nLab

Introduction to Stable homotopy theory -- 1-1

V4D2 – Algebraic Topology II

Stable Homotopy Theory

Dr. Urs Schreiber

Lecture and Seminar

Abstract *We give an introduction to the stable homotopy category and to its key computational tool, the Adams spectral sequence. To that end we introduce the modern tools, such as model categories and highly structured ring spectra. In the accompanying seminar we consider applications to cobordism theory and complex oriented cohomology such as to converge in the end to a glimpse of the modern picture of chromatic homotopy theory.*

Lecture notes.

Main page: *Introduction to Stable homotopy theory*.

Previous section: *Prelude -- Classical homotopy theory*

This section_ *Part 1 -- Stable homotopy theory*

This subsection: **Part 1.1 – Stable homotopy theory – Sequential spectra**

Next subsection: *Part 1.2 -- Stable homotopy theory -- Structured Spectra*

Next section: *Part 2 -- Adams spectral sequences*

Contents

1. **Part 1) Stable homotopy theory**
    - **1.1)** Sequential spectra
        - Sequential pre-spectra
            - Stable homotopy groups
            - Omega-spectra
            - As topological diagrams
            - Suspension and looping
        - The strict model structure on sequential spectra
            - Suspension and looping
            - CW-spectra

Topological enrichment

The stable model structure on sequential spectra
Bousfield localization

Proof of the stable model structure

Stability of the homotopy theory

Cofibrant generation

The stable homotopy category
Additivity

Triangulated structure

Long fiber-cofiber sequences

2. References

1. Part 1) Stable homotopy theory

The *Prelude on Classical homotopy theory* ended with the following phenomenon:

Definition 1.1. The reduced suspension/looping operation on pointed (def.) compactly generated topological spaces (def.) is the smash-tensor/hom-adjunction (cor.) for the standard 1-sphere smash product from the left:

$$(\Sigma \dashv \Omega) \;:\; \mathrm{Top}^{*/}_{\mathrm{cg}} \underset{\mathrm{Maps}(S^1,-)_*}{\overset{S^1 \wedge (-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Top}^{*/}_{\mathrm{cg}} \;.$$

Proposition 1.2. *With respect to the classical model structure on pointed compactly generated topological spaces* $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$ *(thm., prop.)*

1. *the adjunction in def. 1.1 is a Quillen adjunction (def.)*

$$(\Sigma \dashv \Omega) \;:\; (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} \underset{\mathrm{Maps}(S^1,-)_*}{\overset{S^1 \wedge (-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} \,,$$

2. *its induced adjoint pair of derived functors on the classical pointed homotopy category (by this prop.) is the canonical suspension/looping adjunction (according to this prop.)*

$$(\Sigma \dashv \Omega) \;:\; \mathrm{Ho}(\mathrm{Top}^{*/}) \underset{\Omega}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Ho}(\mathrm{Top}^{*/}) \;.$$

See (this prop.).

The *stable homotopy category* $\mathrm{Ho}(\mathrm{Spectra})$ is to be the result of stabilizing the adjunction in prop. 1.2, in the sense of forcing it to become an equivalence of categories in a compatible way, i.e. such as to fit into a diagram of categories of the form

$$\begin{array}{ccc} \mathrm{Ho}(\mathrm{Top}^{*/}) & \underset{\Omega}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \mathrm{Ho}(\mathrm{Top}^{*/}) \\ {}^{\Sigma^\infty}\downarrow \dashv \uparrow^{\Omega^\infty} && {}^{\Sigma^\infty}\downarrow \dashv \uparrow^{\Omega^\infty} \\ \mathrm{Ho}(\mathrm{Spectra}) & \underset{\Omega}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} & \mathrm{Ho}(\mathrm{Spectra}) \end{array} .$$

Moreover, for stable homotopy theory proper we are to refine this situation from homotopy categories to model categories and ask it to be the diagram of derived functors (according to this prop.) of a diagram of Quillen adjunctions (def.)

$$\begin{array}{ccc} (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} \\ {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\ \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq_Q} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \end{array} \,,$$

This we establish in theorem 1.86 below.

The notation Σ^∞ and Ω^∞ is meant to be suggestive of the intuition behind how this stabilization will work: The universal way of making a topological space X become stable under suspension is to pass to its infinite suspension in a suitable sense. That suitable sense is going to be called the *suspension spectrum* of X (def. 1.5 below). Conversely, if an object does not change up to equivalence, by forming its loop spaces, it must give an infinite loop space.

In contrast to the classical homotopy category, the stable homotopy category is a triangulated category (a shadow of the fact that the (∞,1)-category of spectra is a stable (∞,1)-category). As such it may be thought of as a refinement of the derived category of chain complexes (of abelian groups): every chain complex gives rise to a spectrum and every chain map to a map between these spectra (the stable Dold-Kan correspondence), but there are many more spectra and maps between them than arise from chain complexes and chain maps.

There is a variety of different models for the stable homotopy theory of spectra, some of which fits into this hierarchy:

1. sequential spectra with their model structure on sequential spectra

2. symmetric spectra with their model structure on symmetric spectra

3. orthogonal spectra with their model structure on orthogonal spectra

4. excisive functors with their model structure for excisive functors

As one moves down this list, the objects modelling the spectra become richer. This means on the one hand that their abstract properties become better as one moves down the list, on the other hand it means that it is more immediate to construct and manipulate examples as one stays further up in the list.

We start with plain sequential spectra in 1.1) as a transparent means to construct the stable homotopy category. In order to discuss ring spectra below in 1.3), it is convenient to first pass to the richer model of highly structured spectra, this we do in 1.2).

1.1) Sequential spectra

The most lighweight model for spectra are *sequential spectra*. They support most of stable homotopy theory in a straightforward way, and have the advantage that examples tend to be immediate (for instance the proof of the Brown representability theorem spits out sequential spectra).

The key disadvantage of sequential spectra is that they do not support a functorial smash product of spectra before passing to the stable homotopy category, much less a symmetric smash product of spectra. This is the structure needed for a decent discussion of the higher algebra of ring spectra. To accomodate this, further below we enhance sequential spectra to the more highly structured models given by symmetric spectra and orthogonal spectra. But all these models are connected by a free-forgetful adjunction and for working with either it is useful to have the means to pass back and forth between them.

Sequential pre-spectra

The following def. 1.3 is the traditional component-wise definition of sequential spectra. It was first stated in (Lima 58) and became widely appreciated with (Boardman 65).

> *It is generally supposed that G. W. Whitehead also had something to do with it, but the latter takes a modest attitude about that. (Adams 74, p. 131)*

Below in prop. 1.25 we discuss an equivalent definition of sequential spectra as "topological diagram spectra" (Mandell-May-Schwede-Shipley 00), namely as topologically enriched functors (defn.) on a topologically enriched category of n-spheres, which is useful for establishing the stable model category structure (below) and for establishing the symmetric monoidal smash product of spectra (in 1.2).

Throughout, our ambient category of topological spaces is $\mathrm{Top}_{\mathrm{cg}}$, the category of compactly generated topological space (defn.).

Definition 1.3. A **sequential prespectrum in topological spaces**, or just **sequential spectrum** for short (or even just **spectrum), is**

1. an $\mathbb{N}$-graded pointed compactly generated topological space

   $$X_\bullet = (X_n \in \mathrm{Top}_{\mathrm{cg}}^{*/})_{n \in \mathbb{N}}$$

   (the **component spaces**);

2. pointed continuous functions

   $$\sigma_n : S^1 \wedge X_n \to X_{n+1}$$

   for all $n \in \mathbb{N}$ (the **structure maps**) from the smash product (defn.) of one component space with the standard 1-sphere to the next component space.

A homomorphism $f : X \to Y$ of sequential spectra is a sequence $f_\bullet : X_\bullet \to Y_\bullet$ of base point-preserving continuous functions between component spaces, such that these respect the structure maps in that all diagrams of the form

$$\begin{array}{ccc} S^1 \wedge X_n & \overset{S^1 \wedge f_n}{\longrightarrow} & S^1 \wedge Y_n \\ \downarrow^{\sigma_n^X} && \downarrow^{\sigma_n^Y} \\ X_{n+1} & \underset{f_{n+1}}{\longrightarrow} & Y_{n+1} \end{array}$$

commute.

Write $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ for this category of topological sequential spectra.

Due to the classical adjunction

$$\mathrm{Top}_{\mathrm{cg}}^{*/} \underoverset{\underset{\mathrm{Maps}(S^1,-)_*}{\longrightarrow}}{\overset{S^1 \wedge (-)}{\longleftarrow}}{\bot} \mathrm{Top}_{\mathrm{cg}}^{*/}$$

from classical homotopy theory (this prop.), the definition of sequential spectra in def. 1.3 is equivalent to the following definition

Definition 1.4. A **sequential prespectrum in topological spaces**, or just **sequential spectrum** for short (or even just **spectrum), is**

1. an $\mathbb{N}$-graded pointed compactly generated topological space

   $$X_\bullet = (X_n \in \mathrm{Top}_{\mathrm{cg}}^{*/})_{n \in \mathbb{N}}$$

   (the **component spaces**);

2. pointed continuous functions

   $$\tilde{\sigma}_n : X_n \to \mathrm{Maps}(S^1, X_{n+1})_*$$

   for all $n \in \mathbb{N}$ (the **adjunct structure maps**) from one component space to the pointed mapping space (def., exmpl.) out of S^1 into the next component space.

A homomorphism $f : X \to Y$ of sequential spectra is a sequence $\widetilde{f_\bullet} : X_\bullet \to Y_\bullet$ of base point-preserving continuous function, such that all diagrams of the form

$$\begin{array}{ccc} X_n & \overset{f_n}{\longrightarrow} & Y_n \\ {}^{\mathllap{\tilde{\sigma}_n^X}}\downarrow && \downarrow^{\mathrlap{\tilde{\sigma}_n^Y}} \\ \mathrm{Maps}(S^1, X_{n+1})_* & \underset{\mathrm{Maps}(S^1, f_{n+1})_*}{\longrightarrow} & \mathrm{Maps}(S^1, Y_{n+1})_* \end{array}$$

commute.

Example 1.5. For $X \in \mathrm{Top}^{*/\mathrm{cg}}$ a pointed topological space, its **suspension spectrum** $\Sigma^\infty X$ is the sequential spectrum , def. 1.3, with

- $(\Sigma^\infty X)_n \coloneqq S^n \wedge X$ (smash product of X with the n-sphere);
- $\sigma_n : S^1 \wedge S^n \wedge X \overset{\simeq}{\longrightarrow} S^{n+1} X$ (the canonical homeomorphism).

This construction extends to a functor

$$\Sigma^\infty \; : \; \mathrm{Top}_{\mathrm{cg}}^{*/} \longrightarrow \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \; .$$

Example 1.6. The suspension spectrum (example 1.5) of the point is the **standard sequential sphere spectrum**

$$\mathbb{S}_{\mathrm{seq}} \coloneqq \Sigma^\infty S^0 \; .$$

Its nth component space is the standard n-sphere

$$(\mathbb{S}_{\mathrm{seq}})_n = S^n \; .$$

Example 1.7. A fundamental example of a spectrum that is not just a suspension spectrum is the universal real Thom spectrum, denoted MO. For details on this see *Part S – Thom spectra*.

There are are also the universal complex Thom spectrum denoted MU, and the universal symplectic Thom spectrum denoted MSp. Their standard construction first yields an example of a "sequential S^2-spectrum"; which we introduce below in def. 1.78; and then there is an adjunction (prop. 1.80) that canonically turns this into an ordinary sequential spectrum.

Definition 1.8. Let $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ be a sequential spectrum (def. 1.3) and $K \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ a pointed compactly generated topological space. Then

1. $X \wedge K$ (the **smash tensoring** of X with K) is the sequential spectrum given by
   - $(X \wedge K)_n \coloneqq X_n \wedge K$ (smash product on component spaces (defn.))
   - $\sigma_n^{X \wedge K} \coloneqq \sigma_n^X \wedge \mathrm{id}_K$.
2. $\mathrm{Maps}(K,X)_*$ (the **powering** of K into X) is the sequential spectrum with
   - $(\mathrm{Maps}(K,X)_*)_n \;\coloneqq\; \mathrm{Maps}(K,X_n)_*$ (compactly generated pointed mapping space (def., def.))
   - $\sigma_n^{\mathrm{Maps}(K,X)_*} : S^1 \wedge \mathrm{Maps}(K,X_n) \overset{(\mathrm{const},\mathrm{id})}{\longrightarrow} \mathrm{Maps}(K, S^1 \wedge X_n)_* \overset{\mathrm{Maps}(K,\sigma_n)_*}{\longrightarrow} \mathrm{Maps}(K, X_{n+1})_*$,

   where $(\mathrm{const},\mathrm{id}) : [s,\phi] \mapsto [\mathrm{const}_s, \phi]$.

These operations canonically extend to functors

$$(-)\wedge(-) : \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \times \mathrm{Top}_{\mathrm{cg}}^{*/} \longrightarrow \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$$

and

$$\mathrm{Maps}(-,-)_* : (\mathrm{Top}_{\mathrm{cg}}^{*/})^{\mathrm{op}} \times \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \,.$$

Example 1.9. The tensoring (def. 1.8) of the standard sphere spectrum $\mathbb{S}_{\mathrm{std}}$ (def. 1.6) with a space $X \in \mathrm{Top}_{\mathrm{cg}}$ is isomorphic to the suspension spectrum of X (def. 1.5):

$$\mathbb{S}_{\mathrm{std}} \wedge X \simeq \Sigma^\infty X \,.$$

Proposition 1.10. *For any $K \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ the functors of smash tensoring and powering with K, from def. 1.8, constitute a pair of adjoint functors*

$$\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \underoverset{\underset{\mathrm{Maps}(K,-)_*}{\longrightarrow}}{\overset{(-)\wedge K}{\longleftarrow}}{\bot} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \,.$$

Proof. For $X, Y \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ and $K \in \mathrm{Top}_{\mathrm{cg}}^{*/}$, let

$$X \wedge K \overset{f}{\longrightarrow} Y$$

be a morphism, with component maps fitting into commuting squares of the form

$$\begin{array}{ccc} S^1 \wedge X_n \wedge K & \overset{S^1 \wedge f_n}{\longrightarrow} & S^1 \wedge Y_n \\ {}^{\mathllap{\sigma_n^X \wedge K}}\downarrow && \downarrow^{\mathrlap{\sigma_n^Y}} \\ X_{n+1} \wedge K & \overset{f_{n+1}}{\longrightarrow} & Y_{n+1} \end{array} \,.$$

Applying degreewise the adjunction

$$\mathrm{Top}_{\mathrm{cg}}^{*/} \underoverset{\underset{\mathrm{Maps}(K,-)_*}{\longrightarrow}}{\overset{(-)\wedge K}{\longleftarrow}}{\bot} \mathrm{Top}_{\mathrm{cg}}^{*/}$$

from classical homotopy theory (this prop.) gives that these squares are in natural bijection with squares of the form

$$\begin{array}{ccc} S^1 \wedge X_n & \overset{\widetilde{S^1 \wedge f_n}}{\longrightarrow} & \mathrm{Maps}(K, S^1 \wedge Y_n)_* \\ {}^{\mathllap{\sigma_n^X}}\downarrow && \downarrow^{\mathrlap{\mathrm{Maps}(K,\sigma_n^Y)_*}} \\ X_{n+1} & \overset{\widetilde{f_{n+1}}}{\longrightarrow} & \mathrm{Maps}(K, Y_{n+1})_* \end{array}$$

But since the map $S^1 \wedge f_n$ is the smash product of two maps, only one of which involves the smash factor of K, one sees that here the top map factors through the map $(\mathrm{const}, \mathrm{id})$ from def. 1.8.

Hence the commuting square above factors as

$$\begin{array}{ccc} S^1 \wedge X_n & \overset{S^1 \wedge \widetilde{f_n}}{\longrightarrow} & S^1 \wedge \mathrm{Maps}(K, Y_n)_* \\ {}^{\mathllap{\sigma_n^X}}\downarrow && \downarrow^{\mathrlap{\sigma_n^{\mathrm{Maps}(K,Y)_*}}} \\ X_{n+1} & \overset{\widetilde{f_{n+1}}}{\longrightarrow} & \mathrm{Maps}(K, Y_{n+1})_* \end{array} .$$

This gives the structure maps for a homomorphism

$$\tilde f \,:\, X \longrightarrow \mathrm{Maps}(K,Y)_* \,.$$

Running this argument backwards shows that the map $f \mapsto \tilde f$ given thereby is a bijection. ▮

Remark 1.11. For the adjunction of prop. 1.10 it is crucial that the smash tensoring in def. 1.8 is from the *right*, at least as long as the structure maps in def. 1.3 are defined as they are, with the circle smash factor on the left. We could change both jointly: take the structure maps to be from smash products with the circle on the right, and take smash tensoring to be from the left. But having both on the right or both on the left does not work.

Proposition 1.12. *The functor Σ^∞ that forms suspension spectra (def. 1.5) has a right adjoint functor Ω^∞*

$$(\Sigma^\infty \dashv \Omega^\infty) \,:\, \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \underoverset{\underset{\Omega^\infty}{\longrightarrow}}{\overset{\Sigma^\infty}{\longleftarrow}}{\bot} \mathrm{Top}_{\mathrm{cg}}^{*/} \,,$$

given by picking the 0-component space:

$$\Omega^\infty(X) = X_0 \,.$$

Proof. By def. 1.3 the components f_n of a homomorphism of sequential spectra of the form

$$\Sigma^\infty X \overset{f}{\longrightarrow} Y$$

have to make these diagrams commute

$$\begin{array}{ccc} S^1 \wedge S^n X & \overset{S^1 \wedge f_n}{\longrightarrow} & S^1 \wedge Y_n \\ {}^{\simeq}\downarrow && \downarrow^{\sigma^Y_n} \\ S^{n+1} \wedge X & \overset{f_{n+1}}{\longrightarrow} & Y_{n+1} \end{array}$$

for all $n \in \mathbb{N}$. Since here the left vertical map is an isomorphism by def. 1.5, this uniquely fixes f_{n+1} in terms of f_n. Hence the only freedom in specifying f is in the choice of the component $f_0 : X \longrightarrow Y_0$, which is equivalently a morphism

$$X \overset{\tilde f}{\longrightarrow} \Omega^\infty Y \,.$$

∎

Stable homotopy groups

In analogy to how homotopy groups are the fundamental invariants in classial homotopy theory, the fundamental invariants of stable homtopy theory are *stable homtopy groups*:

Definition 1.13. The **stable homotopy groups** of a sequential prespectrum X, def. 1.3, is the $\mathbb{Z}$-graded abelian group given by the colimit of homotopy groups of the component spaces (def.)

$$\pi_\bullet(X) \coloneqq \underset{\longrightarrow_k}{\lim} \pi_{\bullet+k}(X_k) \,,$$

where the colimit is over the sequential diagram whose component morphisms are given in terms of the structure maps of def. 1.3 by

$$\pi_{q+k}(X_k) \overset{\simeq}{\Rightarrow} [S^{q+k}, X_k]_\ast \xrightarrow{(S^1 \wedge (-))_{S^{q+k}, X_k}} [S^{q+k+1}, S^1 \wedge X_k]_\ast \xrightarrow{[S^{q+k+1}, \sigma_k]} [S^{q+k+1}, X_{k+1}]_\ast \overset{\simeq}{\Rightarrow} \pi_{q+k+1}(X_{k+1})$$

and equivalently are given in terms of the adjunct structure maps of def. 1.4 by

$$\pi_{q+k}(X_k) \overset{\simeq}{\longrightarrow} [S^{q+k}, X_k]_\ast \xrightarrow{[S^{q+k}, \tilde\sigma_k]} [S^{q+k}, \mathrm{Maps}(S^1, X_{k+1})_\ast]_\ast \simeq [S^1 \wedge S^{q+k}, X_{k+1}]_\ast \simeq \pi_{q+k+1}(X_{k+1}) \,.$$

The colimit starts at

$$k = \begin{cases} 0 & \text{if}\; q \geq 0 \\ |q| & \text{if}\; q \lt 0 \end{cases}$$

This canonically extends to a functor

$$\pi_\bullet \;:\; \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{Ab}^{\mathbb{Z}} \,.$$

Proposition 1.14. *The two component morphisms given in def. 1.13 indeed agree.*

Proof. Consider the following instance of the defining naturality square of the $(S^1 \wedge (-)) \dashv \mathrm{Maps}(S^1, -)_\ast$-adjunction of prop. 1.2:

$$\begin{array}{ccc} [S^1 \wedge X_k, S^1 \wedge X_k]_\ast & \overset{\simeq}{\longrightarrow} & [X_k, \mathrm{Maps}(S^1, S^1 \wedge X_k)_\ast]_\ast \\ {}^{[S^1 \wedge \alpha, \sigma_k]}\downarrow && \downarrow^{[\alpha, \mathrm{Maps}(S^1, \sigma_k)_\ast]_\ast} \\ [S^1 \wedge S^{q+k}, X_{k+1}]_\ast & \underset{\simeq}{\longrightarrow} & [S^{q+k}, \mathrm{Maps}(S^1, X_{k+1})_\ast]_\ast \end{array}$$

Then consider the identity element in the top left hom-set. Its image under the left vertical map is the first of the two given component morphisms. Its image under going around the other way is the second of the two component morphisms. By the commutativity of the diagram, these two images agree. ■

Example 1.15. Given $X \in \mathrm{Top}_{\mathrm{cg}}^{*/}$, then the stable homotopy groups (def. 1.13) of its suspension spectrum (example 1.5) are given by

$$\begin{aligned} \pi_q^S(X) &\coloneqq \pi_q(\Sigma^\infty X) \\ &= \underset{\longrightarrow_k}{\lim} \pi_{q+k}(S^k \wedge X) \\ &\simeq \underset{\longrightarrow_k}{\lim} \pi_q(\Omega^k(\Sigma^k X)) \end{aligned} .$$

Specifically for $X = S^0$ the 0-sphere, with suspension spectrum the standard sphere spectrum (def. 1.6), its stable homotopy groups are the stable homotopy groups of spheres:

$$\begin{aligned} \pi_q^S(S^0) &\coloneqq \pi_q(\mathbb{S}) \\ &= \underset{\longrightarrow_k}{\lim} \pi_{q+k}(S^k) \end{aligned} .$$

Recall the Freudenthal suspension theorem, which states that if X is an n-connected pointed CW-complex then the comparison map

$$\pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X)$$

is an isomorphism for $q \leq 2n$. This implies first of all that every $\Sigma^k X$ is $(k-1)$-connected

$$\begin{aligned} &\pi_0(\Sigma X) \simeq * \\ &\pi_1(\Sigma^2 X) \simeq \pi_0(\Sigma X) \simeq * \\ &\pi_2(\Sigma^3 X) \simeq \pi_1(\Sigma^2 X) \simeq \pi_0(\Sigma X) \simeq * \\ &\quad \cdots \end{aligned}$$

and then that the qth stable homotopy group of X is attained at stage $k = q+2$ in the colimit:

$$\pi_q^S(X) \simeq \pi_{q+(q+2)}(\Sigma^{q+2} X) .$$

Historically, this fact was one of the motivations for finding a stable homotopy category (def. 1.97 below).

Definition 1.16. A morphism $f : X \longrightarrow Y$ of sequential spectra, def. 1.3, is called a **stable weak homotopy equivalence**, if its image under the stable homotopy group-functor of def. 1.13 is an isomorphism

$$\pi_\bullet(f) : \pi_\bullet(X) \overset{\simeq}{\longrightarrow} \pi_\bullet(Y) .$$

Omega-spectra

In order to motivate Omega-spectra consider the following shadow of the structure they will carry:

Example 1.17. A $\mathbb{Z}$-graded abelian group is equivalently a sequence $\{A_n\}_{n\mathbb{Z}}$ of $\mathbb{N}$-graded abelian groups A_n, together with isomorphisms

$$A_n \simeq A_{n+1}[1] ,$$

(where $[1]$ denotes the operation of shifting all entries in a graded abelian group down in degree by -1). Because this means that the sequence of $\mathbb{N}$-graded abelian groups is of the following form

$$\begin{matrix} \vdots & \vdots & & \\ a_3 & a_2 & a_1 & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ \underbrace{a_0}_{A_0} & \underbrace{a_{-1}}_{A_1} & \underbrace{a_{-2}}_{A_2} & \cdots \end{matrix} \,.$$

This allows to recover the $\mathbb{Z}$-graded abelian group $\{a_n\}_{n \in \mathbb{Z}}$ from an $\mathbb{N}$-sequence of $\mathbb{N}$-graded abelian groups.

Then consider the case that the $\mathbb{N}$-graded abelian groups here are homotopy groups of some topological space. Then shifting the degree of the component groups corresponds to forming loop spaces, because for any topological space X then

$$\pi_\bullet(\Omega X) \simeq \pi_{\bullet+1}(X) \,.$$

(This may be seen concretely in point-set topology or abstractly by looking at the long exact sequence of homotopy groups for the fiber sequence $\Omega X \to \mathrm{Path}_*(X) \to X$.)

We find this kind of behaviour for the stable homotopy groups of Omega-spectra below in example 1.20.

Definition 1.18. An **Omega-spectrum** is a sequential spectrum X of topological spaces, def. 1.3, such that the (smash product $\dashv$ pointed mapping space)-adjuncts $\tilde \sigma_n$ of the structure maps $\sigma_n : \Sigma X_n \to X_{n+1}$ of X are weak homotopy equivalences (def.), hence classical weak equivalences (def.):

$$\tilde \sigma_n \;:\; X_n \overset{\in W_{\mathrm{cl}}}{\longrightarrow} \mathrm{Maps}(S^1, X_{n+1})_*$$

for all $n \in \mathbb{N}$.

Equivalently: an Omega-spectrum is a sequential spectrum in the incarnation of def. 1.4 such that all adjunct structure maps are weak homotopy equivalences.

Example 1.19. The Brown representability theorem (thm.) implies (prop.) that every generalized (Eilenberg-Steenrod) cohomology theory (def.) is represented by an Omega-spectrum (def. 1.18).

Applied to ordinary cohomology with coefficients some abelian group A, this yields the **Eilenberg-MacLane spectra** $H A$ (exmpl.). These are the Omega-spectra whose nth component space is an Eilenberg-MacLane space

$$(H A)_n \simeq K(A,n) \,.$$

A genuinely generalized (i.e. non-ordinary, hence "extra-ordinary") cohomology theory is topological K-theory $K^\bullet(-)$. Applying the Brown representability theorem to topological K-theory yields the K-theory spectrum denoted KU.

Omega-spectra are singled out among all sequential pre-spectra as having good behaviour under forming stable homotopy groups.

Example 1.20. If a sequential spectrum X is an Omega-spectrum, def. 1.18, then its colimiting stable homotopy groups reduce to the actual homotopy groups of the component spaces, in that:

$$X \text{ Omega-spectrum} \quad \Rightarrow \quad \pi_k(X) \simeq \begin{cases} \pi_k X_0 & \text{if } k \geq 0 \\ \pi_0 X_{|k|} & \text{if } k < 0 \end{cases} .$$

(Hence the stable homotopy groups of an Omega-spectrum realize the general pattern discussed in example 1.17.)

Proof. For an Omega-spectrum, the adjunct structure maps $\tilde{\sigma}_X$ are weak homotopy equivalences, by definition, hence are classical weak equivalences. Hence $[S^1, \tilde{\sigma}_n]_*$ is an isomorphism (prop.). Therefore, by prop. 1.14, the sequential colimit in def. 1.13 is entirely over isomorphisms and hence is given already by the first object of the sequence. ■

We now show that every sequential pre-spectrum may be completed to an Omega-spectrum, up to stable weak homotopy equivalence:

Definition 1.21. For $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$, define a spectrum $QX \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ and a morphism

$$\eta_X : X \longrightarrow QX$$

(to be called the **spectrification** of X) as follows.

First introduce for the given components X_k and adjunct structure maps $\tilde{\sigma}_k$ of X (from def. 1.4) the notation

$$Z_{0,k} \coloneqq X_k \,, \qquad \tilde{\sigma}_{0,k} \coloneqq \tilde{\sigma}_k \,.$$

Now assume, by induction, that sets of objects $\{Z_{i,k}\}_{k \in \mathbb{N}}$ and maps $\{Z_{i,k} \overset{\tilde{\sigma}_{i,k}}{\longrightarrow} \Omega Z_{i,k+1}\}_{k \in \mathbb{N}}$ have been constructed for some $i \in \mathbb{N}$.

Then construct $Z_{i+1,k} \in \mathrm{Top}_{\mathrm{cg}}$ by factorizing $\tilde{\sigma}_{i,k}$, with respect to the model structure $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ (thm.) as a classical cofibration followed by a classical weak equivalence. More specifically, apply the small object argument (prop.) with respect to the set of generating cofibrations I_{Top} (def.) to produce functorial factorizations (def.) into a relative cell complex followed by a weak homotopy equivalence (just as in the proof of this lemma):

$$\tilde{\sigma}_{i,k} \;:\; Z_{i,k} \underset{\in I_{\mathrm{Top}}\,\mathrm{Cell}}{\overset{\iota_{i,k}}{\longrightarrow}} Z_{i+1,k} \underset{\in W_{\mathrm{cl}}}{\overset{\phi_{i,k}}{\longrightarrow}} \Omega Z_{i,k+1} \,.$$

Then define $\tilde{\sigma}_{i+1,k}$ as the composite

$$\tilde{\sigma}_{i+1,k} \;:\; Z_{i+1,k} \overset{\phi_{i,k}}{\longrightarrow} \Omega Z_{i,k+1} \overset{\Omega(\iota_{i,k+1})}{\longrightarrow} \Omega Z_{i+1,k+1} \,.$$

This produces for each $i \in \mathbb{N}$ a commuting diagram of the form

$$\begin{array}{ccccccc} X_k = Z_{0,k} & \underset{\in I_{\mathrm{Top}}\,\mathrm{Cell}}{\overset{\iota_{0,k}}{\longrightarrow}} & Z_{1,k} & \underset{\in I_{\mathrm{Top}}\,\mathrm{Cell}}{\overset{\iota_{1,k}}{\longrightarrow}} & Z_{2,k} & \underset{\in I_{\mathrm{Top}}\,\mathrm{Cell}}{\overset{\iota_{2,k}}{\longrightarrow}} & \cdots \\ {}^{\tilde{\sigma}_k = \tilde{\sigma}_{0,k}}\downarrow && {}^{\tilde{\sigma}_{1,k}}\downarrow && {}^{\tilde{\sigma}_{2,k}}\downarrow && \cdots \\ \Omega X_{k+1} = \Omega Z_{0,k+1} & \overset{\Omega(\iota_{0,k+1})}{\longrightarrow} & \Omega Z_{1,k+1} & \overset{\Omega(\iota_{1,k+1})}{\longrightarrow} & \Omega Z_{2,k+1} & \overset{\Omega(\iota_{2,k+1})}{\longrightarrow} & \cdots \end{array} .$$

That this indeed commutes is the identity

$$
\begin{aligned}
\tilde{\sigma}_{i+1,k} \circ \iota_{i,k} &= (\Omega(\iota_{i,k+1}) \circ \phi_{i,k}) \circ \iota_{i,k} \\
&= \Omega(\iota_{i,k+1}) \circ (\phi_{i,k} \circ \iota_{i,k}) \; . \\
&= \Omega(\iota_{i,k+1}) \circ \tilde{\sigma}_{i,k}
\end{aligned}
$$

Now let QX be the spectrum with component spaces the colimit

$$
(QX)_k \coloneqq \underset{\longrightarrow_i}{\lim} Z_{i,k}
$$

and with adjunct structure maps (via def. 1.4) given by the map induced under colimits by the above diagrams

$$
\tilde{\sigma}^{QX}_k \;\coloneqq\; \underset{\longrightarrow}{\lim} \tilde{\sigma}_{i,k} \;:\; QX \longrightarrow \Omega(QX) \; .
$$

Notice that this is indeed well-defined: since each component map $X_{i,k} \to X_{i+1,k}$ is a relative cell complex and since the 1-sphere S^1 is compact, it follows (lemma) that

$$
\begin{aligned}
\underset{\longrightarrow_i}{\lim} \Omega Z_{i,k} &= \underset{\longrightarrow_i}{\lim} \mathrm{Maps}(S^1, Z_{i,k})_* \\
&\simeq \mathrm{Maps}(S^1, \underset{\longrightarrow_i}{\lim} Z_{i,k})_* \\
&= \Omega \underset{\longrightarrow_i}{\lim} Z_{i,k} \\
&\simeq (\Omega QX)
\end{aligned}
\; .
$$

Finally, let

$$
\eta_X : X \to QX
$$

be degreewise the inclusion of the first component ($i = 0$) into the colimit. By construction, this is a homomorphism of sequential spectra (according to def. 1.4).

