



Introduction to Topology -- 1

This page is a detailed introduction to basic [topology](#). Starting from scratch (required background is just a basic concept of [sets](#)), and amplifying motivation from [analysis](#), it first develops standard [point-set topology](#) ([topological spaces](#)). In passing, some basics of [category theory](#) make an informal appearance, used to transparently summarize some conceptually important aspects of the theory, such as [initial](#) and [final topologies](#) and the [reflection](#) into [Hausdorff](#) and [sober topological spaces](#). The second part introduces some basics of [homotopy theory](#), mostly the [fundamental group](#), and ends with their first application to the classification of [covering spaces](#).

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For introduction to more general and abstract [homotopy theory](#) see instead at [Introduction to Homotopy Theory](#).

Point-set Topology

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The idea of [topology](#) is to study “[spaces](#)” with “[continuous functions](#)” between them. Specifically one considers [functions](#) between [sets](#) (whence “[point-set topology](#)”, see [below](#)) such that there is a concept for what it means that these functions depend continuously on their arguments, in that that their values do not “jump”. Such a concept of [continuity](#) is familiar from [analysis](#) on [metric spaces](#), (recalled [below](#)) but the definition in topology generalizes this analytic concept and renders it more foundational, generalizing the concept of [metric spaces](#) to that of [topological spaces](#). (def. [2.3](#) below).

Hence [topology](#) is the study of the [category](#) whose [objects](#) are [topological spaces](#), and whose [morphisms](#) are [continuous functions](#) (see also remark [3.3](#) below). This category is much more flexible than that of [metric spaces](#), for example it admits the construction of arbitrary [quotients](#) and [intersections](#) of spaces. Accordingly, topology underlies or informs many and diverse areas of mathematics, such as [functional analysis](#), [operator algebra](#), [manifold/scheme](#) theory, hence [algebraic geometry](#) and [differential geometry](#), and the study of [topological groups](#), [topological vector spaces](#), [local rings](#), etc.. Not the least, it gives rise to the field of [homotopy theory](#), where one considers also continuous deformations of continuous functions themselves (“[homotopies](#)”). Topology itself has many branches, such as [low-dimensional topology](#) or [topological domain theory](#).

A popular imagery for the concept of a [continuous function](#) is provided by deformations of [elastic](#) physical bodies, which may be deformed by stretching them without tearing. The canonical illustration is a continuous [bijective](#) function from the [torus](#) to the surface of a coffee mug, which maps half of the torus to the handle of the coffee mug, and continuously deforms parts of the other half in order to form the actual cup. Since the [inverse function](#) to this function is itself continuous, the torus and the coffee mug, both regarded as [topological spaces](#), are “[the same](#)” for the purposes of [topology](#), one says they are [homeomorphic](#).

On the other hand, there is *no* [homeomorphism](#) from the [torus](#) to, for instance, the [sphere](#), signifying that these represent two topologically distinct spaces. Part of topology is concerned with studying [homeomorphism-invariants](#) of topological spaces which allow to detect by means of [algebraic](#) manipulations whether two topological spaces are homeomorphic (or more generally [homotopy equivalent](#)). This is called [algebraic topology](#). A basic algebraic invariant is the [fundamental group](#) of a topological space (discussed [below](#)), which measures how many ways there are to wind loops inside a topological space.

Beware that the popular imagery of “[rubber-sheet geometry](#)” only captures part of the full scope of topology, in that it invokes spaces that *locally* still look like [metric spaces](#). But the concept of topological spaces is a good bit more general. Notably [finite topological spaces](#) are either [discrete](#) or very much unlike [metric spaces](#) (example 4.3 below), they play a role in [categorical logic](#). Also in [geometry](#) exotic topological spaces frequently arise when forming non-free [quotients](#). In order to gauge just how many of such “exotic” examples of topological spaces beyond locally [metric spaces](#) one wishes to admit in the theory, extra “[separation axioms](#)” are imposed on topological spaces (see [below](#)), and the flavour of topology as a field depends on this choice.

Among the separation axioms, the [Hausdorff space](#) axiom is most popular (see [below](#)) the weaker axiom of [soberity](#) (see [below](#)) stands out, on the one hand because this is the weakest axiom that is still naturally satisfied in applications to [algebraic geometry](#) ([schemes are sober](#)) and [computer science](#) (Vickers 89) and on the other hand because it fully realizes the strong roots that topology has in [formal logic](#): [sober topological spaces](#) are entirely characterized by the union-, intersection- and inclusion-relations (logical [conjunction](#), [disjunction](#) and [implication](#)) among their [open subsets](#) ([propositions](#)). This leads to a natural and fruitful generalization of [topology](#) to more general “purely logic-determined spaces”, called [locales](#) and in yet more generality [toposes](#) and [higher toposes](#). While the latter are beyond the scope of this introduction, their rich theory and relation to the [foundations](#) of mathematics and geometry provides an outlook on the relevance of the basic ideas of [topology](#).

In this first part we discuss the foundations of the concept of “sets equipped with topology” ([topological spaces](#)) and of [continuous functions](#) between them.

1. Metric spaces

The concept of continuity was first made precise in [analysis](#), in terms of [epsilon-delta analysis](#) on [metric spaces](#), recalled as def. 1.8 below. Then it was realized that this has a more elegant formulation in terms of the more general concept of [open sets](#), this is prop. 1.13 below. Adopting the latter as the definition leads to a more abstract concept of “continuous space”, this is the concept of [topological spaces](#), def. 2.3 below.

Here we briefly recall the relevant basic concepts from [analysis](#), as a motivation for various definitions in [topology](#). The reader who either already recalls these concepts in analysis or is content with ignoring the motivation coming from analysis should skip right away to the section [Topological spaces](#).

Definition 1.1. (metric space)

A [metric space](#) is

1. a [set](#) X (the “underlying set”);

2. a function $d : X \times X \rightarrow [0, \infty)$ (the “distance function”) from the Cartesian product of the set with itself to the non-negative real numbers

such that for all $x, y, z \in X$:

1. (symmetry) $d(x, y) = d(y, x)$
2. (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.
3. (non-degeneracy) $d(x, y) = 0 \Leftrightarrow x = y$

Definition 1.2. Let (X, d) , be a metric space. Then for every element $x \in X$ and every $\epsilon \in \mathbb{R}_+$ a positive real number, we write

$$B_x^\circ(\epsilon) := \{y \in X \mid d(x, y) < \epsilon\}$$

for the open ball of radius ϵ around x . Similarly we write

$$B_x(\epsilon) := \{y \in X \mid d(x, y) \leq \epsilon\}$$

for the *closed ball* of radius ϵ around x . Finally we write

$$S_x(\epsilon) := \{y \in X \mid d(x, y) = \epsilon\}$$

for the sphere of radius ϵ around x .

For $\epsilon = 1$ we also speak of the *unit open/closed ball* and the *unit sphere*.

Definition 1.3. For (X, d) a metric space (def. 1.1) then a subset $S \subset X$ is called a bounded subset if S is contained in some open ball (def. 1.2)

$$S \subset B_x^\circ(r)$$

around some $x \in X$ of some radius $r \in \mathbb{R}$.

A key source of metric spaces are normed vector spaces:

Definition 1.4. (normed vector space)

A normed vector space is

1. a real vector space V ;
2. a function (the norm)

$$\|-\| : V \rightarrow \mathbb{R}_{\geq 0}$$

from the underlying set of V to the non-negative real numbers,

such that for all $c \in \mathbb{R}$ with absolute value $|c|$ and all $v, w \in V$ it holds true that

1. (linearity) $\|cv\| = |c|\|v\|$;

2. ([triangle inequality](#)) $\|v + w\| \leq \|v\| + \|w\|$;

3. (non-degeneracy) if $\|v\| = 0$ then $v = 0$.

Proposition 1.5. Every [normed vector space](#) $(V, \|\cdot\|)$ becomes a [metric space](#) according to [def. 1.1](#) by setting

$$d(x, y) := \|x - y\| .$$

Examples of [normed vector spaces](#) ([def. 1.4](#)) and hence, via [prop. 1.5](#), of [metric spaces](#) include the following:

Example 1.6. For $n \in \mathbb{N}$, the [Cartesian space](#)

$$\mathbb{R}^n = \{\vec{x} = (x_i)_{i=1}^n \mid x_i \in \mathbb{R}\}$$

carries a [norm](#) (the *Euclidean norm*) given by the [square root](#) of the [sum](#) of the [squares](#) of the components:

$$\|\vec{x}\| := \sqrt{\sum_{i=1}^n (x_i)^2} .$$

Via [prop. 1.5](#) this gives \mathbb{R}^n the structure of a [metric space](#), and as such it is called the [Euclidean space](#) of [dimension](#) n .

Example 1.7. More generally, for $n \in \mathbb{N}$, and $p \in \mathbb{R}$, $p \geq 1$, then the [Cartesian space](#) \mathbb{R}^n carries the [p-norm](#)

$$\|\vec{x}\|_p := \sqrt[p]{\sum_i |x_i|^p}$$

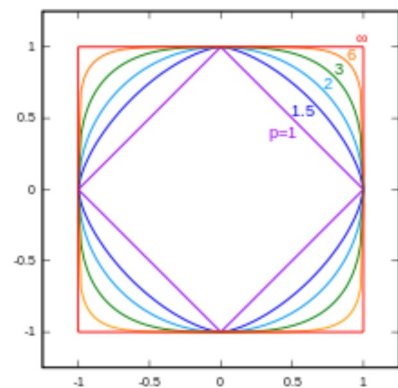
One also sets

$$\|\vec{x}\|_\infty := \max_{i \in I} |x_i|$$

and calls this the [supremum norm](#).

The graphics on the right (grabbed from Wikipedia) shows unit circles ([def. 1.2](#)) in \mathbb{R}^2 with respect to various [p-norms](#).

By the [Minkowski inequality](#), the [p-norm](#) generalizes to non-[finite dimensional vector spaces](#) such as [sequence spaces](#) and [Lebesgue spaces](#).



Continuity

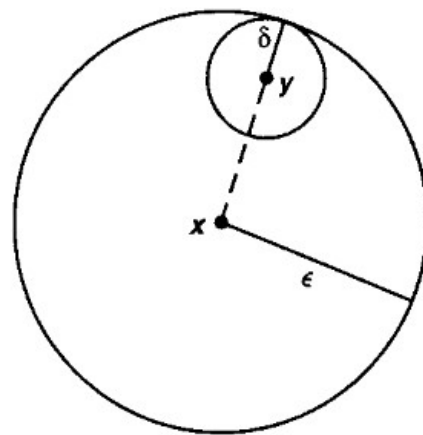
The following is now the fairly obvious definition of continuity for functions between metric spaces.

Definition 1.8. (epsilon definition of continuity)

For (X, d_X) and (Y, d_Y) two [metric spaces](#) (def. [1.1](#)), then a [function](#)

$$f : X \rightarrow Y$$

is said to be *continuous at a point* $x \in X$ if for every [positive real number](#) ϵ there exists a [positive real number](#) δ such that for all $x' \in X$ that are a [distance](#) smaller than δ from x then their image $f(x')$ is a distance smaller than ϵ from $f(x)$:



$$(f \text{ continuous at } x) := \forall_{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}} \left(\exists_{\substack{\delta \in \mathbb{R} \\ \delta > 0}} ((d_X(x, x') < \delta) \Rightarrow (d_Y(f(x), f(x')) < \epsilon)) \right).$$

The function f is said to be *continuous* if it is continuous at every point $x \in X$.

Example 1.9. (polynomials are continuous functions)

Consider the [real line](#) \mathbb{R} regarded as the 1-dimensional [Euclidean space](#) \mathbb{R} from example [1.6](#).

For $P \in \mathbb{R}[X]$ a [polynomial](#), then the function

$$\begin{aligned} f_P &: \mathbb{R} \rightarrow \mathbb{R} \\ x &\mapsto P(x) \end{aligned}$$

is a [continuous function](#) in the sense of def. [1.8](#).

On the other hand, a [step function](#) is continuous everywhere except at the [finite number](#) of points at which it changes its value, see example [1.14](#) below.

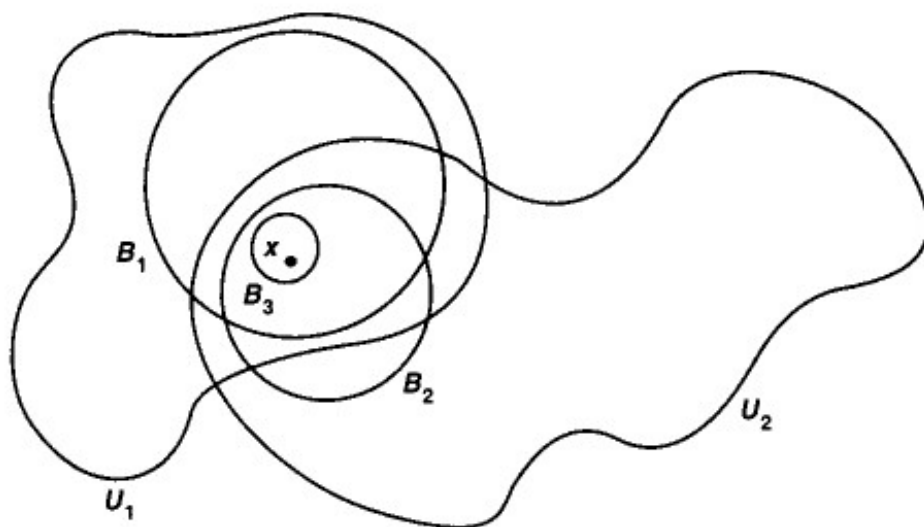
We now reformulate the analytic concept of continuity from def. [1.8](#) in terms of the simple but important concept of [open sets](#):

Definition 1.10. (neighbourhood and open set)

Let (X, d) be a [metric space](#) (def. [1.1](#)). Say that:

1. A [neighbourhood](#) of a point $x \in X$ is a [subset](#) $U_x \subset X$ which contains some [open ball](#) $B_x^\circ(\epsilon) \subset U_x$ around x (def. [1.2](#)).
2. An [open subset](#) of X is a [subset](#) $U \subset X$ such that for every $x \in U$ it also contains an [open ball](#) $B_x^\circ(\epsilon)$ around x (def. [1.2](#)).
3. An [open neighbourhood](#) of a point $x \in X$ is a [neighbourhood](#) U_x of x which is also an open subset, hence equivalently this is any open subset of X that contains x .

The following picture shows a point x , some [open balls](#) B_i containing it, and two of its [neighbourhoods](#) U_i :



graphics grabbed from [Munkres 75](#)

Example 1.11. (the empty subset is open)

Notice that for (X, d) a [metric space](#), then the [empty subset](#) $\emptyset \subset X$ is always an [open subset](#) of (X, d) according to def. [1.10](#). This is because the clause for open subsets $U \subset X$ says that “for every point $x \in U$ there exists...”, but since there is no x in $U = \emptyset$, this clause is always satisfied in this case.

Conversely, the entire set X is always an open subset of (X, d) .

Example 1.12. (open/closed [intervals](#))

Regard the [real numbers](#) \mathbb{R} as the 1-dimensional [Euclidean space](#) (example [1.6](#)).

For $a < b \in \mathbb{R}$ consider the following [subsets](#):

1. $(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$ (*open interval*)
2. $(a, b] := \{x \in \mathbb{R} \mid a < x \leq b\}$ (*half-open interval*)
3. $[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}$ (*half-open interval*)
4. $[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$ (*closed interval*)

The first of these is an open subset according to def. [1.10](#), the other three are not. The first one is called an [open interval](#), the last one a [closed interval](#) and the middle two are called [half-open intervals](#).

