∞.

Example 4.49. If for a spectral sequence there is r_s such that all differentials on pages after r_s vanish, $\partial^{r \geq r_s} = 0$, then $\{E^{r_s}\}_{p,q}$ is a limit term for the spectral sequence. One says in this cases that the spectral sequence **degenerates** at r_s.

Proof. By the defining relation

$$E^{r+1}_{p,q} \simeq \ker(\partial^r_{p-r,q+r-1})/\mathrm{im}(\partial^r_{p,q}) = E^r_{\mathrm{pq}}$$

the spectral sequence becomes constant in r from r_s on if all the differentials vanish, so that $\ker(\partial^r_{p,q}) = E^r_{p,q}$ for all p,q. ▮

Example 4.50. If for a spectral sequence $\{E^r_{p,q}\}_{r,p,q}$ there is $r_s \geq 2$ such that the r_sth page is concentrated in a single row or a single column, then the spectral sequence degenerates on this pages, example 4.49, hence this page is a limit term, def. 4.48. One says in this case that the spectral sequence **collapses** on this page.

Proof. For $r \geq 2$ the differentials of the spectral sequence

$$\partial^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

have domain and codomain necessarily in different rows an columns (while for $r=1$ both are in the same row and for $r=0$ both coincide). Therefore if all but one row or column vanish, then all these differentials vanish. ▮

Definition 4.51. A spectral sequence $\{E^r_{p,q}\}_{r,p,q}$ is said to **converge** to a graded object $H_\bullet$ with filtering $F_\bullet H_\bullet$, traditionally denoted

$$E^r_{p,q} \Rightarrow H_\bullet \,,$$

if the associated graded complex $\{G_p H_{p+q}\}_{p,q} \coloneqq \{F_p H_{p+q}/F_{p-1}H_{p+q}\}$ of H is the limit term of E, def. 4.48:

$$E^\infty_{p,q} \simeq G_p H_{p+q} \qquad \forall_{p,q} \; .$$

Remark 4.52. In practice spectral sequences are often referred to via their first non-trivial page, often also the page at which it collapses, def. 4.50, often already the second page. Then one tends to use notation such as

$$E^2_{p,q} \Rightarrow H_\bullet$$

to be read as "There is a spectral sequence whose second page is as shown on the left and which converges to a filtered object as shown on the right."

Definition 4.53. A spectral sequence $\{E^r_{p,q}\}$ is called a **bounded spectral sequence** if for all $n,r \in \mathbb{Z}$ the number of non-vanishing terms of total degree n, hence of the form $E^r_{k,n-k}$, is finite.

Example 4.54. A spectral sequence $\{E^r_{p,q}\}$ is called

- a **first quadrant spectral sequence** if all terms except possibly for $p,q \geq 0$ vanish;
- a **third quadrant spectral sequence** if all terms except possibly for $p,q \leq 0$ vanish.

Such spectral sequences are bounded, def. 4.53.

Proposition 4.55. *A bounded spectral sequence, def. 4.53, has a limit term, def. 4.48.*

Proof. First notice that if a spectral sequence has at most N non-vanishing terms of total degree n on page r, then all the following pages have at most at these positions non-vanishing terms, too, since these are the homologies of the previous terms.

Therefore for a bounded spectral sequence for each n there is $L(n) \in \mathbb{Z}$ such that $E^r_{p,n-p} = 0$ for all $p \leq L(n)$ and all r. Similarly there is $T(n) \in \mathbb{Z}$ such $E^r_{n-q,q} = 0$ for all $q \leq T(n)$ and all r.

We claim then that the limit term of the bounded spectral sequence is in position (p,q) given by the value $E^r_{p,q}$ for

$$r > \max(p - L(p+q-1), q + 1 - L(p+q+1)) \,.$$

This is because for such r we have

1. $E^r_{p-r,q+r-1} = 0$ because $p - r < L(p+q-1)$, and hence the kernel $\ker(\partial^r_{p-r,q+r-1}) = 0$ vanishes;

2. $E^r_{p+r,q-r+1} = 0$ because $q - r + 1 < T(p+q+1)$, and hence the image $\mathrm{im}(\partial^r_{p,q}) = 0$ vanishes.