Proposition 1.22. *Let $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ be a sequential prespectrum with $j_X : X \to QX$ from def. 1.21. Then:*

1. *QX is an Omega-spectrum (def. 1.18);*
2. *$\eta_X : X \to QX$ is a stable weak homotopy equivalence (def. 1.16):*
3. *η_X is a level weak equivalence (is in W_{strict}, def. 1.40) precisely if X is an Omega-spectrum;*
4. *a morphism $f : X \to Y$ is a stable weak homotopy equivalence (def. 1.16), precisely if $Qf : QX \to QY$ is a level weak equivalence (is in W_{strict}, def. 1.40).*

(Schwede 97, lemma 2.1.3 and remark before section 2.2)

Proof. Since the colimit defining QX is a transfinite composition of relative cell complexes, each component map $X_k \to (QX)_k$ is itself a relative cell complex. Since n-spheres are compact topological spaces, it follows (lemma) that each element of a homotopy group in $\pi_\bullet((QX)_k)$ is in the image of a finite stage $\pi_\bullet(Z_{i,k})$ for some $i \in \mathbb{N}$. From this, all statements follow by inspection at finite stages.

Regarding first statement:

Since each $\tilde{\sigma}_{i,k}$ by construction is a weak homotopy equivalence followed by an inclusion of stages in the colimit, as any element of $\pi_q((QX)_k)$ is sent along $\tilde{\sigma}^{QX}_k$ it passes through one such $\pi_q(\tilde{\sigma}_{i,k})$ at some stage i, hence also through all the following, and is hence identically preserved in the colimit.

Regarding the second statement:

By the previous statement and by example 1.20, the map $\pi_\bullet(\eta_X) : \pi_\bullet(X) \to \pi_\bullet(QX)$ is given in degree $q \geq 0$ by

$$\underset{\simeq \underset{\longrightarrow_k}{\lim} \pi_q(\Omega^k X_k)}{\underbrace{\underset{\longrightarrow_{k \in \mathbb{N}}}{\lim} \pi_{q+k}(X_k)}} \longrightarrow \pi_q((QX)_0)$$

and similarly in degree $q < 0$. Now using the compactness of the spheres and the definition of Q we compute on the right:

$$\begin{aligned} \pi_q((QX)_0) &= \pi_q(\underset{\longrightarrow_k}{\lim} Z_{k,0}) \\ &\simeq \underset{\longrightarrow_k}{\lim} \pi_q(Z_{k,0}) \\ &\simeq \underset{\longrightarrow_k}{\lim} \pi_q(\Omega^k X_k) \end{aligned} ,$$

where the last isomorphism is π_q applied to the composite of the weak homotopy equivalences

$$Z_{k,0} \underset{\in W_{\mathrm{cl}}}{\overset{\phi_{k-1,0}}{\longrightarrow}} \Omega Z_{k-1,1} \to \cdots \to \Omega^k Z_{0,k} = \Omega^k X_k \; .$$

Regarding the third statement:

In one direction:

If X is an Omega-spectrum in that all its adjunct structure maps $\tilde{\sigma}_k$ are weak homotopy equivalences, then by two-out-of-three also the maps $\iota_{i,k}$ in def. 1.21 are weak homotopy equivalences. Hence $(j_X)_k : X_k \to (QX)_k$ is the map into a sequential colimit over acyclic relative cell complexes, and again by the compactness of the spheres, this means that it is itself a weak homotopy equivalence.

In the other direction:

If η_X is degreewise a weak homotopy equivalence, then by applying two-out-of-three (def.) to the compatibility squares for the adjunct structure morphisms (def. 1.4), using that $\tilde{\sigma}_n^{QX}$ is a weak homotopy equivalence by the first point above

$$\begin{array}{ccc} X_n & \underset{\in W_{\mathrm{cl}}}{\overset{(j_X)_n}{\longrightarrow}} & (QX)_n \\ {}^{\tilde{\sigma}_n^X}\downarrow && \downarrow^{\tilde{\sigma}_n^{QX}}_{\in W_{\mathrm{cl}}} \\ \mathrm{Maps}(S^1, X_{n+1}) & \underset{\mathrm{Maps}(S^1,(j_X)_{n+1})}{\overset{\in W_{\mathrm{cl}}}{\longrightarrow}} & \mathrm{Maps}(S^1,(QX)_{n+1}) \end{array}$$

implies that also $\tilde{\sigma}_n^X \in W_{\mathrm{cl}}$, hence that X is an Omega-spectrum.

The fourth statement follows with similar reasoning. ∎

Remark 1.23. In the case that X is a CW-spectrum (def. 1.46) then the sequence of resolutions in the definition of spectrification in def. 1.21 is not necessary, and one may simply consider

$$(Q_{\mathrm{CW}} X)_n \coloneqq \underset{\longrightarrow_k}{\lim} \Omega^k X_{n+k} \,.$$

See for instance (Lewis-May-Steinberger 86, p. 3) and (Weibel 94, 10.9.6 and topology exercise 10.9.2).

As topological diagrams

In order to conveniently understand the stable model category structure on spectra, we now consider an equivalent reformulation of the component-wise definition of sequential spectra, def. 1.3, as topologically enriched functors (defn.).

Definition 1.24. Write

$$\iota \colon \mathrm{StdSpheres} \longrightarrow \mathrm{Top}^{*/}_{\mathrm{cg}}$$

for the non-full topologically enriched subcategory (def.) of that of pointed compactly generated topological spaces (def.) where:

- objects are the standard n-spheres S^n, for $n \in \mathbb{N}$, identified as the smash product powers $S^n \coloneqq (S^1)^{\wedge^n}$ of the standard circle;
- hom-spaces are

$$\mathrm{StdSpheres}(S^n, S^{k+n}) \coloneqq \left\{ \begin{array}{ccc} * & \text{for} & k \lt 0 \\ S^k & \text{otherwise} & \end{array} \right.$$

- composition is induced from composition in $\mathrm{Top}^{*/}_{\mathrm{cg}}$ by regarding the hom-space S^k above as its image in $\mathrm{Maps}(S^n, S^{k+n})_*$ under the adjunct

$$S^k \to \mathrm{Maps}(S^n, S^{k+n})_*$$

of the canonical isomorphism

$$S^k \wedge S^n \overset{\simeq}{\longrightarrow} S^{k+n} \,.$$

This induces the category

$$[\mathrm{StdSpheres}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$$

of topologically enriched functors on $\mathrm{StdSpheres}$ with values in $\mathrm{Top}^{*/}_{\mathrm{cg}}$ (exmpl.).

Proposition 1.25. *There is an equivalence of categories*

$$(-)^{\mathrm{seq}} \;\colon\; [\mathrm{StdSpheres},\; \mathrm{Top}^{*/}_{\mathrm{cg}}] \overset{\simeq}{\longrightarrow} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$$

from the category of topologically enriched functors on the category of standard spheres of def. 1.24 to the category of topological sequential spectra, def. 1.3, which is given on objects by sending $X \in [\mathrm{StdSpheres}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ to the sequential prespectrum X^{seq} with components

$$X^{\mathrm{seq}}_n \coloneqq X(S^n)$$

and with structure maps

$$\frac{S^1 \wedge X_n^{\text{seq}} \overset{\sigma_n}{\longrightarrow} X_n^{\text{seq}}}{S^1 \longrightarrow \text{Maps}(X_n^{\text{seq}}, X_{n+1}^{\text{seq}})_*}$$

being the adjunct of the component map of X on spheres of consecutive dimension.

Proof. First observe that from its components on consecutive spheres the functor X is already uniquely determined. Indeed, by definition the hom-space between non-consecutive spheres $\text{StdSpheres}(S^n, S^{n+k})$ is the smash product of the hom-spaces between the consecutive spheres, for instance:

$$\begin{array}{ccc} S^1 \wedge S^1 & = & \text{StdSpheres}(S^n, S^{n+1}) \wedge \text{StdSpheres}(S^{n+1}, S^{n+2}) \\ {}^{\simeq}\downarrow && {}^{\simeq}\downarrow^{\circ} \\ S^2 & = & \text{StdSpheres}(S^n, S^{n+2}) \end{array},$$

and so functoriality completely fixes the former by the latter.

This means that we actually have a bijection between classes of objects.

Now observe that a natural transformation $f : X \to Y$ between two functors on StdSpheres is equivalently a collection of component maps $f_n : X_n \to Y_n$, such that for each $s \in S^1$ then the following squares commute

$$\begin{array}{ccc} X(S^n) & \overset{f_n}{\longrightarrow} & Y^{S^n} \\ {}^{X_{S^n,S^{n+1}}(s)}\downarrow && \downarrow^{Y_{S^n,S^{n+1}}(s)} \\ X(S^{n+1}) & \underset{f_{n+1}}{\longrightarrow} & Y(S^{n+1}) \end{array}$$

By the smash/hom adjunction, the square equivalently factors as

$$\begin{array}{ccc} X(S^n) & \overset{f_n}{\longrightarrow} & Y^{S^n} \\ {}^{(s,\text{id})}\downarrow && \downarrow^{(s,\text{id})} \\ S^1 \wedge X(S^n) & \underset{\text{id} \times f_n}{\longrightarrow} & S^1 \wedge Y(S^n) \\ {}^{\sigma_n^X}\downarrow && \downarrow^{\sigma_n^Y} \\ X(S^{n+1}) & \text{underset} f_{n+1} \longrightarrow & Y(S^{n+1}) \end{array}.$$

Here the top square commutes in any case, and so the total rectangle commutes precisely if the lower square commutes, hence if under our identification the components $\{f_n\}$ constitute a homomorphism of sequential spectra.

Hence we have an isomorphism on all hom-sets, and hence an equivalence of categories. ▮

Further below we use prop. 1.25 to naturally induce a model structure on the category of topological sequential spectra.

Remark 1.26. Under the equivalence of prop. 1.25, the general concept of tensoring of topologically enriched functors over topological spaces (according to this def.) restricts to the concept of tensoring of sequential spectral over topological spaces according to def. 1.8.

Proposition 1.27. *The category $\text{SeqSpec}(\text{Top}_{\text{cq}})$ of sequential spectra (def. 1.3) has all limits and colimits, and they are computed objectwise:*

Given

$$X_\bullet : I \longrightarrow \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$$

a diagram of sequential spectra, then:

1. *its colimiting spectrum has component spaces the colimit of the component spaces formed in* $\mathrm{Top}_{\mathrm{cg}}$ *(via this prop. and this corollary):*

$$(\underset{\longrightarrow_i}{\lim} X(i))_n \simeq \underset{\longrightarrow_i}{\lim} X(i)_n \,,$$

2. *its limiting spectrum has component spaces the limit of the component spaces formed in* $\mathrm{Top}_{\mathrm{cg}}$ *(via this prop. and this corollary):*

$$(\underset{\longleftarrow_i}{\lim} X(i))_n \simeq \underset{\longleftarrow_i}{\lim} X(i)_n \,;$$

moreover:

1. *the colimiting spectrum has structure maps in the sense of def. 1.3 given by*

$$S^1 \wedge (\underset{\longrightarrow_i}{\lim} X(i)_n) \simeq \underset{\longrightarrow_i}{\lim} (S^1 \wedge X(i)_n) \overset{\underset{\longrightarrow_i}{\lim} \sigma_n^{X(i)}}{\longrightarrow} \underset{\longrightarrow_i}{\lim} X(i)_{n+1}$$

   where the first isomorphism exhibits that $S^1 \wedge (-)$ *preserves all colimits, since it is a left adjoint by prop. 1.2;*

2. *the limiting spectrum has adjunct structure maps in the sense of def. 1.4 given by*

$$\underset{\longleftarrow_i}{\lim} X(i)_n \overset{\underset{\longleftarrow_i}{\lim} \tilde\sigma_n^{X(i)}}{\longrightarrow} \underset{\longleftarrow_i}{\lim} \mathrm{Maps}(S^1, X(i)_n)_* \simeq \mathrm{Maps}(S^1, \underset{\longleftarrow_i}{\lim} X(i)_n)_*$$

   where the last isomorphism exhibits that $\mathrm{Maps}(S^1,-)_*$ *preserves all limits, since it is a right adjoint by prop. 1.2.*

Proof. That the limits and colimits exist and are computed objectwise follows via prop. 1.25 from the general statement for categories of topological functors (prop.). But it is also immediate to directly check the universal property. ▮

Example 1.28. The initial object and the terminal object in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ agree and are both given by the spectrum constant on the point, which is also the suspension spectrum $\Sigma^\infty *$ (def. 1.5) of the point). We will denote this spectrum $*$ or 0 (since it is hence a zero object):

$$\begin{aligned} *_n &= * \\ S^1 \wedge *_n &\simeq * \overset{\simeq}{\to} * \,. \end{aligned}$$

Example 1.29. The coproduct of spectra $X, Y \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$, called the **wedge sum of spectra**

$$X \vee Y \coloneqq X \sqcup Y$$

is componentwise the wedge sum of pointed topological spaces (exmpl.)

$$(X \vee Y)_n = X_n \vee Y_n$$

with structure maps

$$\sigma_n^{X \vee Y} \;:\; S^1 \wedge (X \vee Y) \simeq S^1 \wedge X \,\vee\, S^1 \wedge Y \overset{(\sigma_n^X, \sigma_n^Y)}{\longrightarrow} X_{n+1} \vee Y_{n+1} \,.$$

Example 1.30. For $X \in SeqSpec(Top_{cg})$ a sequential spectrum, def. 1.3, its **standard cylinder spectrum** is its smash tensoring $X \wedge (I_+)$, according to def. 1.8, with the standard interval (def.) with a basepoint freely adjoined (def.). The component spaces of the cylinder spectrum are the standard reduced cylinders (def.) of the component spaces of X:

$$(X \wedge (I_+))_n = X_n \wedge I_+ \,.$$

By the functoriality of the smash tensoring, the factoring

$$\nabla_{S^0} \;:\; S^0 \vee S^0 \longrightarrow I_+ \longrightarrow S^0$$

of the codiagonal on the 0-sphere through the standard interval with a base point adjoined, gives a factoring of the codiagonal of X through its standard cylinder spectrum

$$\nabla_X \;:\; X \vee X \overset{X \wedge (S^0 \vee S^0 \to I_+)}{\longrightarrow} X \wedge (I_+) \overset{X \wedge (I_+ \to S^0)}{\longrightarrow} X$$

(where we are using that wedge sum is the coproduct in pointed topological spaces (exmpl.).)

Suspension and looping

We discuss models for the operation of reduced suspension and forming loop space objects of sequential spectra.

Definition 1.31. For X a sequential spectrum, then

1. the **standard suspension** of X is the smash product-tensoring $X \wedge S^1$ according to def. 1.8;

2. the **standard looping** of X is the smash powering $Maps(S^1, X)_*$ according to def. 1.8.

Proposition 1.32. *For $X \in SeqSpec(Top_{cg})$, the standard suspension $X \wedge S^1$ of def. 1.31 is equivalently the cofiber (formed via prop. 1.27) of the canonical inclusion of boundaries into the standard cylinder spectrum $X \wedge (I_+)$ of example 1.30:*

$$X \wedge S^1 \simeq cofib(X \vee X \to X \wedge (I_+)) \,.$$

Proof. This is immediate from the componentwise construction of the smash tensoring and the componentwise computation of colimits of spectra via prop. 1.27. ▮

This means that once we know that $X \vee X \to X \wedge (I_+)$ is suitably a cofibration (to which we turn below) then the standard suspension is a homotopy-correct model for the suspension operation. However, some properties of suspension are hard to prove directly with the standard suspension model. For such there are two other models for suspension and looping of spectra. These three models are not isomorphic to each other in $SeqSpec(Top_{cg})$, but (this is lemma 1.83 below) they will become isomorphic in the stable homotopy category (def. 1.97).

Definition 1.33. For X a sequential spectrum (def. 1.3) and $k \in \mathbb{Z}$, the k-fold **shifted spectrum** of X is the sequential spectrum denoted $X[k]$ given by

- $(X[k])_n \coloneqq \begin{cases} X_{n+k} & \text{for}\; n+k \geq 0 \\ \ast & \text{otherwise} \end{cases}$;
- $\sigma_n^{X[k]} \coloneqq \begin{cases} \sigma^X_{n+k} & \text{for}\; n+k \geq 0 \\ 0 & \text{otherwise} \end{cases}$.

Definition 1.34. For X a sequential spectrum, def. 1.3, then

1. the **alternative suspension** of X is the sequential spectrum ΣX with
   1. $(\Sigma X)_n \coloneqq S^1 \wedge X_n$ (smash product on the left (defn.))
   2. $\sigma_n^{\Sigma X} \coloneqq S^1 \wedge (\sigma_n^X)$.

   in the sense of def. 1.3;
2. the **alternative looping** of X is the sequential spectrum ΩX with
   1. $(\Omega X)_n \coloneqq \mathrm{Maps}(S^1, X_n)_\ast$;
   2. $\tilde\sigma_n^{\Omega X} \coloneqq \mathrm{Maps}(S^1, \tilde\sigma_n^X)_\ast$

   in the sense of def. 1.4.

Remark 1.35. In various references the "alternative suspension" from def. 1.34 is called the "fake suspension" (e.g. Goerss-Jardine 96, p. 499, Jardine 15, section 10.4).

Remark 1.36. There is no direct natural isomorphism between the standard suspension (def. 1.31) and the alternative suspension (def. 1.34). This is due to the non-trivial graded commutativity (braiding) of smash products of spheres. (We discuss braiding of the smash product more in detail in Part 1.2, this example).

Namely a natural isomorphism $\phi \colon \Sigma X \longrightarrow X \wedge S^1$ (or alternatively the other way around) would have to make the following diagrams commute:

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge X_n & \overset{id_{S^1} \wedge \phi_n}{\longrightarrow} & S^1 \wedge X_n \wedge S^1 \\ {}^{S^1 \wedge \sigma_n}\downarrow & (nc) & \downarrow^{\sigma_n \wedge S^1} \\ S^1 \wedge X_{n+1} & \underset{\phi_{n+1}}{\longrightarrow} & X_{n+1} \wedge S^1 \end{array}$$

and naturally so in X.

The only evident option is to have ϕ be the braiding homomorphisms of the smash product

$$\phi_n = \tau_{S^1, X_n} \;\colon\; S^1 \wedge X_n \overset{\simeq}{\to} X_n \wedge S^1 \,.$$

It may superficially look like this makes the above diagram commute, but it does not. To make this explicit, consider labeling the two copies of the circle appearing here as S^1_a and S^1_b. Then the diagram we are dealing with looks like this:

$$\begin{array}{ccc} S^1_a \wedge S^1_b \wedge X_n & \longrightarrow & S^1_a \wedge X_n \wedge S^1_b \\ {}^{S^1_a \wedge \sigma_n}\downarrow & (nc) & \downarrow^{\sigma_n \wedge S^1_b} \\ S^1_a \wedge X_{n+1} & \longrightarrow & X_{n+1} \wedge S^1_b \end{array}$$

If we had $S^1_a \wedge \sigma_n$ on the left and $\sigma_n \wedge S^1_a$ on the right, then the naturality of the braiding would indeed give a commuting diagram. But since this is not the case, the only way to achieve this would be by exchanging in the top left

$$S^1_a \wedge S^1_b \longrightarrow S^1_b \wedge S^1_a \,.$$

However, this map is non-trivial. It represents -1 in $[S^2, S^2]_* = \pi_2(S^2) = \mathbb{Z}$. Hence inserting this map in the top of the previous diagram still does not make it commute.

But this technical problem points to its own solutions: if we were to restrict to the homotopy category of spectra which had structure maps only of the form $S^2 \wedge X_n \to X_{n+2}$, then the braiding required to make the two models of suspension comparable would be

$$S^2_a \wedge S^1_b \longrightarrow S^1_b \wedge S^2_a$$

and this map is indeed trivial, up to homotopy. This we make precise as lemma 1.83 below.

More generally, the kind of issue encountered here is taken care of by the concept of *symmetric spectra*, to which we turn in *Part 1.2*.

Remark 1.37. The looping and suspension operations in def. 1.31 and def. 1.34 commute with shifting, def. 1.33. Therefore in expressions like $\Sigma(X[1])$ etc. we may omit the parenthesis.

Proposition 1.38. *The constructions from def. 1.31, def. 1.33 and def. 1.34 form pairs of adjoint functors* $\mathrm{SeqSpec} \to \mathrm{SeqSpec}$ *like so:*

1. $(-)[-1] \dashv (-)[1]$;
2. $(-) \wedge S^1 \dashv \mathrm{Maps}(S^1, -)_*$;
3. $\Sigma \dashv \Omega$.

Proof. Regarding the first statement:

A morphism of the form $f : X[-1] \longrightarrow Y$ has components of the form

$$\begin{array}{ccc} \vdots && \vdots \\ X_2 & \overset{f_2}{\longrightarrow} & Y_3 \\ X_1 & \overset{f_2}{\longrightarrow} & Y_2 \\ X_0 & \overset{f_1}{\longrightarrow} & Y_1 \\ * & \overset{f_0 = 0}{\longrightarrow} & Y_0 \end{array}$$

and the compatibility condition with the structure maps in lowest degree is automatically satisfied

$$\begin{array}{ccc} * & \overset{(S^1 \wedge f_0) = 0}{\longrightarrow} & S^1 \wedge Y_0 \\ {}^{\mathllap{\sigma_0^{X[-1]} = 0}}\downarrow && \downarrow^{\mathrlap{\sigma_0^Y}} \\ X_0 & \overset{f_1}{\longrightarrow} & Y_1 \end{array} \,.$$

Therefore this is equivalent to components

$$\begin{matrix} \vdots && \vdots \\ X_2 & \overset{f_2}{\longrightarrow} & Y_3 \\ X_1 & \overset{f_2}{\longrightarrow} & Y_2 \\ X_0 & \overset{f_1}{\longrightarrow} & Y_1 \end{matrix}$$

hence to a morphism $X \longrightarrow Y[1]$.

The second statement is a special case of prop. 1.10.

Regarding the third statement:

This follows by applying the (smash product⊣pointed mapping space)-adjunction isomorphism twice, like so:

Morphisms $f : \Sigma X \to Y$ in the sense of def. 1.3 are in components given by commuting diagrams of this form:

$$\begin{matrix} S^1 \wedge S^1 \wedge X_n & \overset{S^1 \wedge f_n}{\longrightarrow} & S^1 \wedge Y_n \\ {}^{\mathllap{S^1 \wedge \sigma_n^X}}\downarrow && \downarrow^{\mathrlap{\sigma_n^Y}} \\ S^1 \wedge X_{n+1} & \underset{f_{n+1}}{\longrightarrow} & Y_{n+1} \end{matrix} \,.$$

Applying the adjunction isomorphism diagonally gives a natural bijection to diagrams of this form:

$$\begin{matrix} S^1 \wedge X_n & \overset{f_n}{\longrightarrow} & Y_n \\ {}^{\mathllap{\sigma_n^X}}\downarrow && \downarrow^{\mathrlap{\tilde \sigma_n^Y}} \\ X_{n+1} & \underset{\widetilde{f_{n+1}}}{\longrightarrow} & \mathrm{Maps}(S^1, Y_{n+1})_\ast \end{matrix} \,.$$

(To see this in full detail, for instance for the adjunct of the left and bottom morphism: chase the identity $\mathrm{id}_{S^1 \wedge X_{n+1}}$ in both ways

$$\begin{matrix} \mathrm{Hom}(S^1 \wedge X_{n+1}, S^1 \wedge X_{n+1}) & \overset{\simeq}{\longrightarrow} & \mathrm{Hom}(X_{n+1}, \mathrm{Maps}(S^1, S^1 \wedge X_{n+1})_\ast) \\ {}^{\mathllap{\mathrm{Hom}(S^1 \wedge \sigma_n^X, f_{n+1})}}\downarrow && \downarrow^{\mathrlap{\mathrm{Hom}(\sigma_n^X, \mathrm{Maps}(S^1, f_{n+1})_\ast)}} \\ \mathrm{Hom}(S^1 \wedge S^1 \wedge X_n, Y_{n+1}) & \overset{\simeq}{\longrightarrow} & \mathrm{Hom}(S^1 \wedge X_n, \mathrm{Maps}(S^1, Y_{n+1})_\ast) \end{matrix}$$

through the adjunction naturality square. The other cases follow analogously.)

Then applying the adjunction isomorphism diagonally once more gives a further bijection to commuting diagrams of this form:

$$\begin{matrix} X_n & \overset{\widetilde{f_n}}{\longrightarrow} & \mathrm{Maps}(S^1, Y_n)_\ast \\ {}^{\mathllap{\tilde \sigma_n}}\downarrow && \downarrow^{\mathrlap{\mathrm{Maps}(S^1, \tilde \sigma_n^Y)_\ast}} \\ \mathrm{Maps}(S^1, X_{n+1})_\ast & \underset{\mathrm{Maps}(S^1, \widetilde{f_{n+1}})_\ast}{\longrightarrow} & \mathrm{Maps}\left(S^1, \mathrm{Maps}(S^1, Y_{n+1})_\ast\right)_\ast \end{matrix} \,.$$

This, finally, equivalently exhibits homomorphisms of the form

$$X \longrightarrow \Omega Y$$

in the sense of def. 1.4. ▮

Proposition 1.39. *The following diagram of adjoint pairs of functors commutes:*

$$\begin{array}{ccc} \mathrm{Top}^{*/}_{\mathrm{cg}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{Top}^{*/}_{\mathrm{cg}} \\ {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\ \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \end{array} \,,$$

Here the top horizontal adjunction is from prop. 1.2, the vertical adjunction is from prop. 1.10 and the bottom adjunction is from prop. 1.38.

Proof. It is sufficient to check

$$\Sigma^\infty \circ \Sigma \simeq \Sigma \circ \Sigma^\infty \,.$$

From this the statement

$$\Omega^\infty \circ \Omega \simeq \Omega \circ \Omega^\infty$$

follows by uniqueness of adjoints.

So let $X \in \mathrm{Top}^{*/}_{\mathrm{cg}}$. Then

- $(\Sigma \Sigma^\infty X)_n = S^1 \wedge S^n \wedge X$,

- $\sigma_n^{(\Sigma\Sigma^\infty X)} \colon S^1 \wedge S^1 \wedge S^n \wedge X \overset{S^1 \wedge \mathrm{id}}{\longrightarrow} S^1 \wedge S^{1+n} \wedge X$,

while

- $(\Sigma^\infty \Sigma X)_n = S^n \wedge S^1 \wedge X$,

- $\sigma_n^{(\Sigma^\infty \Sigma X)} \colon S^1 \wedge S^n \wedge S^1 \wedge X \overset{\mathrm{id} \wedge S^1 \wedge X}{\longrightarrow} S^{1+n} \wedge S^1 \wedge X$,

where we write "id" for the canonical isomorphism. Clearly there is a natural isomorphism given by the canonical identifications

$$S^1 \wedge S^n \wedge X \overset{\simeq}{\longrightarrow} (S^1)^{\wedge^{n+1}} \wedge X \overset{\simeq}{\longrightarrow} S^n \wedge S^1 \wedge X \,.$$

(As long as we are not smash-permuting the S^1 factor with the S^n factor – and here we are not – then the fact that they get mixed under this isomorphism is irrelevant. The point where this does become relevant is the content of remark 1.36 below.) ▮

The strict model structure on sequential spectra

The model category structure on sequential spectra which presents stable homotopy theory is the "stable model structure" discussed below. Its fibrant-cofibrant objects are (in particular) Omega-spectra, hence are the proper spectrum objects among the pre-spectrum objects.

But for technical purposes it is useful to also be able to speak of a model structure on pre-spectra, which sees their homotopy theory as sequences of simplicial sets equipped with suspension maps, but not their stable structure. This is called the "strict model structure" for sequential spectra. Its main point is that the stable model structure of interest arises from it via left Bousfield localization.

Definition 1.40. Say that a homomorphism $f_\bullet : X_\bullet \to Y_\bullet$ in the category $\mathrm{SeqSpec}(\mathrm{Top})$, def. 1.3 is

- a **strict weak equivalence** if each component $f_n : X_n \to Y_n$ is a weak equivalence in the classical model structure on topological spaces (hence a weak homotopy equivalence);
- a **strict fibration** if each component $f_n : X_n \to Y_n$ is a fibration in the classical model structure on topological spaces (hence a Serre fibration);
- a **strict cofibration** if the maps $f_0 : X_0 \to Y_0$ as well as for all $n \in \mathbb{N}$ the maps

$$(f_{n+1}, \sigma_n^Y) \; : \; X_{n+1} \underset{S^1 \wedge X_n}{\sqcup} S^1 \wedge Y_n \longrightarrow Y_{n+1}$$

are cofibrations in the classical model structure on topological spaces (hence retracts of relative cell complexes);

We write W_{strict}, $\mathrm{Fib}_{\mathrm{strict}}$ and $\mathrm{Cof}_{\mathrm{strict}}$ for these classes of morphisms, respectively.

Recall the sets

$$I_{\mathrm{Top}^{*/}} \coloneqq \{ S_+^{n-1} \overset{(\iota_n)_+}{\longrightarrow} D_+^n \}_{n \in \mathbb{N}}$$
$$J_{\mathrm{Top}^{*/}} \coloneqq \{ D^n \overset{(j_n)_+}{\longrightarrow} D^n \times I \}_{n \in \mathbb{N}}$$

of standard generating (acyclic) cofibrations (def.) of the classical model structure on pointed topological spaces (thm.).