Similarly for $a, b \in \mathbb{R}$ one considers

1. $(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$ (*unbounded open interval*)

2. $(a, \infty) := \{x \in \mathbb{R} \mid a < x\}$ (unbounded open interval)
3. $(-\infty, b] := \{x \in \mathbb{R} \mid x \leq b\}$ (unbounded half-open interval)
4. $[a, \infty) := \{x \in \mathbb{R} \mid a \leq x\}$ (unbounded half-open interval)

The first two of these are open subsets, the last two are not.

For completeness we may also consider

- $(-\infty, \infty) = \mathbb{R}$
- $(a, a) = \emptyset$

which are both open, according to def. 2.3.

We may now rephrase the analytic definition of continuity entirely in terms of open subsets (def. 1.10):

Proposition 1.13. (rephrasing continuity in terms of open sets)

Let (X, d_X) and (Y, d_Y) be two *metric space* (def. 1.1). Then a *function* $f: X \rightarrow Y$ is *continuous* in the *epsilontic* sense of def. 1.8 precisely if it has the property that its *pre-images* of *open subsets* of Y (in the sense of def. 1.10) are open subsets of X :

$$(f \text{ continuous}) \Leftrightarrow ((O_Y \subset Y \text{ open}) \Rightarrow (f^{-1}(O_Y) \subset X \text{ open})) .$$

principle of continuity

Continuous pre-Images of open subsets are open.

Proof. Observe, by direct unwinding the definitions, that the epsilontic definition of continuity (def. 1.8) says equivalently in terms of *open balls* (def. 1.2) that f is continuous at x precisely if for every open ball $B_{f(x)}^\circ(\epsilon)$ around an image point, there exists an open ball $B_x^\circ(\delta)$ around the corresponding pre-image point which maps into it:

$$\begin{aligned} (f \text{ continuous at } x) &\Leftrightarrow \forall_{\epsilon > 0} \left(\exists_{\delta > 0} (f(B_x^\circ(\delta))) \subset B_{f(x)}^\circ(\epsilon) \right) \\ &\Leftrightarrow \forall_{\epsilon > 0} \left(\exists_{\delta > 0} (B_x^\circ(\delta) \subset f^{-1}(B_{f(x)}^\circ(\epsilon))) \right) . \end{aligned}$$

With this observation the proof immediate. For the record, we spell it out:

First assume that f is continuous in the epsilontic sense. Then for $O_Y \subset Y$ any *open subset* and $x \in f^{-1}(O_Y)$ any point in the pre-image, we need to show that there exists an *open neighbourhood* of x in $f^{-1}(O_Y)$.

That O_Y is open in Y means by definition that there exists an [open ball](#) $B_{f(x)}^\circ(\epsilon)$ in O_Y around $f(x)$ for some radius ϵ . By the assumption that f is continuous and using the above observation, this implies that there exists an open ball $B_x^\circ(\delta)$ in X such that $f(B_x^\circ(\delta)) \subset B_{f(x)}^\circ(\epsilon) \subset Y$, hence such that $B_x^\circ(\delta) \subset f^{-1}(B_{f(x)}^\circ(\epsilon)) \subset f^{-1}(O_Y)$. Hence this is an open ball of the required kind.

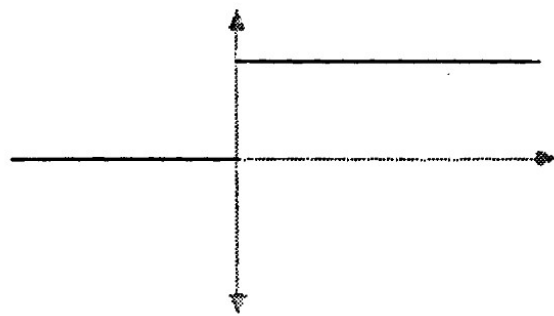
Conversely, assume that the pre-image function f^{-1} takes open subsets to open subsets. Then for every $x \in X$ and $B_{f(x)}^\circ(\epsilon) \subset Y$ an [open ball](#) around its image, we need to produce an open ball $B_x^\circ(\delta) \subset X$ around x such that $f(B_x^\circ(\delta)) \subset B_{f(x)}^\circ(\epsilon)$.

But by definition of open subsets, $B_{f(x)}^\circ(\epsilon) \subset Y$ is open, and therefore by assumption on f its pre-image $f^{-1}(B_{f(x)}^\circ(\epsilon)) \subset X$ is also an open subset of X . Again by definition of open subsets, this implies that it contains an open ball as required. ■

Example 1.14. ([step function](#))

Consider \mathbb{R} as the 1-dimensional [Euclidean space](#) (example 1.6) and consider the [step function](#)

$$\begin{aligned} \mathbb{R} &\xrightarrow{H} \mathbb{R} \\ x &\mapsto \begin{cases} 0 & | x \leq 0 \\ 1 & | x > 0 \end{cases} . \end{aligned}$$



graphics grabbed from [Vickers 89](#)

Consider then for $a < b \in \mathbb{R}$ the [open interval](#) $(a, b) \subset \mathbb{R}$, an [open subset](#) according to example 1.12. The [preimage](#) $H^{-1}(a, b)$ of this open subset is

$$H^{-1} : (a, b) \mapsto \begin{cases} \emptyset & | a \geq 1 \text{ or } b \leq 0 \\ \mathbb{R} & | a < 0 \text{ and } b > 1 \\ \emptyset & | a \geq 0 \text{ and } b \leq 1 \\ (0, \infty) & | 0 \leq a < 1 \text{ and } b > 1 \\ (-\infty, 0] & | a < 0 \text{ and } b \leq 1 \end{cases} .$$

By example 1.12, all except the last of these pre-images listed are open subsets.

The failure of the last of the pre-images to be open witnesses that the step function is not continuous at $x = 0$.

Compactness

A key application of [metric spaces](#) in [analysis](#) is that they allow a formalization of what it means for an infinite [sequence](#) of elements in the metric space (def. [1.15](#) below) to [converge](#) to a [limit of a sequence](#) (def. [1.16](#) below). Of particular interest are therefore those metric spaces for which each sequence has a converging subsequence: the [sequentially compact metric spaces](#) (def. [1.19](#)).

We now briefly recall these concepts from [analysis](#). Then, in the above spirit, we reformulate their epsilonic definition in terms of [open subsets](#). This gives a useful definition that generalizes to [topological spaces](#), the [compact topological spaces](#) discussed further [below](#).

Definition 1.15. ([sequence](#))

Given a [set](#) X , then a [sequence](#) of elements in X is a [function](#)

$$x_{(-)} : \mathbb{N} \rightarrow X$$

from the [natural numbers](#) to X .

A [sub-sequence](#) of such a sequence is a sequence of the form

$$x_{i(-)} : \mathbb{N} \xrightarrow{\iota} \mathbb{N} \xrightarrow{x_{(-)}} X$$

for some [injection](#) ι .

Definition 1.16. ([convergence to limit of a sequence](#))

Let (X, d) be a [metric space](#) (def. [1.1](#)). Then a [sequence](#)

$$x_{(-)} : \mathbb{N} \rightarrow X$$

in the underlying set X (def. [1.15](#)) is said to [converge](#) to a point $x_\infty \in X$, denoted

$$x_i \xrightarrow{i \rightarrow \infty} x_\infty$$

if for every [positive real number](#) ϵ , there exists a [natural number](#) n , such that all elements in the sequence after the n th one have [distance](#) less than ϵ from x_∞ .

$$\left(x_i \xrightarrow{i \rightarrow \infty} x_\infty \right) \Leftrightarrow \left(\forall_{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}} \left(\exists_{n \in \mathbb{N}} \left(\forall_{\substack{i \in \mathbb{N} \\ i > n}} d(x_i, x_\infty) \leq \epsilon \right) \right) \right).$$

Here the point x_∞ is called the [limit of the sequence](#). Often one writes $\lim_{i \rightarrow \infty} x_i$ for this point.

Definition 1.17. ([Cauchy sequence](#))

Given a [metric space](#) (X, d) (def. [1.1](#)), then a [sequence](#) of points in X (def. [1.15](#))

$$x_{(-)} : \mathbb{N} \rightarrow X$$

is called a Cauchy sequence if for every positive real number ϵ there exists a natural number $n \in \mathbb{N}$ such that the distance between any two elements of the sequence beyond the n th one is less than ϵ

$$(x_{(-)} \text{ Cauchy}) \Leftrightarrow \left(\forall_{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}} \left(\exists_{N \in \mathbb{N}} \left(\forall_{\substack{i, j \in \mathbb{N} \\ i, j > N}} d(x_i, x_j) \leq \epsilon \right) \right) \right).$$

Definition 1.18. (complete metric space)

A metric space (X, d) (def. 1.1), for which every Cauchy sequence (def. 1.17) converges (def. 1.16) is called a complete metric space.

A normed vector space, regarded as a metric space via prop. 1.5 that is complete in this sense is called a Banach space.

Finally recall the concept of compactness of metric spaces via epsilon analysis:

Definition 1.19. (sequentially compact metric space)

A metric space (X, d) (def. 1.1) is called sequentially compact if every sequence in X has a subsequence (def. 1.15) which converges (def. 1.16).

The key fact to translate this epsilon definition of compactness to a concept that makes sense for general topological spaces (below) is the following:

Proposition 1.20. (sequentially compact metric spaces are equivalently compact metric spaces)

For a metric space (X, d) (def. 1.1) the following are equivalent:

1. X is sequentially compact;
2. for every set $\{U_i \subset X\}_{i \in I}$ of open subsets U_i of X (def. 1.10) which cover X in that $X = \bigcup_{i \in I} U_i$, then there exists a finite subset $J \subset I$ of these open subsets which still covers X in that also $X = \bigcup_{i \in J} U_i$.

The **proof** of prop. 1.20 is most conveniently formulated with some of the terminology of topology in hand, which we introduce now. Therefore we postpone the proof to below.

In **summary** prop. 1.13 and prop. 1.20 show that the purely combinatorial and in particular non-epsilon concept of open subsets captures a substantial part of the nature of metric spaces in analysis. This motivates to reverse the logic and consider more general "spaces" which are *only* characterized by what counts as their open subsets. These are the topological spaces which we turn to now in def. 2.3 (or, more generally, these are the "locales", which we briefly consider below in remark 5.6).

2. Topological spaces

Due to prop. 1.13 we should pay attention to [open subsets](#) in [metric spaces](#). It turns out that the following closure property, which follow directly from the definitions, is at the heart of the concept:

Proposition 2.1. (closure properties of open sets in a metric space)

The collection of [open subsets](#) of a [metric space](#) (X, d) as in def. 1.10 has the following properties:

1. The [union](#) of any [set](#) of open subsets is again an open subset.
2. The [intersection](#) of any [finite number](#) of open subsets is again an open subset.

Remark 2.2. (empty union and empty intersection)

Notice the degenerate case of [unions](#) $\bigcup_{i \in I} U_i$ and [intersections](#) $\bigcap_{i \in I} U_i$ of [subsets](#) $U_i \subset X$ for the case that they are indexed by the [empty set](#) $I = \emptyset$:

1. the *empty union* is the empty set itself;
2. the *empty intersection* is all of X .

(The second of these may seem less obvious than the first. We discuss the general logic behind these kinds of phenomena [below](#).)

This way prop. 2.1 is indeed compatible with the degenerate cases of examples of open subsets in example 1.11.

Proposition 2.1 motivates the following generalized definition, which abstracts away from the concept of [metric space](#) just its system of [open subsets](#):

Definition 2.3. (topological spaces)

Given a [set](#) X , then a *topology* on X is a collection τ of [subsets](#) of X called the [open subsets](#), hence a [subset](#) of the [power set](#) $P(X)$

$$\tau \subset P(X)$$

such that this is closed under forming

1. finite [intersections](#);
2. arbitrary [unions](#).

In particular (by remark 2.2):

- the [empty set](#) $\emptyset \subset X$ is in τ (being the union of no subsets)

and

- the whole set $X \subset X$ itself is in τ (being the intersection of no subsets).

A set X equipped with such a [topology](#) is called a [topological space](#).

Remark 2.4. In the field of [topology](#) it is common to eventually simply say “[space](#)” as shorthand for “[topological space](#)”. This is especially so as further qualifiers are added, such as “Hausdorff space” (def. [4.1](#) below). But beware that there are other kinds of [spaces](#) in mathematics.

Remark 2.5. The simple definition of [open subsets](#) in def. [2.3](#) and the simple implementation of the *principle of continuity* below in def. [3.1](#) gives the field of [topology](#) its fundamental and universal flavor. The combinatorial nature of these definitions makes [topology](#) be closely related to [formal logic](#). This becomes more manifest still for the “[sober topological space](#)” discussed [below](#). For more on this perspective see the remark on [locales](#) below, remark [5.6](#). An introductory textbook amplifying this perspective is ([Vickers 89](#)).

Before we look at first examples [below](#), here is some common **further terminology** regarding topological spaces:

There is an evident [partial ordering](#) on the set of topologies that a given set may carry:

Definition 2.6. ([finer/coarser topologies](#))

Let X be a [set](#), and let $\tau_1, \tau_2 \in P(X)$ be two [topologies](#) on X , hence two choices of [open subsets](#) for X , making it a [topological space](#). If

$$\tau_1 \subset \tau_2$$

hence if every open subset of X with respect to τ_1 is also regarded as open by τ_2 , then one says that

- the topology τ_2 is [finer](#) than the topology τ_1
- the topology τ_1 is [coarser](#) than the topology τ_2 .

With any kind of [structure](#) on [sets](#), it is of interest how to “[generate](#)” such structures from a small amount of data:

Definition 2.7. ([basis for the topology](#))

Let (X, τ) be a [topological space](#), def. [2.3](#), and let

$$\beta \subset \tau$$

be a [subset](#) of its set of [open subsets](#). We say that

1. β is a [basis for the topology](#) τ if every open subset $O \in \tau$ is a [union](#) of

elements of β ;

2. β is a [sub-basis for the topology](#) if every open subset $O \in \tau$ is a [union of finite intersections](#) of elements of β .

Often it is convenient to *define* topologies by defining some (sub-)basis as in def. 2.7. Examples are the [metric topology](#) below def. 2.9, the [binary product topology](#) in def. 2.18 below, and the [compact-open topology](#) on [mapping spaces](#) below in def. 6.14. To make use of this, we need to recognize sets of open subsets that serve as the basis for some topology:

Lemma 2.8. (*recognition of topological bases*)

Let X be a set.

1. A collection $\beta \subset P(X)$ of [subsets](#) of X is a [basis](#) for some topology $\tau \subset P(X)$ (def. 2.7) precisely if
 1. every point of X is contained in at least one element of β ;
 2. for every two subsets $B_1, B_2 \in \beta$ and for every point $x \in B_1 \cap B_2$ in their intersection, then there exists a $B \in \beta$ that contains x and is contained in the intersection: $x \in B \subset B_1 \cap B_2$.
2. A subset $B \subset \tau$ of opens is a [sub-basis](#) for a topology τ on X precisely if τ is the coarsest topology (def. 2.6) which contains B .

Examples

We discuss here some basic examples of [topological spaces](#) (def. 2.3), to get a feeling for the scope of the concept. But topological spaces are ubiquitous in [mathematics](#), so that there are many more examples and many more classes of examples than could be listed. As we further develop the theory below, we encounter more examples, and more classes of examples. Below in [Universal constructions](#) we discuss a very general construction principle of new topological space from given ones.