Therefore

$$\begin{aligned} E^{r+1}_{p,q} & = \ker(\partial^r_{p-r,q+r-1}) / \mathrm{im}(\partial^r_{p,q}) \\ & \simeq E^r_{p,q}/0 \\ & \simeq E^r_{p,q} \end{aligned} \,.$$

∎

The central statement about the notion of the spectral sequence of a filtered chain complex then is the following proposition. It says that the iterative computation of higher order relative homology indeed in the limit computes the genuine homology.

Definition 4.56. For $F_\bullet C_\bullet$ a filtered complex, write for $p \in \mathbb{Z}$

$$F_p H_\bullet(C) \coloneqq \mathrm{image}(H_\bullet(F_p C) \to H_\bullet(C)) \,.$$

This defines a filtering $F_\bullet H_\bullet(C)$ of the homology, regarded as a graded object.

Proposition 4.57. *If the spectral sequence of a filtered complex $F_\bullet C_\bullet$ of prop. 4.46 has a limit term, def. 4.48 then it converges, def. 4.51, to the chain homology of $C_\bullet$*

$$E^r_{p,q} \Rightarrow H_{p+q}(C_\bullet) \,,$$

i.e. for sufficiently large r we have

$$E^r_{p,q} \simeq G_p H_{p+q}(C) \,,$$

where on the right we have the associated graded object of the filtering of def. 4.56.

Proof. By assumption, there is for each p,q an $r(p,q)$ such that for all $r \geq r(p,q)$ the r-almost cycles and r-almost boundaries, def. 4.41, in $F_p C_{p+q}$ are the ordinary cycles and boundaries. Therefore for $r \geq r(p,q)$ def. 4.45 gives $E^r_{p,q} \simeq G_p H_{p+q}(C)$. ∎

This says what these spectral sequences are converging to. For computations it is also important to know how they start out for low r. We can generally characterize $E^r_{p,q}$ for very low values of r simply as follows:

Proposition 4.58. *We have*

- $E^0_{p,q} = G_p C_{p+q} = F_p C_{p+q} / F_{p-1} C_{p+q}$

  is the associated p-graded piece of C_{p+q};

- $E^1_{p,q} = H_{p+q}(G_p C_\bullet)$

Proof. For $r = 0$ def. 4.45 restricts to

$$E^0_{p,q} = \frac{F_p C_{p+q}}{F_{p-1} C_{p+q}} = G_p C_{p+q}$$

because for $c \in F_p C_{p+q}$ we automatically also have $\partial c \in F_p C_{p+q}$ since the differential respects the filtering degree by assumption.

For $r = 1$ def. 4.45 gives

$$E^1_{p,q} = \frac{\{c \in G_p C_{p+q} \,|\, \partial c = 0 \in G_p C_{p+q}\}}{\partial(F_p C_{p+q})} = H_{p+q}(G_p C_\bullet) \,.$$

∎

Remark 4.59. There is, in general, a decisive difference between the homology of the associated graded complex $H_{p+q}(G_p C_\bullet)$ and the associated graded piece of the genuine homology $G_p H_{p+q}(C_\bullet)$: in the former the differentials of cycles are required to vanish only up to terms in lower degree, but in the latter they are required to vanish genuinely. The latter expression is instead the value of the spectral sequence for $r \to \infty$, see prop. 4.57 below.

These general facts now allow us, as a first simple example for the application of spectral sequences to see

transparently that the cellular homology of a CW complex, def. 4.36, coincides with its genuine singular homology.

First notice that of course the structure of a CW-complex on a topological space X, def. 4.33 naturally induces on its singular simplicial complex $C_\bullet(X)$ the structure of a filtered chain complex, def. 4.39:

Definition 4.60. For $X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X$ a CW complex, and $p \in \mathbb{N}$, write

$$F_p C_\bullet(X) \coloneqq C_\bullet(X_p)$$

for the singular chain complex of $X_p \hookrightarrow X$. The given topological subspace inclusions $X_p \hookrightarrow X_{p+1}$ induce chain map inclusions $F_p C_\bullet(X) \hookrightarrow F_{p+1} C_\bullet(X)$ and these equip the singular chain complex $C_\bullet(X)$ of X with the structure of a bounded filtered chain complex

$$0 \hookrightarrow F_0 C_\bullet(X) \hookrightarrow F_1 C_\bullet(X) \hookrightarrow F_2 C_\bullet(X) \hookrightarrow \cdots \hookrightarrow F_\infty C_\bullet(X) \coloneqq C_\bullet(X) \;.$$

(If X is of finite dimension $\dim X$ then this is a bounded filtration.)