Definition 1.41. Write

$$I_{\mathrm{seq}}^{\mathrm{strict}} \coloneqq \{ y(S^n) \cdot i_+ \}_{\substack{S^n \in \mathrm{StdSpheres}, \\ i_+ \in I_{\mathrm{Top}^{*/}}}} \quad \in [\mathrm{StdSpheres}, \mathrm{Top}^{*/}] \simeq \mathrm{SeqSpec}(\mathrm{Top})$$

and

$$J_{\mathrm{seq}}^{\mathrm{strict}} \coloneqq \{ y(S^n) \cdot j_+ \}_{\substack{S^n \in \mathrm{StdSpheres} \\ j_+ \in J_{\mathrm{Top}^{*/}}}} \quad \in [\mathrm{StdSpheres}, \mathrm{Top}^{*/}] \simeq \mathrm{SeqSpec}(\mathrm{Top}) \,,$$

for the set of morphisms arising as the tensoring (remark 1.26) of a representable (exmpl.) with a generating acyclic cofibration of the classical model structure on pointed topological spaces (def.).

Theorem 1.42. *The classes of morphisms in def. 1.40 give the structure of a model category (def.) to be denoted $\mathrm{SeqSpec}(\mathrm{Top})_{\mathrm{strict}}$ and called the **strict model structure on topological sequential spectra** (or: **level model structure**).*

Moreover, this is a cofibrantly generated model category with generating (acyclic) cofibrations the set $I_{\mathrm{seq}}^{\mathrm{strict}}$ (resp. $J_{\mathrm{seq}}^{\mathrm{strict}}$) from def. 1.41.

Proof. Prop. 1.25 says that the category of sequential spectra is equivalently an enriched functor category

$$\mathrm{SeqSpec}(\mathrm{Top}) \simeq [\mathrm{StdSpheres}, \; \mathrm{Top}_{\mathrm{cg}}^{*/}] \,.$$

Accordingly, this carries the projective model structure on functors (thm.). This immediately gives the statement for the fibrations and the weak equivalences.

It only remains to check that the cofibrations are as claimed. To that end, consider a commuting square of sequential spectra

$$\begin{array}{ccc} X & \overset{h}{\longrightarrow} & A \\ \downarrow^{f} && \downarrow \\ Y & \longrightarrow & B \end{array} .$$

By definition, this is equivalently an $\mathbb{N}$-collection of commuting diagrams in $\mathrm{Top}_{\mathrm{cg}}$ of the form

$$\begin{array}{ccc} X_n & \overset{h_n}{\longrightarrow} & A_n \\ \downarrow^{f_n} && \downarrow \\ Y_n & \longrightarrow & B_n \end{array}$$

such that all structure maps are respected.

$$\begin{array}{ccccccccc} S^1 \wedge X_n & \overset{\sigma_n^X}{\longrightarrow} & X_{n+1} &&&& S^1 \wedge X_n & \overset{\sigma_n^X}{\longrightarrow} & X_{n+1} \\ \downarrow^{S^1 \wedge f_n} && \downarrow^{f_{n+1}} &&&& & \searrow^{S^1\wedge h_n} && \searrow^{h_{n+1}} \\ S^1 \wedge Y_n & \overset{\sigma_n^Y}{\longrightarrow} & Y_{n+1} && = &&& S^1 \wedge A_n & \overset{\sigma_n^A}{\longrightarrow} & A_{n+1} \\ & \searrow && \searrow &&&& \downarrow && \downarrow \\ && S^1 \wedge B_n & \overset{\sigma_n^B}{\longrightarrow} & B_{n+1} &&& S^1 \wedge B_n & \overset{\sigma_n^B}{\longrightarrow} & B_{n+1} \end{array} .$$

Hence a lifting in the original diagram is a lifting in each degree n, such that the lifting in degree $n+1$ makes these diagrams of structure maps commute.

Since components are parameterized over $\mathbb{N}$, this condition has solutions by induction:

First of all there must be an ordinary lifting in degree 0. Since the strict fibrations are degreewise classical fibrations, this gives the condition that for $f_\bullet$ to be a strict cofibration, then f_0 is to be a classical cofibration.

Then assume that a lifting l_n in degree n has been found

$$\begin{array}{ccc} X_n & \overset{h_n}{\longrightarrow} & A_n \\ \downarrow^{f_n} & \nearrow_{l_n} & \downarrow \\ Y_n & \longrightarrow & B_n \end{array} .$$

Now the lifting l_{n+1} in the next degree has to also make the following diagram commute

$$\begin{array}{ccccc} S^1 \wedge X_n & \overset{\sigma_n^X}{\longrightarrow} & X_{n+1} && \\ \downarrow^{S^1 \wedge f_n} && \downarrow^{f_{n+1}} & \searrow^{h_{n+1}} & \\ S^1 \wedge Y_n & \overset{\sigma_n^Y}{\longrightarrow} & Y_{n+1} && \\ & \searrow^{S^1 \wedge l_n} && \searrow^{l_{n+1}} & \downarrow \\ && S^1 \wedge A_n & \overset{\sigma_n^A}{\longrightarrow} & A_{n+1} \end{array} .$$

This is a cocone under the commuting square for the structure maps, and therefore the outer diagram is equivalently a morphism out of the domain of the pushout product $f_n \Box \sigma_n^X$ (def.), while the compatible lift l_{n+1} is equivalently a lift against this pushout product:

$$\begin{array}{ccc} S^1 \wedge Y_n \underset{S^1 \wedge X_n}{\sqcup} X_{n+1} & \overset{(\sigma_n^A \circ S^1 \wedge l_n, h_{n+1})}{\longrightarrow} & A_{n+1} \\ {}^{\mathllap{f_n \Box \sigma_n^X}}\downarrow & {}^{l_{n+1}}\nearrow & \downarrow \\ Y_{n+1} & \longrightarrow & B_{n+1} \end{array} .$$

This shows that $f_\bullet$ is a strict cofibration precisely if, in addition to f_0 being a classical cofibration, all these pushout products are classical cofibrations. ▮

Suspension and looping

Proposition 1.43. *The $(\Sigma^\infty \dashv \Omega^\infty)$-adjunction from prop. 1.12 is a Quillen adjunction (def.) between the classical model structure on pointed topological spaces (thm., prop.) and the strict model structure on topological sequential spectra of theorem 1.42:*

$$(\Sigma^\infty \dashv \Omega^\infty) \;:\; SeqSpec(Top_{cg})_{strict} \underoverset{\underset{\Omega^\infty}{\longrightarrow}}{\overset{\Sigma^\infty}{\longleftarrow}}{\bot} (Top_{cg}^{*/})_{Quillen} .$$

Proof. It is clear that Ω^∞ preserves fibrations and acyclic cofibrations. This is sufficient to deduce a Quillen adjunction.

Just for the record, we spell out a direct argument that also Σ^∞ preserves cofibrations and acyclic cofibrations:

Let $f : X \longrightarrow Y$ be a morphism in $Top_{cg}^{*/}$ and

$$\Sigma^\infty f : \Sigma^\infty X \longrightarrow \Sigma^\infty Y$$

its image.

Since the structure maps in a suspension spectrum, example 1.5, are all isomorphisms, we have for all $n \in \mathbb{N}$ an isomorphism

$$(\Sigma^\infty X)_{n+1} \coprod_{S^1 \wedge (\Sigma^\infty X)_n} S^1 \wedge (\Sigma^\infty Y)_n \simeq S^1 \wedge (\Sigma^\infty Y)_n .$$

Therefore $\Sigma^\infty f$ is a strict cofibration, according to def. 1.40, precisely if $(\Sigma^\infty f)_0 = f$ is a classical cofibration and all the structure maps of $\Sigma^\infty Y$ are classical cofibrations. But the latter are even isomorphisms, so that this is no extra condition (prop.). Hence Σ^∞ sends classical cofibrations of spaces to strict cofibrations of sequential spectra.

Furthermore, since $S^n \wedge (-) : (Top_{cg}^{*/})_{Quillen} \to (Top_{cg}^{*/})_{Quillen}$ is a left Quillen functor for all $n \in \mathbb{N}$ by prop. 1.2 it sends classical acyclic cofibrations to classical acyclic cofibrations. Hence Σ^∞, which is degreewise given by $S^n \wedge (-)$, sends classical acyclic cofibrations to degreewise acyclic cofibrations, hence in particular to degreewise weak equivalences, hence to weak equivalences in the strict model structure on sequential spectra.

This shows that Σ^∞ is a left Quillen functor. ▮

Proposition 1.44. *The $(\Sigma \dashv \Omega)$-adjunction from prop. 1.38 is a Quillen adjunction (def.) with* *respect to the strict model structure on sequential spectra of theorem 1.42.*

$$\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \,.$$

Proof. Since the (acyclic) fibrations of $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ are by definition those morphisms that are degreewise (acylic) fibrations in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$, the statement follows immediately from the fact that the right adjoint Ω is degreewise given by $\mathrm{Maps}(S^1,-)_* \colon (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \to (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$, which is a right Quillen functor by prop. 1.2. ▮

In summary, prop. 1.39, prop. 1.43 and prop. 1.44 say that

Corollary 1.45. *The commuting square of adjunctions in prop. 1.39 is a square of Quillen adjunctions with respect to the classical model structure on pointed compactly generated topological spaces (thm., prop.) and the strict model structure on topological sequential spectra of theorem 1.42:*

$$\begin{array}{ccc} (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \\ {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\ \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \end{array} \,,$$

Further below we pass to the stable model structure in order to make the bottom adjunction in this diagram become a Quillen equivalence. This stable model structure will have more weak equivalences than the strict model structure, but will have the same cofibrations. Therefore we first consider now cofibrancy conditions already in the strict model structure.

CW-spectra

Definition 1.46. A sequential spectrum X (def. 1.3) is called a **cell spectrum** if

1. all component spaces X_n are cell complexes (def.);
2. all structure maps $\sigma_n \colon S^1 \wedge X_n \longrightarrow X_{n+1}$ are relative cell complex inclusions.

A **CW-spectrum** is a cell spectrum such that all component spaces X_n are CW-complexes (def.).

Example 1.47. The suspension spectrum $\Sigma^\infty X$ (example 1.5) for $X \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ a CW-complex is a CW-spectrum (def. 1.46).

Remark 1.48. Since, by definition 1.46, a p-cell of a cell spectrum that appears at stage q shows up as its k-fold suspension at stage $q+k$, its attachment to some spectrum X is reflected by a pushout of spectra of the form

$$\begin{array}{ccccc} \Sigma^\infty S^{-1}_+[-q] & \longrightarrow & X & \longrightarrow & * \\ {}^{\mathllap{\Sigma^\infty (i_p)_+[-q]}}\downarrow && (\mathrm{po}) \;\; \downarrow \;\; (\mathrm{po}) && \downarrow \\ \Sigma^\infty D^p_+[-q] & \longrightarrow & \hat X & \longrightarrow & \Sigma^\infty S^p[-q] \end{array} \,,$$

where the left vertical morphism is the image under the $-q$th shift spectrum functor (def.

1.33) of the image under the suspension spectrum functor (example 1.5) of the basic cell inclusion $(i_p)_+$ of pointed topological spaces (def.). This is a cofibration by prop. 1.43, and so also the middle vertical morphism is a cofibration, by theorem 1.42. Using the pasting law for pushouts, we find that the cofiber of the middle vertical morphisms (hence its homotopy cofiber (def.) in the strict model structure) is $\Sigma^\infty S^p[-q]$ (not $\Sigma^\infty S^p_+[-q]$ (!)). This is a shift of a trunction of the sphere spectrum.

After having set up the stable model category structure in theorem 1.72 below, we find that this means that cell attachments to CW-spectra in the stable model structure are by cofibers of integer shifts of the sphere spectrum $\mathbb{S}$ (def. 1.6), in that in the stable homotopy category (def. 1.97) the above situation is reflected as a homotopy cofiber sequence of the form

$$\Sigma^{p-q-1}\mathbb{S} \longrightarrow X \longrightarrow \hat X \longrightarrow \Sigma^{p-q}\mathbb{S} \,.$$

Lemma 1.49. *Let κ be an regular cardinal and let X be a κ-cell spectrum, hence a cell spectrum (def. 1.46) obtained from at most κ stable cell attachments as in remark 1.48. Then X is κ-small (def.) with respect to morphisms of spectra that are degreewise relative cell complex inclusions.*

Proof. By remark 1.48 the attachment of stable cells is by free spectra (def. 1.87) on compact topological spaces. By prop. 1.89 maps out of them are equivalently maps of component spaces in the lowest nontrivial degree. Since compact topological spaces are small with respect to relative cell complex inclusions (lemma), all these cells are small.

Now notice that κ-filtered colimits of sets commute with κ-small limtis of sets (prop.). By assumption X is a κ-small transfinite composition of pushouts of κ-small coproducts, all three of which are κ-small colimits; and let Y be the codomain of a κ-small relative cell complex inclusion, hence itself a κ-small colimit.

Now if $A = \underset{\longrightarrow_n}{\lim} \sigma_n$ is a κ-small colimit of κ-small objects σ_n, and $Y = \underset{\longrightarrow_i}{\lim} Y_i$ is a κ-small colimit, then

$$\begin{aligned} \mathrm{Hom}(A, \underset{\longrightarrow_i}{\lim} Y_i) & \simeq \mathrm{Hom}(\underset{\longrightarrow_\sigma}{\lim} c_\sigma, \underset{\longrightarrow_i}{\lim} Y_i) \\ & \simeq \underset{\longleftarrow_\sigma}{\lim} \mathrm{Hom}(c_\sigma, \underset{\longrightarrow_i}{\lim} Y_i) \\ & \simeq \underset{\longleftarrow_\sigma}{\lim} \underset{\longrightarrow_i}{\lim} \mathrm{Hom}(c_\sigma, Y_i) \\ & \simeq \underset{\longrightarrow_i}{\lim} \underset{\longleftarrow_\sigma}{\lim} \mathrm{Hom}(c_\sigma, Y_i) \\ & \simeq \underset{\longrightarrow_i}{\lim} \mathrm{Hom}(\underset{\longrightarrow_\sigma}{\lim} c_\sigma, Y_i) \\ & \simeq \underset{\longrightarrow_i}{\lim} \mathrm{Hom}(A, Y_i) \end{aligned} \,.$$

Hence the claim follows. ▮

Proposition 1.50. *The class of CW-spectra is closed under various operations, including*

- *finite wedge sum (def. 1.29)*
- *…*

Proposition 1.51. *A sequential spectrum $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ is cofibrant in the strict model structure $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ of theorem 1.42 precisely if*

1. *X_0 is cofibrant;*

2. each structure map $\sigma_n : S^1 \wedge X_n \to X_{n+1}$ is a cofibration

in the classical model structure $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ on pointed compactly generated topological spaces (thm., prop.).

In particular cell spectra and specifically CW-spectra (def. 1.46) are cofibrant.

Proof. The initial object in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ is the spectrum $*$ that is constant on the point (example 1.28). A morphism $* \to X$ is a cofibration according to def. 1.40 if

1. the morphism $* \to X_0$ is a classical cofibration, hence if the object X_0 is a classical cofibrant object, hence a retract of a cell complex;

2. the morphisms

$$*_{n+1} \underset{S^1 \wedge *_n}{\sqcup} S^1 \wedge X_n \longrightarrow X_{n+1}$$

are classical cofibrations. But since $S^1 \wedge * \simeq * \overset{\simeq}{\to} *$ is an isomorphism in this case the pushout reduces to just its second summand, and so this is now equivalent to

$$S^1 \wedge X_n \longrightarrow X_{n+1}$$

being classical cofibrations; hence retracts of relative cell complexes.

▮

Proposition 1.52. *For $X \in \mathrm{SeqSpec}(\mathrm{Top})_{\mathrm{stable}}$ a CW-spectrum, def. 1.46, then its standard cylinder spectrum $X \wedge (I_+)$ of def. 1.30 satisfies the conditions on an abstract cylinder object (def.) in that the inclusion*

$$X \vee X \longrightarrow X \wedge (I_+)$$

(of the wedge sum of X with itself, example 1.29) is a cofibration in $\mathrm{SeqSpec}(\mathrm{Top})_{\mathrm{stable}}$.

Proof. According to def. 1.40 we need to check that for all n the morphism

$$(X \vee X)_{n+1} \underset{S^1 \wedge (X \vee X)_n}{\sqcup} S^1 \wedge (X \wedge (I_+))_n \longrightarrow (X \wedge (I_+))_{n+1}$$

is a retract of a relative cell complex. After distributing indices and smash products over wedge sums, this is equivalently

$$(X_{n+1} \vee X_{n+1}) \underset{(S^1 \wedge X_n) \vee (S^1 \wedge X_n))}{\sqcup} S^1 \wedge X_n \wedge (I_+) \longrightarrow X_{n+1} \wedge (I_+)\,.$$

Now by the assumption that X is a CW-spectrum, each X_n is a CW-complex, and this implies that $X_n \wedge (I_+)$ is a relative cell complex in $\mathrm{Top}^{*/}$. With this, inspection shows that also the above morphism is a relative cell complex. ▮

We now turn to discussion of CW-approximation of sequential spectra. First recall the relative version of CW-approximation for topological spaces.

For the following, recall that a continuous function $f : X \to Y$ between topological spaces is called an **n-connected map** if the induced morphism on homotopy groups $\pi_\bullet(f) : \pi_\bullet(X, x) \to \pi_\bullet(Y, f(x))$ is

1. an isomorphism in degree $< n$;

2. an epimorphism in degree n.

(Hence an weak homotopy equivalence is an "∞-connected map".)

Lemma 1.53. *Let $f : A \longrightarrow X$ be a continuous function between topological spaces. Then there exists for each $n \in \mathbb{N}$ a relative CW-complex $\hat{f}: A \hookrightarrow \hat{Y}$ together with an extension $\phi: Y \to X$, i.e.*

$$\begin{array}{ccc} A & \overset{f}{\longrightarrow} & X \\ {}^{\mathllap{\hat{f}}}\downarrow & \nearrow_{\mathrlap{\phi}} & \\ \hat{X} & & \end{array}$$

such that ϕ is n-connected.

Moreover:

- *if f itself is k-connected, then the relative CW-complex $\hat{f}$ may be chosen to have cells only of dimension $k+1 \leq \mathrm{dim} \leq n$.*
- *if A is already a CW-complex, then $\hat{f}: A \to X$ may be chosen to be a subcomplex inclusion.*

(tomDieck 08, theorem 8.6.1)

Proposition 1.54. *For every continuous function $f: A \longrightarrow X$ out of a CW-complex A, there exists a relative CW-complex $\hat{f}: A \longrightarrow \hat{X}$ that factors f followed by a weak homotopy equivalence*

$$\begin{array}{ccccc} A & & \overset{f}{\longrightarrow} & & X \\ & {}_{\mathllap{\hat{f}}}\searrow & & \nearrow_{\mathrlap{\underset{\in \mathrm{WHE}}{\phi}}} & \\ & & \hat{X} & & \end{array}$$

Proof. Apply lemma 1.53 iteratively for $n \in \mathbb{N}$ to produce a sequence with cocone of the form

$$\begin{array}{ccccccc} A & \overset{f_0}{\longrightarrow} & X_0 & \overset{f_2}{\longrightarrow} & X_1 & \longrightarrow & \cdots \\ & {}_{\mathllap{f}}\searrow & \downarrow^{\mathrlap{\phi_0}} & \swarrow_{\mathrlap{\phi_1}} & \cdots & & \\ & & X & & & & \end{array} \,,$$

where each f_n is a relative CW-complex adding cells exactly of dimension n, and where ϕ_n in n-connected.

Let then $\hat{X}$ be the colimit over the sequence (its transfinite composition) and $\hat{f}: A \to X$ the induced component map. By definition of relative CW-complexes, this $\hat{f}$ is itself a relative CW-complex.

By the universal property of the colimit this factors f as

$$\begin{array}{ccccccc} A & \overset{f_0}{\longrightarrow} & X_0 & \overset{f_1}{\longrightarrow} & X_1 & \longrightarrow & \cdots \\ & \searrow & \downarrow & \swarrow & \cdots \\ && \hat X \\ && \downarrow^{\phi} \\ && X \end{array} \,.$$

Finally to see that ϕ is a weak homotopy equivalence: since n-spheres are compact topological spaces, then every map $\alpha : S^n \to \hat X$ factors through a finite stage $i \in \mathbb{N}$ as $S^n \to X_i \to \hat X$ (by this lemma). By possibly including further into higher stages, we may choose $i > n$. But then the above says that further mapping along $\hat X \to X$ is the same as mapping along ϕ_i, which is $(i > n)$-connected and hence an isomorphism on the homotopy class of α. ▮

Proposition 1.55. *For X any topological sequential spectrum (def.1.3), then there exists a CW-spectrum $\hat X$ (def. 1.46) and a homomorphism*

$$\phi : \hat X \overset{\in W_{\text{strict}}}{\longrightarrow} X \,.$$

which is degreewise a weak homotopy equivalence, hence a weak equivalence in the strict model structure of theorem 1.42.

Proof. First let $\hat X_0 \longrightarrow X_0$ be a CW-approximation of the component space in degree 0, via prop. 1.54. Then proceed by induction: suppose that for $n \in \mathbb{N}$ a CW-approximation $\phi_{k \leq n} : \hat X_{k \leq n} \to X_{k \leq n}$ has been found such that all the structure maps in degrees $< n$ are respected. Consider then the composite continuous function

$$S^1 \wedge \hat X_n \overset{S^1 \wedge \phi_n}{\longrightarrow} S^1 \wedge X_n \overset{\sigma_n}{\longrightarrow} X_{n+1} \,.$$

Applying prop. 1.54 to this function factors it as

$$S^1 \wedge \hat X_n \hookrightarrow \hat X_{n+1} \overset{\phi_{n+1}}{\longrightarrow} X_{n+1} \,.$$

Hence we have obtained the next stage $\hat X_{n+1}$ of the CW-approximation. The respect for the structure maps is just this factorization property:

$$\begin{array}{ccc} S^1 \wedge \hat X_n & \overset{S^1 \wedge \phi_n}{\longrightarrow} & S^1 \wedge X_n \\ {}^{\mathllap{\text{incl}}}\downarrow && \downarrow^{\mathrlap{\sigma_n}} \\ \hat X_{n+1} & \underset{\phi_{n+1}}{\longrightarrow} & X_{n+1} \end{array} \,.$$

▮

Topological enrichment

We discuss here how the hom-set of homomorphisms between any two sequential spectra is naturally equipped with a topology, and how these *hom-spaces* interact well with the strict model structure on sequential spectra from theorem 1.42. This is in direct analogy to the compatibility of compactly generated mapping spaces (def.) with the classical model structure on compactly generated topological spaces discussed at *Classical homotopy theory – Topological enrichment*. It gives an improved handle on the analysis of morphisms of spectra below in the proof of the stable model structure and it paves the way to the discussion of fully fledge *mapping spectra* below in *part 1.2*. There we will give a fully general account of the principles underlying the following. Here we just consider a pragmatic minimum that allows us to proceed.

Definition 1.56. For $X, Y \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ two sequential spectra (def. 1.3) let

$$\mathrm{SeqSpec}(X,Y) \in \mathrm{Top}_{\mathrm{cg}}^{*/}$$

be the pointed topological space whose underlying set is the hom-set $\mathrm{Hom}_{\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})}(X,Y)$ of homomorphisms from X to Y, and which is equipped with the final topology (def.) generated by those functions

$$\phi : K \longrightarrow \mathrm{Hom}_{\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})}(X,Y)$$

out of compact Hausdorff spaces K, for which there exists a homomorphism of spectra

$$\tilde{\phi} : X \wedge K \longrightarrow Y$$

out of the smash tensoring of X with K (def. 1.8) such that for all $y \in K$, $n \in \mathbb{N}$, $x \in X_n$

$$\phi(y)_n(x) = \tilde{\phi}_n(x,y) \,.$$

By construction this makes $\mathrm{SeqSpec}(X,Y)$ indeed into a compactly generated topological space, and it gives a natural bijection

$$\mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}(K, \mathrm{SeqSpec}(X,Y)) \;\simeq\; \mathrm{Hom}_{\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}^{*/})}(X \wedge K, Y) \,.$$

In *Prelude -- Classical homotopy theory* we discussed, in the section *Topological enrichment*, that the classical model structure on topological spaces (when restricted to compactly generated topological spaces) interacts well with forming smash products and pointed mapping spaces. Concretely, the smash pushout product of two classical cofibrations is a classical cofibration, and is acyclic if either of the factors is:

$$\mathrm{Cof}_{\mathrm{cl}} \,\square\, \mathrm{Cof}_{\mathrm{cl}} \subset \mathrm{Cof}_{\mathrm{cl}} \;,\qquad (\mathrm{Cof}_{\mathrm{cl}} \cap W_{\mathrm{cl}}) \,\square\, \mathrm{Cof}_{\mathrm{cl}} \subset \mathrm{Cof}_{\mathrm{cl}} \cap W_{\mathrm{cl}} \,.$$

We also saw that, by Joyal-Tierney calculus (prop.), this is equivalent to the *pullback powering* satisfying the dual relations

$$\mathrm{Fib}_{\mathrm{cl}}^{\square \mathrm{Cof}_{\mathrm{cl}}} \subset \mathrm{Fib}_{\mathrm{cl}} \;,\quad \mathrm{Fib}_{\mathrm{cl}}^{\square (\mathrm{Cof}_{\mathrm{cl}} \cap W_{\mathrm{cl}})} \subset \mathrm{Fib}_{\mathrm{cl}} \cap W_{\mathrm{cl}} \;,\quad (\mathrm{Fib}_{\mathrm{cl}} \cap W_{\mathrm{cl}})^{\square \mathrm{Cof}_{\mathrm{cl}}} \subset \mathrm{Fib}_{\mathrm{cl}} \cap W_{\mathrm{cl}} \,.$$

Now that we passed from spaces to spectra, def. 1.8 generalizes the smash product of spaces to the smash tensoring of sequential spectra by spaces, and generalizes the pointed mapping space construction for spaces to the powering of a space into a sequential spectrum. Accordingly there is now the analogous concept of *pushout product* with respect to smash tensoring, and of *pullback powering* with respect to smash powering.

From the way things are presented, it is immediate that these operations on spectra satisfy the analogous compatibility condition with the strict model structure on spectra from theorem 1.42, in fact this follows generally for topologically enriched functor categories and is inherited via prop. 1.25. But since this will be important for some of the discussion to follow, we here make it explicit:

Definition 1.57. Let $f : X \to Y$ be a morphism in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ (def. 1.3) and let $i : A \to B$ a morphism in $\mathrm{Top}_{\mathrm{cg}}^{*/}$.

Their **pushout product with respect to smash tensoring** is the universal morphism

$$f \Box i \coloneqq ((\mathrm{id}, i), (f, \mathrm{id}))$$

in

$$\begin{array}{ccccc} && X \wedge A && \\ & {}^{(f,\mathrm{id})}\swarrow && \searrow^{(\mathrm{id},i)} & \\ Y \wedge A && (\mathrm{po}) && X \wedge B \\ & \searrow && \swarrow & \\ && (Y \wedge A) \underset{X \wedge A}{\sqcup} (X \wedge B) && \\ && \downarrow^{((\mathrm{id},i),(f,\mathrm{id}))} && \\ && Y \wedge B && \end{array},$$

where $(-) \wedge (-)$ denotes the smash tensoring from def. 1.8.

Dually, their **pullback powering** is the universal morphism

$$f^{\Box i} \coloneqq (\mathrm{Maps}(B, f)_*, \mathrm{Maps}(i, X)_*)$$

in

$$\begin{array}{ccccc} && \mathrm{Maps}(B,X)_* && \\ && \downarrow^{(\mathrm{Maps}(B,f)_*, \mathrm{Maps}(i,X)_*)} && \\ && \mathrm{Maps}(B,Y)_* \underset{\mathrm{Maps}(A,Y)_*}{\times} \mathrm{Maps}(A,X)_* && \\ & \swarrow && \searrow & \\ \mathrm{Maps}(B,Y)_* && (\mathrm{pb}) && \mathrm{Maps}(A,X)_* \\ & {}_{\mathrm{Maps}(i,Y)_*}\searrow && \swarrow_{\mathrm{Maps}(A,p)_*} & \\ && \mathrm{Maps}(A,Y)_* && \end{array},$$

where $\mathrm{Maps}(-,-)_*$ denotes the smash powering from def. 1.8.

Similarly, for $f : X \to Y$ and $i : A \to B$ both morphisms of sequential spectra, then their pullback powering is the universal morphism

$$f^{\Box i} \coloneqq (\mathrm{SeqSpec}(B, f), \mathrm{SeqSpec}(i, X))$$

in

$$\begin{array}{ccccc} && \mathrm{SeqSpec}(B,X)_* && \\ && \downarrow^{(\mathrm{SeqSpec}(B,f)_*, \mathrm{SeqSpec}(i,X)_*)} && \\ && \mathrm{SeqSpec}(B,Y)_* \underset{\mathrm{SeqSpec}(A,Y)_*}{\times} \mathrm{SeqSpec}(A,X)_* && \\ & \swarrow && \searrow & \\ \mathrm{SeqSpec}(B,Y)_* && (\mathrm{pb}) && \mathrm{SeqSpec}(A,X)_* \\ & {}_{\mathrm{SeqSpec}(i,Y)_*}\searrow && \swarrow_{\mathrm{SeqSpec}(A,p)_*} & \\ && \mathrm{SeqSpec}(A,Y)_* && \end{array},$$

where now $\mathrm{SeqSpec}(-,-)$ is the hom-space functor from def. 1.56.

Proposition 1.58. *The operation of forming pushout products with respect to smash* *tensoring in def. 1.57 is compatible with the strict model structure on sequential spectra from theorem 1.42 and with the classical model structure on compactly generated pointed topological spaces (thm., prop.) in that it takes two cofibrations to a cofibration, and to an acyclic cofibration if at least one of the inputs is acyclic:*

$$\begin{aligned} \mathrm{Cof}_{\mathrm{strict}} \,\square\, \mathrm{Cof}_{\mathrm{cl}} &\subset \mathrm{Cof}_{\mathrm{strict}} \\ \mathrm{Cof}_{\mathrm{strict}} \,\square\, (\mathrm{Cof}_{\mathrm{cl}} \,\square\, W_{\mathrm{cl}}) &\subset \mathrm{Cof}_{\mathrm{strict}} \cap W_{\mathrm{strict}} \\ (\mathrm{Cof}_{\mathrm{strict}} \cap W_{\mathrm{strict}}) \,\square\, \mathrm{Cof}_{\mathrm{cl}} &\subset \mathrm{Cof}_{\mathrm{strict}} \cap W_{\mathrm{strict}} \end{aligned} \,.$$

Dually, the pullback powering satisfies

$$\begin{aligned} \mathrm{Fib}_{\mathrm{strict}}^{\square \mathrm{Cof}_{\mathrm{cl}}} &\subset \mathrm{Fib}_{\mathrm{strict}} \\ \mathrm{Fib}_{\mathrm{strict}}^{\square (\mathrm{Cof}_{\mathrm{cl}} \cap W_{\mathrm{cl}})} &\subset \mathrm{Fib}_{\mathrm{strict}} \cap W_{\mathrm{strict}} \\ (\mathrm{Fib}_{\mathrm{strict}} \cap W_{\mathrm{strict}})^{\square \mathrm{Cof}_{\mathrm{cl}}} &\subset \mathrm{Fib}_{\mathrm{strict}} \cap W_{\mathrm{strict}} \end{aligned} \,.$$

Proof. The statement concering the pullback powering follows directly form the analogous statement for topological spaces (prop.) by the fact that via theorem 1.42 the fibrations and weak equivalences in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ are degree-wise those in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$. From this the statement about the pushout product follows dually by Joyal-Tierney calculus (prop.). ■

Remark 1.59. In the language of model category-theory, prop. 1.58 says that $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ is an *enriched model category*, the enrichment being over $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$. This is often referred to simply as a “topological model category”.