First of all, our motivating example from [above](#) now reads as follows:

Example 2.9. ([metric topology](#))

Let (X, d) be a [metric space](#) (def. 1.1). Then the collection of its [open subsets](#) in def. 1.10 constitutes a [topology](#) on the set X , making it a [topological space](#) in the sense of def. 2.3. This is called the [metric topology](#).

The [open balls](#) in a metric space constitute a [basis of a topology](#) (def. 2.7) for the [metric topology](#).

While the example of [metric space](#) topologies (example 2.9) is the motivating

example for the concept of [topological spaces](#), it is important to notice that the concept of topological spaces is considerably more general, as some of the following examples show.

The following simplistic example of a (metric) topological space is important for the theory (for instance in prop. [2.34](#)):

Example 2.10. (the [point](#))

On a [singleton](#) set $\{1\}$ there exists a unique topology τ making it a [topological space](#) according to def. [2.3](#), namely

$$\tau := \{\emptyset, \{1\}\}.$$

We write

$$* := (\{1\}, \tau := \{\emptyset, \{1\}\})$$

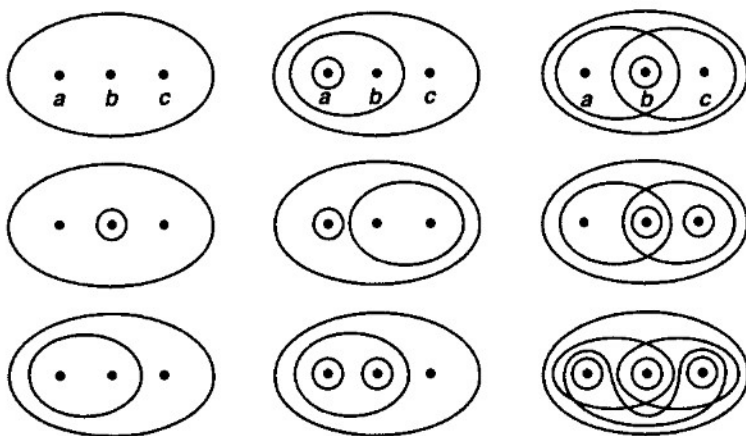
for this topological space and call it *the [point](#)*.

This is equivalently the [metric topology](#) (example [2.9](#)) on \mathbb{R}^0 , regarded as the 0-dimensional [Euclidean space](#) (example [1.6](#)).

Example 2.11. On the 2-element set $\{0, 1\}$ there are (up to [permutation](#) of elements) three distinct topologies:

1. the [codiscrete topology](#) (def. [2.13](#)) $\tau = \{\emptyset, \{0, 1\}\}$;
2. the [discrete topology](#) (def. [2.13](#)), $\tau = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$;
3. the [Sierpinski space](#) topology $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$.

Example 2.12. The following shows all the topologies on the 3-element set (up to [permutation](#) of elements)



graphics grabbed from [Munkres 75](#)

Example 2.13. (discrete and co-discrete topology)

Let S be any [set](#). Then there are always the following two extreme possibilities of equipping X with a topology $\tau \subset P(X)$ in the sense of [def. 2.3](#), and hence making it a [topological space](#):

1. $\tau := P(S)$ the set of *all* open subsets;

this is called the [discrete topology](#) on S , it is the [finest topology](#) ([def. 2.6](#)) on X ,

we write $\text{Disc}(S)$ for the resulting topological space;

2. $\tau := \{\emptyset, S\}$ the set containing only the [empty](#) subset of S and all of S itself;

this is called the [codiscrete topology](#) on S , it is the [coarsest topology](#) ([def. 2.6](#)) on X ,

we write $\text{CoDisc}(S)$ for the resulting topological space.

The reason for this terminology is best seen when considering [continuous functions](#) into or out of these (co-)discrete topological spaces, we come to this in [example 3.8](#) below.

Example 2.14. ([cofinite topology](#))

Given a [set](#) X , then the [cofinite topology](#) or *finite complement topology* on X is the [topology](#) ([def. 2.3](#)) whose [open subsets](#) are precisely

1. all [cofinite subsets](#) $S \subset X$ (i.e. those such that the [complement](#) $X \setminus S$ is a [finite set](#));
2. the [empty set](#).

If X is itself a [finite set](#) (but not otherwise) then the cofinite topology on X coincides with the [discrete topology](#) on X ([example 2.13](#)).

We now consider basic construction principles of new topological spaces from given ones:

1. [disjoint union spaces](#) ([example 2.15](#))
2. [subspaces](#) ([example 2.16](#)),
3. [quotient spaces](#) ([example 2.17](#))
4. [product spaces](#) ([example 2.18](#)).

Below in [Universal constructions](#) we will recognize these as simple special cases of a general construction principle.

Example 2.15. ([disjoint union](#))

For $\{(X_i, \tau_i)\}_{i \in I}$ a [set](#) of topological spaces, then their [disjoint union](#)

$$\bigsqcup_{i \in I} (X_i, \tau_i)$$

is the topological space whose underlying set is the [disjoint union](#) of the underlying sets of the summand spaces, and whose open subsets are precisely the disjoint unions of the open subsets of the summand spaces.

In particular, for I any index set, then the disjoint union of I copies of the [point](#) (example [2.10](#)) is equivalently the [discrete topological space](#) (example [2.13](#)) on that index set:

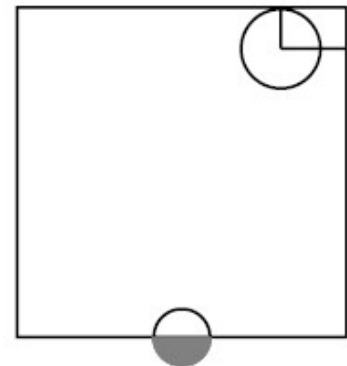
$$\bigsqcup_{i \in I} * = \text{Disc}(I) .$$

Example 2.16. ([subspace topology](#))

Let (X, τ_X) be a [topological space](#), and let $S \subset X$ be a [subset](#) of the underlying set. Then the corresponding [topological subspace](#) has S as its underlying set, and its open subsets are those subsets of S which arise as restrictions of open subsets of X .

$$(U_S \subset S \text{ open}) \Leftrightarrow \left(\exists_{U_X \in \tau_X} (U_S = U_X \cap S) \right) .$$

(This is also called the [initial topology](#) of the inclusion map. We come back to this below in def. [7.1](#).)



The picture on the right shows two open subsets inside the [square](#), regarded as a [topological subspace](#) of the [plane](#) \mathbb{R}^2 :

graphics grabbed from [Munkres 75](#)

Example 2.17. ([quotient topological space](#))

Let (X, τ_X) be a [topological space](#) (def. [2.3](#)) and let

$$R_\sim \subset X \times X$$

be an [equivalence relation](#) on its underlying set. Then the [quotient topological space](#) has

- as underlying set the [quotient set](#) X / \sim , hence the set of [equivalence classes](#),

and

- a subset $O \subset X / \sim$ is declared to be an [open subset](#) precisely if its [preimage](#) $\pi^{-1}(O)$ under the canonical [projection map](#)

$$\pi : X \rightarrow X / \sim$$

is open in X .

(This is also called the *final topology* of the projection π . We come back to this below in def. [7.1](#).)

Often one considers this with input datum not the equivalence relation, but any [surjection](#)

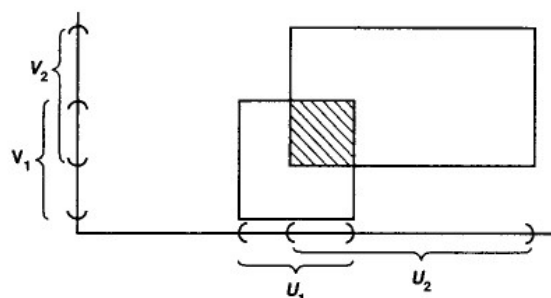
$$\pi : X \rightarrow Y$$

of sets. Of course this identifies $Y = X / \sim$ with $(x_1 \sim x_2) \Leftrightarrow (\pi(x_1) = \pi(x_2))$. Hence the *quotient topology* on the codomain set of a function out of any topological space has as open subsets those whose pre-images are open.

To see that this indeed does define a topology on X / \sim it is sufficient to observe that taking pre-images commutes with taking unions and with taking intersections.

Example 2.18. ([binary product topological space](#))

For (X_1, τ_{X_1}) and (X_2, τ_{X_2}) two [topological spaces](#), then their [binary product topological space](#) has as underlying set the [Cartesian product](#) $X_1 \times X_2$ of the corresponding two underlying sets, and its topology is generated from the [basis](#) (def. [2.7](#)) given by the Cartesian products $U_1 \times U_2$ of the opens $U_i \in \tau_i$.



graphics grabbed from [Munkres 75](#)

Beware that for non-[finite](#) products, the descriptions of the product topology is not as simple. This we turn to below in example [7.7](#), after introducing the general concept of [limits](#) in the [category of topological spaces](#).

The following example illustrates how all these ingredients and construction principles may be combined. We will examine this example in more detail below in example [3.28](#), after we have introduced the concept of [homeomorphisms](#) below.

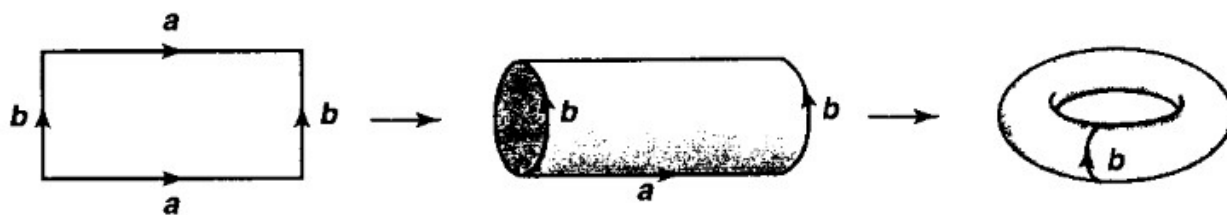
Example 2.19. Consider the [real numbers](#) \mathbb{R} as the 1-dimensional [Euclidean space](#) (example [1.6](#)) and hence as a [topological space](#) via the corresponding [metric topology](#) (example [2.9](#)). Moreover, consider the [closed interval](#) $[0, 1] \subset \mathbb{R}$ from example [1.12](#), regarded as a [subspace](#) (def. [2.16](#)) of \mathbb{R} .

The [product space](#) (example [2.18](#)) of this interval with itself

$$[0, 1] \times [0, 1]$$

is a topological space modelling the closed square. The [quotient space](#) (example [2.17](#)) of that by the relation which identifies a pair of opposite sides is a model for the [cylinder](#). The further quotient by the relation that identifies the

remaining pair of sides yields a model for the [torus](#).



graphics grabbed from [Munkres 75](#)

Closed subsets

The [complements](#) of [open subsets](#) in a [topological space](#) are called [closed subsets](#) (def. [2.20](#) below). This simple definition indeed captures the concept of closure in the [analytic](#) sense of [convergence](#) of [sequences](#) (prop. [2.27](#) below). Of particular interest for the theory of topological spaces in the discussion of [separation axioms](#) below are those closed subsets which are “[irreducible](#)” (def. [2.28](#) below). These happen to be equivalently the “[frame](#) homomorphisms” (def. [2.31](#)) to the [frame of opens](#) of the point (prop. [2.34](#) below).

Definition 2.20. ([closed subsets](#))

Let (X, τ) be a [topological space](#) (def. [2.3](#)). Then a [subset](#) S of X is called a [closed subset](#) if its [complement](#) $X \setminus S$ is an [open subset](#):



open



closed



neither

$$(S \subset X \text{ is closed}) \quad \Leftrightarrow \quad (X \setminus S \subset X \text{ is open}) .$$

graphics grabbed from [Vickers 89](#)

If a [singleton](#) subset $\{x\} \subset X$ is closed, one says that x is a [closed point](#) of X .

Given any subset $S \subset X$, then its [topological closure](#) $\text{Cl}(S)$ is the smallest closed subset containing S :

$$\text{Cl}(S) := \bigcap_{\substack{C \subset X \text{ closed} \\ S \subset C}} (C) .$$

Example 2.21. Regard the [real numbers](#) as the 1-dimensional [Euclidean space](#) and equipped with the corresponding [metric topology](#). Let $a < b \in \mathbb{R}$. Then the [topological closure](#) (def. [2.20](#)) of the [open interval](#) $(a, b) \subset \mathbb{R}$ (example [1.12](#)) is the [closed interval](#) $[a, b] \subset \mathbb{R}$:

$$\text{Cl}((a, b)) = [a, b] .$$

Remark 2.22. ([de Morgan's law](#))

In reasoning about closed subsets in topology we are concerned with complements of unions and intersections as well as with unions/intersections of complements. Recall therefore that taking complements of subsets exchanges unions with intersections (de Morgan's law):

Given a set X and a set of subsets

$$\{S_i \subset S\}_{i \in I}$$

then

$$X \setminus \left(\bigcup_{i \in I} S_i \right) = \bigcap_{i \in I} (X \setminus S_i)$$

and

$$X \setminus \left(\bigcap_{i \in I} S_i \right) = \bigcup_{i \in I} (X \setminus S_i).$$

Also notice that taking complements reverses inclusion relations:

$$(S_1 \subset S_2) \Leftrightarrow (X \setminus S_2 \subset X \setminus S_1).$$

Often it is useful to reformulate def. 2.20 of closed subsets as follows:

Lemma 2.23. *Let (X, τ) be a topological space and let $S \subset X$ be a subset of its underlying set. Then a point $x \in X$ is contained in the topological closure $\text{Cl}(S)$ (def. 2.20) precisely if every open neighbourhood $U_x \subset X$ of x intersects S :*

$$(x \in \text{Cl}(S)) \Leftrightarrow \neg \left(\bigcap_{\substack{U \subset X \\ U \text{ open}}} \exists S \text{ open } (x \in U) \right).$$

Proof. In view of remark 2.22 we may rephrase the definition of the topological closure as follows:

$$\begin{aligned} \text{Cl}(S) &:= \bigcap_{\substack{C \subset X \\ C \text{ closed}}} (C) \\ &= \bigcap_{\substack{U \subset X \setminus S \\ U \text{ open}}} (X \setminus U) \\ &= X \setminus \left(\bigcup_{\substack{U \subset X \setminus S \\ U \text{ open}}} U \right) \end{aligned}$$

Definition 2.24. (topological interior)

Let (X, τ) be a topological space (def. 2.3) and let $S \subset X$ be a subset. Then the topological interior of S is the largest open subset $\text{Int}(S) \in \tau$ still contained in S , $\text{Int}(S) \subset S \subset X$:

$$\text{Int}(S) := \bigcup_{\substack{O \subset S \\ O \subset X \text{ open}}} (O) .$$

Example 2.25. Regard the [real numbers](#) as the 1-dimensional [Euclidean space](#) and equipped with the corresponding [metric topology](#). Let $a < b \in \mathbb{R}$. Then the [topological interior](#) (def. 2.24) of the [closed interval](#) $[a, b] \subset \mathbb{R}$ (example 1.12) is the [open interval](#) $(a, b) \subset \mathbb{R}$:

$$\text{Int}([a, b]) = (a, b) .$$

Lemma 2.26. (duality between closure and interior)

Let (X, τ) be a [topological space](#) and let $S \subset X$ be a [subset](#). Then the [topological interior](#) of S (def. 2.24) is the same as the [complement](#) of the [topological closure](#) $\text{Cl}(X \setminus S)$ of the complement of S :

$$X \setminus \text{Int}(S) = \text{Cl}(X \setminus S)$$

and conversely

$$X \setminus \text{Cl}(S) = \text{Int}(X \setminus S) .$$

Proof. Using remark 2.22, we compute as follows:

$$\begin{aligned} X \setminus \text{Int}(S) &= X \setminus \left(\bigcup_{\substack{U \subset S \\ U \subset X \text{ open}}} U \right) \\ &= \bigcap_{\substack{U \subset S \\ U \subset X \text{ open}}} (X \setminus U) \\ &= \bigcap_{\substack{C \supset X \setminus S \\ C \text{ closed}}} (C) \\ &= \text{Cl}(X \setminus S) \end{aligned}$$

■

The terminology “closed” subspace for complements of opens is justified by the following statement, which is a further example of how the combinatorial concept of open subsets captures key phenomena in analysis:

Proposition 2.27. (convergence in closed subspaces)

Let (X, d) be a [metric space](#) (def. 1.1), regarded as a [topological space](#) via example 2.9, and let $V \subset X$ be a [subset](#). Then the following are equivalent:

1. $V \subset X$ is a [closed subspace](#) according to def. 2.20.
2. For every [sequence](#) $x_i \in V \subset X$ (def. 1.15) with elements in V , which [converges](#) as a sequence in X (def. 1.16) to some $x_\infty \in X$, then $x_\infty \in V \subset X$.