Write $\{E^r_{p,q}(X)\}$ for the spectral sequence of a filtered complex corresponding to this filtering.

Proposition 4.61. *The spectral sequence $\{E^r_{p,q}(X)\}$ of singular chains in a CW complex X, def. 4.60 converges, def. 4.51, to the singular homology of X:*

$$E^r_{p,q}(X) \Rightarrow H_\bullet(X) \;.$$

Proof. The spectral sequence $\{E^r_{p,q}(X)\}$ is clearly a first-quadrant spectral sequence, def. 4.54. Therefore it is a bounded spectral sequence, def. 4.53 and hence has a limit term, def. 4.55. So the statement follows with prop. 4.57. ▮

We now identify the low-degree pages of $\{E^r_{p,q}(X)\}$ with structures in singular homology theory.

Proposition 4.62.

- $r = 0$ – $E^0_{p,q}(X) \simeq C_{p+q}(X_p)/C_{p+q}(X_{p-1})$ *is the group of X_{p-1}-relative (p+q)-chains, def. 4.16, in X_p;*
- $r = 1$ – $E^1_{p,q}(X) \simeq H_{p+q}(X_p, X_{p-1})$ *is the X_{p-1}-relative singular homology, def. 4.16, of X_p;*
- $r = 2$ – $E^2_{p,q}(X) \simeq \begin{cases} H^{\mathrm{CW}}_p(X) & \text{for } q = 0 \\ 0 & \text{otherwise} \end{cases}$
- $r = \infty$ – $E^\infty_{p,q}(X) \simeq F_p H_{p+q}(X)/F_{p-1} H_{p+q}(X)$.

Proof. By straightforward and immediate analysis of the definitions. ▮

As a result of these general considerations we now obtain the promised isomorphism between the cellular homology and the singular homology of a CW-complex X:

Corollary 4.63. *For $X \in$ Top a CW complex, def. 4.33, its cellular homology, def. 4.36 $H^{\mathrm{CW}}_\bullet(X)$ coincides with its singular homology $H_\bullet(X)$, def. 1.48:*

$$H^{\mathrm{CW}}_\bullet(X) \simeq H_\bullet(X) \;.$$

Proof. By the third item of prop. 4.62 the $(r = 2)$-page of the spectral sequence $\{E^r_{p,q}(X)\}$ is concentrated in the $(q = 0)$-row and hence it collapses there, def. 4.50. Accordingly we have

$$E^\infty_{p,q}(X) \simeq E^2_{p,q}(X)$$

for all p, q. By the third and fourth item of prop. 4.62 this non-trivial only for $q = 0$ and there it is equivalently

$$G_p H_p(X) \simeq H^{\mathrm{CW}}_p(X) \;.$$

Finally observe that $G_p H_p(X) \simeq H_p(X)$ by the definition of the filtering on the homology, def. 4.56, and using prop. 4.35. ▮

This concludes our discussion of how relative homology theory of cellularly filtered objects allows to efficiently compute the genuine homology of these objects, and how this motivates the general concept of spectral sequences as organizing higher order relative homology groups. In the next section we consider an important special special class of filtered objects – the total complexes of double complexes – and apply these tools to analyze them.

13) Total complexes of double complexes

In 6) we had discussed basic properties of double complexes. A central aspect of double complexes is that by a kind fo amalgamation they induce an ordinary chain complex, called their total complex. The conceptual relevance of this construction rests in the fact, which we indate below in V, that a double complex is a *diagram of complex* and its total complex is the correspoding homotopy colimit of this diagram,

hence the universal way of "gluing" the rows in the double complex to a single complex. Here we just focus on examples of the explicit construction and analyse the homology of total chain complexes using the tool of spectral sequences introduced above.

Definition 4.64. For $C_{\bullet,\bullet} \in \mathrm{Ch}_\bullet(\mathrm{Ch}_\bullet(R\,\mathrm{Mod}))$ a double chain complex, the corresponding **total chain complex** is the chain complex $\mathrm{Tot}(C)_\bullet \in \mathrm{Ch}_\bullet(R\,\mathrm{Mod})$ whose degree-n module is the direct sum of all entries of *total* degree n:

$$\mathrm{Tot}(C)_n \coloneqq \oplus_{n_1+n_2=n} C_{n_1,n_2}\,,$$

and whose differential is the sum, with column-degree-weighted sign, of the horizontal and vertical differentials of the double complex, hence on a direct summand C_{n_1,n_2} given by

$$\partial^{\mathrm{Tot}} \coloneqq \partial^C_{\mathrm{hor}} + (-1)^p \partial^C_{\mathrm{vert}}\ .$$

One important example of this we have already seen.