Proposition 1.60. *For $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ a sequential spectrum, $f \in \mathrm{Mor}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}))$ any morphism of sequential spectra, and for $g \in \mathrm{Mor}(\mathrm{Top}_{\mathrm{cpt}}^{*/})$ a morphism of compact Hausdorff spaces, then the hom-spaces of def. 1.56 interact with the pushout-product and pullback-powering from def. 1.57 in that there is a natural isomorphism*

$$\mathrm{SeqSpec}(f \,\square\, g, X) \simeq \mathrm{SeqSpec}(f, X)^{\square g} \,.$$

Proposition 1.61. *For $X, Y \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ two sequential spectra with X a CW-spectrum (def. 1.46), then there is a natural bijection*

$$\pi_0 \,\mathrm{SeqSpec}(X, Y) \simeq [X, Y]_{\mathrm{strict}}$$

between the connected components of the hom-space from def. 1.56 and the hom-set in the homotopy category (def.) of the strict model structure from theorem 1.42.

Proof. By def. 1.56 the path components of the hom-space are the left homotopy classes of morphisms of spectra with respect to the standard cylinder spectrum of def. 1.30:

$$\frac{I_+ \longrightarrow \mathrm{SeqSpec}(X, Y)}{X \wedge (I_+) \longrightarrow Y} \,.$$

By prop. 1.52, for X a CW-spectrum then the standard cylinder spectrum $X \wedge (I_+)$ is a good cyclinder object (def.) on a cofibrant object.

Since moreover every object in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ is fibrant, the statement follows (with this lemma). ■

The stable model structure on sequential spectra

The actual spectrum objects of interest in stable homotopy theory are not the pre-spectra of def. 1.3, but the Omega-spectra of def. 1.18 among them. Hence we need to equip the category of sequential pre-spectra of def. 1.3 with a model structure (def.) whose fibrant-cofibrant objects are, in particular Omega-spectra. More in detail, it is plausible to require that every pre-spectrum is weakly equivalent to a fibrant-cofibrant one which is both an Omega-spectrum and a CW-spectrum as in def. 1.46. By prop. 1.51 this suggests to construct a model category structure on $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ that has the same cofibrations as the strict model structure of theorem 1.42, but more weak equivalences (and hence less fibrations), such as to make every sequential pre-spectrum weakly equivalent to an Omega cell spectrum.

Such a situation is called a *Bousfield localization of a model category*.

Bousfield localization

In plain category theory, a *localization* of a category $\mathcal{C}$ is equivalently a full subcategory

$$i : \mathcal{C}_{\mathrm{loc}} \hookrightarrow \mathcal{C}$$

such that the inclusion functor has a left adjoint L

$$\mathcal{C}_{\mathrm{loc}} \underoverset{\underset{i}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{C} \,.$$

The adjunction unit $\eta_X : X \to L(X)$ “reflects” every object X of $\mathcal{C}$ into one in the $\mathcal{C}_{\mathrm{loc}}$, and therefore this is also called a *reflective subcategory* inclusion.

It is a classical fact (Gabriel-Zisman 67, prop.) that in this situation

$$\mathcal{C}_{\mathrm{loc}} \simeq \mathcal{C}[W_L^{-1}]$$

is equivalently the localization (def.) of $\mathcal{C}$ at the “L-equivalences”, namely at those morphisms f such that $L(f)$ is an isomorphism. Hence one also speaks of *reflective localizations*.

The following concept of *Bousfield localization of model categories* is the evident lift of this concept of reflective localization from the realm of categories to the realm of model categories (def.), where isomorphism is genereralized to weak equivalence and where adjoint functors are taken to exhibit Quillen adjunctions.

Definition 1.62. A **left Bousfield localization** $\mathcal{C}_{\mathrm{loc}}$ of a model category $\mathcal{C}$ (def.) is another model category structure on the same underlying category with the same cofibrations,

$$\mathrm{Cof}_{\mathrm{loc}} = \mathrm{Cof}$$

but more weak equivalences

$$W_{\mathrm{loc}} \supset W \,.$$

Notice that:

Proposition 1.63. *Given a left Bousfield localization $\mathcal{C}_{\mathrm{loc}}$ of $\mathcal{C}$ as in def. 1.62, then*

1. $\mathrm{Fib}_{\mathrm{loc}} \subset \mathrm{Fib}$;
2. $W_{\mathrm{loc}} \cap \mathrm{Fib}_{\mathrm{loc}} = W \cap \mathrm{Fib}$;
3. *the identity functors constitute a Quillen adjunction*

$$\mathcal{C}_{loc} \underoverset{\underset{id}{\longrightarrow}}{\overset{id}{\longleftarrow}}{\bot} \mathcal{C} \,.$$

4. *the induced adjunction of derived functors (prop.) exhibits a reflective subcategory inclusion of homotopy categories (def.)*

$$Ho(\mathcal{C}_{loc}) \underoverset{\underset{\mathbb{R} id}{\longrightarrow}}{\overset{\mathbb{L} id}{\longleftarrow}}{\bot} Ho(\mathcal{C}) \,.$$

Proof. Regarding the first two items:

Using the properties of the weak factorization systems (def.) of (acyclic cofibrations, fibrations) and (cofibrations, acyclic fibrations) for both model structures we get

$$\begin{aligned} Fib_{loc} & = (Cof_{loc} \cap W_{loc}) Inj \\ & \subset (Cof_{loc} \cap W) Inj \\ & = Fib \end{aligned}$$

and

$$\begin{aligned} Fib_{loc} \cap W_{loc} & = Cof_{loc} Inj \\ & = Cof\, Inj \\ & = Fib \cap W \end{aligned} \,.$$

Regarding the third point:

By construction, $id \colon \mathcal{C} \to \mathcal{C}_{loc}$ preserves cofibrations and acyclic cofibrations, hence is a left Quillen functor.

Regarding the fourth point:

Since $Cof_{loc} = Cof$ the notion of left homotopy in $\mathcal{C}_{loc}$ is the same as that in $\mathcal{C}$, and hence the inclusion of the subcategory of local cofibrant-fibrant objects into the homotopy category of the original cofibrant-fibrant objects is clearly a full inclusion. Since $Fib_{loc} \subset Fib$ by the first statement, on these cofibrant-fibrant objects the right derived functor of the identity is just the identity and hence does exhibit this inclusion. The left adjoint to this inclusion is given by $\mathbb{L} id$, by the general properties of Quillen adjunctions (prop). ▮

In plain category theory, given a reflective subcategory

$$\mathcal{C}_{loc} \underoverset{\underset{i}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{C}$$

then the composite

$$Q \coloneqq i \circ L \;\colon\; \mathcal{C} \longrightarrow \mathcal{C}$$

is an idempotent monad on $\mathcal{C}$, hence, in particular, an endofunctor equipped with a natural transformation $\eta_X \colon X \to L X$ (the adjunction unit) – which "reflects" every object into one in the image of L – such that this reflection is a projection in that each $L(\eta_X)$ is an isomorphism. This characterizes the reflective subcategory $\mathcal{C}_{loc} \hookrightarrow \mathcal{C}$ as the subcategory of those objects X for which η_X is an isomorphism.

The following is the lift of this alternative perspective of reflective localization via idempotent monads from category theory to model category theory.

Definition 1.64. Let $\mathcal{C}$ be a model category (def.) which is right proper (def.), in that pullback along fibrations preserves weak equivalences.

Say that a **Quillen idempotent monad** on $\mathcal{C}$ is

1. an endofunctor

   $Q : \mathcal{C} \longrightarrow \mathcal{C}$

2. a natural transformation

   $\eta : \mathrm{id}_{\mathcal{C}} \longrightarrow Q$

such that

1. (homotopical functor) Q preserves weak equivalences;

2. (idempotency) for all $X \in \mathcal{C}$ the morphisms

   $$Q(\eta_X) : Q(X) \overset{\in W}{\longrightarrow} Q(Q(X))$$

   and

   $$\eta_{Q(X)} : Q(X) \overset{\in W}{\longrightarrow} Q(Q(X))$$

   are weak equivalences;

3. (right-properness of the localization) if in a pullback square in $\mathcal{C}$

   $$\begin{array}{ccc} f^* Z & \overset{f^* h}{\longrightarrow} & X \\ \downarrow & (\mathrm{pb}) & \downarrow^{f} \\ Z & \underset{h}{\longrightarrow} & Y \end{array}$$

   we have that

   1. f is a fibration;

   2. η_X, η_Y, and $Q(h)$ are weak equivalences

   then $Q(f^* h)$ is a weak equivalence.

Definition 1.65. For $Q : \mathcal{C} \longrightarrow \mathcal{C}$ a Quillen idempotent monad according to def. 1.64, say that a morphism f in $\mathcal{C}$ is

1. a **Q-weak equivalence** if $Q(f)$ is a weak equivalence;

2. **a Q-cofibation** if it is a cofibration.

3. **a Q-fibration** if it has the right lifting property against the morphisms that are both (Q-)cofibrations as well as Q-weak equivalences.

Write

$$\mathcal{C}_Q$$

for $\mathcal{C}$ equipped with these classes of morphisms.

Since Q preserves weak equivalences (by def. 1.64) then if the classes of morphisms in def. 1.65 do constitute a model category structure, then this is a left Bousfield localization of $\mathcal{C}$, according to def. 1.62.

We establish a couple of lemmas that will prove that the model structure indeed exists (prop. 1.68 below).

Lemma 1.66. *In the situation of def. 1.65, a morphism is an acyclic fibration in $\mathcal{C}_Q$ precisely if it is an acyclic fibration in $\mathcal{C}$.*

Proof. Let f be a fibration and a weak equivalence. Since Q preserves weak equivalences by condition 1 in def. 1.64, f is also a Q-weak equivalence. Since Q-cofibrations are cofibrations, the acyclic fibration f has right lifting against Q-cofibrations, hence in particular against against Q-acyclic Q-cofibrations, hence is a Q-fibration.

In the other direction, let $f : X \longrightarrow Y$ be a Q-acyclic Q-fibration. Consider its factorization into a cofibration followed by an acyclic fibration

$$f : X \underset{\in \mathrm{Cof}}{\overset{i}{\longrightarrow}} Z \underset{\in W \cap \mathrm{Fib}}{\overset{p}{\longrightarrow}} Y \,.$$

Observe that Q-equivalences satisfy two-out-of-three (def.), by functoriality and since the plain equivalences do. Now the assumption that Q preserves weak equivalences together with two-out-of-three implies that i is a Q-weak equivalence, hence a Q-acyclic Q-cofibration. This implies that f has the right lifting property against i (since f is assumed to be a Q-fibration, which is defined by this lifting property). Hence the retract argument (prop.) implies that f is a retract of the acyclic fibration p, and so is itself an acyclic fibration. ■

Lemma 1.67. *In the situation of def. 1.65, if a morphism $f : X \longrightarrow Y$ is a fibration, and if η_X, η_Y are weak equivalences, then f is a Q-fibration.*

(e.g. Goerss-Jardine 96, chapter X, lemma 4.4)

Proof. We need to show under the given assumptions that for every commuting square of the form

$$\begin{array}{ccc} A & \overset{\alpha}{\longrightarrow} & X \\ {}^{\in W_Q \cap \mathrm{Cof}_Q}{}^{i}\downarrow && \downarrow^{f} \\ B & \underset{\beta}{\longrightarrow} & Y \end{array}$$

there exists a lifting.

To that end, first consider a factorization of the image under Q of this square as follows:

$$\begin{array}{ccc} Q(A) & \overset{Q(\alpha)}{\longrightarrow} & Q(X) \\ {}^{Q(i)}\downarrow && \downarrow^{Q(f)} \\ Q(B) & \underset{Q(\beta)}{\longrightarrow} & Q(Y) \end{array} \;\;\simeq\;\; \begin{array}{ccccc} Q(A) & \underset{\in W \cap \mathrm{Cof}}{\overset{j_\alpha}{\longrightarrow}} & Z & \underset{\in \mathrm{Fib}}{\overset{p_\alpha}{\longrightarrow}} & Q(X) \\ {}^{Q(i)}\downarrow && \downarrow^{\pi} && \downarrow^{Q(f)} \\ Q(B) & \underset{j_\beta}{\overset{\in W \cap \mathrm{Cof}}{\longrightarrow}} & W & \underset{p_\beta}{\overset{\in \mathrm{Fib}}{\longrightarrow}} & Q(Y) \end{array}$$

(This exists even without assuming functorial factorization: factor the bottom morphism, form the pullback of the resulting p_β, observe that this is still a fibration, and then factor (through j_α) the universal morpism from the outer square into this pullback.)

Now consider the pullback of the right square above along the naturality square of $\eta : \mathrm{id} \to Q$, take this to be the right square in the following diagram

$$\begin{array}{ccccccc} \alpha : & A & \overset{(j_\alpha \circ \eta_A, \alpha)}{\longrightarrow} & Z \underset{Q(X)}{\times} X & \longrightarrow & X \\ & {}^{\mathllap{i}}\downarrow && \downarrow^{\mathrlap{(\pi,f)}} && \downarrow^{\mathrlap{f}} \\ \beta : & B & \underset{(j_\beta \circ \eta_B, \beta)}{\longrightarrow} & W \underset{Q(Y)}{\times} Y & \longrightarrow & Y \end{array} \,,$$

where the left square is the universal morphism into the pullback which is induced from the naturality squares of η on α and β.

We claim that (π, f) here is a weak equivalence. This implies that we find the desired lift by factoring (π, f) into an acyclic cofibration followed by an acyclic fibration and then lifting consecutively as follows

$$\begin{array}{ccccccc} \alpha : & A & \overset{(j_\alpha \circ \eta_A, \alpha)}{\longrightarrow} & Z \underset{Q(X)}{\times} X & \longrightarrow & X \\ & {}^{\mathllap{id}}\downarrow && {}^{\mathllap{\in W \cap Cof}}\downarrow & {}^{\exists}\nearrow & \downarrow^{\mathrlap{f}}_{\mathrlap{\in Fib}} \\ & A & \longrightarrow & & \longrightarrow & Y \\ & {}^{\mathllap{i}}_{\mathllap{\in Cof}}\downarrow & {}^{\exists}\nearrow & \downarrow^{\mathrlap{\in W \cap Fib}} && \downarrow^{\mathrlap{id}} \\ \beta : & B & \underset{(j_\beta \circ \eta_B, \beta)}{\longrightarrow} & W \underset{Q(Y)}{\times} Y & \longrightarrow & Y \end{array} \,.$$

To see that (ϕ, f) indeed is a weak equivalence:

Consider the diagram

$$\begin{array}{ccccc} Q(A) & \underoverset{\in W \cap Cof}{j_\alpha}{\longrightarrow} & Z & \underoverset{\in W}{pr_1}{\longleftarrow} & Z \underset{Q(X)}{\times} X \\ {}^{\mathllap{Q(i)}}_{\mathllap{\in W}}\downarrow && \downarrow^{\mathrlap{\pi}} && \downarrow^{\mathrlap{(\pi,f)}} \\ Q(B) & \underoverset{j_\beta}{\in W \cap Cof}{\longrightarrow} & Z & \underoverset{pr_2}{\in W}{\longleftarrow} & W \underset{Q(X)}{\times} X \end{array} \,.$$

Here the projections are weak equivalences as shown, because by assumption in def. 1.64 the ambient model category is right proper and these projections are the pullbacks along the fibrations p_α and p_β of the morphisms η_X and η_Y, respectively, where the latter are weak equivalences by assumption. Moreover $Q(i)$ is a weak equivalence, since i is a Q-weak equivalence.

Hence now it follows by two-out-of-three (def.) that π and then (π, f) are weak equivalences. ▮

Proposition 1.68. ***(Bousfield-Friedlander theorem)***

> *Let $\mathcal{C}$ be a right proper model category. Let $Q : \mathcal{C} \longrightarrow \mathcal{C}$ be a Quillen idempotent monad on $\mathcal{C}$, according to def. 1.64.*
>
> *Then the Bousfield localization model category $\mathcal{C}_Q$ (def. 1.62) at the Q-weak equivalences (def. 1.65) exists, in that the model structure on $\mathcal{C}$ with the classes of morphisms in def. 1.65 exists.*

(Bousfield-Friedlander 78, theorem 8.7, Bousfield 01, theorem 9.3, Goerss-Jardine 96, chapter X, lemma 4.5, lemma 4.6, theorem 4.1)

Proof. The existence of limits and colimits is guaranteed since $\mathcal{C}$ is already assumed to be a model category. The two-out-of-three poperty for Q-weak equivalences is an immediate consequence of two-out-of-three for the original weak equivalences of $\mathcal{C}$. Moreover, according to lemma 1.66 the pair of classes $(\mathrm{Cof}_Q, W_Q \cap \mathrm{Fib}_Q)$ equals the pair $(\mathrm{Cof}, W \cap \mathrm{Fib})$, and this is a weak factorization system by the model structure $\mathcal{C}$.

Hence it remains to show that $(W_Q \cap \mathrm{Cof}_Q,\ \mathrm{Fib}_Q)$ is a weak factorization system. The condition $\mathrm{Fib}_Q = \mathrm{RLP}(W_Q \cap \mathrm{Cof}_Q)$ holds by definition of Fib_Q. Once we show that every morphism factors as $W_Q \cap \mathrm{Cof}_Q$ followed by Fib_Q, then the condition $W_Q \cap \mathrm{Cof}_Q = \mathrm{LLP}(\mathrm{Fib}_Q)$ follows from the retract argument (lemma) and the fact that the classes W_Q and Cof_Q are closed under retracts, because W and $\mathrm{Cof} = \mathrm{Cof}_Q$ are (by this prop. and this prop., respectively).

So we may conclude by showing the existence of $(W_Q \cap \mathrm{Cof}_Q,\ \mathrm{Fib}_Q)$ factorizations:

First we consider the case of morphisms of the form $f : Q(Y) \to Q(Y)$. These may be factored with respect to $\mathcal{C}$ as

$$f \;:\; Q(X) \underset{\in W \cap \mathrm{Cof}}{\overset{\in i}{\longrightarrow}} Z \underset{\in \mathrm{Fib}}{\overset{p}{\longrightarrow}} Q(Y)\,.$$

Here i is already a Q-acyclic Q-cofibration, since Q preserves weak equivalences by the first clause in def. 1.64. Now apply $\mathrm{id} \overset{\eta}{\to} Q$ to obtain

$$\begin{array}{ccccccc} f: & Q(X) & \underset{\in W \cap \mathrm{Cof}}{\overset{i}{\longrightarrow}} & Z & \underset{\in \mathrm{Fib}}{\overset{p}{\longrightarrow}} & Q(Y) \\ & {}^{\mathllap{}}\downarrow^{\eta_{Q(X)}}_{\in W} && \downarrow^{\eta_Z} && \downarrow^{\eta_{Q(Y)}}_{\in W}, \\ & Q(Q(X)) & \underset{Q(i)}{\overset{\in W}{\longrightarrow}} & Q(Z) & \longrightarrow & Q(Q(Y)) \end{array}$$

where $\eta_{Q(X)}$ and $\eta_{Q(Y)}$ are weak equivalences by idempotency (the second clause in def. 1.64), and $Q(i)$ is a weak equivalence since Q preserves weak equivalences. Hence by two-out-of-three also η_Z is a weak equivalence. Therefore lemma 1.67 gives that p is a Q-fibration, and hence the above factorization is already as desired

$$f \;:\; Q(X) \underset{\in W_Q \cap \mathrm{Cof}_Q}{\overset{\in i}{\longrightarrow}} Z \underset{\in \mathrm{Fib}_Q}{\overset{p}{\longrightarrow}} Q(Y)\,.$$

Now for an arbitrary morphism $g : X \to Y$, form a factorization of $Q(g)$ as above and then decompose the naturality square for η on g into the pullback of the resulting Q-fibration along η_Y:

$$\begin{array}{ccccccc} g: & X & \overset{\tilde i}{\longrightarrow} & Z \underset{Q(Y)}{\times} Y & \overset{\tilde p \in \mathrm{Fib}_Q}{\longrightarrow} & Y \\ & {}^{\eta_X}_{\in W_Q}\downarrow && \downarrow^{\eta'} & (\mathrm{pb}) & \downarrow^{\eta_Y}_{\in W_Q} \\ Q(g): & Q(Y) & \underset{i}{\overset{\in W_Q}{\longrightarrow}} & Z & \underset{p}{\overset{\in \mathrm{Fib}_Q}{\longrightarrow}} & Q(Y) \end{array}$$

This exhibits η' as the pullback of a Q-weak equivalence along a fibration between objects on which η is a weak equivalence. Then the third clause in def. 1.64 says that η' is itself as a Q-weak equivalence. This way, two-out-of-three implies that $\tilde i$ is a Q-weak equivalence.

Observe that $\tilde p$ is a Q-fibration, because it is the pullback of a Q-fibration and because Q-fibrations are defined by a right lifting property (def. 1.65) and hence closed under pullback (prop.) Finally, apply factorization in $(\mathrm{Cof},\ W \cap \mathrm{Fib})$ to $\tilde i$ to obtain the desired factorization

$$f \,:\, \underset{W_Q \cap \mathrm{Cof}}{\overset{\tilde i_L}{\longrightarrow}} \underset{W \cap \mathrm{Fib} = W_Q \cap \mathrm{Fib}_Q}{\overset{\tilde i_R}{\longrightarrow}} \underset{\mathrm{Fib}_Q}{\overset{\tilde p}{\longrightarrow}} \,.$$

▮

While this establishes the Q-model structure, so far this leaves open a more explicit description of the Q-fibrations. This is provided by the next statement.

Proposition 1.69. *For $Q : \mathcal{C} \longrightarrow \mathcal{C}$ a Quillen idempotent monad according to def. 1.64, then a morphism $f : X \to Y$ in $\mathcal{C}$ is a Q-fibration (def. 1.65) precisely if*

1. *f is a fibration;*

2. *the η-naturality square on f*

$$\begin{matrix} X & \overset{\eta_X}{\longrightarrow} & Q(X) \\ {}^{\mathllap{f}}\downarrow & (\mathrm{pb})^h & \downarrow^{\mathrlap{Q(f)}} \\ Y & \underset{\eta_Y}{\longrightarrow} & Q(Y) \end{matrix}$$

exhibits a homotopy pullback in $\mathcal{C}$ (def.), in that for any factorization of $Q(f)$ through a weak equivalence followed by a fibration p, then the universally induced morphism

$$X \longrightarrow p^* Y$$

is weak equivalence (in $\mathcal{C}$).

(e.g. Goerss-Jardine 96, chapter X, theorem 4.8)

Proof. First consider the case that f is a fibration and that the square is a homotopy pullback. We need to show that then f is a Q-fibration.

Factor $Q(f)$ as

$$Q(f) \,:\, Q(X) \underset{\in W \cap \mathrm{Cof}}{\overset{i}{\longrightarrow}} Z \underset{\in \mathrm{Fib}}{\overset{p}{\longrightarrow}} Q(Y) \,.$$

By the proof of prop. 1.68, the morphism p is also a Q-fibration. Hence by the existence of the Q-local model structure, also due to prop. 1.68, its pullback $\tilde p$ is also a Q-fibration

$$\begin{matrix} X & \overset{\eta_X}{\longrightarrow} & Q(X) \\ {}^{\mathllap{\in W}}{}^{\tilde i}\downarrow && \downarrow^{i}_{\mathrlap{\in W}} \\ Y \underset{Q(Y)}{\times} Z & \overset{p^* \eta_Y}{\longrightarrow} & Z \\ {}^{\mathllap{\in \mathrm{Fib}_Q}}{}^{\tilde p}\downarrow & (\mathrm{pb}) & \downarrow^{p}_{\mathrlap{\in \mathrm{Fib}_Q}} \\ Y & \underset{\eta_Y}{\longrightarrow} & Q(Y) \end{matrix} \,.$$

Here $\tilde i$ is a weak equivalence by assumption that the diagram exhibits a homotopy pullback. Hence it factors as

$$\tilde i \,:\, X \underset{\in W \cap \mathrm{Cof}}{\overset{j}{\longrightarrow}} \hat X \underset{\in W \cap \mathrm{Fib} = W_Q \cap \mathrm{Fib}_Q}{\overset{\pi}{\longrightarrow}} Y \underset{Q(Y)}{\times} Z \,.$$

This yields the situation

$$\begin{array}{ccccc} X & \overset{=}{\longrightarrow} & X \\ {}^{\mathllap{j}}_{\mathllap{\in W \cap Cof}}\downarrow & {}^{\exists}\nearrow & \downarrow^{\mathrlap{f}}_{\mathrlap{\in Fib}} \\ \hat X & \underset{\tilde p \circ \pi}{\overset{\in Fib_Q}{\longrightarrow}} & Y \end{array} \;\;\;\;\; \leftrightarrow \;\;\;\;\; \begin{array}{ccccc} X &\overset{j}{\longrightarrow}& \hat X &\overset{\exists}{\longrightarrow}& X \\ {}^{\mathllap{f}}\downarrow && \downarrow^{\mathrlap{\tilde p \circ \pi}} && \downarrow^{\mathrlap{f}} \\ Y &=& Y &=& Y \end{array} \,.$$

As in the retract argument (prop.) this diagram exhibits f as a retract (in the arrow category, rmk.) of the Q-fibration $\tilde p \circ \pi$. Hence by the existence of the Q-model structure (prop. 1.68) and by the closure properties for fibrations (prop.), also f is a Q-fibration.

Now for the converse. Assume that f is a Q-fibration. Since $\mathcal{C}_Q$ is a left Bousfield localization of $\mathcal{C}$ (prop. 1.68), f is also a fibration (prop. 1.63). We need to show that the η-naturality square on f exhibits a homotopy pullback.

So factor $Q(f)$ as before, and consider the pasting composite of the factorization of the given square with the naturality squares of η:

$$\begin{array}{ccccc} X &\underset{\in W_Q}{\overset{\eta_X}{\longrightarrow}}& Q(X) &\underset{\in W \subset W_Q}{\overset{\eta_{Q(X)}}{\longrightarrow}}& Q(Q(X)) \\ {}^{\mathllap{\tilde i}}_{\mathllap{\in W_Q}}\downarrow && {}^{\mathllap{i}}_{\mathllap{\in W \subset W_Q}}\downarrow && \downarrow^{\mathrlap{Q(i)}}_{\mathrlap{\in W}} \\ Y \underset{Q(Y)}{\times} Z &\underset{\in W_Q}{\overset{p^\ast \eta_Y}{\longrightarrow}}& Z &\underset{\in W}{\overset{\eta_Z}{\longrightarrow}}& Q(Z) \\ {}^{\mathllap{\tilde p}}_{\mathllap{\in Fib_Q}}\downarrow &(pb)& \downarrow^{\mathrlap{p}}_{\mathrlap{\in Fib_Q \subset Fib}} && \downarrow^{\mathrlap{Q(p)}} \\ Y &\underset{\eta_Y}{\overset{\in W_Q}{\longrightarrow}}& Q(Y) &\underset{\eta_{Q(Y)}}{\overset{\in W \subset W_Q}{\longrightarrow}}& Q(Q(Y)) \end{array} \,.$$

Here the top and bottom horizontal morphisms are weak (Q-)equivalences by the idempotency of Q, and $Q(i)$ is a weak equivalence since Q preserves weak equivalences (first and second clause in def. 1.64). Hence by two-out-of-three also η_Z is a weak equivalence. From this, lemma 1.67 gives that p is a Q-fibration. Then $p^\ast \eta_Y$ is a Q-weak equivalence since it is the pullback of a Q-weak equivalence along a fibration between objects whose η is a weak equivalence, via the third clause in def. 1.64. Finally two-out-of-three implies that $\tilde \imath$ is a Q-weak equivalence.

In particular, the bottom right square is a homotopy pullback (since two opposite edges are weak equivalences, by this prop.), and since the left square is a genuine pullback of a fibration, hence a homotopy pullback, the total bottom rectangle here exhibits a homotopy pullback by the pasting law for homotopy pullbacks (prop.).

Now by naturality of η, that total bottom rectangle is the same as the following rectangle

$$\begin{array}{ccccc} Y \underset{Q(Y)}{\times} Z &\overset{\eta\left(Y \underset{Q(Y)}{\times} Z\right)}{\longrightarrow}& Q(Y \underset{Q(Y)}{\times} Z) &\underset{\in W}{\overset{Q(p^\ast \eta_Y)}{\longrightarrow}}& Q(Z) \\ {}^{\mathllap{\tilde p}}_{\mathllap{\in Fib_Q}}\downarrow && \downarrow^{\mathrlap{Q(\tilde p)}} && \downarrow^{\mathrlap{Q(p)}} \\ Y &\underset{\eta_Y}{\longrightarrow}& Q(Y) &\underset{Q(\eta_Y)}{\overset{\in W}{\longrightarrow}}& Q(Q(Y)) \end{array} \,,$$

where now $Q(p^\ast \eta_Y) \in W$ since $p^\ast \eta_Y \in W_Q$, as we had just established. This means again that the right square is a homotopy pullback (prop.), and since the total rectangle still is a homotopy pullback itself, by the previous remark, so is now also the left square, by the other direction of the pasting law for homotopy pullbacks (prop.).

So far this establishes that the η-naturality square of $\tilde p$ is a homotopy pullback. We still need to show that also the η-naturality square of f is a homotopy pullback.