Proof. First assume that $V \subset X$ is closed and that $x_i \xrightarrow{i \rightarrow \infty} x_\infty$ for some $x_\infty \in X$. We

need to show that then $x_\infty \in V$. Suppose it were not, hence that $x_\infty \in X \setminus V$. Since, by assumption on V , this [complement](#) $X \setminus V \subset X$ is an [open subset](#), it would follow that there exists a [real number](#) $\epsilon > 0$ such that the [open ball](#) around x of radius ϵ were still contained in the complement: $B_x^\circ(\epsilon) \subset X \setminus V$. But since the sequence is assumed to converge in X , this would mean that there exists N_ϵ such that all $x_{i > N_\epsilon}$ are in $B_x^\circ(\epsilon)$, hence in $X \setminus V$. This contradicts the assumption that all x_i are in V , and hence we have [proved by contradiction](#) that $x_\infty \in V$.

Conversely, assume that for all sequences in V that converge to some $x_\infty \in X$ then $x_\infty \in V \subset X$. We need to show that then V is closed, hence that $X \setminus V \subset X$ is an open subset, hence that for every $x \in X \setminus V$ we may find a real number $\epsilon > 0$ such that the [open ball](#) $B_x^\circ(\epsilon)$ around x of radius ϵ is still contained in $X \setminus V$. Suppose on the contrary that such ϵ did not exist. This would mean that for each $k \in \mathbb{N}$ with $k \geq 1$ then the [intersection](#) $B_x^\circ(1/k) \cap V$ were [non-empty](#). Hence then we could choose points $x_k \in B_x^\circ(1/k) \cap V$ in these intersections. These would form a sequence which clearly converges to the original x , and so by assumption we would conclude that $x \in V$, which violates the assumption that $x \in X \setminus V$. Hence we [proved by contradiction](#) $X \setminus V$ is in fact open. ■

A special role in the theory is played by the “irreducible” closed subspaces:

Definition 2.28. ([irreducible closed subspace](#))

A [closed subset](#) $S \subset X$ (def. 2.20) of a [topological space](#) X is called [irreducible](#) if it is [non-empty](#) and not the [union](#) of two closed proper (i.e. smaller) subsets. In other words, a [non-empty](#) closed subset $S \subset X$ is irreducible if whenever $S_1, S_2 \subset X$ are two [closed subspace](#) such that

$$S = S_1 \cup S_2$$

then $S_1 = S$ or $S_2 = S$.

Example 2.29. ([closures of points are irreducible](#))

For $x \in X$ a [point](#) inside a [topological space](#), then the [closure](#) $\text{Cl}(\{x\})$ of the [singleton subset](#) $\{x\} \subset X$ is [irreducible](#) (def. 2.28).

Sometimes it is useful to re-express the condition of irreducibility of closed subspaces in terms of complementary open subsets:

Proposition 2.30. ([irreducible closed subsets in terms of prime open subsets](#))

Let (X, τ) be a [topological space](#), and let $P \in \tau$ be a proper [open subset](#), so that the [complement](#) $F := X \setminus P$ is an [non-empty closed subspace](#). Then F is [irreducible](#) in the sense of def. 2.28 precisely if whenever $U_1, U_2 \in \tau$ are open subsets with $U_1 \cap U_2 \subset P$ then $U_1 \subset P$ or $U_2 \subset P$:

$$(X \setminus P \text{ irreducible}) \Leftrightarrow \left(\forall_{U_1, U_2 \in \tau} ((U_1 \cap U_2 \subset P) \Rightarrow (U_1 \subset P \text{ or } U_2 \subset P)) \right)$$

Proof. Every closed subset $F_i \subset F$ may be exhibited as the complement

$$F_i = F \setminus U_i$$

for some open subset $U_i \in \tau$. Observe that under this identification the condition that $U_1 \cap U_2 \subset P$ is equivalent to the condition that $F_1 \cup F_2 = F$, because it is equivalent to the equation labeled $(*)$ in the following sequence of equations:

$$\begin{aligned} F_1 \cup F_2 &= (F \setminus U_1) \cup (F \setminus U_2) \\ &= (X \setminus (P \cup U_1)) \cup (X \setminus (P \cup U_2)) \\ &= X \setminus ((P \cup U_1) \cap (P \cup U_2)) \\ &= X \setminus (P \cup (U_1 \cap U_2)) \\ &\stackrel{(*)}{=} X \setminus P \\ &= F. \end{aligned}$$

Similarly, the condition that $U_i \subset P$ is equivalent to the condition that $F_i = F$, because it is equivalent to the equality $(*)$ in the following sequence of equalities:

$$\begin{aligned} F_i &= F \setminus U_i \\ &= X \setminus (P \cup U_i) \\ &\stackrel{(*)}{=} X \setminus P \\ &= F \end{aligned}$$

Under these identifications, the two conditions are manifestly the same. ■

We will consider yet another equivalent characterization of irreducible closed subsets. Stating this requires the following concept of “frame” homomorphism, the natural kind of homomorphisms between topological spaces if we were to forget the underlying set of points of a topological space, and only remember the set τ_X with its operations induced by taking finite intersections and arbitrary unions:

Definition 2.31. (frame homomorphisms)

Let (X, τ_X) and (Y, τ_Y) be topological spaces (def. 2.3). Then a function

$$\tau_X \leftarrow \tau_Y : \phi$$

between their sets of open subsets is called a frame homomorphism if it preserves

1. arbitrary unions;
2. finite intersections.

In other words, ϕ is a frame homomorphism precisely if

1. for every [set](#) I and every I -indexed set $\{U_i \in \tau_Y\}_{i \in I}$ of elements of τ_Y , then

$$\phi\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} \phi(U_i) \in \tau_X,$$

2. for every [finite set](#) J and every J -indexed set $\{U_j \in \tau_Y\}_{j \in J}$ of elements in τ_Y , then

$$\phi\left(\bigcap_{j \in J} U_j\right) = \bigcap_{j \in J} \phi(U_j) \in \tau_X.$$

Remark 2.32. (frame homomorphisms preserve inclusions)

A [frame homomorphism](#) ϕ as in def. [2.31](#) necessarily also preserves inclusions in that

- for every inclusion $U_1 \subset U_2$ with $U_1, U_2 \in \tau_Y \subset P(Y)$ then

$$\phi(U_1) \subset \phi(U_2) \in \tau_X.$$

This is because inclusions are witnessed by unions

$$(U_1 \subset U_2) \Leftrightarrow (U_1 \cup U_2 = U_2)$$

or alternatively because inclusions are witnessed by finite intersections:

$$(U_1 \subset U_2) \Leftrightarrow (U_1 \cap U_2 = U_1).$$

Example 2.33. (pre-images of continuous functions are frame homomorphisms)

For

$$f : (X, \tau_X) \rightarrow (Y, \tau_Y)$$

a [continuous function](#), then its function of [pre-images](#)

$$\tau_X \leftarrow \tau_Y : f^{-1}$$

is a frame homomorphism according to def. [2.31](#).

For the following recall from example [2.10](#) the [point](#) topological space

$$* = (\{1\}, \tau_* = \{\emptyset, \{1\}\}).$$

Proposition 2.34. (irreducible closed subsets are equivalently frame homomorphisms to frame of opens of the point)

For (X, τ) a [topological space](#), then there is a [bijection](#) between the [irreducible closed subspaces](#) of (X, τ) (def. [2.28](#)) and the [frame homomorphisms](#) from τ_X to τ_* , and this bijection is given by

$$\begin{aligned} \text{Hom}_{\text{Frame}}(\tau_X, \tau_*) &\xrightarrow{\cong} \text{IrrClSub}(X) \\ \phi &\mapsto X \setminus (U_\emptyset(\phi)) \end{aligned}$$

where $U_\emptyset(\phi)$ is the [union](#) of all elements $U \in \tau_X$ such that $\phi(U) = \emptyset$:

$$U_\emptyset(\phi) := \bigcup_{\substack{U \in \tau_X \\ \phi(U) = \emptyset}} (U).$$

See also ([Johnstone 82, II 1.3](#)).

Proof. First we need to show that the function is well defined in that given a frame homomorphism $\phi: \tau_X \rightarrow \tau_*$ then $X \setminus U_\emptyset(\phi)$ is indeed an irreducible closed subspace.

To that end observe that:

(*) *If there are two elements $U_1, U_2 \in \tau_X$ with $U_1 \cap U_2 \subset U_\emptyset(\phi)$ then $U_1 \subset U_\emptyset(\phi)$ or $U_2 \subset U_\emptyset(\phi)$.*

This is because

$$\begin{aligned} \phi(U_1 \cap U_2) &= \phi(U_1) \cap \phi(U_2) \\ &\subset \phi(U_\emptyset) \quad , \\ &= \emptyset \end{aligned}$$

where the first equality holds because ϕ preserves finite intersections by def. [2.31](#), the inclusion holds because ϕ respects inclusions by remark [2.32](#), and the second equality holds because ϕ preserves arbitrary unions by def. [2.31](#). But in $\tau_* = \{\emptyset, \{1\}\}$ the intersection of two open subsets is empty precisely if at least one of them is empty, hence $\phi(U_1) = \emptyset$ or $\phi(U_2) = \emptyset$. But this means that $U_1 \subset U_\emptyset(\phi)$ or $U_2 \subset U_\emptyset(\phi)$, as claimed.

Now according to prop. [2.30](#) the condition (*) identifies the [complement](#) $X \setminus U_\emptyset(\phi)$ as an [irreducible closed subspace](#) of (X, τ) .

Conversely, given an irreducible closed subset $X \setminus U_0$, define ϕ by

$$\phi : U \mapsto \begin{cases} \emptyset & | \text{if } U \subset U_0 \\ \{1\} & | \text{otherwise} \end{cases}.$$

This does preserve

1. arbitrary unions

because $\phi(\bigcup_i U_i) = \{0\}$ precisely if $\bigcup_i U_i \subset U_0$ which is the case precisely if all $U_i \subset U_0$, which means that all $\phi(U_i) = \emptyset$ and $\bigcup_i \emptyset = \emptyset$;

while $\phi(\bigcup_i U_i) = \{1\}$ as soon as one of the U_i is not contained in U_0 , which

means that one of the $\phi(U_i) = \{1\}$ which means that $\bigcup_i \phi(U_i) = \{1\}$;

2. finite intersections,

because if $U_1 \cap U_2 \in U_0$, then by (*) $U_1 \in U_0$ or $U_2 \in U_0$, whence $\phi(U_1) = \emptyset$ or $\phi(U_2) = \emptyset$, whence with $\phi(U_1 \cap U_2) = \emptyset$ also $\phi(U_1) \cap \phi(U_2) = \emptyset$;

while if $U_1 \cap U_2$ is not contained in U_0 then neither U_1 nor U_2 is contained in U_0 and hence with $\phi(U_1 \cap U_2) = \{1\}$ also $\phi(U_1) \cap \phi(U_2) = \{1\} \cap \{1\} = \{1\}$.

Hence this is indeed a frame homomorphism $\tau_X \rightarrow \tau_*$.

Clearly these two operations are inverse to each other. ■

3. Continuous functions

With the concept of [topological spaces](#) (def. 2.3) it is now immediate to formally implement in abstract generality the statement of prop. 1.13:

principle of continuity

Continuous pre-Images of open subsets are open.

Definition 3.1. (continuous function)

A [continuous function](#) between [topological spaces](#) (def. 2.3)

$$f: (X, \tau_X) \rightarrow (Y, \tau_Y)$$

is a [function](#) between the underlying sets,

$$f: X \rightarrow Y$$

such that [pre-images](#) under f of open subsets of Y are open subsets of X .

We may equivalently state this in terms of [closed subsets](#):

Proposition 3.2. *Let (X_1, τ_X) and (Y, τ_Y) be two [topological spaces](#) (def. 2.3). Then a [function](#)*

$$f: X \rightarrow Y$$

between the underlying [sets](#) is [continuous](#) in the sense of def. 3.1 precisely if [pre-images](#) under f of [closed subsets](#) of Y (def. 2.20) are closed subsets of X .

Proof. This follows since taking [pre-images](#) commutes with taking [complements](#). ■

Remark 3.3. (the category of topological spaces)

For X_1, X_2, X_3 three [topological spaces](#) and for

$$X_1 \xrightarrow{f} X_2 \quad \text{and} \quad X_2 \xrightarrow{g} X_3$$

two [continuous functions](#) (def. 3.1) then their [composition](#)

$$f_2 \circ f_1 : X_1 \xrightarrow{f} X_2 \xrightarrow{f_2} X_3$$

is clearly itself again a continuous function from X_1 to X_3 . Moreover, this composition operation is clearly [associative](#), in that for

$$X_1 \xrightarrow{f} X_2 \quad \text{and} \quad X_2 \xrightarrow{g} X_3 \quad \text{and} \quad X_3 \xrightarrow{h} X_4$$

three [continuous functions](#), then

$$f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1 : X_1 \rightarrow X_3 .$$

Finally, the composition operation is also clearly [unital](#), in that for each topological space X there exists the [identity](#) function $\text{id}_X : X \rightarrow X$ and for $f : X_1 \rightarrow X_2$ any continuous function then

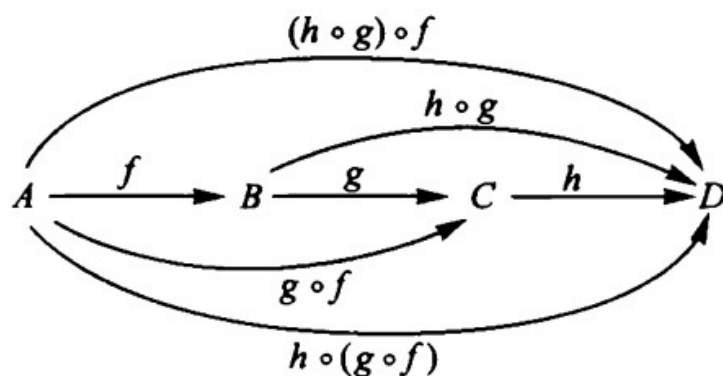
$$\text{id}_{X_2} \circ f = f = f \circ \text{id}_{X_1} .$$

One summarizes this situation by saying that:

1. [topological spaces](#) constitute the [objects](#)
2. [continuous functions](#) constitute the [morphisms](#) (homomorphisms)

of a [category](#), called the [category of topological spaces](#) (“Top” for short).