Example 4.65. Let $X_\bullet, Y_\bullet \in \mathrm{Ch}_\bullet(R\,\mathrm{Mod})$ be two chain complexes. Write $X_\bullet \otimes_R Y_\bullet$ for the double complex which in degree (n_1,n_2) is the tensor product of modules $X_{n_1} \otimes_R Y_{n_2}$, whose horizontal differential is $\partial^X \otimes \mathrm{id}_Y$ and whose vertical differential is $\mathrm{id}_X \otimes \partial^Y$.

Then the corresponding total complex $\mathrm{Tot}(X_\bullet \otimes_R Y_\bullet)_\bullet$, def. 4.64, is the tensor product of chain complexes $(X \otimes Y)_\bullet$ of def. 2.59:

$$(X \otimes_R Y)_\bullet \simeq \mathrm{Tot}(X_\bullet \otimes_R Y_\bullet)_\bullet\ .$$

Proposition 4.66. *The total complex $\mathrm{Tot}(C)_\bullet$ of a double complex $C_{\bullet,\bullet}$, def. 4.64, becomes a filtered chain complex, def. 4.39 either by filtering by row degree*

$$F_p^{\mathrm{hor}}\,\mathrm{Tot}(C)_n \coloneqq \bigoplus_{\substack{n_1+n_2=n \\ n_1 \leq p}} C_{n_1,n_2}$$

or by column degree

$$F_p^{\mathrm{vert}}\,\mathrm{Tot}(C)_n \coloneqq \bigoplus_{\substack{n_1+n_2=n \\ n_2 \leq p}} C_{n_1,n_2}\ .$$

The spectral sequence of a filtered complex induced by either F^{hor} or F^{vert} on the total complex of a double complex is accordingly called the *spectral sequence of a double complex*.

Proposition 4.67. *Let $\{E^r_{p,q}\}_{r,p,q}$ be the spectral sequence of a double complex $C_{\bullet,\bullet}$, according to def. \ref{SpectralSequenceOfDoubleComplex}, with respect to the horizontal filtration. Then the first few pages are for all $p,q \in \mathbb{Z}$ given by*

- $E^0_{p,q} \simeq C_{p,q}$;
- $E^1_{p,q} \simeq H_q(C_{p,\bullet})$;
- $E^2_{p,q} \simeq H_p(H_q^{\mathrm{vert}}(C))$.

Moreover, if $C_{\bullet,\bullet}$ is concentrated in the first quadrant ($0 \leq p,q$), then the spectral sequence converges to the chain homology of the total complex:

$$E^\infty_{p,q} \simeq G_p H_{p+q}(\mathrm{Tot}(C)_\bullet)\ .$$

Proof. This is a matter of unwinding the definition, using prop. 4.58. We display equations for the horizontal filtering, the other case works analogously.

The 0th page is by definition the associated graded piece

$$\begin{aligned} E^0_{p,q} &\coloneqq G_p\,\mathrm{Tot}(C)_{p+q} \\ &\coloneqq F_p\,\mathrm{Tot}(C)_{p+q} / F_{p-1}\,\mathrm{Tot}(C)_{p+q} \\ &\coloneqq \frac{\oplus_{\substack{n_1+n_2=p+q \\ n_1 \leq p}} C_{n_1,n_2}}{\oplus_{\substack{n_1+n_2=p+q \\ n_1 < p}} C_{n_1,n_2}} \\ &\simeq C_{p,q}\ . \end{aligned}$$

The first page is the chain homology of the associated graded chain complex:

$$\begin{aligned} E^1_{p,q} &\simeq H_{p+q}(G_p\,\mathrm{Tot}(C)_\bullet) \\ &\simeq H_{p+q}(C_{p,\bullet}) \\ &\simeq H_q(C_{p,\bullet}) \end{aligned}\ .$$

In particular this means that representatives of $[c] \in E^1_{p,q}$ are given by $c \in C_{p,q}$ such that $\partial^{\mathrm{vert}} c = 0$. It follows that $\partial^1 : E^1_{p,q} \to E^1_{p-1,q}$, which by the definition of a total complex acts as $\partial^{\mathrm{hor}} \pm \partial^{\mathrm{vert}}$, acts on these representatives just as ∂^{hor} and this gives the second page

$$E^2_{p,q} \simeq \ker(\partial^1_{p-1,q}) / \mathrm{im}(\partial^1_{p,q}) \simeq H_p(H^{\mathrm{vert}}_q(C_{\bullet,\bullet})) \,.$$