Factor $\tilde\iota$ as a cofibration followed by an acyclic fibration. Since $\tilde\iota$ is also a Q-weak equivalence, by the above, two-out-of-three for Q-fibrations gives that this factorization is of the form

$$X \underset{\in W_Q \cap \mathrm{Cof} = W_Q \cap \mathrm{Cof}_Q}{\overset{j}{\longrightarrow}} \hat X \underset{\in W \cap \mathrm{Fib} = W_Q \cap \mathrm{Fib}_Q}{\overset{\pi}{\longrightarrow}} Y \underset{Q(Y)}{\times} Z.$$

As in the first part of the proof, but now with $(W \cap \mathrm{Cof}, \mathrm{Fib})$ replaced by $(W_Q \cap \mathrm{Cof}_Q, \mathrm{Fib}_Q)$ and using lifting in the Q-model structure, this yields the situation

$$\array{ X &\overset{=}{\longrightarrow}& X \\ {}^{\mathllap{j}}_{\mathllap{\in W_Q \cap \mathrm{Cof}_Q}}\downarrow &{}^{\exists}\nearrow& \downarrow^{\mathrlap{f}}_{\mathrlap{\in \mathrm{Fib}_Q}} \\ \hat X &\underset{\tilde p\circ \pi}{\longrightarrow}& Y } \quad \leftrightarrow \quad \array{ X &\overset{j}{\longrightarrow}& \hat X &\overset{\exists}{\longrightarrow}& X \\ {}^{\mathllap{f}}\downarrow && \downarrow^{\mathrlap{\tilde p \circ \pi}} && \downarrow^{\mathrlap{f}} \\ Y &=& Y &=& Y } \,.$$

As in the retract argument (prop.) this diagram exhibits f as a retract (in the arrow category, rmk.) of $\tilde p \circ \pi$.

Observe that the η-naturality square of the weak equivalence π is a homotopy pullback, since Q preserves weak equivalences (first clause of def. 1.64) and since a square with two weak equivalences on opposite sides is a homotopy pullback (prop.). It follows that also the η-naturality square of $\tilde p \circ \pi$ is a homotopy pullback, by the pasting law for homotopy pullbacks (prop.).

In conclusion, we have exhibited f as a retract (in the arrow category, rmk.) of a morphism $\tilde p \circ \pi$ whose η-naturality square is a homotopy pullback. By naturality of η, this means that the whole η-naturality square of f is a retract (in the category of commuting squares in $\mathcal{C}$) of a homotopy pullback square. This means that it is itself a homotopy pullback square (prop.). ▮

Proof of the stable model structure

We show now that the operation of Omega-spectrification of topological sequental spectra, from def. 1.21, is a Quillen idempotent monad in the sense of def. 1.64. Via the Bousfield-Friedlander theorem (prop. 1.68) this establishes the stable model structure on topological sequential spectra in theorem 1.72 below.

Lemma 1.70. *The Omega-spectrification (Q,η) from def. 1.21 preserves homotopy pullbacks (def.) in the strict model structure $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ from theorem 1.42.*

(Schwede 97, lemma 2.1.3 (e))

Proof. Since, by prop. 1.22, Q preserves weak equivalences, it is sufficient to show that every pullback square in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ of a fibration

$$\array{ B \underset{Y}{\times} X &\longrightarrow& X \\ \downarrow &(\mathrm{pb})& \downarrow^{\mathrlap{\in \mathrm{Fib}}} \\ B &\longrightarrow& Y }$$

is taken by Q to a homotopy pullback square. By prop. 1.27 we need to check that this is the case for the kth component space of the sequential spectra in the diagram, for all $k \in \mathbb{N}$.

Let $Z^X_{i,k}$, $Z^Y_{i,k}$ etc. denote the objects appearing in the definition of $(QX)_k \coloneqq \underset{\longrightarrow_i}{\lim} Z^X_{i,k}$, $(QY)_k \coloneqq \underset{\longrightarrow_i}{\lim} Z^Y_{i,k}$, etc. (def. 1.21).

Use the small object argument (prop.) for the set $J_{(\mathrm{Top}^{*/})}$ of acyclic generating cofibrations in $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$ (def.) to construct a functorial factorization (def.) through acyclic relative cell complex inclusions (def.) followed by Serre fibrations (def.) in each degree:

$$Z^X_{i,k} \overset{\in J_{\mathrm{Top}}\,\mathrm{Cell}}{\longrightarrow} W_i \overset{\in \mathrm{Fib}_{\mathrm{cl}}}{\longrightarrow} Z^Y_{i,k} \,.$$

Notice that by construction $Z^K_{\bullet,k}$ and $Z^Y_{\bullet,k}$ are sequences of relative cell complexes. This implies, by the way the small object argument works and by the commutativity of each

$$\begin{array}{ccc} Z^X_{i,k} & \overset{\in J_{(\mathrm{Top}^{*/})}\,\mathrm{Cell}}{\longrightarrow} & W_i \\ {}^{\mathllap{\in I_{(\mathrm{Top}^{*/})}\,\mathrm{Cell}}}\downarrow && \downarrow \\ Z^X_{i+1,k} & \overset{\in J_{(\mathrm{Top}^{*/})}\,\mathrm{Cell}}{\longrightarrow} & W_{i+1} \end{array} \,,$$

that also $W_\bullet$ is a sequence of relative cell complex inclusions: a cell in W_i is given by the top square in the following diagram, and the total rectangle is the image of that cell as a cell in W_{i+1}:

$$\begin{array}{ccc} S^{n-1} & \overset{i_n}{\longrightarrow} & D^{n-1} \\ \downarrow && \downarrow \\ Z^X_{i,k} & \overset{\in J_{(\mathrm{Top}^{*/})}\,\mathrm{Cell}}{\longrightarrow} & W_i \\ {}^{\mathllap{\in I_{(\mathrm{Top}^{*/})}\,\mathrm{Cell}}}\downarrow && \downarrow \\ Z^X_{i+1,k} & \overset{\in J_{(\mathrm{Top}^{*/})}\,\mathrm{Cell}}{\longrightarrow} & W_{i+1} \end{array} \,,$$

Therefore, forming the colimit over $i \in I$ of these sequences sends the degreewise Serre fibration to a Serre fibration (prop.): because we test for a Serre fibration by lifting against the morphism in $J_{\mathrm{Top}^{*/}}$, which have compact domain and codomain, and these may be taken inside the colimit over relative cell complex inclusions (by this lemma)). So we have a Serre fibration

$$\underset{\longrightarrow_i}{\lim} W_i \overset{\in W_{\mathrm{cl}}}{\longrightarrow} (QY)_k$$

for each $k \in \mathbb{N}$.

Consider then the commuting diagrams

$$\begin{array}{ccccccc} Z^B_{i,k} & \longrightarrow & Z^Y_{i,k} & \overset{\in \mathrm{Fib}_{\mathrm{cl}}}{\longleftarrow} & W_i & \overset{\in W_{\mathrm{cl}} \in \mathrm{Cof}_{\mathrm{cl}}}{\longleftarrow} & Z^X_{i,k} \\ \downarrow^{\mathrlap{\phi^B}}_{\mathrlap{\in W_{\mathrm{cl}}}} && \downarrow^{\mathrlap{\phi^Y}}_{\mathrlap{\in W_{\mathrm{cl}}}} && & {}^{\mathllap{\exists \in W_{\mathrm{cl}}}}\searrow & \downarrow^{\mathrlap{\phi^X}}_{\mathrlap{\in W_{\mathrm{cl}}}} \\ \Omega^i B_{k+i} & \longrightarrow & \Omega^i Y_{k+i} & \longleftarrow && \underset{\in \mathrm{Fib}_{\mathrm{cl}}}{\longleftarrow} & \Omega^i X_{k+i} \end{array}$$

where the vertical morphisms are composites of the weak equivalences $\phi_{i,k} \colon Z_{i+1,k} \overset{\phi_{i,k}}{\longrightarrow} \Omega Z_{i,k+1}$ from def. 1.21.

The diagonal is a chosen lift (where we use that $\Omega = \mathrm{Maps}(S^1, -)_*$ preserves Serre fibrations by prop. 1.2). This lift is a weak equivalence by two-out-of-three. On the left of the diagram this exhibits now a weak equivalence of cospan-diagrams with right leg a fibration. Therefore, since forming the limit over these cospan diagrams is a homotopy pullback (def., all objects here being fibrant), this induces a weak equivalence on these limits (prop.)

$$\kappa : Z^B_{i,k} \underset{Z^Y_{i,k}}{\times} W_i \overset{\in W_{\mathrm{cl}}}{\longrightarrow} \Omega^i B_{k+i} \underset{\Omega^i Y_{k+i}}{\times} \Omega^i X_{k+i} \simeq \Omega^i (B_{k+i} \underset{Y_{k+i}}{\times} X_{k+i}) \,.$$

By universality of the pullback there is a commuting triangle

$$\begin{array}{ccccc} Z^{B \times_Y X}_{i,k} && \overset{\rho_i}{\longrightarrow} && Z^B_{i,k} \underset{Z^Y_{i,k}}{\times} W_i \\ & {}_{\phi \in W_{\mathrm{cl}}}\searrow && \swarrow_{\kappa \in W_{\mathrm{cl}}} \\ && \Omega^i (B_{i+k} \underset{Y_{i+k}}{\times} X_{i+k}) \end{array}$$

and hence by two-out-of-three also the top morphism is a weak equivalence.

Now observe that colimits over sequences of relative cell inclusions preserve finite limits up to weak equivalence (prop.). This follows again by using that n-spheres may be taken inside the colimits from the classical fact that filtered colimits preserve finite limits. In conclusion then, we have a weak equivalence of the form

$$(Q(B \underset{Y}{\times} X))_k = \underset{\longrightarrow_i}{\lim} Z^{B \times_Y X}_{i,k} \underset{\in W_{\mathrm{cl}}}{\overset{\underset{\longrightarrow_i}{\lim} \rho_i}{\longrightarrow}} \underset{\longrightarrow_i}{\lim} \left(Z^B_{i,k} \underset{Z^Y_{i,k}}{\times} W_i \right) \overset{\in W_{\mathrm{cl}}}{\longrightarrow} \left(\underset{\longrightarrow_i}{\lim} Z^B_{i,k} \right) \underset{\underset{\longrightarrow_i}{\lim} Z^Y_{i,k}}{\times} \left(\underset{\longrightarrow_i}{\lim} W_i \right) = (Q B)_k \underset{(Q Y)_k}{\times} \left(\underset{\longrightarrow_i}{\lim} W_i \right) \,.$$

This exhibits (degreewise and hence globally) the homotopy pullback property to be show. ▮

Proposition 1.71. *The Omega-spectrification (Q, η) from def. 1.21 is a Quillen idempotent monad in the sense of def. 1.64 on the strict model structre theorem 1.42:*

$$Q \;:\; \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \longrightarrow \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \,.$$

(Schwede 97, prop. 2.1.5)

Proof. First notice that the strict model structure is indeed right proper, as demanded in def. 1.64: Since every object in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ is fibrant (this being so degreewise in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$) this follows from this lemma.

The first two conditions required on a Quillen idempotent monad in def. 1.64 are explicit in prop. 1.22.

The third condition follows from lemma 1.70: A pullback of a Q-equivalence along a fibration is a homotopy pullback and is hence sent by Q to another homotopy pullback square.

$$\begin{array}{ccc} f^* Z & \overset{f^* h}{\longrightarrow} & X \\ \downarrow & (\mathrm{pb}) & \downarrow^{f \in \mathrm{Fib}} \\ Z & \underset{h \in W_Q}{\longrightarrow} & Y \end{array} \quad \Rightarrow \quad \begin{array}{ccc} Q(f^* Z) & \overset{Q(f^* h) \in W}{\longrightarrow} & Q(X) \\ \downarrow & (\mathrm{pb})^h & \downarrow^{Q(f)} \\ Q(Z) & \underset{Q(h) \in W}{\longrightarrow} & Q(Y) \end{array}$$

By definition of Q-equivalence that resulting homotopy pullback square has the bottom edge a weak equivalence, and hence also the top edge is a weak equivalence (prop.). ▮

Theorem 1.72. *The left Bousfield localization of the strict model structure on sequential* *spectra (theorem 1.42) at the class of stable weak homotopy equivalences (def. 1.16) exists, called the **stable model structure** on topological sequential spectra*

$$\mathrm{SeqSpec}(\mathrm{Top}_{cg})_{\mathrm{stable}} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{SeqSpec}(\mathrm{Top}_{cg})_{\mathrm{strict}} \,.$$

Moreover, its fibrant objects are precisely the Omega-spectra (def.1.18).

Proof. Let (Q, η) be the Omega-spectrification operation from def. 1.21. According to prop. 1.71 this is a Quillen-idempotent monad (def. 1.64) on $\mathrm{SeqSpec}(\mathrm{Top}_{cg})_{\mathrm{strict}}$. Hence the Bousfield-Friedlander theorem (prop. 1.68) asserts that the Bousfield localization of the strict model structure at the Q-equivalences exists. By prop. 1.22 these are precisely the stable weak homotopy equivalences.

Finally, by prop. 1.69 an object $X \in \mathrm{SeqSpec}(\mathrm{Top}_{cg})_{\mathrm{stable}}$ is fibrant in $\mathrm{SeqSpec}(\mathrm{Top}_{cg})_{\mathrm{stable}}$ precisely if

$$\begin{array}{ccc} X & \overset{\eta_X}{\longrightarrow} & Q(X) \\ \downarrow && \downarrow \\ * & \longrightarrow & * \end{array}$$

exhibits a homotopy pullback in $\mathrm{SeqSpec}(\mathrm{Top}_{cg})_{\mathrm{strict}}$. Since every object in $\mathrm{SeqSpec}(\mathrm{Top}_{cg})_{\mathrm{strict}}$ is fibrant, the vertical morphisms here are fibrations. The pullback of $Q(X)$ along id_* is just $Q(X)$ itself, and the universally induced morphism into this pullback is just η_X itself. Hence the square is a homotopy pullback precisely if η_X is a weak equivalence in $\mathrm{SeqSpec}(\mathrm{Top}_{cg})_{\mathrm{strict}}$, hence degreewise a weak homotopy equivalence. Since $Q(X)$ is an Omega-spectrum by prop. 1.22, this means precisely that X is an Omega-spectrum. ∎

Stability of the homotopy theory

We discuss that the stable model structure $\mathrm{SeqSpec}(\mathrm{Top}_{cg})_{\mathrm{stable}}$ of theorem 1.72 is indeed a *stable model category*, in that the canonical reduced suspension operation is an equivalence of categories from the stable homotopy category (def. 1.97) to itself. This is theorem 1.84 below.

Definition 1.73. A pointed model category $\mathcal{C}$ (exmpl.) is called a **stable model category** if the canonically induced reduced suspension and loop space object-functors (prop.) on its homotopy category (defn.) constitute an equivalence of categories

$$(\Sigma \dashv \Omega) \,:\, \mathrm{Ho}(\mathcal{C}) \underset{\Omega}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \mathrm{Ho}(\mathcal{C}) \,.$$

Literature (Jardine 15, sections 10.3 and 10.4)

First we observe that the *alternative suspension* induces an equivalence of homotopy categories:

Lemma 1.74. *With Σ and Ω the alternative suspension and alternative looping functors from def. 1.34:*

1. *Ω preserves Omega-spectra (def. 1.18);*

2. *Σ preserves stable weak homotopy equivalences (def. 1.16).*

Proof. Regarding the first statement:

By prop. 1.2, Ω acts on component spaces and adjunct structure maps as the right Quillen functor

$$\mathrm{Maps}(S^1,-)_* : (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \longrightarrow (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$$

on the classical model structure on pointed compactly generated topological spaces (thm., prop.). Since in this model structure all objects are fibrant, Ken Brown's lemma (prop.) implies that with $\tilde\sigma_n^X$ a weak homotopy equivalence, so is $\tilde\sigma_n^{\Omega X} = \mathrm{Maps}(S^1, \tilde\sigma_n^X)$.

Regarding the second point:

Let $f : X \to Y$ be a stable weak homotopy equivalence. By the existence of the model structure $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ from theorem 1.72, Σf is a stable weak homotopy equivalence precisely if its image in the homotopy category $\mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}})$ is an isomorphism (prop.). By the Yoneda lemma (fully faithfulness of the Yoneda embedding), this is the case if for all $Z \in \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}})$ the function

$$[\Sigma f, Z]_{\mathrm{stable}} : [\Sigma Y, Z]_{\mathrm{stable}} \longrightarrow [\Sigma X, Z]_{\mathrm{stable}}$$

is a bijection. By the fact that the stable model structure is a left Bousfield localization of the strict model structure with fibrant objects the Omega-spectra, this is the case equivalently (using this lemma) if

$$[\Sigma f, Z]_{\mathrm{strict}} : [\Sigma Y, Z]_{\mathrm{strict}} \longrightarrow [\Sigma X, Z]_{\mathrm{strict}}$$

is a bijection for all Omega-spectra Z. Now by the Quillen adjunction $\Sigma \dashv \Omega$ on the strict model category (prop. 1.44) this is equivalent to

$$[f, \Omega Z]_{\mathrm{strict}} : [Y, \Omega Z]_{\mathrm{strict}} \longrightarrow [X, \Omega Z]_{\mathrm{strict}}$$

being a bijection for all Omega-spectra Z. But since Ω preserves Omega-spectra by the first point above, this is still maps into a fibrant objects, hence is again equivalent (using again the property of the left Bousfield localization) to the hom in the strict model structure

$$[f, \Omega Z]_{\mathrm{stable}} : [Y, \Omega Z]_{\mathrm{stable}} \longrightarrow [X, \Omega Z]_{\mathrm{stable}}$$

being a bijection for all ΩZ. But this is indeed a bijection, since f is a stable weak homotopy equivalence, hence an isomorphism in the homotopy category. ▮

Lemma 1.75. *For X a sequential spectrum, then (using remark 1.37 to suppress parenthesis)*

1. *the structure maps constitute a homomorphism*

$$\Sigma X[-1] \longrightarrow X$$

(from the shift, def. 1.33, of the alternative suspension, def. 1.34) and this is a stable weak homotopy equivalence,

2. *the adjunct structure maps constitute a homomorphism*

$$X \longrightarrow \Omega X[1]$$

(to the shift, def. 1.33, of the alternative looping, def. 1.34)

If X is an Omega-spectrum (def. 1.18) then this is a weak equivalence in the strict model structure (def. 1.40), hence in particular a stable weak homotopy equivalence.

Proof. The diagrams that need to commute for the structure maps to give a homomorphism as claimed are in degree 0 this one

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge \ast & \overset{0}{\longrightarrow} & X_0 \\ {}^{S^1 \wedge 0}\downarrow && \downarrow^{\sigma_0} \\ S^1 \wedge X_0 & \underset{\sigma_0}{\longrightarrow} & X_1 \end{array}$$

and in degree $n \geq 1$ these:

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge X_{n-1} & \overset{S^1 \wedge \sigma_{n-1}}{\longrightarrow} & X_n \\ {}^{S^1 \wedge \sigma_{n-1}}\downarrow && \downarrow^{\sigma_n} \\ S^1 \wedge X_n & \underset{\sigma_n}{\longrightarrow} & X_{n+1} \end{array} \,.$$

But in all these cases commutativity it trivially satisfied.

That the adjunct structure maps constitute a morphism $X \to \Omega X[1]$ follows dually.

If X is an Omega-spectrum, then by definition this last morphism is already a weak equivalence in the strict model structure, hence in particular a weak equivalence in the stable model structure.

From this it follows that also $\Sigma X[-1] \to X$ is a stable weak homotopy equivalence, because for every Omega-spectrum Y then by the adjunctions in prop. 1.38 we have a commuting diagram of the form

$$\begin{array}{ccc} [X,Y]_{\text{strict}} & \longrightarrow & [\Sigma X[-1], Y]_{\text{strict}} \\ {}^{\text{id}}\downarrow && \downarrow^{\simeq} \\ [X,Y]_{\text{strict}} & \underset{\simeq}{\longrightarrow} & [X, \Omega Y[1]]_{\text{strict}} \end{array} \,.$$

(To see the commutativity of this diagram in detail, consider for any $[f] \in [X,Y]_{\text{strict}}$ chasing the element σ_n^Y in the two possible ways through the natural adjunction isomorphism:

$$\begin{array}{ccc} [S^1 \wedge Y_{n-1}, Y_n] & \simeq & [Y_{n-1}, \Omega Y_n] \\ {}^{[S^1 \wedge f_{n-1}, Y_n]}\downarrow && \downarrow^{[f_{n-1}, \Omega Y_n]} \\ [S^1 \wedge X_{n-1}, Y_n] & \simeq & [X_{n-1}, \Omega Y_n] \end{array}$$

Sending σ_n^Y down gives $\sigma_n^Y \circ S^1 \wedge f_{n-1}$ which equals (by the homomorphism property) $f_n \circ \sigma_n^X$. Instead sending σ_n^Y to the right yields $\tilde{\sigma}_n^Y$ and then down yields $\tilde{\sigma}_n^Y \circ f_{n-1}$. By commutativity this is adjunct to $f_n \circ \sigma_n^X$.)

Hence

$$[X,Y]_{\text{strict}} \longrightarrow [\Sigma X[-1], Y]_{\text{strict}}$$

is a bijection for all Omega-spectra Y, and so the conclusion that $\Sigma X[-1] \to X$ is a stable weak homotopy equivalence follows as in the proof of lemma 1.74. ∎

Lemma 1.76. *The total derived functor of the alternative suspension operation Σ of def.*

1.34 exists and constitutes an equivalence of categories from the stable homotopy category to itself:

$$\Sigma : \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top})_{\mathrm{stable}}) \xrightarrow{\simeq} \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top})_{\mathrm{stable}}) \,.$$

Proof. The total derived functor of Σ exists, because by lemma 1.74 Σ preserves stable weak homotopy equivalences. Also the shift functor $[-1]$ from def. 1.33 clearly preserves stable equivalences, hence both descend to the homotopy category. There, by prop. 1.75 and remark 1.37, they are inverses of each other, up to isomorphism. ▮

Lemma 1.77. *The canonical suspension functor on the homotopy category of any model category (from this prop.) in the case of the stable homotopy category (def. 1.97) $\mathrm{Ho}(\mathrm{Spectra}) = \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}})$ is represented by the "standard suspension" operation of def. 1.31.*

Proof. By CW-approximation (prop. 1.55), every object in the stable homotopy category is represented by a CW-spectrum. By prop. 1.52, on CW-spectra the canonical suspension functor on the homotopy category (from this prop.) is represented by the "standard suspension" operation of def. 1.31. ▮

The combination of lemma 1.76 with lemma 1.77 gives that in order to show that $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ is indeed a stable model category according to def. 1.73, we are reduced to showing that in the homotopy category the alternative suspension operation (which we know gives an equivalence) is naturally isomorphic to the standard suspension operation (which we know is the correct suspension operation). This we turn to now.

According to remark 1.36, both should be directly comparable and isomorphic in the homotopy category "in even degrees", but non-comparable in odd degree. In order to make this precise, we now introduce the concept of sequential spectra with components only in even degree and then use an adjunction back to ordinary sequential spectra.

Observe that the definition of the category $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ of sequential spectra in def. 1.3 does not require anything specific of the circle S^1: the same kind of definition may be considered for any other pointed topological space T in place of S^1. The construction of the stable model structure $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ in theorem 1.72 does depend on the nature of S^1, but only in that it uses that the n-spheres $S^n = (S^1)^{\wedge n}$

1. co-represent homotopy groups in the classical pointed homotopy category: $[S^n, -]_* \simeq \pi_n(-)$;

2. are compact, so that maps out of them factor through finite stages of transfinite compositions of relative cell complex inclusions.

Both points still hold with S^1 replaced by $S^1 \wedge K_+$, for K any contractible compact topological space. Moreover, since only the stable homotopy groups matter for the construction of the stable model category, one could replace S^1 by any S^k: While the smash powers $(S^k)^{\wedge n}$ co-represent only every kth homotopy group, this is still sufficient for co-represent all the stable homotopy groups.

The following is an immediate variant of the definition 1.3 of sequential spectra:

Definition 1.78. Let $T = K_+ \in \mathrm{Top}^{*/}_{\mathrm{cg}}$ be a compact contractible topological space with a basepoint freely adjoined, and let $k \in \mathbb{N}$, $k \geq 1$.

A **sequential $T \wedge S^k$-spectrum** is a sequence of component spaces $X_{kn} \in \mathrm{Top}_{\mathrm{cg}}$ for $n \in \mathbb{N}$, and a sequence of structure maps of the form

$$\sigma_{k,n} \;:\; T \wedge S^k \wedge X_{kn} \longrightarrow X_{k(n+1)} \,.$$

A homomorphism of sequential $T \wedge S^k$-spectra $f : X \to Y$ is a sequence of component maps $f_{kn} \;:\; X_{kn} \to Y_{kn}$ such that all these diagrams commute:

$$\array{ T \wedge S^k \wedge X_{kn} &\overset{T \wedge S^k \wedge f_{kn}}{\longrightarrow}& T \wedge S^k \wedge Y_{kn} \\ {}^{\mathllap{\sigma^X_{k,n}}}\downarrow && \downarrow^{\mathrlap{\sigma^Y_{k,n}}} \\ X_{k(n+1)} &\underset{f_{k(n+1)}}{\longrightarrow}& Y_{k(n+1)} } \,.$$

Write

$$Seq_{T \wedge S^k}Spec(Top_{cg})$$

for the resulting category of sequential $T \wedge S^k$-spectra.

Proposition 1.79. *For any $T \wedge S^k$ as in def. 1.78, there exists a model category structure*

$$Seq_{T \wedge S^k}Spec(Top_{cg})_{stable}$$

on the category of sequential $T \wedge S^k$-spectra, where

- *the weak equivalences are the morphisms that induce isomorphisms under $\underset{\longrightarrow_{kn \in k \mathbb{N}}}{\lim} \pi_{kn}(-)$;*
- *the fibrations are the morphisms whose η_k-naturality square is a homotopy pullback, where $\eta_K : id \to Q_k$ is the $K \wedge S^k$-spectrification functor defined as in def. 1.21 but with S^1 replaced by $T \wedge S^k$ throughout.*

Proof. The proof is verbatim that of theorem 1.72, with S^1 replaced by $T \wedge S^k$ throughout. ▮

Lemma 1.80. *For $k \in \mathbb{N}$, $k \geq 1$, there is a pair of adjoint functors*

$$SeqSpec(Top_{cg}) \underoverset{\underset{R_k}{\longrightarrow}}{\overset{L_k}{\longleftarrow}}{\bot} Seq_{S^k}Spec(Top_{cg})$$

between sequential spectra (def. 1.3) and sequential S^k-spectra (def. 1.78)

- *where $(R_k X)_{kn} \coloneqq X_{kn}$ and*

$$\sigma_n^{R_k X} \;:\; S^k X_{kn} \simeq S^{k-1} \wedge S^1 \wedge X_{kn} \overset{S^1 \wedge \sigma^X_{kn}}{\longrightarrow} S^{k-1} \wedge X_{kn+1} \longrightarrow \cdots \longrightarrow S^1 \wedge X_{kn + (k-1)} \overset{\sigma^X_{kn + (k-1)}}{\longrightarrow} X_{k(n+1)}$$

- *and where*

$$(L_k \mathcal{X})_n \coloneqq \left\{ \array{ \mathcal{X}_n & if \; n \in k \mathbb{N} \\ S^q \wedge \mathcal{X}_{n-q} & if \; q \lt k \; and \; n-q \in k \mathbb{N} } \right.$$

and

$$\sigma_n^{L_k \mathcal{X}} = \left\{ \array{ \sigma^{\mathcal{X}}_{n-(k-1)} & if \; n+1 \in k\mathbb{N} \\ id_{S^1 \wedge \mathcal{X}_n} & otherwise } \right. \,.$$

Moreover, for each $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$, the adjunction unit

$$L_k R_k X \longrightarrow X$$

is a stable weak homotopy equivalence (def. 1.16).

Proof. For ease of notation we discuss this for $k = 2$. The general case is directly analogous. To see that we have an adjunction, consider a homomorphism

$$f : L_2 \mathcal{X} \longrightarrow Y \,.$$

Given its even-graded component maps, then its odd-graded component maps f_{2n+1} need to fit into commuting squares of the form

$$\begin{array}{ccc} S^1 \wedge \mathcal{X}_{2n} & \overset{S^1 \wedge f_{2n}}{\longrightarrow} & S^1 \wedge Y_{2n} \\ {}^{\mathllap{\mathrm{id}}}\downarrow && \downarrow^{\mathrlap{\sigma^Y_{2n}}} \\ S^1 \wedge \mathcal{X}_{2n} & \underset{f_{2n+1}}{\longrightarrow} & Y_{2n+1} \end{array} \,.$$

Since here the left map is an identity, this uniquely fixes the odd-graded components f_{2n+1} in terms of the even-graded components. Moreover, these components then make the following pasting rectangles comute

$$\begin{array}{ccc} S^2 \wedge \mathcal{X}_{2n} & \overset{S^2 \wedge f_{2n}}{\longrightarrow} & S^2 \wedge Y_{2n} \\ {}^{\mathllap{\simeq}}\downarrow && \downarrow^{\mathrlap{S^1 \wedge \sigma^Y_{2n}}} \\ S^2 \wedge \mathcal{X}_{2n} & \overset{S^1 \wedge f_{2n+1}}{\longrightarrow} & S^1 \wedge Y_{2n+1} \\ {}^{\mathllap{\sigma^{\mathcal{X}}_{2n}}}\downarrow && \downarrow^{\mathrlap{\sigma^Y_{2n+1}}} \\ \mathcal{X}_{2n+2} & \overset{f_{2n+2}}{\longrightarrow} & Y_{2n+2} \end{array} \,.$$

This equivalently exhibits f as a homomorphism of the form

$$\tilde f : \mathcal{X} \longrightarrow R_2 Y$$

and hence establishes the adjunction isomorphism.

Finally to see that the adjunction unit is a stable weak homotopy equivalence: for $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ then the morphism of stable homotopy groups induced from

$$L_2 R_2 X \longrightarrow X$$

is in degree q given by

$$\begin{array}{ccccccccc} \underset{\longrightarrow}{\lim}(\cdots & \to & \pi_{q+2k}(X_{2k}) & & \longrightarrow & & \pi_{q+2k+2}(X_{q+2k+2}) & \to & \cdots) & = & \pi_q(L_2 R_2 X) \\ && {}^{\mathllap{\simeq}}\downarrow &&&& {}^{\mathllap{\simeq}}\downarrow &&&& \downarrow \\ \underset{\longrightarrow}{\lim}(\cdots & \to & \pi_{q+2k}(X_{2k}) & \longrightarrow & \pi_{2+2k+1}(X_{2k+1}) & \longrightarrow & \pi_{q+2k+2}(X_{q+2k+2}) & \to & \cdots) & = & \pi_q(X) \end{array} \,.$$

From this it is clear by inspection that the induced vertical map on the right is an isomorphism. Stated more abstractly: the inclusion of partially ordered sets $\mathbb{N}^{\leq}_{\mathrm{even}} \hookrightarrow \mathbb{N}^{\leq}$ is a cofinal functor and hence restriction along it preserves colimits. ▮

Definition 1.81. For

$$\alpha : T_1 \wedge S^k \longrightarrow T_2 \wedge S^k$$

any morphism, write

$$\alpha^* : \mathrm{Seq}_{T_2 \wedge S^k}\mathrm{Spect}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{Seq}_{T_1 \wedge S^k}\mathrm{Spect}(\mathrm{Top}_{\mathrm{cg}})$$

for the functor from the category of sequential $T_2 \wedge S^k$-spectra (def. 1.78) to that of $T_1 \wedge S^k$-spectra which sends any X to $\alpha^* X$ with

$$(\alpha^* X)_{kn} \coloneqq X_{kn}$$

and

$$\sigma^{\alpha^* X}_{k,n} : T_1 \wedge S^k \wedge X_{kn} \overset{\alpha \wedge \mathrm{id}}{\longrightarrow} T_2 \wedge S^k \wedge X_{kn} \overset{\sigma^X_{k,n}}{\longrightarrow} X_{k(n+1)} \,.$$

Lemma 1.82. *For $T \coloneqq K_+$ a compact contractible topological space with base point adjoined, and for $k \in \mathbb{N}$, write $i : S^k \longrightarrow T \wedge S^k$ for the canonical inclusion. Then the induced functor i^* from def. 1.81 is the right adjoint in a Quillen equivalence (def.)*

$$\mathrm{Seq}_{T \wedge S^1}\mathrm{Spec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \underoverset{\underset{i^*}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\simeq_{\mathrm{Qu}}} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$$

between the stable model structures of sequential S^k-spectra and of sequential $T \wedge S^k$-spectra (prop. 1.79), respectively.