It is useful to depict collections of [objects](#) with [morphisms](#) between them by [diagrams](#), like this one:



graphics grabbed from Lawvere-Schanuel 09.

Example 3.4. (product topological space construction is functorial)

Let (X_1, τ_{X_1}) , (X_2, τ_{X_2}) , (Y_1, τ_{Y_1}) and (Y_2, τ_{Y_2}) be [topological spaces](#). Then for all pairs of [continuous functions](#)

$$f_1 : (X_1, \tau_{X_1}) \rightarrow (Y_1, \tau_{Y_1})$$

and

$$f_2 : (X_2, \tau_{X_2}) \rightarrow (Y_2, \tau_{Y_2})$$

the canonically induced function on [Cartesian products](#) of sets

$$\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{f_1 \times f_2} & Y_1 \times Y_2 \\ (x_1, x_2) & \mapsto & (f_1(x_1), f_2(x_2)) \end{array}$$

is a [continuous function](#) with respect to the [binary product space topologies](#) (def. [2.18](#))

$$f_1 \times f_2 : (X_1 \times X_2, \tau_{X_1 \times X_2}) \rightarrow (Y_1 \times Y_2, \tau_{Y_1 \times Y_2}) .$$

Moreover, this construction respects [identity functions](#) and [composition](#) of functions in both arguments.

In the language of [category theory](#) (remark [3.3](#)), this is summarized by saying that the [product topological space](#) construction $(-) \times (-)$ extends to a [functor](#) from the [product category](#) of the [category Top](#) with itself to itself:

$$(-) \times (-) : \text{Top} \times \text{Top} \rightarrow \text{Top} .$$

Examples

We discuss some basic examples of [continuous functions](#) (def. [3.1](#)) between [topological spaces](#) (def. [2.3](#)).

Example 3.5. (point space is [terminal](#))

For (X, τ) any topological space, then there is a unique continuous function (which we denote by the same symbol)

$$X \rightarrow *$$

from X to the [point](#) (def. [2.10](#)).

In the language of [category theory](#) (remark [3.3](#)), example [3.5](#) says that the point $*$ is the [terminal object](#) in the [category Top](#) of topological spaces.

Example 3.6. (points as continuous functions)

For (X, τ) a [topological space](#) then for $x \in X$ any element of the underlying set, there is a unique continuous function

$$x : * \rightarrow X$$

from the [point](#) (def. 2.10), whose image in X is that element. Hence there is a [natural bijection](#)

$$\left\{ * \xrightarrow{f} X \mid f \text{ continuous} \right\} \simeq X$$

between the continuous functions from the point to any topological space, and the underlying set of that topological space.

Definition 3.7. ([locally constant function](#))

A [continuous function](#) $f: X \rightarrow Y$ (def. 3.1) is called [locally constant](#) if every point $x \in X$ has a [neighbourhood](#) on which the function is constant.

Example 3.8. Let S be a [set](#) and let (X, τ) be a [topological space](#). Recall from example 2.13

1. the [discrete topological space](#) $\text{Disc}(S)$;
2. the [co-discrete topological space](#) $\text{CoDisc}(S)$

on the underlying set S . Then [continuous functions](#) (def. 3.1) into/out of these satisfy:

1. every [function](#) (of sets) $\text{Disc}(S) \rightarrow X$ out of a discrete space is [continuous](#);
2. every [function](#) (of sets) $X \rightarrow \text{CoDisc}(S)$ into a codiscrete space is [continuous](#).

Also:

- every [continuous function](#) $(X, \tau) \rightarrow \text{Disc}(S)$ into a discrete space is [locally constant](#) (def. 3.7).

Example 3.9. ([diagonal](#))

For X a [set](#), its [diagonal](#) Δ_X is the [function](#)

$$\begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \times X \\ x & \mapsto & (x, x) \end{array}$$

For (X, τ) a [topological space](#), then the diagonal is a [continuous function](#) to the [product topological space](#) (def. 2.18) of X with itself.

$$\Delta_X : (X, \tau) \rightarrow (X \times X, \tau_{X \times X}) .$$

To see this, it is sufficient to see that the [preimages](#) of [basic opens](#) $U_1 \times U_2$ in $\tau_{X \times X}$ are in τ_X . But these pre-images are the [intersections](#) $U_1 \cap U_2 \subset X$, which are open by the axioms on the topology τ_X .

Example 3.10. ([image factorization](#))

Let $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ be a [continuous function](#).

Write $f(X) \subset Y$ for the [image](#) of f on underlying sets, and consider the resulting factorization of f through $f(X)$ on underlying sets:

$$f : X \xrightarrow{\text{surjective}} f(X) \xrightarrow{\text{injective}} Y .$$

There are the following two ways to topologize the [image](#) $f(X)$ such as to make this a sequence of two continuous functions:

1. By example [2.16](#) $f(X)$ inherits a [subspace topology](#) from (Y, τ_Y) which makes the inclusion $f(X) \rightarrow Y$ a [continuous function](#).

Observe that this also makes $X \rightarrow f(X)$ a continuous function: An open subset of $f(X)$ in this case is of the form $U_Y \cap f(X)$ for $U_Y \in \tau_Y$, and $f^{-1}(U_Y \cap f(X)) = f^{-1}(U_Y)$, which is open in X since f is continuous.

2. By example [2.17](#) $f(X)$ inherits a [quotient topology](#) from (X, τ_X) which makes the surjection $X \rightarrow f(X)$ a [continuous function](#).

Observe that this also makes $f(X) \rightarrow Y$ a continuous function: The preimage under this map of an open subset $U_Y \in \tau_Y$ is the restriction $U_Y \cap f(X)$, and the pre-image of that under $X \rightarrow f(X)$ is $f^{-1}(U_Y)$, as before, which is open since f is continuous, and therefore $U_Y \cap f(X)$ is open in the quotient topology.

Beware that in general a continuous function itself (as opposed to its [pre-image](#) function) neither preserves [open subsets](#), nor [closed subsets](#), as the following examples show:

Example 3.11. Regard the [real numbers](#) \mathbb{R} as the 1-dimensional [Euclidean space](#) (def. [1.6](#)) equipped with the [metric topology](#) (def. [2.9](#)). For $a \in \mathbb{R}$ the [constant function](#)

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\text{const}_a} & \mathbb{R} \\ x & \mapsto & a \end{array}$$

maps every [open subset](#) $U \subset \mathbb{R}$ to the [singleton set](#) $\{a\} \subset \mathbb{R}$, which is not open.

Example 3.12. Write $\text{Disc}(\mathbb{R})$ for the set of [real numbers](#) equipped with its [discrete topology](#) (def. [2.13](#)) and \mathbb{R} for the set of [real numbers](#) equipped with its [Euclidean metric topology](#) (def. [1.6](#), def. [2.9](#)). Then the [identity function](#) on the underlying sets

$$\text{id}_{\mathbb{R}} : \text{Disc}(\mathbb{R}) \rightarrow \mathbb{R}$$

is a [continuous function](#) (see also example [3.8](#)). A [singleton subset](#) $\{a\} \in \text{Disc}(\mathbb{R})$

is open, but regarded as a subset $\{a\} \in \mathbb{R}$ it is not open.

Example 3.13. Consider the set of [real numbers](#) \mathbb{R} equipped with its [Euclidean metric topology](#) (def. [1.6](#), def. [2.9](#)). The [exponential function](#)

$$\exp(-) : \mathbb{R} \rightarrow \mathbb{R}$$

maps all of \mathbb{R} (which is a closed subset, since $\mathbb{R} = \mathbb{R} \setminus \emptyset$) to the [open interval](#) $(0, \infty) \subset \mathbb{R}$, which is not closed.

Those continuous functions that do happen to preserve open or closed subsets get a special name:

Definition 3.14. ([open maps](#) and [closed maps](#))

A [continuous function](#) $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ (def. [3.1](#)) is called

- an [open map](#) if the [image](#) under f of an [open subset](#) of X is an open subset of Y ;
- a [closed map](#) if the [image](#) under f of a [closed subset](#) of X (def. [2.20](#)) is a closed subset of Y .

Example 3.15. ([projections are open](#))

For (X_1, τ_{X_1}) and (X_2, τ_{X_2}) two [topological spaces](#), then the projection maps

$$\pi_i : (X_1 \times X_2, \tau_{X_1 \times X_2}) \rightarrow (X_i, \tau_{X_i})$$

out of their [product topological space](#) (def. [2.18](#)) are [open maps](#) (def. [3.14](#)).

Below in prop. [6.20](#) we find a large supply of [closed maps](#).

Sometimes it is useful to recognize [quotient topological space](#) projections via [saturated subsets](#) (essentially another term for pre-images of underlying sets):

Definition 3.16. ([saturated subset](#))

Let $f : X \rightarrow Y$ be a [function](#) of [sets](#). Then a [subset](#) $S \subset X$ is called an [f-saturated subset](#) (or just [saturated subset](#), if f is understood) if S is the [pre-image](#) of its [image](#):

$$(S \subset X \text{ f-saturated}) \Leftrightarrow (S = f^{-1}(f(S))).$$

Here $f^{-1}(f(S))$ is also called the [f-saturation](#) of S .

Example 3.17. ([pre-images are saturated subsets](#))

For $f : X \rightarrow Y$ any [function](#) of [sets](#), and $S_Y \subset Y$ any [subset](#) of Y , then the [pre-image](#) $f^{-1}(S_Y) \subset X$ is an [f-saturated subset](#) of X (def. [3.16](#)).

Observe that:

Lemma 3.18. *Let $f: X \rightarrow Y$ be a [function](#). Then a [subset](#) $S \subset X$ is f -saturated (def. 3.16) precisely if its [complement](#) $X \setminus S$ is saturated.*

Proposition 3.19. (recognition of quotient topologies)

A [continuous function](#) (def. 3.1)

$$f : (X, \tau_X) \rightarrow (Y, \tau_Y)$$

whose underlying function $f: X \rightarrow Y$ is [surjective](#) exhibits τ_Y as the corresponding [quotient topology](#) (def. 2.17) precisely if f sends open and f -saturated subsets in X (def. 3.16) to open subsets of Y . By lemma 3.18 this is the case precisely if it sends closed and f -saturated subsets to closed subsets.

We record the following technical lemma about saturated subspaces, which we will need below to prove prop. 6.24.

Lemma 3.20. (saturated open neighbourhoods of saturated closed subsets under closed maps)

Let

1. $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ be a [closed map](#) (def. 3.14);
2. $C \subset X$ be a [closed subset](#) of X (def. 2.20) which is f -saturated (def. 3.16);
3. $U \supset C$ an [open subset](#) containing C ;

then there exists a smaller open subset V still containing C

$$U \supset V \supset C$$

and such that V is f -saturated.

Proof. We claim that the [complement](#) of X by the f -saturation (def. 3.16) of the complement of X by U

$$V := X \setminus (f^{-1}(f(X \setminus U)))$$

has the desired properties. To see this, observe first that

1. the [complement](#) $X \setminus U$ is closed, since U is assumed to be open;
2. hence the image $f(X \setminus U)$ is closed, since f is assumed to be a closed map;
3. hence the pre-image $f^{-1}(f(X \setminus U))$ is closed, since f is continuous (using prop. 3.2), therefore its complement V is indeed open;
4. this pre-image $f^{-1}(f(X \setminus U))$ is saturated (example 3.17) and hence also its complement V is saturated, by lemma 3.18.

Therefore it now only remains to see that $U \supset V \supset C$.

The inclusion $U \supset V$ means equivalently that $f^{-1}(f(X \setminus U)) \supset X \setminus U$, which is clearly the case.

The inclusion $V \supset C$ means that $f^{-1}(f(X \setminus U)) \cap C = \emptyset$. Since C is saturated by assumption, this means that $f^{-1}(f(X \setminus U)) \cap f^{-1}(f(C)) = \emptyset$. This in turn holds precisely if $f(X \setminus U) \cap f(C) = \emptyset$. Since C is saturated, this holds precisely if $X \setminus U \cap C = \emptyset$, and this is true by the assumption that $U \supset C$. ■

Homeomorphisms

With the [objects](#) ([topological spaces](#)) and the [morphisms](#) ([continuous functions](#)) of the [category Top](#) thus defined (remark 3.3), we obtain the concept of “sameness” in topology. To make this precise, one says that a [morphism](#)

$$X \xrightarrow{f} Y$$

in a [category](#) is an [isomorphism](#) if there exists a morphism going the other way around

$$X \xleftarrow{g} Y$$

which is an [inverse](#) in the sense that

$$f \circ g = \text{id}_Y \quad \text{and} \quad g \circ f = \text{id}_X .$$

Since such g is unique if it exists, one often writes “ f^{-1} ” for this [inverse morphism](#). However, in the context of [topology](#) then f^{-1} usually refers to the [pre-image](#) function of a given [function](#) f , and in these notes we will stick to this usage.

Definition 3.21. (homeomorphisms)

An [isomorphism](#) in the [category Top](#) of [topological spaces](#) with [continuous functions](#) between them is called a [homeomorphism](#).

Hence a [homeomorphism](#) is a [continuous function](#)

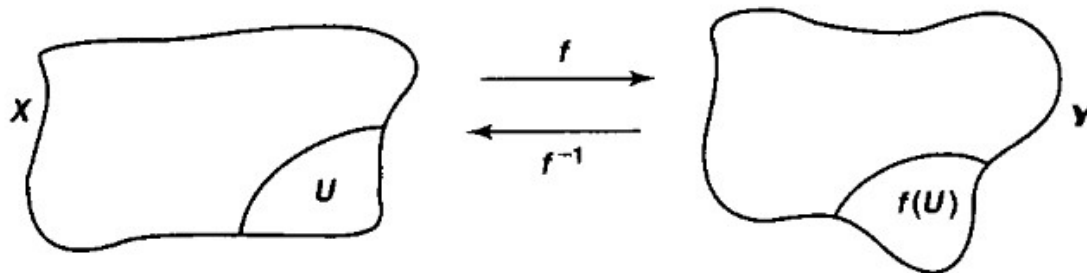
$$f : (X, \tau_X) \rightarrow (Y, \tau_Y)$$

such that there exists a [continuous function](#) the other way around

$$(X, \tau_X) \leftarrow (Y, \tau_Y) : g$$

such that their [composites](#) are the [identity functions](#) on X and Y , respectively:

$$f \circ g = \text{id}_Y \quad \text{and} \quad g \circ f = \text{id}_X .$$



graphics grabbed from [Munkres 75](#)

Remark 3.22. If $f: (X, \tau_X) \rightarrow (Y, \tau_Y)$ is a [homeomorphism](#) (def. 3.21) with inverse continuous function g , then of course also g is a homeomorphism, with inverse continuous function f .

The underlying function of sets $f: X \rightarrow Y$ of a homeomorphism f is necessarily a [bijection](#).

But beware that not every [continuous function](#) which is [bijective](#) on underlying sets is a homeomorphism. While an inverse g will exist on the level of functions of sets, this inverse may fail to be continuous:

Counter Example 3.23. Consider the [continuous function](#)

$$\begin{aligned} [0, 2\pi) &\rightarrow S^1 \subset \mathbb{R}^2 \\ t &\mapsto (\cos(t), \sin(t)) \end{aligned}$$

from the [half-open interval](#) (def. 1.12) to the unit circle $S^1 := S_0(1) \subset \mathbb{R}^2$ (def. 1.2), regarded as a [topological subspace](#) (example 2.16) of the [Euclidean plane](#) (def. 1.6).