Finally, for $C_{\bullet,\bullet}$ concentrated in $0 \leq p,q$ the corresponding filtered chain complex $F_p \,\mathrm{Tot}(C)_\bullet$ is a non-negatively graded chain complex with filtration bounded below. Therefore the spectral sequence converges as claimed by prop. 4.57. ∎

As a first example application we can tie up a loose end of section 10 b) (remark 3.102): we show that forming the derived functor of the tensor product in the first argument yields the same result as deriving in the second argument.

Proposition 4.68. *Let R be a commutative ring. For $A, B \in R$Mod, the two ways of computing the Tor left derived functor coincide*

$$(L_n((-) \otimes_R B))(A) \simeq (L_n(A \otimes_R (-)))(B)$$

and hence we can consistently write $\mathrm{Tor}_n(A,B)$ for either.

Proof. Let $Q^A_\bullet \overset{\simeq \mathrm{qi}}{\longrightarrow} A$ and $Q^B_\bullet \overset{\simeq \mathrm{qi}}{\longrightarrow} B$ be projective resolutions of A and B, respectively, def. 3.25. The corresponding tensor product of chain complexes $\mathrm{Tot}(Q^A_\bullet \otimes Q^B_\bullet)$, hence by prop. \ref{AsTotalComplex} the total complex of the degreewise tensor product of modules double complex carries the filtration by horizontal degree as well as that by vertical degree.

Accordingly there are the corresponding two spectral sequences of a double complex, to be denoted here $\{{}^A E^r_{p,q}\}_{r,p,q}$ (for the filtering by A-degree) and $\{{}^B E^r_{p,q}\}_{r,p,q}$ (for the filtering by B-degree). By the discussion there, both converge to the chain homology of the total complex.

We find the value of both spectral sequences on low degree pages according to prop. 4.67:

The 0th page for both is

$${}^A E^0_{p,q} = {}^B E^0_{p,q} \coloneqq Q^A_p \otimes_R Q^B_q \,.$$

For the first page we have

$$\begin{aligned} {}^A E^1_{p,q} & \simeq H_q(C_{p,\bullet}) \\ & \simeq H_q(Q^A_p \otimes Q^B_\bullet) \end{aligned}$$

and

$$\begin{aligned} {}^B E^1_{p,q} & \simeq H_q(C_{\bullet,p}) \\ & \simeq H_q(Q^A_\bullet \otimes Q^B_p) \end{aligned} \,.$$

Now using the universal coefficient theorem in homology, theorem 4.6, and the fact that $Q^A_\bullet$ and $Q^B_\bullet$ is a resolution by projective objects, by construction, hence of tensor acyclic objects for which all Tor-modules vanish, this simplifies to

$$\begin{aligned} {}^A E^1_{p,q} & \simeq Q^A_p \otimes H_q(Q^B_\bullet) \\ & \simeq \begin{cases} Q^A_p \otimes_R B & \text{if}\ q = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and similarly

$$\begin{aligned} {}^B E^1_{p,q} & \simeq H_q(Q^A_\bullet) \otimes_R Q^B_p \\ & \simeq \begin{cases} A \otimes_R Q^B_p & \text{if}\ q = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \,.$$

It follows for the second pages that

$$\begin{aligned} {}^A E^2_{p,q} & \simeq H_p(H^{\mathrm{vert}}_q(Q^A_\bullet \otimes Q^B_\bullet)) \\ & \simeq \begin{cases} (L_p((-) \otimes_R B))(A) & \text{if}\ q = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} {}^B E^2_{p,q} & \simeq H_p(H^{\mathrm{hor}}_q(Q^A_\bullet \otimes Q^B_\bullet)) \\ & \simeq \begin{cases} (L_p(A \otimes_R (-)))(B) & \text{if}\ q = 0 \\ 0 \ \text{otherwise} & \end{cases} \end{aligned} \,.$$

Now both of these second pages are concentrated in a single row and hence have converged on that page already. Therefore, since they both converge to the same value:

$$L_p((-) \otimes_R B)(A) \simeq {}^A E^2_{p,0} \simeq {}^A E^\infty_{p,0} \simeq {}^B E^2_{p,0} \simeq L_p(A \otimes_R (-))(B) \; .$$

∎

The total complexes of double complexes are ubiquituous in homological algebra for a general abstract reason, and hence so are their spectral sequences. Accordingly there are many names for many spectral sequences of particular filtered and notably of total complexes. The interested reader may find further pointers at *Spectral sequences - list of examples*.