(Jardine 15, theorem 10.40)

Proof. Write $p : T \wedge S^1 \to S^1$ for the canonical projection.

A morphism

$$f : X \longrightarrow i^* Y$$

is given by components fitting into commuting squares of the form

$$\begin{array}{ccc} S^1 \wedge X_n & \overset{S^1 \wedge f_n}{\longrightarrow} & S^1 \wedge Y_n \\ {}^{\mathrm{id}}\downarrow && \downarrow^{i \wedge \mathrm{id}} \\ S^1 \wedge X_n && T \wedge S^1 \wedge Y_n \,. \\ {}^{\sigma^X_n}\downarrow && \downarrow^{\sigma^Y_n} \\ X_{n+1} & \underset{f_{n+1}}{\longrightarrow} & Y_{n+1} \end{array}$$

Since $p \circ i = \mathrm{id}$, every such diagram factors as

$$
\array{
S^1 \wedge X_n & \overset{S^1 \wedge f_n}{\longrightarrow} & S^1 \wedge Y_n
\\
{}^{\mathllap{i \wedge id}}\downarrow && \downarrow^{\mathrlap{i \wedge id}}
\\
T \wedge S^1 \wedge X_n & \overset{T \wedge S^1 \wedge f_n}{\longrightarrow} & T \wedge S^1 \wedge Y_n
\\
{}^{\mathllap{p \wedge id}}\downarrow && \downarrow
\\
S^1 \wedge X_n &&
\\
{}^{\mathllap{\sigma_n^X}}\downarrow && \downarrow^{\mathrlap{\sigma_n^Y}}
\\
X_{n+1} & \underset{f_{n+1}}{\longrightarrow} & Y_{n+1}
}
\,.
$$

Here the bottom square exhibits the components of a morphism

$$\tilde f \;:\; p^\ast X \longrightarrow Y$$

and this correspondence is clearly naturally bijective

This establishes the adjunction $p^\ast \dashv i^\ast$. This is a Quillen equivalence because for every $Z \in Top_{cg}^{\ast/}$ then by the contractibility of K there is an equivalence

$$[T \wedge S^q, Z]_\ast \simeq [S^q, Z]_\ast$$

and hence the concept of stable weak homotopy equivalences in both categories agrees. Hence any $\tilde f : p^\ast X \to Y$ is a stable weak homotopy equivalence precisely if $f : X \to i^\ast y$ is. ▮

With this in hand, we now finally state the comparison between standard and alternative suspension:

Lemma 1.83. *There is a natural isomorphism in the homotopy category $Ho(SeqSpec(Top_{cg})_{stable})$ of the stable model structure, between the total derived functors (prop.) of the standard suspension (def. 1.31) and of the alternative suspension (def. 1.34):*

$$\Sigma(-) \;\simeq\; (-) \wedge S^1 \qquad \in Ho(SeqSpec(Top_{cg})_{stable})$$

Notice that we agreed in Part P to suppress the notation $\mathbb{L}$ for left derived functors of the suspension functor, not to clutter the notation. If we re-instantiate this then the above says that there is a natural isomorphism

$$\mathbb{L}\Sigma \;\simeq\; \mathbb{L}((-)\wedge S^1) \,.$$

(Jardine 15, corollary 10.42, prop. 10.53)

Proof. Consider the adjunction $(L_2 \dashv R_2) : SeqSpec(Top) \leftrightarrow Seq_2Spec(Top)$ from lemma 1.80. We claim that there is a natural isomorphism

$$\tau \;:\; R_2(\Sigma(-)) \simeq R_2((-)\wedge S^1) \,,$$

in $Ho(Seq_{S^2}Spec(Top_{cg})_{stable})$.

This implies the statement, since by lemma 1.80 the adjunction unit is a stable weak equivalence, so that we get natural isomorphisms

$$\Sigma X \simeq L_2 R_2 (\Sigma X) \underoverset{\simeq}{L_2 \tau}{\longrightarrow} L_2 R_2(X \wedge S^1) \simeq X \wedge S^1$$

in $\mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}})$ (where we are using that R_2 evidently preserves cofibrant spectra, so that L_2 applied to τ represents the correct derived functor of L_2 and hence preserves this isomorphism).

Now to see that the isomorphism τ exists. Write

$$\tau_{S^2,S^1} : S^2 \wedge S^1 \overset{\simeq}{\longrightarrow} S^1 \wedge S^2$$

for the braiding isomorphism, which swaps the first two canonical coordinates with the third. Since the homotopy class of this map is trivial in that

$$[\tau_{S^2,S^1}] = 1 \in \mathbb{Z} \simeq \pi_3(S^3)$$

is the trivial element in the homotopy groups of spheres (and that is the point of passing to S^2-spectra here, because for S^1-spectra the analogous map τ_{S^1,S^1} has non-trivial class, remark 1.36) it follows that there is a left homotopy (def.) of the form

$$\begin{array}{ccccc} S^3 & \overset{i_0}{\longrightarrow} & (I_+) \wedge S^3 & \overset{i_1}{\longleftarrow} & S^3 \\ & {}_{\mathrm{id}}\searrow & \downarrow & \swarrow_{\tau_{S^2,S^1}} \\ && S^3 \end{array}.$$

By forming the smash product of the entire diagram with X_{2n} and pasting on the right the naturality square for the braiding with S^1

$$\begin{array}{ccc} S^1 \wedge S^2 \wedge X_{2n} & \overset{\tau_{S^2 \wedge X_{2n}, S^1}}{\longleftarrow} & S^2 \wedge X_{2n} \wedge S^1 \\ {}^{S^1 \wedge (\sigma_{2n+1} \circ (S^1 \wedge \sigma_{2n}))}\downarrow && \downarrow^{(\sigma_{2n+1} \circ (S^1 \wedge \sigma_{2n})) \wedge S^1} \\ S^1 \wedge X_{2(n+1)} & \underset{\tau_{X_{2n}, S^1}}{\longleftarrow} & X_{2n} \wedge S^1 \end{array}$$

this yields the diagram

$$\begin{array}{ccccccc} S^3 \wedge X_{2n} & \overset{i_0}{\longrightarrow} & (I_+) \wedge S^3 \wedge X_{2n} & \overset{i_1}{\longleftarrow} & S^3 \wedge X_{2n} & \underset{\simeq}{\overset{S^2 \wedge \tau_{X_{2n}, S^1}}{\longleftarrow}} & S^2 \wedge X_{2n} \wedge S^1 \\ & {}_{\mathrm{id}}\searrow & \downarrow & \swarrow_{\tau_{S^2,S^1} \wedge X_n} && & \downarrow \\ && S^3 \wedge X_{2n} &&&& \downarrow^{(\sigma_{2n+1} \circ (S^1 \wedge \sigma_{2n})) \wedge S^1} \\ && {}_{S^1 \wedge (\sigma_{2n+1} \circ (S^1 \wedge \sigma_{2n}))} & \searrow &&& \downarrow \\ &&&& S^1 \wedge X_{2n} & \underset{\tau_{X_{2n}, S^1}}{\overset{\simeq}{\longleftarrow}} & X_{2n} \wedge S^1 \end{array}$$

Here the left diagonal composite is the structure map of $R_2(\Sigma X)$ in degree n, while the right vertical morphism is the structure map of $R_2(X \wedge S^1)$ in degree n. In the middle we have the structure map of an auxiliary $(I_+) \wedge S^2$-spectrum (def. 1.78)

$$Z \in \mathrm{Seq}_{I_+ \wedge S^2}\mathrm{Spec}(\mathrm{Top}_{\mathrm{cg}}) ,$$

and the horizontal morphisms exhibit the functors of def. 1.81 from $(I_+) \wedge S^2$-spectra to S^2-spectra with

$$i_0^\ast Z = R_2(\Sigma X) \quad , \quad i_1^\ast Z = R_2(X \wedge S^1) \; .$$

By lemma 1.82 and since I is contractible, these functors are equivalences of categories on the $\mathrm{Ho}(\mathrm{Seq}_{S^2}\mathrm{Spec}(\mathrm{Top}_{\mathrm{cg}}))$, and moreover they have the same inverse, namely p^* for $p : I_+ \wedge S^2 \to S^2$ the canonical projection. This implies the isomorphism.

Explicitly, due to the equivalence there exists V with $Z \simeq p^* V$ and with this we may form the composite isomorphism

$$R_2(\Sigma X) \simeq i_0^* Z \simeq i_0^* p^* V \simeq V \simeq i_1^* p^* V \simeq i_1^* Z \simeq R_2(X \wedge S^1) \,.$$

▮

We conclude:

Theorem 1.84. *The stable model structure $\mathrm{SeqSpec}(\mathrm{Top})_{\mathrm{stable}}$ from theorem 1.72 indeed gives a stable model category in the sense of def. 1.73, in that the canonically induced reduced suspension functor (prop.) on its homotopy category is an equivalence of categories*

$$\Sigma \;:\; \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top})_{\mathrm{stable}}) \overset{\simeq}{\longrightarrow} \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top})_{\mathrm{stable}}) \,.$$

Proof. By lemma 1.77, the canonical suspension functor is represented, on fibrant-cofibrant objects, by the standard suspension functor of def. 1.31. By prop. 1.83 this is naturally isomorphic – on the level of the homotopy category – to the alternative suspension operation of def. 1.34. Therefore the claim follows with prop. 1.76. ▮

In fact this lifts to a Quillen equivalence:

Proposition 1.85. *The $(\Sigma \dashv \Omega)$-adjunction from prop. 1.38 is a Quillen equivalence (def.) with respect to the stable model structure of theorem 1.72:*

$$\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq_Q} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \,.$$

Its derived functors (prop.) exhibit the canonical reduced suspension and looping operation as an adjoint equivalence on the stable homotopy category

$$\mathrm{Ho}(\mathrm{Spectra}) \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq} \mathrm{Ho}(\mathrm{Spectra}) \,.$$

Proof. By prop. 1.44 and the fact that the stable model structure has the same cofibrations as the strict model structure, Σ preserves stable cofibrations. Moreover, by lemma 1.74 Σ preserves in fact all stable weak equivalences. Hence Σ is a left Quillen functor and so $(\Sigma \dashv \Omega)$ is a Quillen adjunction. Finally lemma 1.76 gives that this Quillen adjunction is a Quillen equivalence. ▮

In summary, this concludes the characterization of the stable homotopy category as the result of stabilizing the canonical $(\Sigma \dashv \Omega)$-adjunction on the classical homotopy category:

Theorem 1.86. *The classical model structure $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ on pointed compactly generated topological spaces (thm., prop.) and the stable model structure on topological sequential spectra $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ (theorem 1.72) sit in a commuting diagram of Quillen adjunctions of the form*

$$
\begin{array}{ccc}
(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \\
{}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\
\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \\
{}^{\mathllap{\mathrm{id}}}\downarrow \dashv \uparrow^{\mathrlap{\mathrm{id}}} && {}^{\mathllap{\mathrm{id}}}\downarrow \dashv \uparrow^{\mathrlap{\mathrm{id}}} \\
\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq_Q} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}
\end{array}
,
$$

where the top parts is from corollary 1.45, the bottom vertical Quillen adjunction is the Bousfield localization of theorem 1.72 and the bottom horizontal adjunction is the Quillen equivalence of prop. 1.85.

Hence (by this prop.) the derived functors of the functors in this diagram yield a commuting square of adjoint functors between the classical homotopy category (def.) and the stable homotopy category (def. 1.97) of the form

$$
\begin{array}{ccc}
\mathrm{Ho}(\mathrm{Top}^{*/}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{Ho}(\mathrm{Top}^{*/}) \\
{}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\
\mathrm{Ho}(\mathrm{Spectra}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq} & \mathrm{Ho}(\mathrm{Spectra})
\end{array}
,
$$

where the horizontal adjunctions are the canonically induced (via this prop.)suspension/looping functors by prop. 1.2 and by lemma 1.77 and theorem 1.84.

Cofibrant generation

We show that the stable model structure $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ from theorem 1.72 is a cofibrantly generated model category (def.).

We will not use the result of this section in the remainder of part 1.1, but the following argument is the blueprint for the proof of the model structure on orthogonal spectra that we consider in part 1.2, in the section *The stable model structure on structured spectra*, and it will be used in the proof of the Quillen equivalence of $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ to the stable model structure on orthogonal spectra (thm.).

Moreover, that $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ is cofibrantly generated means that for $\mathcal{C}$ any topologically enriched category (def.) then there exists a projective model structure on functors $[\mathcal{C}, \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}]_{\mathrm{proj}}$ on the category of topologically enriched functors $\mathcal{C} \to \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ (def.), in direct analogy to the projective model structure $[\mathcal{C}, (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}]_{\mathrm{proj}}$ (thm.). This is the model structure for *parameterized stable homotopy theory*. Just as the stable homotopy theory discussed here is the natural home of generalized (Eilenberg-Steenrod) cohomology theories (example 1.102) so parameterized stable homotopy theory is the natural home of twisted cohomology theories.

In order to express the generating (acyclic) cofibrations, we need the following simple but important concept.

Definition 1.87. For $K \in \mathrm{Top}_{\mathrm{cg}}^{*/}$, and $n \in \mathbb{N}$, write $F_n K \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ for the **free spectrum** on K at n, with components

$$(F_n K)_q \coloneqq \begin{cases} * & \text{for } q < n \\ S^{q-n} \wedge K & \text{for } q \geq n \end{cases}$$

and with structure maps σ_q the canonical identifications for $q \geq n$

$$\sigma_q \;:\; S^1 \wedge (F_n K)_q = S^1 \wedge S^{q-n} \wedge K \overset{\simeq}{\longrightarrow} S^{q+1-n} \wedge K = (F_n K)_{q+1} \;.$$

For $n \in \mathbb{N}$, write

$$k_n \;:\; F_{n+1} S^1 \longrightarrow F_n S^0$$

for the canonical morphisms of free sequential spectra with the following components

$$\begin{array}{cccc} & \vdots && \vdots \\ (k_n)_{n+3} & S^3 & \overset{\mathrm{id}}{\longrightarrow} & S^3 \\ (k_n)_{n+2} & S^2 & \overset{\mathrm{id}}{\longrightarrow} & S^2 \\ (k_n)_{n+1} & S^1 & \overset{\mathrm{id}}{\longrightarrow} & S^1 \\ (k_n)_n\colon & * & \overset{0}{\longrightarrow} & S^0 \\ & * & \longrightarrow & * \\ & \vdots && \vdots \\ & * & \longrightarrow & * \\ & \underbrace{} && \underbrace{} \\ k_n\colon & F_{n+1}S^1 & \longrightarrow & F_n S^0 \end{array}$$

Example 1.88. The free spectrum $F_0 S^0$ (def. 1.87) is the standard sequential sphere spectrum from def. 1.6

$$F_0 S^0 \simeq \mathbb{S}_{\mathrm{std}} \;.$$

Generally the free spectrum $F_0 K$ is the suspension spectrum (def. 1.5) on K:

$$F_0 K \simeq \Sigma^\infty K \;.$$

Just as forming suspension spectra is left adjoint to extracting the 0th component space of a sequential spectrum (prop. 1.12), so forming the nth free spectrum is left adjoint to extracting the nth component space:

Proposition 1.89. *For $n \in \mathbb{N}$, let*

$$\mathrm{Ev}_n \;:\; \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}$$

be the functor from sequential spectra (def. 1.3) to pointed topological spaces given by extracting the nth component space

$$\mathrm{Ev}_n(X) \coloneqq X_n \;.$$

Then this functor is right adjoint to forming nth free spectra (def. 1.87):

$$(F_n \dashv \mathrm{Ev}_n) \;:\; \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \underoverset{\underset{\mathrm{Ev}_n}{\longrightarrow}}{\overset{F_n}{\longleftarrow}}{\bot} \mathrm{Top}_{\mathrm{cg}}^{*/}\,.$$

Proof. The proof is verbatim as that of prop. 1.12, just with n zeros inserted at the bottom of the sequences of components maps. ∎

Definition 1.90. Write

$$I_{\mathrm{seq}}^{\mathrm{stable}} \coloneqq I_{\mathrm{seq}}^{\mathrm{strict}} \quad \in \mathrm{SeqSpec}(\mathrm{Top})$$

for the set of morphisms appearing already in def. 1.41, and write

$$J_{\mathrm{seq}}^{\mathrm{stable}} \coloneqq J_{\mathrm{seq}}^{\mathrm{strict}} \;\sqcup\; \{k_n \,\Box\, i_+\}_{n \in \mathbb{N}, i_+ \in \left(I_{\mathrm{Top}^{*/}}\right)}$$

for the disjoint union of the other set of morphisms appearing in def. 1.41 with the set $\{k_n \,\Box\, i_+\}_{n,i_+}$ of pushout-products under smash tensoring (according to def. 1.57) of the morphisms k_n from def. 1.87 with the generating cofibrations of the classical model structure on pointed topological spaces (def.).

Theorem 1.91. *The stable model structure $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ from theorem 1.72 is cofibrantly generated (def.) with generating (acyclic) cofibrations the sets $I_{\mathrm{seq}}^{\mathrm{stable}}$ (and $J_{\mathrm{seq}}^{\mathrm{stable}}$) from def. 1.90.*

This is one of the cofibrantly model categories considered in (Mandell-May-Schwede-Shipley 01) .

Proof. It is clear (as in theorem 1.42) that the two classes have small domains (def.). Moreover, since $I_{\mathrm{seq}}^{\mathrm{stable}} = I_{\mathrm{seq}}^{\mathrm{strict}}$ and $\mathrm{Cof}_{\mathrm{stable}} = \mathrm{Cof}_{\mathrm{strict}}$ by definition, the fact that the ccofibrations are the retracts of relative $I_{\mathrm{seq}}^{\mathrm{stable}}$-cell complexes is part of theorem 1.42. It only remains to show that the stable acyclic cofibrations are precisely the retracts of relative $J_{\mathrm{seq}}^{\mathrm{stable}}$-cell complexes. This we is the statement of lemma 1.96 below. ∎

Lemma 1.92. *The morphisms of free spectra $\{k_n\}_{n \in \mathbb{N}}$ from def. 1.87 co-represent the adjunct structure maps of sequential spectra from def. 1.4, in that for $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$, then*

$$\begin{array}{ccc} \mathrm{SeqSpec}(F_n S^0, X) & \simeq & X_n \\ {}^{\mathllap{\mathrm{SeqSpec}(k_n, X)}}\downarrow && \downarrow^{\mathrlap{\tilde \sigma_n^X}} \,, \\ \mathrm{SeqSpec}(F_{n+1} S^1, X) & \simeq & \Omega X_{n+1} \end{array}$$

where on the left we have the hom-spaces of def. 1.60, and where the horizontal equivalences are via prop. 1.89.

Proof. Recall that we are precomposing with

$$
\begin{array}{cccc}
 & \vdots & & \vdots \\
(k_n)_{n+3} & S^3 & \overset{\mathrm{id}}{\longrightarrow} & S^3 \\
(k_n)_{n+2} & S^2 & \overset{\mathrm{id}}{\longrightarrow} & S^2 \\
(k_n)_{n+1} & S^1 & \overset{\mathrm{id}}{\longrightarrow} & S^1 \\
(k_n)_n : & * & \overset{0}{\longrightarrow} & S^0 \\
 & * & \longrightarrow & * \\
 & \vdots & & \vdots \\
 & * & \longrightarrow & * \\
 & \underbrace{} & & \underbrace{} \\
k_n : & F_{n+1}S^1 & \longrightarrow & F_n S^0
\end{array}
$$

Now for X any sequential spectrum, then a morphism $f : F_n S^0 \to X$ is uniquely determined by its nth component $f_n : S^0 \to X_n$: the compatibility with the structure maps forces the next component, in particular, to be $\sigma_n^X \circ \Sigma f$:

$$
\begin{array}{ccc}
\Sigma S^0 & \overset{\Sigma f}{\longrightarrow} & \Sigma X_n \\
\downarrow^{\simeq} & & \downarrow^{\sigma_n^X} \\
S^1 & \overset{\sigma_n^X \circ \Sigma f}{\longrightarrow} & X_n
\end{array}
.
$$

But that $(n+1)$st component is just the component that similarly determines the precompositon of f with k_n, hence $f \circ k_n$ is uniquely determined by the map $\sigma_n^X \circ \Sigma f$. Therefore $\mathrm{SeqSpec}(k_n, -)$ is the function

$$
\mathrm{SeqSpec}(k_n, -) \;:\; X_n = \mathrm{SeqSpec}(S^0, X_n) \overset{f \mapsto \sigma_n^X \circ \Sigma f}{\longrightarrow} \mathrm{Maps}\,(S^1, X_{n+1})_* = \Omega X_{n+1} \;.
$$

It remains to see that this is indeed the $(\Sigma \dashv \Omega)$-adjunct of σ_n^X. By the general formula for adjuncts, this is

$$
\tilde{\sigma}_n^X \;:\; X_n \overset{\eta}{\longrightarrow} \Omega \Sigma X_n \overset{\Omega \sigma_n^X}{\longrightarrow} \Omega X_{n+1} \;.
$$

To compare to the above, we check what this does on points: $S^0 \overset{f}{\longrightarrow} X_n$ is sent to the composite

$$
S^0 \overset{f}{\longrightarrow} X_n \overset{\eta}{\longrightarrow} \Omega \Sigma X_n \overset{\Omega \sigma_0^X}{\longrightarrow} \Omega X_{n+1} \;.
$$

To identify this as a map $S^1 \to X_{n+1}$, we use the adjunction isomorphism once more to throw all the Ω-s on the right back to Σ-s the left, to finally find that this is indeed

$$
\sigma_n^X \circ \Sigma f \;:\; S^1 = \Sigma S^0 \overset{\Sigma f}{\longrightarrow} \Sigma X_n \overset{\sigma_n^X}{\longrightarrow} X_{n+1} \;.
$$

■

Lemma 1.93. *Every element in* $J_{\mathrm{seq}}^{\mathrm{stable}}$ *(def. 1.90) is an acyclic cofibration in the model structure* $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ *from theorem 1.72.*

Proof. For the elements in $J_{\mathrm{seq}}^{\mathrm{strict}}$ this is part of theorem 1.42. It only remains to see that the morphisms $k_n \,\square\, i_+$ are stable acyclic cofibrations.

To see that they are stable cofibrations, hence strict cofibrations:

By Joyal-Tierney calculus (prop.) $k_n \Box i_+$ has left lifting against any strict acyclic fibration f precisely if k_n has left lifting against the pullback powering $f^{\Box i_+}$ (def. 1.57). By prop. 1.58 the latter is still a strict acyclic fibration. Since k_n is evidently a strict cofibration, the lifting follows and hence also $k_n \Box i_+$ is a strict cofibration, hence a stable cofibration.

To see that they are stable weak equivalences: For each q the morphisms $k_n \wedge S^{q-1}$ are stable acyclic cofibrations, and since stable acyclic cofibrations are preserved under pushout, it follows by two-out-of-three that also $k_n \Box i_+$ is a stable weak equivalence. ■

The reason for considering the set $\{k_n \Box i_+\}$ is to make the following true:

Lemma 1.94. *A morphism $f : X \to Y$ in $\mathrm{SeqSpec(Top)}$ is a $J^{\mathrm{stable}}_{\mathrm{seq}}$-injective morphism (def.) precisely if*

1. *it is fibration in the strict model structure (hence degreewise a fibration);*

2. *for all $n \in \mathbb{N}$ the commuting squares of structure map compatibilities on the underlying sequential spectra*

$$\begin{array}{ccc} X_n & \overset{\tilde \sigma_n^X}{\longrightarrow} & \Omega X_{n+1} \\ {}^{\mathllap{f_n}}\downarrow && \downarrow^{\mathrlap{\Omega f_{n+1}}} \\ Y_n & \underset{\tilde \sigma_n^Y}{\longrightarrow} & \Omega Y_{n+1} \end{array}$$

exhibit homotopy pullbacks (def.) in $\mathrm{SeqSpec(Top_{cg})_{strict}}$, in that the comparison map

$$X_n \longrightarrow Y_n \underset{\Omega Y_{n+1}}{\times} \Omega X_{n-1}$$

is a weak homotopy equivalence (notice that Ωf_{n+1} is a fibration by the previous item and since $\Omega = \mathrm{Maps}(S^1, -)_*$ is a right Quillen functor by prop. 1.2).

In particular, the $J^{\mathrm{stable}}_{\mathrm{seq}}$-injective objects are precisely the Omega-spectra, def. 1.18.

Proof. By theorem 1.42, lifting against $J^{\mathrm{stric}}_{\mathrm{seq}}$ alone characterizes strict fibrations, hence degreewise fibrations. Lifting against the remaining pushout product morphism $k_n \Box i_+$ is, by Joyal-Tierney calculus (prop.), equivalent to left lifting i_+ against the pullback powering $f^{\Box k_n}$ from def. 1.57. Since the $\{i_+\}$ are the generating cofibrations in $\mathrm{Top}^{*/}_{\mathrm{cg}}$ such lifting means that $f^{\Box k_n}$ is a weak equivalence in the strict model sructure. But by lemma 1.92, $f^{\Box k_n}$ is precisely the comparison morphism in question. ■

Lemma 1.95. *A morphism in $\mathrm{SeqSpec(Top)}$ which is both*

1. *a stable weak homotopy equivalence (def. 1.16);*

2. *a $J^{\mathrm{stable}}_{\mathrm{seq}}$-injective morphism (def. 1.90, def.)*

is an acyclic fibration in the strict model structure, hence is degreewise a weak homotopy equivalence and Serre fibration of topological spaces;

Proof. Let $f : X \to B$ be both a stable weak homotopy equivalence as well as a K-injective morphism. Since K contains the generating acyclic cofibrations for the strict model structure, f is in particular a strict fibration, hence a degreewise fibration.

Consider the fiber F of f, hence the morphism $F \to *$ which is the pullback of f along $* \to B$. Notice that since f is a strict fibration, this is the homotopy fiber (def.) of f in the strict model structure.

We claim that

1. F is an Omega-spectrum;
2. $F \to *$ is a stable weak homotopy equivalence.

The first item follows since F, being the pullback of a K-injective morphisms, is a K-injective object (prop.), so that, by lemma 1.94, F it is an Omega-spectrum.

For the second item:

Since $F \to X \overset{f}{\to} B$ is degreewise a homotopy fiber sequence, there are degreewise its long exact sequences of homotopy groups (exmpl.)

$$\cdots \to \pi_{\bullet+1}(B_n) \longrightarrow \pi_\bullet(F_n) \longrightarrow \pi_\bullet(X_n) \overset{(f_n)_*}{\longrightarrow} \pi_\bullet(B_n) \to \cdots \to \pi_1(B_n) \longrightarrow \pi_0(F_n) \longrightarrow \pi_0(X_n) \longrightarrow \pi_0(B)_n$$

Since in the category Ab of abelian group forming filtered colimits is an exact functor (prop.), it follows that after passing to stable homotopy groups the resulting sequence

$$\cdots \pi_{\bullet+1}(X) \overset{f_*}{\longrightarrow} \pi_{\bullet+1}(B) \longrightarrow \pi_\bullet(F) \longrightarrow \pi_\bullet(X) \overset{(f_*}{\longrightarrow} \pi_\bullet(B) \to \cdots$$

is still a long exact sequence.

Since, by assumption, f_* is an isomorphism, this exactness implies that $\pi_\bullet(F) = 0$, and hence that $F \to *$ is a stable weak homotopy equivalence. But since, by the first item above, F is an Omega-spectrum, it follows (via example 1.20) that $F \to *$ is even a degreewise weak homotopy equivalence, hence that $\pi_\bullet(F_n) \simeq 0$ for all $n \in \mathbb{N}$.

Feeding this back into the above degreewise long exact sequence of homotopy groups now implies that $\pi_{\bullet \geq 1}(f_n)$ is a weak homotopy equivalence for all n and for each homotopy group in positive degree.

To deduce the remaining case that also $\pi_0(f_0)$ is an isomorphism, observe that by assumption of K-injectivity, lemma 1.94 gives that f_0 is the pullback (in topological spaces) of $\Omega(f_1)$. But by the above Ωf_1 is a weak homotopy equivalence, and since $\Omega = \mathrm{Maps}(S^1, -)_*$ is a right Quillen functor (prop. 1.2) it is also a Serre fibration. Therefore f_0 is the pullback of an acyclic Serre fibration and hence itself a weak homotopy equivalence. ■

Lemma 1.96. *The retracts (rmk.) of $J_{\mathrm{seq}}^{\mathrm{stable}}$-relative cell complexes are precisely the stable acyclic cofibrations.*

Proof. Since all elements of $J_{\mathrm{seq}}^{\mathrm{stable}}$ are stable weak equivalences and strict cofibrations by lemma 1.93, it follows that every retract of a relative $J_{\mathrm{seq}}^{\mathrm{stable}}$-cell complex has the same property.

In the other direction, let f be a stable acyclic cofibration. Apply the small object argument (prop.) to factor it

$$f : \underset{J_{\mathrm{seq}}^{\mathrm{stable}}\,\mathrm{Cell}}{\overset{i}{\longrightarrow}} \underset{J_{\mathrm{seq}}^{\mathrm{stable}}\,\mathrm{Inj}}{\overset{p}{\longrightarrow}}$$

as a $J_{\text{seq}}^{\text{stable}}$-relative cell complex i followed by a $J_{\text{seq}}^{\text{stable}}$-injective morphism p. By the previous statement i is a stable weak homotopy equivalence, and hence by assumption and by two-out-of-three so is p. Therefore lemma 1.95 implies that p is a strict acyclic fibration. But then the assumption that f is a strict cofibration means that it has the left lifting property against p, and so the retract argument (prop.) implies that f is a retract of the relative $J_{\text{seq}}^{\text{stable}}$-cell complex i. ▮

This completes the proof of theorem 1.91.