The underlying function of sets of f is a [bijection](#). The [inverse function](#) of sets however fails to be continuous at $(1, 0) \in S^1 \subset \mathbb{R}^2$. Hence this f is *not* a [homeomorphism](#).

Indeed, below we see that the two topological spaces $[0, 2\pi)$ and S^1 are distinguished by topological invariants and hence not homeomorphic. For example S^1 is a [compact topological space](#) (def. 6.4) while $[0, 2\pi)$ is not, and S^1 has a non-trivial [fundamental group](#), while that of $[0, 2\pi)$ is trivial (def. [FundamentalGroup](#)).

Below in example 6.25 we discuss a criterion under which continuous bijections are homeomorphisms after all.

Now we consider some actual examples of [homeomorphisms](#):

Example 3.24. Let (X, τ_X) be a [non-empty topological space](#), and let $x \in X$ be any point. Regard the corresponding [singleton subset](#) $\{x\} \subset X$ as equipped with its [subspace topology](#) $\tau_{\{x\}}$ (example 2.16). Then this is [homeomorphic](#) to the abstract [point](#) space from example 2.10:

$$(\{x\}, \tau_{\{x\}}) \simeq * .$$

Example 3.25. (open interval homeomorphic to the real line)

Regard the [real line](#) as the 1-dimensional [Euclidean space](#) (example 1.6).

The open [interval](#) $(-1, 1)$ (def. 1.12) is [homeomorphic](#) to all of the [real line](#)

$$(0, 1) \underset{\text{homeo}}{\simeq} \mathbb{R}^1 .$$

An [inverse](#) pair of [continuous functions](#) is for instance given by

$$\begin{aligned} f &: \mathbb{R}^1 \rightarrow (-1, +1) \\ x &\mapsto \frac{x}{\sqrt{1+x^2}} \end{aligned}$$

and

$$\begin{aligned} f^{-1} &: (-1, +1) \rightarrow \mathbb{R}^1 \\ x &\mapsto \frac{x}{\sqrt{1-x^2}} . \end{aligned}$$

Generally, every [open ball](#) in \mathbb{R}^n (def. 1.2) is [homeomorphic](#) to all of \mathbb{R}^n .

Similarly, for all $a < b \in \mathbb{R}$

1. the [open intervals](#) $(a, b) \subset \mathbb{R}$ (example 1.12) equipped with their [subspace topology](#) are all homeomorphic to each other,
2. the closed intervals $[a, b]$ are all homeomorphic to each other,
3. the half-open intervals of the form $[a, b)$ are all homeomorphic to each other;
4. the half-open intervals of the form $(a, b]$ are all homeomorphic to each other.

Example 3.26. Let (X, τ_X) , (Y, τ_Y) and (Z, τ_Z) be [topological spaces](#).

Then:

1. There is a [homeomorphism](#) between the two ways of bracketing the three factors when forming their [product topological space](#) (def. 2.18), called the [associator](#):

$$\alpha_{X,Y,Z} : ((X, \tau_X) \times (Y, \tau_Y)) \times (Z, \tau_Z) \xrightarrow{\simeq} (X, \tau_X) \times ((Y, \tau_Y) \times (Z, \tau_Z)) .$$

2. There are [homeomorphism](#) between (X, τ) and its [product topological space](#) (def. 2.18) with the [point](#) $*$ (example 2.10), called the left and right [unitors](#):

$$\lambda_X : * \times (X, \tau_X) \xrightarrow{\simeq} (X, \tau_X)$$

and

$$\rho_X : (X, \tau_X) \times * \xrightarrow{\cong} (X, \tau_X) .$$

3. There is a [homeomorphism](#) between the results of the two orders in which to form their [product topological spaces](#) (def. 2.18), called the [braiding](#):

$$\beta_{X,Y} : (X, \tau_X) \times (Y, \tau_Y) \xrightarrow{\cong} (Y, \tau_Y) \times (X, \tau_X) .$$

Moreover, all these homeomorphisms are compatible with each other, in that they make the following [diagrams commute](#):

1. (triangle identity)

$$\begin{array}{ccc} (X \times *) \times Y & \xrightarrow{\alpha_{X,*,Y}} & X \times (* \times Y) \\ \rho_x \times \text{id}_Y \searrow & & \swarrow \text{id}_X \times \lambda_Y \\ & X \times Y & \end{array}$$

2. ([pentagon identity](#))

$$\begin{array}{ccc} & (W \times X) \times (Y \times Z) & \\ \alpha_{W \times X, Y, Z} \nearrow & & \searrow \alpha_{W, X, Y \times Z} \\ ((W \times X) \times Y) \times Z & & (W \times (X \times (Y \times Z))) \\ \alpha_{W, X, Y} \times \text{id}_Z \downarrow & & \uparrow \text{id}_W \times \alpha_{X, Y, Z} \\ (W \times (X \times Y)) \times Z & \xrightarrow{\alpha_{W, X \times Y, Z}} & W \times ((X \times Y) \times Z) \end{array}$$

3. (hexagon identities)

$$\begin{array}{ccccc} (X \times Y) \times Z & \xrightarrow{\alpha_{X, Y, Z}} & X \times (Y \times Z) & \xrightarrow{\beta_{X, Y \times Z}} & (Y \times Z) \times X \\ \downarrow \beta_{X, Y} \times \text{id}_Z & & & & \downarrow \alpha_{Y, Z, X} \\ (Y \times X) \times Z & \xrightarrow{\alpha_{Y, X, Z}} & Y \times (X \times Z) & \xrightarrow{\text{id}_Y \times \beta_{X, Y}} & Y \times (Z \times X) \end{array}$$

and

$$\begin{array}{ccccc} X \times (Y \times Z) & \xrightarrow{\alpha_{X, Y, Z}^{\text{inv}}} & (X \times Y) \times Z & \xrightarrow{\beta_{X \times Y, Z}} & Z \times (X \times Y) \\ \downarrow \text{id}_X \times \beta_{Y, Z} & & & & \downarrow \alpha_{Z, X, Y}^{\text{inv}} \\ X \times (Z \times Y) & \xrightarrow{\alpha_{X, Z, Y}^{\text{inv}}} & (X \times Z) \times Y & \xrightarrow{\beta_{X, Z} \times \text{id}} & (Z \times X) \times Y \end{array} ,$$

4. (symmetry)

$$\beta_{Y,X} \circ \beta_{X,Y} = \text{id} : (X_1 \times X_2 \tau_{X_1 \times X_2}) \rightarrow (X_1 \times X_2 \tau_{X_1 \times X_2}) .$$

In the language of [category theory](#) (remark 3.3), this is summarized by saying that the the [functorial](#) construction $(-) \times (-)$ of [product topological spaces](#) (example 3.4) gives the [category Top](#) of [topological spaces](#) the [structure](#) of a

[monoidal category](#) which is [symmetrically braided](#).

From this, a basic result of [category theory](#), the [MacLane coherence theorem](#), guarantees that there is no essential ambiguity in re-bracketing arbitrary iterations of the binary product topological space construction. Accordingly, we may write

$$(X_1, \tau_1) \times (X_2, \tau_2) \times \cdots \times (X_n, \tau_n)$$

without putting parenthesis.

Example 3.27. (product of closed intervals homeomorphic to hypercubes)

Let $n \in \mathbb{N}$, and let $[a_i, b_i] \subset \mathbb{R}$ for $i \in \{1, \dots, n\}$ be n [closed intervals](#) in the [real line](#) (example 1.12), regarded as [topological subspaces](#) of the 1-dimensional [Euclidean space](#). Then the [product topological space](#) (def. 2.18, example 3.26) of all these intervals is [homeomorphic](#) (def. 3.21) to the corresponding [topological subspace](#) of the n -dimensional [Euclidean space](#) (def. 1.6):

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \simeq \left\{ \vec{x} \in \mathbb{R}^n \mid \forall_i a_i \leq x_i \leq b_i \right\} \subset \mathbb{R}^n .$$

Proof. There is a canonical [bijection](#) between the underlying sets. It remains to see that this as well and its inverse are [continuous functions](#). For this it is sufficient to see that under this bijection the defining [basis](#) for the [product topology](#) is also a basis for the [subspace topology](#). But this is immediate from lemma 2.8. ■

Example 3.28. (interval glued at endpoints is homeomorphic to the circle)

As topological spaces, the [closed interval](#) $[0, 1]$ (def. 1.12) with its two endpoints identified is [homeomorphic](#) (def. 3.21) to the standard [circle](#):

$$[0, 1]_{/(0 \sim 1)} \underset{\text{homeo}}{\simeq} S^1 .$$

More in detail: let

$$S^1 \hookrightarrow \mathbb{R}^2$$

be the unit [circle](#) in the [plane](#)

$$S^1 = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 = 1\}$$

equipped with the [subspace topology](#) (example 2.16) of the plane \mathbb{R}^2 , which itself equipped with its standard [metric topology](#) (example 2.9).

Moreover, let

$$[0, 1]_{/(0 \sim 1)}$$

be the [quotient topological space](#) (example 2.17) obtained from the [interval](#) $[0, 1] \subset \mathbb{R}^1$ with its [subspace topology](#) by applying the [equivalence relation](#) which

identifies the two endpoints (and nothing else).

Consider then the function

$$f : [0, 1] \rightarrow S^1$$

given by

$$t \mapsto (\cos(t), \sin(t)) .$$

This has the property that $f(0) = f(1)$, so that it descends to the [quotient topological space](#)

$$\begin{array}{ccc} [0, 1] & \rightarrow & [0, 1]_{/(0 \sim 1)} \\ f \searrow & & \downarrow \tilde{f} \\ & & S^1 \end{array} .$$

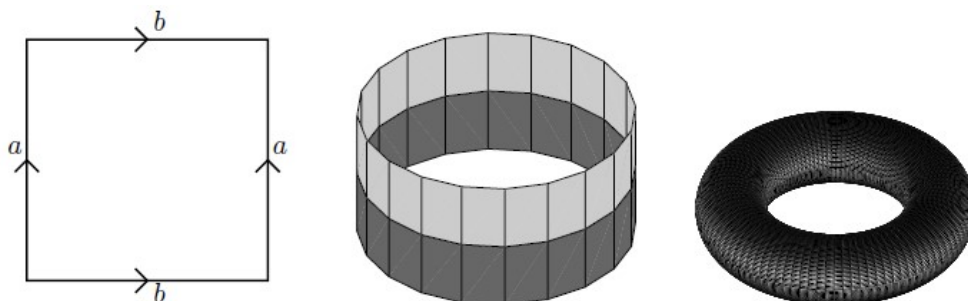
We claim that \tilde{f} is a [homeomorphism](#) (definition [3.21](#)).

First of all it is immediate that \tilde{f} is a [continuous function](#). This follows immediately from the fact that f is a [continuous function](#) and by definition of the [quotient topology](#) (example [2.17](#)).

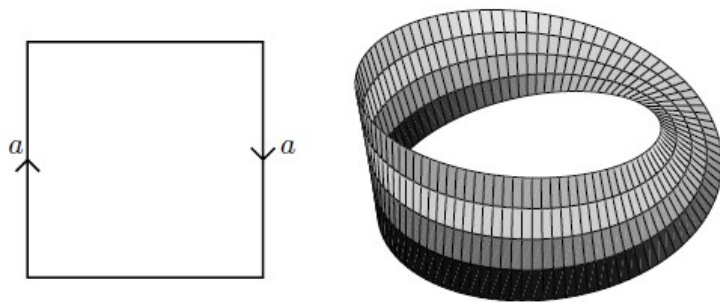
So we need to check that \tilde{f} has a continuous inverse function. Clearly the restriction of f itself to the open interval $(0, 1)$ has a continuous inverse. It fails to have a continuous inverse on $[0, 1)$ and on $(0, 1]$ and fails to have an inverse at all on $[0, 1]$, due to the fact that $f(0) = f(1)$. But the relation quotiented out in $[0, 1]_{/(0 \sim 1)}$ is exactly such as to fix this failure.

Similarly:

The [square](#) $[0, 1]^2$ with two of its sides identified is the [cylinder](#), and with also the other two sides identified is the [torus](#):



If the sides are identified with opposite orientation, the result is the [Möbius strip](#):



graphics grabbed from [Lawson 03](#)

Important examples of pairs of spaces that are *not* homeomorphic include the following:

Theorem 3.29. ([topological invariance of dimension](#))

For $n_1, n_2 \in \mathbb{N}$ but $n_1 \neq n_2$, then the [Cartesian spaces](#) \mathbb{R}^{n_1} and \mathbb{R}^{n_2} are not [homeomorphic](#).

More generally, an [open set](#) in \mathbb{R}^{n_1} is never homeomorphic to an open set in \mathbb{R}^{n_2} if $n_1 \neq n_2$.

The proof of theorem [3.29](#) is surprisingly hard, given how obvious the statement seems intuitively. It requires tools from a field called [algebraic topology](#) (notably [Brouwer's fixed point theorem](#)).

We showcase some basic tools of [algebraic topology](#) now and demonstrate the nature of their usage by proving two very simple special cases of the [topological invariance of dimension](#) (prop.

[\ref{TopologicalInvarianceOfDimensionFirstSimpleCase}](#) and prop. [\ref{topologicalInvarianceOfDimensionSecondSimpleCase}](#) below).

Example 3.30. (homeomorphism classes of surfaces)

The [2-sphere](#) $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is *not* [homeomorphic](#) to the [torus](#) $T^2 = S^1 \times S^1$.

Generally the [homeomorphism](#) class of a [closed orientable surface](#) is determined by the number of “holes” it has, its [genus](#).

4. Separation axioms

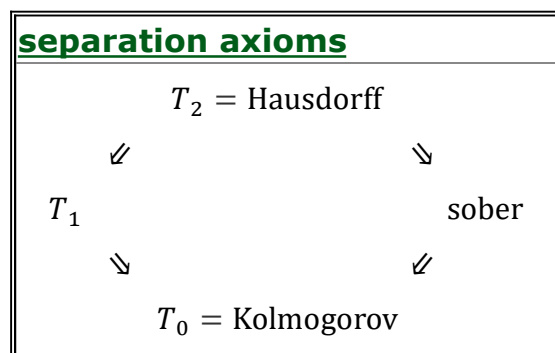
The plain definition of [topological space](#) happens to allow examples where distinct points or distinct subsets of the underlying set of a topological space appear as as more-or-less unseparable as seen by the topology on that set. In many applications one wants to exclude at least some of such degenerate examples

from the discussion. The relevant conditions to be imposed on top of the plain [axioms](#) of a [topological space](#) are hence known as [separation axioms](#).

These axioms are all of the form of saying that two subsets (of certain forms) in the topological space are 'separated' from each other in one sense if they are 'separated' in a (generally) weaker sense. For example the weakest axiom (called T_0) demands that if two points are distinct as elements of the underlying set of points, then there exists at least one [open subset](#) that contains one but not the other.

In this fashion one may impose a hierarchy of stronger axioms. For example demanding that given two distinct points, then each of them is contained in some open subset not containing the other (T_1) or that such a pair of open subsets around two distinct points may in addition be chosen to be disjoint (T_2). This last condition, T_2 , also called the [Hausdorff condition](#) is the most common among all separation axioms. Often in [topology](#), this axiom is considered by default.