5. V) Outlook

It turns out that the chain complexes in homological algebra discussed here are a shadow of the richer concept of *spectra* in *stable homotopy theory*. For an introduction to this subject see

- *Introduction to Stable homotopy theory*.

Under this generalization the spectral sequence of a filtered complex discussed here generalizes to the *spectral sequence of a filtered spectrum*. Important examples of these are the *Atiyah-Hirzebruch spectral sequence* and the *Adams spectral sequence*. These are discussed in

- *Introduction to Stable homotpy theory -- Applications: Complex oriented cohomology*

and

- *Introduction to Stable homotopy theory -- Part 2: Adams spectral sequence*,

respectively.

6. Notation index

- $\mathcal{A}$: the basis abelian category, assumed (without serious restriction of generality) to be $\simeq R$Mod, throughout, for some commutative ring R;
- $\mathrm{Ch}_\bullet(\mathcal{A})$ category of chain complexes in $\mathbb{A}$ in degrees $0, 1, 2, \cdots$ with differential decreasing the degree;
- $\mathrm{Ch}^\bullet(\mathcal{A})$ category of cochain complexes in degrees $0, 1, 2, \cdots$ with differential increasing the degree;
- $\mathrm{Ch}^{\mathrm{ub}}_\bullet(\mathcal{A})$ category of chain complexes with degree in $\mathbb{Z}$ and differential decreasing the degree.
- $\mathcal{K}_\bullet(\mathcal{A})$ homotopy category of chain complexes, obtained from $\mathrm{Ch}_\bullet(\mathcal{A})$ by quotienting out chain homotopy
- $\mathcal{K}^\bullet(\mathcal{A})$ homotopy category of cochain complexes, obtained from $\mathrm{Ch}^\bullet(\mathcal{A})$ by quotienting out cochain homotopy
- $\mathcal{D}_\bullet(\mathcal{A})$ derived category, obtained from $\mathcal{K}_\bullet(\mathcal{A})$ as the full subcategory on the degreewise projective objects;
- $\mathcal{D}^\bullet(\mathcal{A})$ derived category, obtained from $\mathcal{K}^\bullet(\mathcal{A})$ as the full subcategory on the degreewise injective objects;

7. References

Here are some recommended further references to go with the above material. (For a fairly comprehensive list of related literature see also at *homological algebra - References*.)

From our chapter II on we follow material in outline as in chapters 1, 2, 3 and 5 of the classical textbook:

- Charles Weibel, *An Introduction to Homological Algebra*, Cambridge University Press (1994).

This book focuses on explicit component constructions. The novice reader happy with such can entirely stick to this book as parallel reading and safely ignore all of the following pointers.

The more systematic theory which we briefly allude to in chapter III is well exposed for instance in the textbook

- Masaki Kashiwara, Pierre Schapira, *Categories and Sheaves*, Grundlehren der Mathematischen Wissenschaften **332**, Springer (2006)

Therefore the ambitious novice desiring more conceptual background might profit from at least browsing through the following lecture notes that accompany this book:

- Pierre Schapira, *Categories and homological algebra* (2011) (pdf)

The basic algebraic topology that we use in chapter I) for motivational purposes is nicely discussed in

- Alan Hatcher, *Algebraic Topology*

Similarly, a good place to *look up* the notions that we mention in chapter I and chapter V is

- Paul Goerss, Rick Jardine, *Simplicial homotopy theory*, Progress in Mathematics, Birkhäuser (1996)

A homological algebra textbook which amplifies the relation to homotopy theory as in our chapters I) and V) is

- Sergei Gelfand, Yuri Manin, *Methods of homological algebra*, Springer (1997)

For the refinement of homological algebra to stable homotopy theory see

- Urs Schreiber, *Introduction to Stable homotopy theory*

8. Thanks

I thank Todd Trimble for technical discussion while this page was being created. Notably, Todd kindly wrote up some of the proofs on the nLab that are not shown here but linked to.

And many thanks to Danny Stevenson for comments on the writeup and for catching a bunch of typos.

Revised on November 28, 2016 23:08:50 by Matt Carmonag?