The stable homotopy category

Definition 1.97. Write

$$\text{Ho}(\text{Spectra}) \coloneqq \text{Ho}(\text{SeqSpec}(\text{Top}_{\text{cg}})_{\text{stable}})$$

for the homotopy category (defn.) of the stable model structure on topological sequential spectra from theorem 1.72.

This is called the **stable homotopy category**.

The stable homotopy category of def. 1.97 inherits particularly nice properties that are usefully axiomatized for themselves. This axiomatics is called *triangulated category* structure (def. 1.111 below) where the "triangles" are referring to the structure of the long fiber sequences and long cofiber sequences (prop.) which happen to coincide in stable homotopy theory.

Additivity

The stable homotopy category $\text{Ho}(\text{Spectra})$ is the analog in homotopy theory of the category Ab of abelian groups in homological algebra. While the stable homotopy category is *not* an *abelian category*, as Ab is, but a homotopy-theoretic version of that to which we turn below, it *is* an additive category.

Lemma 1.98. *The stable homotopy category (def. 1.97) has finite coproducts. They are represented by wedge sums (example 1.29) of CW-spectra (def. 1.46).*

Proof. Having finite coproducts means

1. having empty coproducts, hence initial objects,
2. and having binary coproducts.

Regarding the initial object:

The spectrum $\Sigma^\infty *$ (suspension spectrum (example 1.5) on the point) is both an initial object and a terminal object in $\text{SeqSpec}(\text{Top}_{\text{cg}})$. This implies in particular that it is both fibrant and cofibrant. Finally its standard cylinder spectrum (example 1.30) is trivial $(\Sigma^\infty *) \wedge (I_+) \simeq \Sigma^\infty *$. All together with means that for X any fibrant-cofibrant spectrum, then

$$\text{Hom}_{\text{Ho}(\text{Spectra})}(\Sigma^\infty *, Z) \simeq \text{Hom}_{\text{SeqSpec}}(\Sigma^\infty *, Z)/_\sim \simeq *$$

and so $\Sigma^\infty *$ also represents the initial object in the stable homotopy category.

Now regarding binary coproducts:

By prop. 1.55 and prop. 1.51, every spectrum has a cofibrant replacement by a CW-spectrum. By prop. 1.50 the wedge sum $X \vee Y$ of two CW-spectra is still a CW-spectrum, hence still cofibrant.

Let P and Q be fibrant and cofibrant replacement functors, respectively, as in the section Classical homotopy theory – The homotopy category.

We claim now that $P(X \vee Y) \in \mathrm{Ho}(\mathrm{Spectra})$ is the coproduct of PX with PY in $\mathrm{Ho}(\mathrm{Spectra})$. By definition of the homotopy category (def.) this is equivalent to claiming that for Z any stable fibrant spectrum (hence an Omega-spectrum by theorem 1.72) then there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{SeqSpec}}(P(X \vee Y), QZ)/_\sim \simeq \mathrm{Hom}_{\mathrm{SeqSpec}}(PX, QZ)/_\sim \times \mathrm{Hom}_{\mathrm{SeqSpec}}(PY, QZ)/_\sim$$

between left homotopy-classes of morphisms of sequential spectra.

But since $X \vee Y$ is cofibrant and Z is fibrant, there is a natural isomorphism (prop.)

$$\mathrm{Hom}_{\mathrm{SeqSpec}}(P(X \vee Y), QZ)/_\sim \overset{\simeq}{\longrightarrow} \mathrm{Hom}_{\mathrm{SeqSpec}}(X \vee Y, Z)/_\sim \,.$$

Now the wedge sum $X \vee Y$ is the coproduct in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$, and hence morphisms out of it are indeed in natural bijection with pairs of morphisms out of the two summands. But we need this property to hold still after dividing out left homotopy. The key is that smash tensoring (def. 1.8) distributes over wedge sum

$$(X \vee Y) \wedge (I_+) \simeq (X \wedge (I_+)) \vee (Y \wedge (I_+))$$

(due to the fact that the smash product of compactly generated pointed topological spaces distributes this way over wedge sum of pointed spaces). This means that also left homotopies out of $X \vee Y$ are in natural bijection with pairs of left homotopies out of the summands separately, and hence that there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{SeqSpec}}(X \vee Y, Z)/_\sim \overset{\simeq}{\longrightarrow} \mathrm{Hom}_{\mathrm{SeqSpec}}(X, Z)/_\sim \times \mathrm{Hom}_{\mathrm{SeqSpec}}(Y, Z)/_\sim \,.$$

Finally we may apply the inverse of the natural isomorphism used before (prop.) to obtain in total

$$\mathrm{Hom}_{\mathrm{SeqSpec}}(X, Z)/_\sim \times \mathrm{Hom}_{\mathrm{SeqSpec}}(Y, Z)/_\sim \overset{\simeq}{\longrightarrow} \mathrm{Hom}_{\mathrm{SeqSpec}}(PX, QZ)/_\sim \times \mathrm{Hom}_{\mathrm{SeqSpec}}(PY, QZ)/_\sim \,.$$

The composite of all these isomorphisms proves the claim. ▮

Definition 1.99. Define group structure on the pointed hom-sets of the stable homotopy category (def. 1.97)

$$[X, Y] \in \mathrm{Grp}$$

induced from the fact (prop.) that the hom-sets of any homotopy category into an object in the image of the canonical loop space functor Ω inherit group structure, together with the fact (theorem 1.84) that on the stable homotopy category Ω and Σ are inverse to each other, so that

$$[X, Y] \simeq [X, \Omega\Sigma Y]\,,$$

Lemma 1.100. *The group structure on $[X, Y]$ in def. 1.99 is abelian and composition in $\mathrm{Ho}(\mathrm{Spectra})$ is bilinear with respect to this group structure. (Hence this makes $\mathrm{Ho}(\mathrm{Spectra})$ an Ab-enriched category.)*

Proof. Recall (prop, rmk.) that the group structure is given by concatenation of loops

$$X \overset{\Delta_X}{\to} X \times X \overset{(f,g)}{\longrightarrow} \Omega\Sigma X \times \Omega\Sigma X \longrightarrow \Omega\Sigma X \,.$$

That the group structure is abelian follows via the Eckmann-Hilton argument from the fact that there is always a compatible second (and indeed arbitrarily many compatible) further group structures, since, by stability

$$[X,Y] \simeq [X, \Omega\Sigma Y] \simeq [X, \Omega \circ (\Omega\Sigma) \circ \Sigma Y] = [X, \Omega^2 \Sigma^2 Y] \,.$$

That composition of morphisms distributes over the operation in this group is evident for precomposition. Let $f : W \to X$ then clearly

$$f^\ast \;:\; [X, \Omega\Sigma Y] \longrightarrow [W, \Omega\Sigma Y]$$

preserves the group structure induced by the group structure on $\Omega\Sigma Y$. That the same holds for postcomposition may be immediately deduced from noticing that this group structure is also the same as that induced by the cogroup structure on $\Sigma\Omega X$, so that with $g : Y \to Z$ then

$$g_\ast \;:\; [\Sigma\Omega X, Y] \longrightarrow [\Sigma\Omega X, Z]$$

preserves group structure.

More explicitly, we may see the respect for groupstructure structure of the postcomposition opeation from the naturality of the loop composition map which is manifest when representing loop spectra via the standard topological loop space object $\Omega X = \mathrm{fib}(\mathrm{Maps}(I_+, X) \to X \times X)$ (rmk.) under smash powering (def. 1.8).

To make this fully explicit, consider the following diagram in $\mathrm{Ho}(\mathrm{Spectra})$:

$$\array{ Y \times Y &\overset{\simeq}{\to}& \Omega\Sigma Y \times \Omega\Sigma Y &\overset{\simeq}{\to}& Q(\mathrm{Maps}(S^1, \Sigma Y)_\ast \times \mathrm{Maps}(S^1,\Sigma Y)_\ast) &\longrightarrow& Q(\mathrm{Maps}(S^1_{[0,2]}, \Sigma Y))_\ast &\simeq& \Omega \Sigma Y &\simeq& Y \\ {}^{\mathllap{g \times g}}\downarrow && \downarrow^{\mathrlap{\Omega\Sigma g \times \Omega \Sigma g}} && \downarrow^{\mathrlap{Q(\mathrm{Maps}(S^1,\Sigma g)_\ast \times \mathrm{Maps}(S^1,\Omega\Sigma g)_\ast)}} && \downarrow^{\mathrlap{Q(\mathrm{Maps}(S^1_{[0,2]}, \Sigma g)_\ast)}} && \downarrow^{\mathrlap{\Omega\Sigma g}} && \downarrow^{\mathrlap{g}} \\ Z \times Z &\overset{\simeq}{\to}& \Omega\Sigma Z \times \Omega\Sigma Z &\overset{\simeq}{\to}& Q(\mathrm{Maps}(S^1, \Sigma Z)_\ast \times \mathrm{Maps}(S^1,\Sigma Z)_\ast) &\longrightarrow& Q(\mathrm{Maps}(S^1_{[0,2]}, \Sigma Z)_\ast) &\simeq& \Omega \Sigma Z &\simeq& Z }\,,$$

where $S^1_{[0,2]}$ denotes the sphere of length 2.

Here the leftmost square and the rightmost square are the naturality squares of the equivalence of categories $(\Sigma \dashv \Omega)$ (theorem 1.84).

The second square from the left and the second square from the right exhibit the equivalent expression of Ω as the right derived functor of (either the standard or the alternative, by lemma 1.83) degreewise loop space functor. Here we let ΣX denote any fibrant representative, for notational brevity, and use that the derived functor of a right Quillen functor is given on fibrant objects by the original functor followed by cofibrant replacement (prop.).

The middle square is the image under Q of the evident naturality square for concatenation of loops. This is where we use that we have the standard model for forming loop spaces and concatenation of loops (rmk.): the diagram commutes because the loops are always poinwise pushed forward along the map f. ▮

It is conventional (Adams 74, p. 138) to furthermore make the following definition:

Definition 1.101. For $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ two spectra, define the $\mathbb{Z}$-graded abelian group

$$[X,Y]_\bullet \in \mathrm{Ab}^{\mathbb{Z}}$$

to be in degree n the abelian hom group of lemma 1.100 out of X into the n-fold suspension of Y (lemma 1.83):

$$[X, Y]_n \;\coloneqq\; [X, \Sigma^{-n} Y] \;.$$

Defining the composition of $f_1 \in [X, Y]_{n_1}$ with $f_2 \in [Y, Z]_{n_2}$ to be the composite

$$X \overset{f_1}{\longrightarrow} \Sigma^{-n_1} Y \overset{\Sigma^{-n_2}(f_2)}{\longrightarrow} \Sigma^{-n_2}(\Sigma^{-n_1} Z) \simeq \Sigma^{-n_1-n_2} Z$$

gives the stable homotopy category the structure of an $\mathrm{Ab}^{\mathbb{Z}}$-enriched category.

Example 1.102. **(generalized cohomology groups)**

Let $E \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ be an Omega-spectrum (def. 1.18) and let $X \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ be a pointed topological space with $\Sigma^\infty X$ its suspension spectrum (example 1.5). Then the graded abelian group (by prop. 1.100, def. 1.101)

$$\begin{aligned} \tilde E^\bullet(X) &\coloneqq [\Sigma^\infty X, E]_{-\bullet} \\ &= [\Sigma^\infty X, \Sigma^\bullet E] \\ &\simeq [X, \Omega^\infty \Sigma^\bullet E]_* \\ &\simeq [X, E_\bullet]_* \end{aligned}$$

is also called the **reduced cohomology of** X in the **generalized (Eilenberg-Steenrod) cohomology theory** that is represented by E.

Here the equivalences used are

1. the adjunction isomorphism of $(\Sigma^\infty \dashv \Omega^\infty)$ from theorem 1.86;
2. the isomorphism $\Sigma \simeq [1]$ of suspension with the shift spectrum (def. 1.33) on $\mathrm{Ho}(\mathrm{Spectra})$ of lemma 1.75, together with the nature of Ω^∞ from prop. 1.12.

The latter expression

$$\tilde E^n(X) \simeq [X, E_n]_*$$

(on the right the hom in in the classical homotopy category $\mathrm{Ho}(\mathrm{Top}^{*/})$ of pointed topological spaces) is manifestly the definition of reduced generalized (Eilenberg-Steenrod) cohomology as discussed in part S in the section on the Brown representability theorem.

Suppose E here is not necessarily given as an Omega-spectrum. In general the hom-groups $[X, E] = [X, E]_{\mathrm{stable}}$ in the stable homotopy category are given by the naive homotopy classes of maps out of a cofibrant resolution of X into a fibrant resolution of E (by this lemma). By theorem 1.72 a fibrant replacement of E is given by Omega-spectrification $Q E$ (def. 1.21). Since the stable model structure of theorem 1.72 is a left Bousfield localization of the strict model structure from theorem 1.42, and since for the latter all objects are fibrant, it follows that

$$[X, E]_{\mathrm{stable}} \simeq [X, Q E]_{\mathrm{strict}} \,,$$

and hence

$$\begin{aligned} E^0(X) &\coloneqq [\Sigma^\infty X, E]_{\mathrm{stable}} \\ &\simeq [\Sigma^\infty X, Q E]_{\mathrm{strict}} \\ &\simeq [X, \Omega^\infty Q E]_* \\ &= [X, (Q E)_0]_* \end{aligned},$$

where the last two hom-sets are again those of the classical homotopy category. Now if E happens to be a CW-spectrum, then by remark 1.23 its Omega-spectrification is given simply by $(Q E)_n \simeq \underset{\longrightarrow_k}{\lim} \Omega^k E_{n+k})$ and hence in this case

$$E^0(X) \simeq [X, \underset{\longrightarrow_k}{\lim} \Omega^k E_k]_* \,.$$

If X here is moreover a compact topological space, then it may be taken inside the colimit (e.g. Weibel 94, topology exercise 10.9.2), and using the $(\Sigma \dashv \Omega)$-adjunction this is rewritten as

$$\begin{aligned} E^0(X) &\simeq \underset{\longrightarrow_k}{\lim} [X, \Omega^k E_k]_* \\ &\simeq \underset{\longrightarrow_k}{\lim} [\Sigma^k X, E_k]_* \end{aligned}.$$

(e.g. Adams 74, prop. 2.8).

This last expression is sometimes used to define cohomology with coefficients in an arbitrary spectrum. For examples see in the part S the section *Orientation in generalized cohomology*.

More generally, it is immediate now that there is a concept of E-cohomology not only for spaces and their suspension spectra, but also for general spectra: for $X \in \mathrm{Ho}(\mathrm{Spectra})$ be any spectrum, then

$$\tilde E^\bullet(X) \coloneqq [X, \Sigma^\bullet E]$$

is called the reduced E-cohomology of the spectrum X.

Beware that here one usually drops the tilde sign.

In summary, lemma 1.98 and lemma 1.100 state that the stable homotopy category is an Ab-enriched category with finite coproducts. This is called an *additive category*:

Definition 1.103. An **additive category** is a category which is

1. an Ab-enriched category;

   (sometimes called a pre-additive category–this means that each hom-set carries the structure of an abelian group and composition is bilinear)

2. which admits finite coproducts

   (and hence, by prop. 1.104 below, finite products which coincide with the coproducts, hence finite biproducts).

Proposition 1.104. *In an Ab-enriched category, a finite product is also a coproduct, and dually.*

This statement includes the zero-ary case: any terminal object is also an initial object, hence a zero object (and dually), hence every additive category (def. 1.103) has a zero *object.*

More precisely, for $\{X_i\}_{i \in I}$ a finite set of objects in an Ab-enriched category, then the unique morphism

$$\coprod_{i \in I} X_i \longrightarrow \prod_{j \in I} X_j \,,$$

whose components are identities for $i = j$ and are zero otherwise, is an isomorphism.

Proof. Consider first the zero-ary case. Given an initial object $\emptyset$ and a terminal object $*$, observe that since the hom-sets $\mathrm{Hom}(\emptyset, \emptyset)$ and $\mathrm{Hom}(*, *)$ by definition contain a single element, this element has to be the zero element in the abelian group structure. But it also has to be the identity morphism, and hence $\mathrm{id}_\emptyset = 0$ and $\mathrm{id}_* = 0$. It follows that the 0-element in $\mathrm{Hom}(*, \emptyset)$ is a left and right inverse to the unique element in $\mathrm{Hom}(\emptyset, *)$, and so this is an isomorphism

$$0 : \emptyset \overset{\simeq}{\longrightarrow} * \,.$$

Consider now the case of binary (co-)products. Using the existence of the zero object, hence of zero morphisms, then in addition to its canonical projection maps $p_i : X_1 \times X_2 \to X_i$, any binary product also receives "injection" maps $X_i \to X_1 \times X_2$, and dually for the coproduct:

$$\begin{array}{ccccc} X_1 && && X_2 \\ & {}_{\searrow}{}^{(\mathrm{id},0)} && {}^{(0,\mathrm{id})}{}_{\swarrow} & \\ {}^{\mathrm{id}_{X_1}}\downarrow && X_1 \times X_2 && \downarrow^{\mathrm{id}_{X_2}} \\ & \swarrow_{p_{X_1}} && {}_{p_{X_2}}\searrow & \\ X_1 && && X_2 \end{array} \,, \qquad \begin{array}{ccccc} X_1 && && X_2 \\ & {}_{\searrow}{}^{i_{X_1}} && {}^{i_{X_2}}{}_{\swarrow} & \\ {}^{\mathrm{id}_{X_1}}\downarrow && X_1 \sqcup X_2 && \downarrow^{\mathrm{id}_{X_2}} \\ & \swarrow_{(\mathrm{id},0)} && {}_{(0,\mathrm{id})}\searrow & \\ X_1 && && X_2 \end{array}$$

Observe some basic compatibility of the Ab-enrichment with the product:

First, for $(\alpha_1, \beta_1), (\alpha_2, \beta_2) : R \to X_1 \times X_2$ then

$$(\star) \qquad (\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2,\ \beta_1 + \beta_2)$$

(using that the projections p_1 and p_2 are linear and by the universal property of the porduct).

Second, $(\mathrm{id}, 0) \circ p_1$ and $(0, \mathrm{id}) \circ p_2$ are two projections on $X_1 \times X_2$ whose sum is the identity:

$$(\star\star) \qquad (\mathrm{id}, 0) \circ p_1 + (0, \mathrm{id}) \circ p_2 = \mathrm{id}_{X_1 \times X_2} \,.$$

(We may check this, via the Yoneda lemma on generalized elements: for $(\alpha, \beta) : R \to X_1 \times X_2$ any morphism, then $(\mathrm{id}, 0) \circ p_1 \circ (\alpha, \beta) = (\alpha, 0)$ and $(0, \mathrm{id}) \circ p_2 \circ (\alpha, \beta) = (0, \beta)$, so the statement follows with equation $(\star)$.)

Now observe that for $f_i : X_i \to Q$ any two morphisms, the sum

$$\phi \;\coloneqq\; f_1 \circ p_1 + f_2 \circ p_2 \;:\; X_1 \times X_2 \longrightarrow Q$$

gives a morphism of cocones

$$
\begin{array}{ccccc}
X_1 && && X_2 \\
& {}^{\mathllap{}}\searrow^{(\mathrm{id},0)} && {}^{(0,\mathrm{id})}\swarrow & \\
{}^{\mathllap{\mathrm{id}_{X_1}}}\downarrow && X_1 \times X_2 && \downarrow^{\mathrlap{\mathrm{id}_{X_2}}} \\
X_1 && \downarrow^{\phi} && X_2 \\
& {}_{f_1}\searrow && \swarrow_{f_2} & \\
&& Q &&
\end{array}
.
$$

Moreover, this is unique: suppose ϕ' is another morphism filling this diagram, then, by using equation $(\star\star)$, we get

$$
\begin{aligned}
(\phi - \phi') & = (\phi - \phi') \circ \mathrm{id}_{X_1 \times X_2} \\
& = (\phi - \phi') \circ ((\mathrm{id}_{X_1}, 0) \circ p_1 + (0, \mathrm{id}_{X_2}) \circ p_2) \\
& = \underset{= 0}{\underbrace{(\phi - \phi') \circ (\mathrm{id}_{X_1}, 0)}} \circ p_1 + \underset{= 0}{\underbrace{(\phi - \phi') \circ (0, \mathrm{id}_{X_2})}} \circ p_2 \\
& = 0
\end{aligned}
$$

and hence $\phi = \phi'$. This means that $X_1 \times X_2$ satisfies the universal property of a coproduct.

By a dual argument, the binary coproduct $X_1 \sqcup X_2$ is seen to also satisfy the universal property of the binary product. By induction, this implies the statement for all finite (co-)products. ▮

Remark 1.105. Finite coproducts coinciding with products as in prop. 1.104 are also called *biproducts* or *direct sums*, denoted

$$X_1 \oplus X_2 \coloneqq X_1 \sqcup X_2 \simeq X_1 \times X_2 \,.$$

The zero object is denoted “0”, of course.

Conversely:

Definition 1.106. A **semiadditive category** is a category that has all finite products which, moreover, are biproducts in that they coincide with finite coproducts as in def. 1.104.

Proposition 1.107. *In a semiadditive category, def. 1.106, the hom-sets acquire the structure of commutative monoids by defining the sum of two morphisms $f,g : X \longrightarrow Y$ to be*

$$f + g \;\coloneqq\; X \overset{\Delta_X}{\to} X \times X \simeq X \oplus X \overset{f \oplus g}{\longrightarrow} Y \oplus Y \simeq Y \sqcup Y \overset{\nabla_X}{\to} Y \,.$$

With respect to this operation, composition is bilinear.

Proof. The associativity and commutativity of $+$ follows directly from the corresponding properties of $\oplus$. Bilinearity of composition follows from naturality of the diagonal Δ_X and codiagonal ∇_X:

$$\begin{array}{ccccccccccc} W & \overset{\Delta_W}{\longrightarrow} & W \times W & \overset{\simeq}{\longrightarrow} & W \oplus W \\ \downarrow^{e} && \downarrow^{e \times e} && \downarrow^{e \oplus e} \\ X & \overset{\Delta_X}{\to} & X \times X & \simeq & X \oplus X & \overset{f \oplus g}{\longrightarrow} & Y \oplus Y & \simeq & Y \sqcup Y & \overset{\nabla_X}{\to} & Y \\ &&&&&& \downarrow^{h \oplus h} && \downarrow^{h \sqcup h} && \downarrow^{h} \\ &&&&&& Z \oplus Z & \simeq & Z \sqcup Z & \overset{\nabla_Z}{\to} & Z \end{array}$$

■

Proposition 1.108. *Given an additive category according to def. 1.103, then the enrichement in commutative monoids which is induced on it via prop. 1.104 and prop. 1.107 from its underlying semiadditive category structure coincides with the original enrichment.*

Proof. By the proof of prop. 1.104, the codiagonal on any object in an additive category is the sum of the two projections:

$$\nabla_X \,:\, X \oplus X \overset{p_1 + p_2}{\longrightarrow} X \,.$$

Therefore (checking on generalized elements, as in the proof of prop. 1.104) for all morphisms $f, g : X \to Y$ we have commuting squares of the form

$$\begin{array}{ccc} X & \overset{f+g}{\longrightarrow} & Y \\ {}^{\Delta_X}\downarrow && \uparrow^{\nabla_Y = }_{p_1 + p_2} \\ X \oplus X & \underset{f \oplus g}{\longrightarrow} & Y \oplus Y \end{array}$$

■

Remark 1.109. Prop. 1.108 says that being an additive category is an extra property on a category, not extra structure. We may ask whether a given category is additive or not, without specifying with respect to which abelian group structure on the hom-sets.

In conclusion we have:

Proposition 1.110. *The stable homotopy category (def. 1.97) is an additive category (def. 1.103).*

Hence prop. 1.104 implies that in the stable homotopy category finite coproducts (wedge sums) and finite products agree, in that they are finite biproducts (direct sums).

$$\vee \;\simeq\; \times \;\simeq\; \oplus \quad \in \; \mathrm{Ho}(\mathrm{Spectra}) \,.$$

Proof. By lemma 1.98 and lemma 1.100. ■

Triangulated structure

We have seen above that the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ is an additive category. In the context of homological algebra, when faced with an additive category one next asks for the existence of kernels (fibers) and cokernels (cofibers) to yield a pre-abelian category, and then asks that these are suitably compatible, to yield an abelian category.

Now here in stable homotopy theory, the concept of kernels and cokernels is replaced by that of homotopy fibers and homotopy cofibers. That these certainly exist for homotopy theories presented by model categories was the topic of the general discussion in the section

Homotopy theory – Homotopy fibers. Various of the properties they satisfy was the topic of the sections *Homotopy theory – Long sequences* and *Homotopy theory – Homotopy pullbacks.*. For the special case of *stable homotopy theory* we will find a crucial further property relating homotopy fibers to homotopy cofibers.

The axiomatic formulation of a subset of these properties of stable homotopy fibers and stable homotopy cofibers is called a *triangulated category* structure. This is the analog in stable homotopy theory of abelian category structure in homological algebra.

	category of abelian groups	**stable homotopy category**
direct sums and hom-abelian groups	additive category	additive category
(homotopy) fibers and cofibers exist	pre-additive category	homotopy category of a model category
(homotopy) fibers and cofibers are compatible	abelian category	triangulated category

Literature (Hubery, Schwede 12, II.2)

Definition 1.111. A **triangulated category** is

1. an additive category Ho (def. 1.103);
2. a functor, called the **suspension functor** or **shift functor**

   $$\Sigma : \mathrm{Ho} \overset{\simeq}{\to} \mathrm{Ho}$$

   which is required to be an equivalence of categories;
3. a sub-class $\mathrm{CofSeq} \subset \mathrm{Mor}(\mathrm{Ho}^{\Delta[3]})$ of the class of triples of composable morphisms, called the class of **distinguished triangles**, where each element that starts at A ends at ΣA; we write these as

   $$A \longrightarrow B \longrightarrow B/A \longrightarrow \Sigma A\,,$$

   or

   $$\begin{array}{ccccc} A && \overset{f}{\longrightarrow} && B \\ & {}_{[1]}\nwarrow && \swarrow \\ && B/A \end{array}$$

   (whence the name *triangle*);

such that the following conditions hold:

- **T0** For every morphism $f : A \to B$, there does exist a distinguished triangle of the form

  $$A \overset{f}{\longrightarrow} B \longrightarrow B/A \longrightarrow \Sigma A\,.$$

  If (f,g,h) is a distinguished triangle and there is a commuting diagram in Ho of the form

$$\begin{array}{ccccccc} A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \\ \downarrow^{\in Iso} && \downarrow^{\in Iso} && \downarrow^{\in Iso} && \downarrow^{\in Iso} \\ A' & \overset{f'}{\longrightarrow} & B' & \overset{g'}{\longrightarrow} & B'/A' & \overset{h'}{\longrightarrow} & \Sigma A' \end{array}$$

(with all vertical morphisms being isomorphisms) then (f', g', h') is also a distinguished triangle.

- **T1** For every object $X \in \mathrm{Ho}$ then $(0, \mathrm{id}_X, 0)$ is a distinguished triangle

$$0 \longrightarrow X \overset{\mathrm{id}_X}{\longrightarrow} X \longrightarrow 0\,;$$

- **T2** If (f, g, h) is a distinguished triangle, then so is $(g, h, -\Sigma f)$; hence if

$$A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} B/A \overset{h}{\longrightarrow} \Sigma A$$

is, then so is

$$B \overset{g}{\longrightarrow} B/A \overset{h}{\longrightarrow} \Sigma A \overset{-\Sigma f}{\longrightarrow} \Sigma B\,.$$

- **T3** Given a commuting diagram in Ho of the form

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & B/A & \longrightarrow & \Sigma A \\ \downarrow^{\phi_A} && \downarrow^{\phi_B} \\ A' & \longrightarrow & B' & \longrightarrow & B'/A' & \longrightarrow & \Sigma A' \end{array}$$

where the top and bottom are distinguished triangles, then there exists a morphism $B/A \to B'/A'$ such as to make the completed diagram commute

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & B/A & \longrightarrow & \Sigma A \\ \downarrow^{\phi_A} && \downarrow^{\phi_B} && \downarrow^{\exists} && \downarrow^{\Sigma\phi_A} \\ A' & \longrightarrow & B' & \longrightarrow & B'/A' & \longrightarrow & \Sigma A' \end{array}$$

- **T4 (octahedral axiom)** For every pair of composable morphisms $f : A \to B$ and $f' : B \to D$ then there is a commutative diagram of the form

$$\begin{array}{ccccccc} A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \\ {}^{=}\downarrow && {}^{f'}\downarrow && \downarrow^{x} && \downarrow^{=} \\ A & \underset{f' \circ f}{\longrightarrow} & D & \underset{g''}{\longrightarrow} & D/A & \underset{h''}{\longrightarrow} & \Sigma A \\ && {}^{g'}\downarrow && \downarrow^{y} \\ && D/B & \underset{\simeq}{\longrightarrow} & D/B \\ && {}^{h'}\downarrow && \downarrow^{(\Sigma g)\circ h'} \\ && \Sigma B & \underset{\Sigma g}{\longrightarrow} & \Sigma B/A \end{array}$$

such that the two top horizontal sequences and the two middle vertical sequences each are distinguished triangles.

Proposition 1.112. *The stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ from def. 1.97, equipped with the canonical suspension functor $\Sigma : \mathrm{Ho}(\mathrm{Spectra}) \overset{\simeq}{\longrightarrow} \mathrm{Ho}(\mathrm{Spectra})$ (according to this prop.) is a triangulated category (def. 1.111) for the distinguished triangles being the closure under isomorphism of triangles of the images (under localization*

$\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \to \mathrm{Ho}(\mathrm{Spectra})$ *(prop.) of the stable model category of theorem 1.72) of the canonical long homotopy cofiber sequences (prop.)*

$$A \overset{f}{\longrightarrow} B \longrightarrow \mathrm{hocofib}(f) \longrightarrow \Sigma A \,.$$

(e.g. Schwede 12, chapter II, theorem 2.9)

Proof. By prop. 1.110 the stable homotopy category is additive, by theorem 1.84 the functor Σ is an equivalence.

The axioms T0 and T1 are immediate from the definition of homotopy cofiber sequences.

The axiom T2 is the very characterization of long homotopy cofiber sequences (from this prop.).

Regarding axiom T3:

By the factorization axioms of the model category we may represent the morphisms $A \to A'$ and $B \to B'$ in the homotopy category by cofibrations in the model category. Then $B \to B/A$ and $B' \to B'/A'$ are represented by their ordinary cofibers (def., prop.).

We may assume without restriction (lemma) that the commuting square

$$\begin{array}{ccc} A & \overset{f}{\longrightarrow} & B \\ {}^{\mathllap{\phi_A}}\downarrow && \downarrow^{\mathrlap{\phi_B}} \\ A' & \underset{f'}{\longrightarrow} & B' \end{array}$$

in the homotopy category is the image of a commuting square (not just commuting up to homotopy) in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$. In this case then the morphism $B/A \to B'/A'$ is induced by the universal property of ordinary cofibers. To see that this also completes the last vertical morphism, observe that by the small object argument (prop.) we have functorial factorization (def.).