However, there are respectable areas of mathematics that involve topological spaces where the Hausdorff axiom fails, but a weaker axiom is still satisfied, called [soberity](#). This is the case notably in [algebraic geometry](#) ([schemes are sober](#)) and in [computer science](#) (Vickers 89). These [sober topological spaces](#) are singled out by the fact that they are entirely characterized by their [partially ordered sets of open subsets](#) and may hence be understood independently from their underlying sets of points.



All separation axioms are satisfied by [metric spaces](#) (def. 1.1), from whom the concept of topological space was originally abstracted [above](#). Hence imposing some of them may also be understood as gauging just how far one allows topological spaces to generalize away from metric spaces

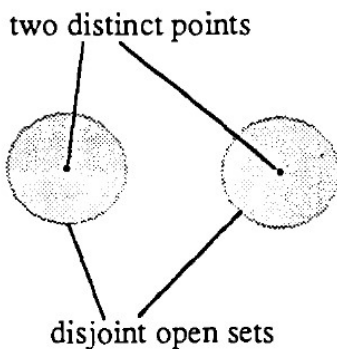
T_n spaces

Definition 4.1. (the first three [separation axioms](#))

Let (X, τ) be a [topological space](#) (def. 2.3).

For $x \neq y \in X$ any two points in the underlying set of X which are not [equal](#) as elements of this set, consider the following [propositions](#):

- **(T0)** *There exists a [neighbourhood](#) of one of the two points which does not*



contain the other point.

- **(T1)** There exist neighbourhoods of both points which do not contain the other point.
- **(T2)** There exists neighbourhoods of both points which do not intersect each other.

graphics grabbed from Vickers 89

The topological space X is called a T_n -topological space or just T_n -space, for short, if it satisfies condition T_n above for all pairs of distinct points.

A T_0 -topological space is also called a Kolmogorov space.

A T_2 -topological space is also called a Hausdorff topological space.

Notice that these propositions evidently imply each other as

$$T_2 \Rightarrow T_1 \Rightarrow T_0 .$$

For definiteness, we re-state these conditions formally. Write $x, y \in X$ for points in X , write $U_x, U_y \in \tau$ for open neighbourhoods of these points. Then:

- **(T0)** $\forall_{x \neq y} \left(\left(\exists_{U_y} (\{x\} \cap U_y = \emptyset) \right) \vee \left(\exists_{U_x} (U_x \cap \{y\} = \emptyset) \right) \right)$
- **(T1)** $\forall_{x \neq y} \left(\exists_{U_x, U_y} \left((\{x\} \cap U_y = \emptyset) \wedge (U_x \cap \{y\} = \emptyset) \right) \right)$
- **(T2)** $\forall_{x \neq y} \left(\exists_{U_x, U_y} (U_x \cap U_y = \emptyset) \right)$

Example 4.2. (metric spaces are Hausdorff)

Every metric space (def 1.1), regarded as a topological space via its metric topology (def. 2.9) is a Hausdorff topological space (def. 4.1).

Example 4.3. (finite T_1 -spaces are discrete)

For a finite topological space (X, τ) , hence one for which the underlying set X is a finite set, the following are equivalent:

1. (X, τ) is T_1 (def. 4.1);
2. (X, τ) is a discrete topological space (def. 2.13)

Proposition 4.4. Let (X, τ) be a topological space satisfying the T_i separation

axiom according to def. 4.1. Then also every *topological subspace* $S \subset X$ (example 2.16) satisfies T_i .

Proof. Let $x, y \in S \subset X$ be two distinct points. We need to construct various open neighbourhoods of these in S not containing the other point and possibly (for T_2) not intersecting each other. Now by assumptions that the ambient space (X, τ) satisfies the given axiom, there exist open neighbourhoods with the analogous properties in X . By the nature of the subspace topology, their restriction to S are still open, and clearly still satisfy these properties. ■

Separation in terms of topological closures

The conditions T_0 , T_1 and T_2 have the following equivalent form in terms of *topological closures* (def. 2.20):

Proposition 4.5. (T_0 in terms of topological closures)

A *topological space* (X, τ) is T_0 (def. 4.1) precisely if the function $\text{Cl}(\{-\})$ from the underlying set of X to the set of *irreducible closed subsets* of X (def. 2.28, which is well defined according to example 2.29), is *injective*:

$$\text{Cl}(\{-\}) : X \hookrightarrow \text{IrrClSub}(X)$$

Proof. Assume first that X is T_0 . Then we need to show that if $x, y \in X$ are such that $\text{Cl}(\{x\}) = \text{Cl}(\{y\})$ then $x = y$. Hence assume that $\text{Cl}(\{x\}) = \text{Cl}(\{y\})$. Since the closure of a point is the *complements* of the union of the open subsets not containing the point, this means that the union of open subsets that do not contain x is the same as the union of open subsets that do not contain y . Hence every open subset that does not contain x also does not contain y , and vice versa. By T_0 this is not the case when $x \neq y$, hence it follows that $x = y$.

Conversely, assume that if $x, y \in X$ are such that $\text{Cl}\{x\} = \text{Cl}\{y\}$ then $x = y$. We need to show that if $x \neq y$ then there exists an open neighbourhood around one of the two points not containing the other. Hence assume that $x \neq y$. By assumption it follows that $\text{Cl}(\{x\}) \neq \text{Cl}(\{y\})$. Since the closure of a point is the *complements* of the union of the open subsets not containing the point, this means that there must be at least one open subset which contains x but not y , or vice versa. By definition this means that (X, τ) is T_0 . ■

Proposition 4.6. (T_1 in terms of topological closures)

A *topological space* (X, τ) is T_1 (def. 4.1) precisely if all its points are *closed points* (def. 2.20).

Proof. Assume first that (X, τ) is T_1 . We need to show that for every point $x \in X$ we have $\text{Cl}(\{x\}) = \{x\}$. Since the closure of a point is the *complement* of the union of all open subsets not containing this point, this is the case precisely if the union of

all open subsets not containing x is $X \setminus \{x\}$, hence if every point $y \neq x$ is member of at least one open subset not containing x . This is true by T_1 .

Conversely, assume that for all $x \in X$ then $\text{Cl}(\{x\}) = \{x\}$. Then for $x \neq y \in X$ two distinct points we need to produce an open subset of y that does not contain x . But as before, since $\text{Cl}(\{x\})$ is the complement of the union of all open subsets that do not contain x , and the assumption $\text{Cl}\{x\} = \{x\}$ means that y is member of one of these open subsets that do not contain x . ■

Proposition 4.7. (T_2 in terms of topological closures)

A [topological space](#) (X, τ_X) is T_2 = [Hausdorff](#) (def. 4.1) precisely if the [diagonal function](#) $\Delta_X: (X, \tau_X) \rightarrow (X \times X, \tau_{X \times X})$ (example 3.9) is a [closed map](#) (def. 3.14).

Proof. If (X, τ_X) is Hausdorff, then by definition for every pair of distinct points $x \neq y \in X$ there exists open neighbourhoods $U_x, U_y \in \tau_X$ such that $U_x \cap U_y = \emptyset$. In terms of the [product topology](#) (example 2.18) this means that every point $(x, y) \in X \times X$ which is not on the diagonal has an open neighbourhood $U_x \times U_y$ which still does not contain the diagonal. By definition, this means that in fact every [subset](#) of the diagonal is a [closed subset](#) of $X \times X$, hence in particular those that are in the image under Δ_X of closed subsets of X . Hence Δ_X is a closed map.

Conversely, if Δ_X is a closed map, then the full diagonal (i.e. the image of X under Δ_X) is closed in $X \times X$, and hence this means that every points $(x, y) \in X \times X$ not on the diagonal has an open neighbourhood $U_x \times U_y$ not containing the diagonal, i.e. such that $U_x \cap U_y = \emptyset$. Hence X is Hausdorff. ■

Further separation axioms

Clearly one may and does consider further variants of the separation axioms T_0 , T_1 and T_2 from def. 4.1.

Definition 4.8. Let (X, τ) be [topological space](#) (def. 4.1).

Consider the following conditions

- **(T3)** (X, τ) is T_1 (def. 4.1) and for $x \in X$ a point and $C \subset X$ a [closed subset](#) (def. 2.20) not containing x , then there exist disjoint [open neighbourhoods](#) $U_x \supset \{x\}$ and $U_C \supset C$.
- **(T4)** (X, τ) is T_1 (def. 4.1) and for $C_1, C_2 \subset X$ disjoint [closed subsets](#) (def. 2.20) then there exist disjoint [open neighbourhoods](#) $U_{C_i} \supset C_i$.

If (X, τ) satisfies T_3 it is said to be a T_3 -space also called a [regular Hausdorff topological space](#).

If (X, τ) satisfies T_4 it is to be a T_4 -space also called a [normal Hausdorff](#)

topological space.

Observe that:

Proposition 4.9. *The separation axioms imply each other as*

$$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0 .$$

Proof. The implications

$$T_2 \Rightarrow T_1 \Rightarrow T_0$$

and

$$T_4 \Rightarrow T_3$$

are immediate from the definitions. The remaining implication $T_3 \Rightarrow T_2$ follows with prop. 4.6. ■

Hence instead of saying “ X is T_1 and ...” one could just as well phrase the conditions T_3 and T_4 as “ X is T_2 and ...”, which would render the proof of prop. 4.9 even more trivial.

In summary:

the main Separation Axioms

number	name	statement	reformulation
T_0	<u>Kolmogorov</u>	given two distinct points, at least one of them has an open neighbourhood not containing the other point	every irreducible closed subset is the closure of at most one point
T_1		given two distinct points, both have an open neighbourhood not containing the other point	all points are closed
T_2	<u>Hausdorff</u>	given two distinct points, they have disjoint open neighbourhoods	the diagonal is a closed map
T_3	<u>regular Hausdorff</u>	all points are closed; and given two disjoint closed subsets, at least one of them has an open neighbourhood disjoint from the other closed subset	
T_4	<u>normal Hausdorff</u>	all points are closed; and given two disjoint closed subsets, both of them have open neighbourhoods not intersecting the other closed subset	

Notice that there is a whole zoo of further variants of separation axioms that are

considered in the literature. But the above are maybe the main ones. Specifically $T_2 = \text{Hausdorff}$ is the most popular one, often considered by default in the literature, when topological spaces are considered.

T_n reflection

Not every [universal construction](#) of [topological spaces](#) applied to T_n -spaces results again in a T_n topological space, notably [quotient space](#) constructions need not (example [4.10](#) below).

But at least for T_0 , T_1 and T_2 there is a universal way, called [reflection](#) (prop. [4.12](#) below), to approximate any topological space “from the left” by a T_n topological spaces.

Hence if one wishes to work withing the [full subcategory](#) of the T_n among all [topological space](#), then the correct way to construct quotients and other [colimits](#) (see [below](#)) is to first construct them as usual for topological spaces, and then apply the T_n -reflection to the result.

Example 4.10. ([line with two origins](#))

Consider the [disjoint union](#) $\mathbb{R} \sqcup \mathbb{R}$ of two copies of the [real line](#) \mathbb{R} regarded as the 1-dimensional [Euclidean space](#) (def. [1.6](#)) with its [metric topology](#) (def. [2.9](#)). Moreover, consider the [equivalence relation](#) on the underlying set which identifies every point x_i in the i th copy of \mathbb{R} ($i \in \{0, 1\}$) with the corresponding point in the other, the $(1 - i)$ th copy, except when $x = 0$:

$$(x_i \sim y_j) \Leftrightarrow ((x = y) \text{ and } ((x \neq 0) \text{ or } (i = j))) .$$

The [quotient topological space](#) by this equivalence relation (def. [2.17](#))

$$(\mathbb{R} \sqcup \mathbb{R}) / \sim$$

is called the **line with two origins**.

This is a basic example of a topological space which is a [non-Hausdorff topological space](#):

Because by definition of the [quotient space topology](#), the [open neighbourhoods](#) of $0_i \in (\mathbb{R} \sqcup \mathbb{R}) / \sim$ are precisely those that contain subsets of the form

$$(-\epsilon, \epsilon)_i := (-\epsilon, 0) \cup \{0_i\} \cup (0, \epsilon) .$$

But this means that the “two origins” 0_0 and 0_1 may not be [separated by neighbourhoods](#), since the intersection of $(-\epsilon, \epsilon)_0$ with $(-\epsilon, \epsilon)_i$ is always non-empty:

$$(-\epsilon, \epsilon)_0 \cap (-\epsilon, \epsilon)_1 = (-\epsilon, 0) \cup (0, \epsilon) .$$

Example 4.11. Consider the [real line](#) \mathbb{R} regarded as the 1-dimensional [Euclidean space](#) (def. 1.6) with its [metric topology](#) (def. 2.9) and consider the [equivalence relation](#) \sim on \mathbb{R} which identifies two [real numbers](#) if they differ by a [rational number](#):

$$(x \sim y) \Leftrightarrow \left(\exists_{p/q \in \mathbb{Q} \subset \mathbb{R}} x = y + p/q \right) .$$

Then the [quotient topological space](#) (def. 2.17)

$$\mathbb{R}/\mathbb{Q} := \mathbb{R}/\sim$$

is a [codiscrete topological space](#) (def. 2.13), hence in particular a [non-Hausdorff topological space](#) (def. 4.1).

Proposition 4.12. (T_n -reflection)

Let $n \in \{0, 1, 2\}$. Then for every [topological space](#) X there exists a T_n -topological space $T_n X$ for and a [continuous function](#)

$$t_n(X) : X \rightarrow T_n X$$

which is the "closest approximation from the left" to X by a T_n -topological space, in that for Y any T_n -space, then [continuous functions](#) of the form

$$f : X \rightarrow Y$$

are in [bijection](#) with [continuous function](#) of the form

$$\tilde{f} : T_n X \rightarrow Y$$

and such that the bijection is constituted by

$$f = \tilde{f} \circ t_n(X) : X \xrightarrow{t_n(X)} T_n X \xrightarrow{\tilde{f}} Y .$$

Here $X \xrightarrow{t_n(X)} T_n(X)$ may be called the T_n -reflection of X . For $n = 0$ this is known as the [Kolmogorov quotient](#) construction (see prop. 4.15 below). For $n = 2$ it is known as [Hausdorff reflection](#) or Hausdorffication or similar.

Moreover, the operation $T_n(-)$ extends to [continuous functions](#) $f : X \rightarrow Y$

$$(X \xrightarrow{f} Y) \mapsto (T_n X \xrightarrow{T_n f} T_n Y)$$

such as to preserve [composition](#) of functions as well as [identity functions](#):

$$T_n g \circ T_n f = T_n (g \circ f) \quad , \quad T_n \text{id}_X = \text{id}_{T_n X}$$

Finally, the comparison map is compatible with this in that the follows [squares commute](#):

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ h_X \downarrow & & \downarrow h_Y \\ T_n X & \xrightarrow{T_n f} & T_n Y \end{array}$$

Remark 4.13. (reflective subcategories)

In the language of category theory (remark 3.3) the T_n -reflection of prop. 4.12 says that

1. $T_n(-)$ is a functor $T_n : \text{Top} \rightarrow \text{Top}_{T_n}$ from the category Top of topological spaces to the full subcategory $\text{Top}_{T_n} \overset{\iota}{\hookrightarrow} \text{Top}$ of Hausdorff topological spaces;
2. $t_n(X) : X \rightarrow T_n X$ is a natural transformation from the identity functor on Top to the functor $\iota \circ T_n$
3. T_n -topological spaces form a reflective subcategory of all topological spaces in that T_n is left adjoint to the inclusion functor ι ; this situation is denoted as follows:

$$\text{Top}_{T_n} \overset{H}{\underset{\iota}{\rightleftarrows}} \text{Top} .$$

There are various ways to see the existence and to construct the T_n -reflections. The following is the quickest way to see the existence, even though it leaves the actual construction rather implicit.