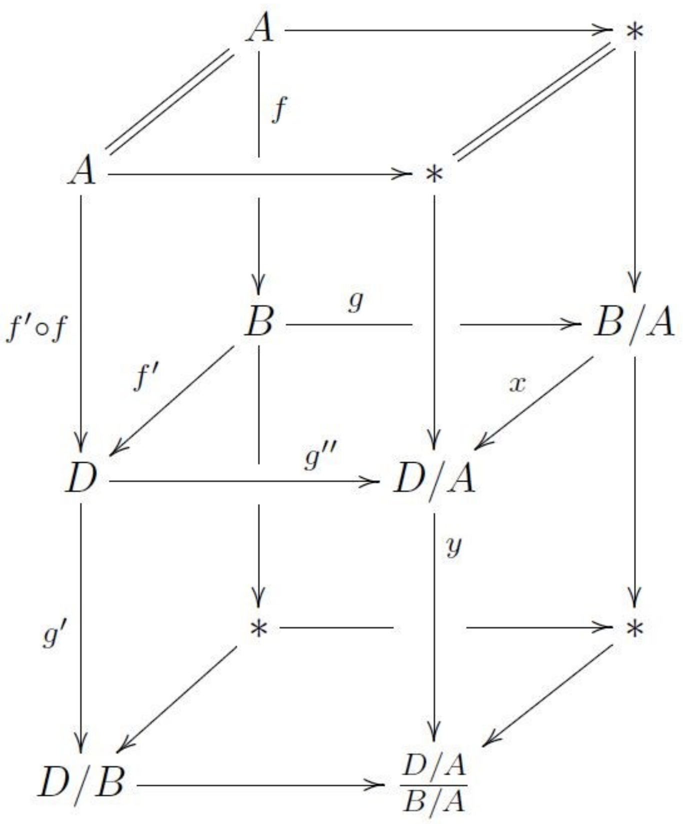
With this, again the universal property of the ordinary cofiber gives the fourth vertical morphism needed for T3.

Axiom T4 follows in the same fashion: we may represent all spectra by CW-spectra and represent f and f', hence also $f' \circ f$, by cofibrations. Then forming the functorial mapping cones as above produces the commuting diagram

$$\begin{array}{ccccccc} A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \\ {}^{\mathllap{=}}\downarrow & (1) & {}^{\mathllap{f'}}\downarrow & (2) & \downarrow^{\mathrlap{x}} && \downarrow^{\mathrlap{=}} \\ A & \underset{f' \circ f}{\longrightarrow} & D & \underset{g''}{\longrightarrow} & D/A & \underset{h''}{\longrightarrow} & \Sigma A \\ && {}^{\mathllap{g'}}\downarrow & (3) & \downarrow^{\mathrlap{y}} \\ && D/B & \underset{\simeq}{\longrightarrow} & D/B \\ && {}^{\mathllap{h'}}\downarrow && \downarrow^{\mathrlap{(\Sigma g)\circ h'}} \\ && \Sigma B & \underset{\Sigma g}{\longrightarrow} & \Sigma B/A \end{array}$$

The fact that the second horizontal morphism from below is indeed an isomorphism follows by applying the pasting law for homotopy pushouts twice (prop.):

Draw all homotopy cofibers as homotopy pushout squares (def.) with one edge going to the



point. Then assemble the squares (1)-(3) in the pasting composite of two cubes on top of each other: (1) as the left face of the top cube, (2) as the middle face where the two cubes touch, and (3) as the front face of the bottom cube. All remaining edges are points. This way the rear and front face of the top cube and the left and right face of the bottom cube are homotopy pushouts by construction. Also the top face

$$\begin{array}{ccc} A & \longrightarrow & * \\ {}^{\simeq}\downarrow && \downarrow \\ A & \longrightarrow & * \end{array}$$

is a homotopy pushout, since two opposite edges of it are weak equivalences (prop.). From this the pasting law for homotopy pushouts (prop.) gives that also the middle square (2) is a homotopy pushout. Applying the pasting law once more this way, now for the bottom cube, gives that the bottom square

$$\begin{array}{ccc} * & \longrightarrow & * \\ \downarrow && \downarrow \\ D/B & \longrightarrow & (D/A)/(B/A) \end{array}$$

is a homotopy pushout. Since here the left edge is a weak equivalence, necessarily, so is the right edge (prop.), which hence exhibits the claimed identification

$$D/B \simeq (D/A)/(B/A) \,.$$

▮

Remark 1.113. All we used in the proof (of prop. 1.112) of the octahedral axiom (T4) is the existence and nature of homotopy pushouts. In fact one may show that the octahedral axiom is equivalent to the existence of homotopy pushouts, in the sense of axiom B in (Hubery).

Long fiber-cofiber sequences

In homotopy theory there are generally long homotopy fiber sequences to the left and long homotopy cofiber sequences to the right, as discussed in the section *Homotopy theory – Long sequences*. We prove now, in the generality of the axiomatics of triangulated categories (since the stable homotopy category is triangulated by prop. 1.112), that in stable homotopy theory both these sequences are long in both directions, and in fact coincide.

Literature (Schwede 12, II.2)

Lemma 1.114. *For* $(\mathrm{Ho}, \Sigma, \mathrm{CofSeq})$ *a triangulated category, def. 1.111, and*

$$A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} B/A \overset{h}{\longrightarrow} \Sigma A$$

a distinguished triangle, then

$$g \circ f = 0$$

is the zero morphism.

Proof. Consider the commuting diagram

$$\begin{array}{ccccccc} A & \overset{\mathrm{id}}{\longrightarrow} & A & \longrightarrow & 0 & \longrightarrow & \Sigma A \\ \downarrow^{\mathrm{id}} && \downarrow^{f} &&&& \\ A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \end{array} .$$

Observe that the top part is a distinguished triangle by axioms T1 and T2 in def. 1.111. Hence by T3 there is an extension to a commuting diagram of the form

$$\begin{array}{ccccccc} A & \overset{\mathrm{id}}{\longrightarrow} & A & \longrightarrow & 0 & \longrightarrow & \Sigma A \\ \downarrow^{\mathrm{id}} && \downarrow^{f} && \downarrow && \downarrow^{\Sigma f} \\ A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \end{array} .$$

Now the commutativity of the middle square proves the claim. ▮

Proposition 1.115. *Let* $(\mathrm{Ho}, \Sigma, \mathrm{CofSeq})$ *be a triangulated category, def. 1.111, with hom-functor denoted by* $[-,-]_\ast : \mathrm{Ho}^{\mathrm{op}} \times \mathrm{Ho} \to \mathrm{Ab}$. *For* $X \in \mathrm{Ho}$ *any object, and for* $D \in \mathrm{CofSeq}$ *any distinguished triangle*

$$D = (A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} B/A \overset{h}{\longrightarrow} \Sigma A)$$

then the sequences of abelian groups

1. *(long cofiber sequence)*

$$[\Sigma A, X]_\ast \overset{[h,X]_\ast}{\longrightarrow} [B/A, X]_\ast \overset{[g,X]_\ast}{\longrightarrow} [B, X]_\ast \overset{[f,X]_\ast}{\longrightarrow} [A, X]_\ast$$

2. *(long fiber sequence)*

$$[X, A]_\ast \overset{[X,f]_\ast}{\longrightarrow} [X, B]_\ast \overset{[X,g]_\ast}{\longrightarrow} [X, B/A]_\ast \overset{[X,h]_\ast}{\longrightarrow} [X, \Sigma A]_\ast$$

are long exact sequences.

Proof. Regarding the first case:

Since $g \circ f = 0$ by lemma 1.114, we have an inclusion $\mathrm{im}([g,X]_\ast) \subset \ker([f,X]_\ast)$. Hence it is sufficient to show that if $\psi : B \to X$ is in the kernel of $[f,X]_\ast$ in that $\psi \circ f = 0$, then there is $\phi : B/A \to X$ with $\phi \circ g = \psi$. To that end, consider the commuting diagram

$$\begin{array}{ccccccc} A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \\ \downarrow && {}^{\psi}\downarrow &&&& \\ 0 & \longrightarrow & X & \overset{\mathrm{id}}{\longrightarrow} & X & \longrightarrow & 0 \end{array} ,$$

where the commutativity of the left square exhibits our assumption.

The top part of this diagram is a distinguished triangle by assumption, and the bottom part is by condition $T1$ in def. 1.111. Hence by condition T3 there exists ϕ fitting into a commuting diagram of the form

$$\begin{array}{ccccccc} A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \\ \downarrow && {}^{\psi}\downarrow && \downarrow^{\phi} && \downarrow \\ 0 & \longrightarrow & X & \overset{id}{\longrightarrow} & X & \longrightarrow & 0 \end{array} .$$

Here the commutativity of the middle square exhibits the desired conclusion.

This shows that the first sequence in question is exact at $[B,X]_*$. Applying the same reasoning to the distinguished triangle $(g, h, -\Sigma f)$ provided by T2 yields exactness at $[B/A,X]_*$.

Regarding the second case:

Again, from lemma 1.114 it is immediate that

$$\mathrm{im}([X,f]_*) \subset \ker([X,g]_*)$$

so that we need to show that for $\psi : X \to B$ in the kernel of $[X,g]_*$, hence such that $g \circ \psi = 0$, then there exists $\phi : X \to A$ with $f \circ \phi = \psi$.

To that end, consider the commuting diagram

$$\begin{array}{ccccccc} X & \longrightarrow & 0 & \longrightarrow & \Sigma X & \overset{-\Sigma\, id}{\longrightarrow} & \Sigma X \\ \downarrow^{\psi} && \downarrow &&&& \\ B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A & \overset{-\Sigma f}{\longrightarrow} & \Sigma B \end{array} ,$$

where the commutativity of the left square exhibits our assumption.

Now the top part of this diagram is a distinguished triangle by conditions T1 and T2 in def. 1.111, while the bottom part is a distinguished triangle by applying T2 to the given distinguished triangle. Hence by T3 there exists $\tilde{\phi} : \Sigma X \to \Sigma A$ such as to extend to a commuting diagram of the form

$$\begin{array}{ccccccc} X & \longrightarrow & 0 & \longrightarrow & \Sigma X & \overset{-\Sigma\, id}{\longrightarrow} & \Sigma X \\ \downarrow^{\psi} && \downarrow && \downarrow^{\tilde{\phi}} && \downarrow^{\Sigma \psi} \\ B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A & \overset{-\Sigma f}{\longrightarrow} & \Sigma B \end{array} ,$$

At this point we appeal to the condition in def. 1.111 that $\Sigma : \mathrm{Ho} \to \mathrm{Ho}$ is an equivalence of categories, so that in particular it is a fully faithful functor. It being a full functor implies that there exists $\phi : X \to A$ with $\tilde{\phi} = \Sigma \phi$. It being faithful then implies that the whole commuting square on the right is the image under Σ of a commuting square

$$\begin{array}{ccc} X & \overset{-id}{\longrightarrow} & X \\ {}^{\phi}\downarrow && \downarrow^{\psi} \\ A & \underset{-f}{\longrightarrow} & B \end{array} .$$

This concludes the exactness of the second sequence at $[X,B]_*$. As before, exactness at $[X,B/A]_*$ follows with the same argument applied to the shifted triangle, via T2. ▮

Lemma 1.116. *Consider a morphism of distinguished triangles in a triangulated category (def. 1.111):*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \\ \downarrow^{a} && \downarrow^{b} && \downarrow^{c} && \downarrow^{\Sigma a} \\ A' & \longrightarrow & B' & \longrightarrow & B'/A' & \longrightarrow & \Sigma A' \end{array}.$$

If two out of $\{a,b,c\}$ are isomorphisms, then so is the third.

Proof. Consider the image of the situation under the hom-functor $[X,-]_*$ out of any object X:

$$\begin{array}{ccccccccc} [X,A]_* & \longrightarrow & [X,B]_* & \overset{g}{\longrightarrow} & [X,B/A]_* & \overset{h}{\longrightarrow} & [X,\Sigma A]_* & \longrightarrow & [X,\Sigma B]_* \\ \downarrow^{a_*} && \downarrow^{b_*} && \downarrow^{c_*} && \downarrow^{(\Sigma a)_*} && \downarrow^{(\Sigma b)_*} \\ [X,A']_* & \longrightarrow & [X,B']_* & \longrightarrow & [X,B'/A']_* & \longrightarrow & [X,\Sigma A']_* & \longrightarrow & [X,\Sigma B']_* \end{array},$$

where we extended one step to the right using axiom T2 (def. 1.111).

By prop. 1.115 here the top and bottom are exact sequences.

So assume the case that a and b are isomorphisms, hence that a_*, b_*, $(\Sigma a)_*$ and $(\Sigma b)_*$ are isomorphisms. Then by exactness of the horizontal sequences, the five lemma implies that c_* is an isomorphism. Since this holds naturally for all X, the Yoneda lemma (fully faithfulness of the Yoneda embedding) then implies that c is an isomorphism.

If instead b and c are isomorphisms, apply this same argument to the triple $(b,c,\Sigma a)$ to conclude that Σa is an isomorphism. Since Σ is an equivalence of categories, this implies then that a is an isomorphism.

Analogously for the third case. ▮

Lemma 1.117. *If $(g,h,-\Sigma f)$ is a distinguished triangle in a triangulated category (def. 1.111), then so is (f,g,h).*

Proof. By T0 there is some distinguished triangle of the form (f,g',h'). By T2 this gives a distinguished triangle $(-\Sigma f,-\Sigma g',-\Sigma h')$. By T3 there is a morphism c' giving a commuting diagram

$$\begin{array}{ccccccc} \Sigma A & \overset{-\Sigma f}{\longrightarrow} & \Sigma B & \overset{-\Sigma g}{\longrightarrow} & \Sigma C & \overset{-\Sigma h}{\longrightarrow} & \Sigma^2 A \\ {}^{=}\downarrow && {}^{=}\downarrow && {}^{c'}\downarrow && {}^{=}\downarrow \\ \Sigma A & \overset{-\Sigma f}{\longrightarrow} & \Sigma B & \overset{-\Sigma g'}{\longrightarrow} & \Sigma C & \overset{-\Sigma h'}{\longrightarrow} & \Sigma^2 A \end{array}.$$

Now lemma 1.116 gives that c' is an isomorphism. Since Σ is an equivalence of categories, there is an isomorphism c such that $c' = \Sigma c$. Since Σ is in particular a faithful functor, this c exhibits an isomorphism between (f,g,h) and (f,g',h'). Since the latter is distinguished, so is the former, by T0. ▮

In conclusion:

Proposition 1.118. *Let*

$$X \stackrel{f}{\longrightarrow} Y \stackrel{g}{\longrightarrow} Z$$

be a homotopy cofiber sequence (def.) of spectra in the stable homotopy category (def. 1.97) $\mathrm{Ho}(\mathrm{Spectra})$. Let $A \in \mathrm{Ho}(\mathrm{Spectra})$ be any other spectrum. Then the abelian hom-groups of the stable homotopy category (def. 1.99, lemma 1.100) sit in long exact sequences of the form

$$\cdots \longrightarrow [A, \Omega Y] \stackrel{-(\Omega g)_*}{\longrightarrow} [A, \Omega Z] \longrightarrow [A, X] \stackrel{f_*}{\longrightarrow} [A, Y] \stackrel{g_*}{\longrightarrow} [A, Z] \longrightarrow [A, \Sigma X] \stackrel{-(\Sigma f)_*}{\longrightarrow} [A, \Sigma Y] \longrightarrow \cdots \,.$$

Proof. By prop. 1.112 the above abstract reasoning in triangulated categories applies. By prop. 1.115 we have long exact sequences to the right as shown. By lemma 1.117 these also extend to the left as shown. ▮

This suggests that homotopy cofiber sequences coincide with homotopy fiber sequence in the stable homotopy category. This is indeed the case:

Proposition 1.119. *In the stable homotopy category, a sequence of morphisms is a homotopy cofiber sequence precisely if it is a homotopy fiber sequence.*

Specifically for $f : X \longrightarrow Y$ any morphism in $\mathrm{Ho}(\mathrm{Spectra})$, then there is an isomorphism

$$\phi \;:\; \mathrm{hofib}(f) \stackrel{\simeq}{\longrightarrow} \Omega\, \mathrm{hocof}(f)$$

between the homotopy fiber and the looping of the homotopy cofiber, which fits into a commuting diagram in the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ of the form

$$\begin{array}{ccccc} \Omega Y & \longrightarrow & \mathrm{hofib}(f) & \longrightarrow & X \\ {}^{\mathllap{=}}\downarrow && \downarrow^{\mathrlap{\phi}}_{\mathrlap{\simeq}} && \downarrow^{\mathrlap{\simeq}} \\ \Omega Y & \longrightarrow & \Omega\, \mathrm{hocof}(f) & \longrightarrow & \Omega \Sigma X \end{array} \,,$$

where the top row is the homotopy fiber sequence of f, while the bottom row is the image under the looping functor Ω of the homotopy cofiber sequence of f.

(Lewis-May-Steinberger 86, chapter III, theorem 2.4)

Proof. Label the diagram in question as follows

$$\begin{array}{ccccc} \Omega Y & \stackrel{a}{\longrightarrow} & \mathrm{hofib}(f) & \stackrel{b}{\longrightarrow} & X \\ {}^{\mathllap{=}}\downarrow & (1) & \downarrow^{\mathrlap{\phi}}_{\mathrlap{\simeq}} & (2) & \downarrow^{\mathrlap{\simeq}} \\ \Omega Y & \underset{c}{\longrightarrow} & \Omega\, \mathrm{hocof}(f) & \underset{d}{\longrightarrow} & \Omega \Sigma X \end{array} \,.$$

Let X be represented by a CW-spectrum (by prop. 1.55), hence in particular by a cofibrant sequential spectrum (by prop. 1.51). By prop. 1.52 and the factorization lemma (lemma) this implies that the standard mapping cone construction on f (def.) is a model for the homotopy cofiber of f (exmpl.):

$$\mathrm{hocof}(f) \simeq \mathrm{Cone}(f) \,.$$

By construction of mapping cones, this sits in the following commuting squares in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$.

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Cone}(X) \\ \downarrow & (po) & \downarrow \\ Y & \longrightarrow & \mathrm{Cone}(f) \\ \downarrow & (po) & \downarrow \\ * & \longrightarrow & \Sigma X \end{array} \,.$$

Consider then the commuting diagram

$$\begin{array}{ccccccc} \Omega Y & \overset{a}{\longrightarrow} & \mathrm{hofib}(f) & \overset{\phi}{\longrightarrow} & \Omega \,\mathrm{hocof}(f) & \overset{d}{\longrightarrow} & \Omega \Sigma X \simeq X \\ \downarrow && \downarrow && \downarrow && \downarrow \\ * & \longrightarrow & X & \longrightarrow & \mathrm{Cone}(X) & \longrightarrow & \mathrm{Cone}(X) \\ \downarrow && \downarrow^{f} && \downarrow && \downarrow \\ Y & \underset{=}{\longrightarrow} & Y & \longrightarrow & \mathrm{Cone}(f) & \longrightarrow & \Sigma X \end{array} \,,$$

in the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ (def. 1.97). Here the bottom commuting squares are the images under localization $\gamma : \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{Ho}(\mathrm{Spectra})$ (thm.) of the above commuting squares in the definition of the mapping cone, and the top row of squares are the morphisms induced via the universal property of fibers by forming homotopy fibers of the bottom vertical morphisms (fibers of fibration replacements, which may be chosen compatibly, either by pullback or by invoking the small object argument).

First of all, this exhibits the composition of the left two horizontal morphisms $\phi \circ a \simeq c$ in the above diagram as the left part (1) of the commuting diagram to be proven.

Now observe that the pasting composite of the two rectangles on the right of the previous diagram is isomorphic, in $\mathrm{Ho}(\mathrm{Spectra})$, to the following pasting composite:

$$\begin{array}{ccccc} \mathrm{hofib}(f) & \overset{b}{\longrightarrow} & X & \underset{\simeq}{\overset{\eta}{\longrightarrow}} & \Omega \Sigma X \simeq X \\ \downarrow && \downarrow && \downarrow \\ X & \overset{\mathrm{id}}{\longrightarrow} & X & \longrightarrow & \mathrm{Cone}(X) \\ \downarrow && \downarrow && \downarrow \\ Y & \longrightarrow & * & \longrightarrow & \Sigma X \end{array} \,.$$

This is because the pasting composite of the bottom squares is isomorphic already in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ by the above commuting diagrams for the mapping cone and the suspension, and then using again the universal property of homotopy fibers.

Hence the top composite morphisms coincide, by universality of homotopy fibers, with the previous top composite:

$$\eta \circ b \simeq d \circ \phi \,.$$

This is the commutativity of the right part (2) of the diagram to be proven.

So far we have shown that

$$\begin{array}{ccccc} \Omega Y & \longrightarrow & \mathrm{hofib}(f) & \longrightarrow & X \\ {}^{=}\downarrow && \downarrow^{\phi} && \downarrow^{=} \\ \Omega Y & \longrightarrow & \Omega \,\mathrm{hocof}(f) & \longrightarrow & X \end{array}$$

commutes in the stable homotopy category. It remains to see that ϕ is an isomorphism.

To that end, consider for any $A \in \mathrm{Ho}(\mathrm{Spectra})$ the image of this commuting diagram, prolonged to the left and right, under the hom-functor $[A,-]_*$ of the stable homotopy category:

$$\begin{array}{ccccccccc} [A,\Omega X] & \longrightarrow & [A,\Omega Y] & \longrightarrow & [A,\mathrm{hofib}(f)] & \longrightarrow & [A,X] & \longrightarrow & [A,Y] \\ {}^{=}\downarrow && {}^{=}\downarrow && \downarrow^{[A,\phi]} && \downarrow^{\simeq} && \downarrow^{\simeq} \\ [A,\Omega X] & \longrightarrow & [A,\Omega Y] & \longrightarrow & [A,\Omega\,\mathrm{hocof}(f)] & \longrightarrow & [A,\Omega\Sigma X] & \longrightarrow & [A,\Omega\Sigma Y] \end{array} \,.$$

Here the top row is long exact, since it is the long homotopy fiber sequence to the left that holds in the homotopy category of any model catgeory (prop.). Moreover, the bottom sequence is long exact by prop. 1.118. Hence the five lemma implies that $[A,\phi]_*$ is an isomorphism. Since this is the case for all A, the Yoneda lemma (faithfulness of the Yoneda embedding) implies that ϕ itself is an isomorphism. ▮

Remark 1.120. Prop. 1.119 is the homotopy theoretic analog of the clause that makes a pre-abelian category into an abelian category:

A pre-abelian category is an additive category in which fibers (kernels) and cofibers (cokernels) exist. This is an abelian category if the cofiber of the fiber of any morphism equals coincides with the fiber of the cofiber of that morphism.

Here we see that in stable homotopy theory, whose homotopy category is additive, and in which homotopy fibers and homotopy cofibers exist, the analogous statement is true even in a stronger form: the homotopy cofiber projection of the homotopy fiber inclusion of any morphism coincides with that morphism, and so does the homotopy fiber projection of the homotopy cofiber inclusion.

In particular there are long exact sequences of stable homotopy groups extending in both directions:

Lemma 1.121. *Let $X \in \mathrm{SeqSpec}(\mathrm{Top}_{cg})$ be any sequential spectrum, then there is an isomorphism*

$$\pi_0(X) \simeq [\mathbb{S}, X]$$

between its stable homotopy group in degree 0 (def. 1.13) and the hom-group (according to def. 1.103, prop. 1.110) in the stable homotopy category (def. 1.97) from the sphere spectrum (def. 1.6) into X.

Generally, with respect to the graded hom-groups of def. 1.101 we have

$$\pi_\bullet(X) \simeq [\mathbb{S}, X]_\bullet \,.$$

Proof. The hom-set in the homotopy category is equivalently given by the left homotopy-equivalence classes out of a cofibrant representative of $\mathbb{S}$ into a fibrant representative of X (lemma).

The standard sphere spectrum $\mathbb{S}_{\mathrm{std}} \coloneqq \Sigma^\infty S^0$ is a CW-spectrum and hence cofibrant, by prop. 1.51. Moreover, this implies by prop. 1.52 that left homotopies out of $\mathbb{S}_{\mathrm{str}}$ are represented by the standard sequential cylinder spectrum

$$\mathbb{S}_{\mathrm{std}} \wedge (I_+) \simeq \Sigma^\infty (I_+) \,.$$

By theorem 1.72, fibrant replacement for X is provided by its spectrification QX according to def. 1.21.

So it follows that $[\mathbb{S}, X]_*$ is given by left homotopy classes of morphisms

$$\Sigma^\infty S^0 = \mathbb{S}_{\mathrm{std}} \longrightarrow QX$$

in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$. By the $(\Sigma^\infty \dashv \Omega^\infty)$-adjunction (prop. 1.12) these are equivalently morphisms

$$S^0 \longrightarrow (QX)_0$$

in $\mathrm{Top}^{*/}_{\mathrm{cg}}$. Hence equivalently morphisms

$$* \longrightarrow (QX)_0$$

in $\mathrm{Top}_{\mathrm{cg}}$, hence equivalently points in $(QX)_0$. Analogously, a left homotopy

$$\Sigma^\infty(I_+) \longrightarrow (QX)_0$$

in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ is equivalently a path

$$I \longrightarrow (QX)_0$$

in $\mathrm{Top}_{\mathrm{cg}}$.

In conclusion this establishes an isomorphism

$$[\mathbb{S}, X]_* \simeq \pi_0((QX)_0)$$

with π_0 of the 0-component of QX. With this the statement follows with example 1.20, since QX is an Omega-spectrum, by prop. 1.22.

From this the last statement follows from the identity

$$\pi_0(\Sigma^{-n}(-)) \simeq \pi_n(-) \;.$$

■

As a consequence:

Proposition 1.122. *Let*

$$X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z$$

be a homotopy cofiber sequence (def.) in the stable homotopy category (def. 1.97). Then there is induced a long exact sequence of stable homotopy groups (def. 1.13) of the form

$$\cdots \longrightarrow \pi_{\bullet+1}(Z) \longrightarrow \pi_\bullet(X) \overset{f_*}{\longrightarrow} \pi_\bullet(Y) \overset{g_*}{\longrightarrow} \pi_\bullet(Z) \longrightarrow \pi_{\bullet-1}(X) \longrightarrow \cdots \;.$$

Proof. Via lemmma 1.121 this is a special case of prop. 1.118. ■

As an example, we check explicitly what we already deduced abstractly in prop. 1.110, that in the stable homotopy category wedge sum and Cartesian product of spectra agree and constitute a biproduct/direct sum:

Example 1.123. For $X, Y \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$, then the canonical morphism

$$X \vee Y \longrightarrow X \times Y$$

out of the coproduct (wedge sum, example 1.29) into the product (via prop. 1.27), given by

$$
\array{
  X && && Y
  \\
  & {}_{\mathllap{i_X}}\searrow && \swarrow_{\mathrlap{i_X}}
  \\
  {}^{\mathllap{id_X}}\downarrow && X \sqcup Y && \downarrow^{\mathrlap{id_Y}}
  \\
  & \swarrow_{(id,0)} && {}_{(0,id)}\searrow
  \\
  X && && Y
  \\
  & \searrow^{(id,0)} && {}^{(0,id)}\swarrow
  \\
  {}^{\mathllap{id_X}}\downarrow && X \times Y && \downarrow^{\mathrlap{id_Y}}
  \\
  & \swarrow_{p_X} && {}_{p_Y}\searrow
  \\
  X && && Y
}
$$

represents an isomorphism in the stable homotopy category.

Proof. By prop. 1.55, we may represent both X and Y by CW-spectra (def. 1.46) in $(SeqSpec(Top_{cg})_{stable})_c[W_{st}^{-1}]$. Then the canonical morphism

$$i_X : X \longrightarrow X \vee Y$$

is a cofibration according to theorem 1.42, because $X_{n+1} \underset{S^1 \wedge X_n}{\sqcup} S^1 (X \vee Y) \simeq X_{n+1} \vee S^1 \wedge Y_n$.

Hence its ordinary cofiber, which is Y, is its homotopy cofiber (def.), and so we have a homotopy cofiber sequence

$$X \longrightarrow X \vee Y \longrightarrow Y \,.$$

Moreover, under forming stable homotopy groups (def. 1.13), the inclusion map evidently gives an injection, and the projection map gives a surjection. Hence the long exact sequence of stable homotopy groups from prop. 1.122 gives the short exact sequence

$$0 \to \pi_\bullet(X) \longrightarrow \pi_\bullet(X \vee Y) \longrightarrow \pi_\bullet(Y) \to 0 \,.$$

Finally, due to the fact that the inclusion and projection for one of the two summands constitute a retraction, this is a split exact sequence, hence exhibits an isomorphism

$$\pi_k(X \vee Y) \overset{\simeq}{\longrightarrow} \pi_k(X) \oplus \pi_k(Y) \simeq \pi_k(X) \times \pi_k(Y) \simeq \pi_k(X \times Y)$$

for all k. But this just says that $X \vee Y \to X \times Y$ is a stable weak homotopy equivalence. ▮

Final Remark 1.124. For a tower of fibrations of spectra, the long sequences of stable homotopy groups associated with any (co-)fiber sequence of spectra, from prop. 1.122, combine to an exact couple. The induced spectral sequence of a tower of fibrations is the central tool of computation in stable homotopy theory.

We discuss how these spectral sequences arise in the section *Interlude -- Spectral sequences*.

We discuss in detail the special case of the Adams spectral sequences in the section *Part 2 -- Adams spectral sequences*.

But for handling any of these spectral sequences it is convenient, or, in many cases, necessary to have multiplicative structure available, induced from a symmetric monoidal smash product of spectra. This we turn to in *part 1.2 -- Structured spectra*.

2. References

We give the modern picture of the stable homotopy category, for which a quick survey may be found in

- Cary Malkiewich, *The stable homotopy category*, 2014 (pdf).

A classical textbook on stable homotopy theory for "unstructured" spectra is

- Frank Adams, part III sections 2, 4-7 of *Stable homotopy and generalized homology*, Chicago Lectures in mathematics, 1974

For establishing the stable model structure on spectra we use the Bousfield-Friedlander theorem as discussed in

- Paul Goerss, Rick Jardine, section X.4 of *Simplicial homotopy theory*, (1996)

and as applied for general Omega-spectrification functors in

- Stefan Schwede, *Spectra in model categories and applications to the algebraic cotangent complex*, Journal of Pure and Applied Algebra 120 (1997) 77-104 (pdf)

For the discussion of the stability of the homotopy theory of sequential spectra we follow

- John F. Jardine, sections 10.3 and 10.4 of *Local homotopy theory*, 2016

For the definition of triangulated categories and a discussion of various equivalent versions of the octahedral axiom the following brief note is useful:

- Andrew Hubery, *Notes on the octahedral axiom*, (pdf)

For the discussion of the triangulated structure of the stable homotopy category we follow

- Stefan Schwede, section II.2 of *Symmetric spectra*, 2012 (pdf)

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