Proposition 4.14. *Let $n \in \{0, 1, 2\}$. Let (X, τ) be a topological space and consider the equivalence relation \sim on the underlying set X for which $x \sim y$ precisely if for every surjective continuous function $f : X \rightarrow Y$ into any T_n -topological space Y we have $f(x) = f(y)$.*

Then the set of equivalence classes

$$T_n X := X / \sim$$

equipped with the quotient topology is a T_n -topological space, and the quotient map $t_n(X) : X \rightarrow X / \sim$ exhibits the T_n -reflection of X , according to prop. 4.12.

Proof. First we observe that every continuous function $f : X \rightarrow Y$ into a T_n -topological space Y factors uniquely via $t_n(X)$ through a continuous function \tilde{f}

$$f = \tilde{f} \circ h_X$$

where

$$\tilde{f} : [x] \mapsto f(x) .$$

To see this, first factor f through its image $f(X)$

$$f : X \rightarrow f(X) \hookrightarrow Y$$

equipped with its [subspace topology](#) as a subspace of Y (example [3.10](#)). By prop. [4.4](#) also $f(X)$ is a T_n -topological space if Y is.

It follows by definition of $t_n(X)$ that the factorization exists at the level of sets as stated, since if $x_1, x_2 \in X$ have the same [equivalence class](#) $[x_1] = [x_2]$ in $T_n X$, then by definition they have the same image under all continuous surjective functions to a T_n -space, hence in particular under $X \rightarrow f(X)$. This means that \tilde{f} as above is well defined.

What remains to be seen is that $T_n X$ as constructed is indeed a T_n -topological space. Hence assume that $[x] \neq [y] \in T_n X$ are two distinct points. We need to open neighbourhoods around one or both of these point not containing the other point and possibly disjoint to each other.

Now by definition of $T_n X$ this means that there exists a T_n -topological space Y and a surjective continuous function $f : X \rightarrow Y$ such that $f(x) \neq f(y) \in Y$. Accordingly, since Y is T_n , there exist the respective kinds of neighbourhoods around these image points in Y . Moreover, by the previous statement there exists a continuous function $\tilde{f} : T_n X \rightarrow Y$ with $\tilde{f}([x]) = f(x)$ and $\tilde{f}([y]) = f(y)$. By the nature of continuous functions, the pre-images of these open neighbourhoods in Y are still open in X and still satisfy the required disjunction properties. Therefore $T_n X$ is a T_n -space. ■

Here are alternative constructions of the reflections:

Proposition 4.15. ([Kolmogorov quotient](#))

Let (X, τ) be a [topological space](#). Consider the [relation](#) on the underlying set by which $x_1 \sim x_2$ precisely if neither x_i has an [open neighbourhood](#) not containing the other. This is an [equivalence relation](#). The [quotient topological space](#) $X \rightarrow X / \sim$ by this equivalence relation (def. [2.17](#)) exhibits the T_0 -reflection of X according to prop. [4.12](#).

Example 4.16. The [Hausdorff reflection](#) (T_2 -reflection, prop. [4.12](#))

$$T_2 : \text{Top} \rightarrow \text{Top}_{\text{Haus}}$$

of the [line with two origins](#) from example [4.10](#) is the [real line](#) itself:

$$T_2((\mathbb{R} \sqcup \mathbb{R}) / \sim) \simeq \mathbb{R}.$$

5. Sober spaces

The alternative characterization of the T_0 -condition in prop. [4.5](#) immediately suggests the following strengthening, different from the T_1 -condition:

Definition 5.1. ([sober topological space](#))

A [topological space](#) (X, τ) is called a [sober topological space](#) precisely if every [irreducible closed subspace](#) (def. 2.29) is the [topological closure](#) (def. 2.20) of a unique point, hence precisely if the function

$$\text{Cl}(\{-\}) : X \rightarrow \text{IrrClSub}(X)$$

from the underlying set of X to the set of [irreducible closed subsets](#) of X (def. 2.28, well defined according to example 2.29) is [bijective](#).

Proposition 5.2. (*sober implies T_0*)

Every [sober topological space](#) (def. 5.1) is T_0 (def. 4.1).

Proof. By prop. 4.5. ■

Proposition 5.3. (*Hausdorff implies sober*)

Every [Hausdorff topological space](#) (def. 4.1) is a [sober topological space](#) (def. 5.1).

More specifically, in a Hausdorff topological space the [irreducible closed subspaces](#) (def. 2.28) are precisely the [singleton subspaces](#) (def. 7.2).

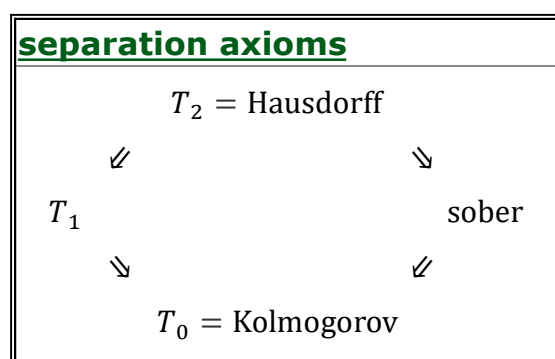
Proof. The second statement clearly implies the first. To see the second statement, suppose that F is an irreducible closed subspace which contained two distinct points $x \neq y$. Then by the Hausdorff property there are disjoint neighbourhoods U_x, U_y , and hence it would follow that the relative [complements](#) $F \setminus U_x$ and $F \setminus U_y$ were distinct proper closed subsets of F with

$$F = (F \setminus U_x) \cup (F \setminus U_y)$$

in contradiction to the assumption that F is irreducible.

This [proves by contradiction](#) that every irreducible closed subset is a singleton. Conversely, generally the [topological closure](#) of every singleton is irreducible closed, by example 2.29. ■

By prop. 5.2 and prop. 5.3 we have the implications on the right of the following diagram:



But there there is no implication between T_1 and sobriety:

Proposition 5.4. The [intersection](#) of the [classes](#) of [sober topological spaces](#) (def. [5.1](#)) and T_1 -topological spaces (def. [4.1](#)) is not [empty](#), but neither class is contained within the other.

That the intersection is not empty follows from prop. [5.3](#). That neither class is contained in the other is shown by the following counter-examples:

Example.

- The [Sierpinski space](#) (def. [2.11](#)) is sober, but not T_1 .
- The [cofinite topology](#) (example [2.14](#)) on a non-[finite set](#) is T_1 but not sober.

Frames of opens

What makes the concept of [sober topological spaces](#) special is that for them the concept of [continuous functions](#) may be expressed entirely in terms of the relations between their [open subsets](#), disregarding the underlying set of points of which these open are in fact subsets.

Recall from example [2.33](#) that for very [continuous function](#) $f:(X, \tau_X) \rightarrow (Y, \tau_Y)$ the [pre-image](#) function $f^{-1}:\tau_Y \rightarrow \tau_X$ is a [frame homomorphism](#) (def. [2.31](#)).

For sober topological spaces the converse holds:

Proposition 5.5. *If (X, τ_X) and (Y, τ_Y) are [sober topological spaces](#) (def. [5.1](#)), then for every [frame homomorphism](#) (def. [2.31](#))*

$$\tau_X \leftarrow \tau_Y : \phi$$

there is a unique [continuous function](#) $f:X \rightarrow Y$ such that ϕ is the function of forming [pre-images](#) under f :

$$\phi = f^{-1} .$$

Proof. We first consider the special case of frame homomorphisms of the form

$$\tau_* \leftarrow \tau_X : \phi$$

and show that these are in bijection to the underlying set X , identified with the continuous functions $* \rightarrow (X, \tau)$ via example [3.6](#).

By prop. [2.34](#), the frame homomorphisms $\phi:\tau_X \rightarrow \tau_*$ are identified with the irreducible closed subspaces $X \setminus U_\emptyset(\phi)$ of (X, τ_X) . Therefore by assumption of [sobriety](#) of (X, τ) there is a unique point $x \in X$ with $X \setminus U_\emptyset = \text{Cl}(\{x\})$. In particular this means that for U_x an open neighbourhood of x , then U_x is not a subset of $U_\emptyset(\phi)$, and so it follows that $\phi(U_x) = \{1\}$. In conclusion we have found a unique $x \in X$ such that

$$\phi : U \mapsto \begin{cases} \{1\} & | \text{if } x \in U \\ \emptyset & | \text{otherwise} \end{cases} .$$

This is precisely the [inverse image](#) function of the continuous function $* \rightarrow X$ which sends $1 \mapsto x$.

Hence this establishes the bijection between frame homomorphisms of the form $\tau_* \leftarrow \tau_X$ and continuous functions of the form $* \rightarrow (X, \tau)$.

With this it follows that a general frame homomorphism of the form $\tau_X \xleftarrow{\phi} \tau_Y$ defines a function of sets $X \xrightarrow{f} Y$ by [composition](#):

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ (\tau_* \leftarrow \tau_X) & \mapsto & (\tau_* \leftarrow \tau_X \xleftarrow{\phi} \tau_Y) \end{array} .$$

By the previous analysis, an element $U_Y \in \tau_Y$ is sent to $\{1\}$ under this composite precisely if the corresponding point $* \rightarrow X \xrightarrow{f} Y$ is in U_Y , and similarly for an element $U_X \in \tau_X$. It follows that $\phi(U_Y) \in \tau_X$ is precisely that subset of points in X which are sent by f to elements of U_Y , hence that $\phi = f^{-1}$ is the [pre-image](#) function of f . Since ϕ by definition sends open subsets of Y to open subsets of X , it follows that f is indeed a continuous function. This proves the claim in generality. ■

Remark 5.6. ([locales](#))

Proposition [5.5](#) is often stated as saying that sober topological spaces are [equivalently](#) the "[locales with enough points](#)" ([Johnstone 82, II 1.](#)). Here "[locale](#)" refers to a concept akin to topological spaces where one considers *just* a "[frame of open subsets](#)" τ_X , without requiring that its elements be actual [subsets](#) of some ambient set. The natural notion of [homomorphism](#) between such generalized topological spaces are clearly the [frame](#) homomorphisms $\tau_X \leftarrow \tau_Y$ as above. From this perspective, prop. [5.5](#) says that sober topological spaces (X, τ_X) are entirely characterized by their [frames of opens](#) τ_X and just so happen to "have enough points" such that these are actual open subsets of some ambient set, namely of X .

Sober reflection

We saw above in prop. [4.12](#) that every topological space has a "best approximation from the left" by a [Hausdorff topological space](#). We now discuss the analogous statement for [sober topological spaces](#).

Recall again the [point](#) topological space $* := (\{1\}, \tau_* = \{\emptyset, \{1\}\})$ (example [2.10](#)).

Definition 5.7. Let (X, τ) be a [topological space](#).

Define SX to be the set

$$SX := \text{Hom}_{\text{Frame}}(\tau_X, \tau_*)$$

of [frame homomorphisms](#) from the [frame of opens](#) of X to that of the point.

Define a [topology](#) $\tau_{SX} \subset P(SX)$ on this set by declaring it to have one element \tilde{U} for each element $U \in \tau_X$ and given by

$$\tilde{U} := \{\phi \in SX \mid \phi(U) = \{1\}\}.$$

Consider the function

$$\begin{aligned} X &\xrightarrow{s_X} SX \\ x &\mapsto (\text{const}_x)^{-1} \end{aligned}$$

which sends an element $x \in X$ to the function which assigns [inverse images](#) of the [constant function](#) $\text{const}_x : \{1\} \rightarrow X$ on that element.

Lemma 5.8. *The construction (SX, τ_{SX}) in def. 5.7 is a [topological space](#), and the function $s_X : X \rightarrow SX$ is a [continuous function](#)*

$$s_X : (X, \tau_X) \rightarrow (SX, \tau_{SX})$$

Proof. To see that $\tau_{SX} \subset P(SX)$ is closed under arbitrary unions and finite intersections, observe that the function

$$\begin{aligned} \tau_X &\xrightarrow{\widetilde{(-)}} \tau_{SX} \\ U &\mapsto \tilde{U} \end{aligned}$$

in fact preserves arbitrary unions and finite intersections. Whith this the statement follows by the fact that τ_X is closed under these operations.

To see that $\widetilde{(-)}$ indeed preserves unions, observe that (e.g. [Johnstone 82, II 1.3 Lemma](#))

$$\begin{aligned} p \in \bigcup_{i \in I} \tilde{U}_i &\Leftrightarrow \exists_{i \in I} p(U_i) = \{1\} \\ &\Leftrightarrow \bigcup_{i \in I} p(U_i) = \{1\} \\ &\Leftrightarrow p\left(\bigcup_{i \in I} U_i\right) = \{1\} \\ &\Leftrightarrow p \in \widetilde{\bigcup_{i \in I} U_i} \end{aligned}$$

where we used that the frame homomorphism $p : \tau_X \rightarrow \tau_*$ preserves unions. Similarly for intersections, now with I a [finite set](#):

$$\begin{aligned}
 p \in \bigcap_{i \in I} \widetilde{U}_i &\Leftrightarrow \forall_{i \in I} p(U_i) = \{1\} \\
 &\Leftrightarrow \bigcap_{i \in I} p(U_i) = \{1\} \\
 &\Leftrightarrow p\left(\bigcap_{i \in I} U_i\right) = \{1\}' \\
 &\Leftrightarrow p \in \overline{\bigcap_{i \in I} U_i}
 \end{aligned}$$

where now we used that the frame homomorphism p preserves finite intersections.

To see that s_X is continuous, observe that $s_X^{-1}(\tilde{U}) = U$, by construction. ■

Lemma 5.9. For (X, τ_X) a topological space, the function $s_X: X \rightarrow SX$ from def. 5.7 is

1. an injection precisely if X is T0;
2. a bijection precisely if X is sober.

In this case s_X is in fact a homeomorphism.

Proof. By lemma 2.34 there is an identification $SX \simeq \text{IrrClSub}(X)$ and via this s_X is identified with the map $x \mapsto \text{Cl}(\{x\})$.

Hence the second statement follows by definition, and the first statement by this prop.

That in the second case s_X is in fact a homeomorphism follows from the definition of the opens \tilde{U} : they are identified with the opens U in this case (...expand...). ■

Lemma 5.10. For (X, τ) a topological space, then the topological space (SX, τ_{SX}) from def. 5.7, lemma 5.8 is sober.

(e.g. Johnstone 82, lemma II 1.7)

Proof. Let $SX \setminus \tilde{U}$ be an irreducible closed subspace of (SX, τ_{SX}) . We need to show that it is the topological closure of a unique element $\phi \in SX$.

Observe first that also $X \setminus U$ is irreducible.

To see this use this prop., saying that irreducibility of $X \setminus U$ is equivalent to $U_1 \cap U_2 \subset U \Rightarrow (U_1 \subset U) \text{ or } (U_2 \subset U)$. But if $U_1 \cap U_2 \subset U$ then also $\tilde{U}_1 \cap \tilde{U}_2 \subset \tilde{U}$ (as in the proof of lemma 5.8) and hence by assumption on \tilde{U} it follows that $\tilde{U}_1 \subset \tilde{U}$ or $\tilde{U}_2 \subset \tilde{U}$. By lemma 2.34 this in turn implies $U_1 \subset U$ or $U_2 \subset U$. In conclusion, this shows that also $X \setminus U$ is irreducible .

By lemma 2.34 this irreducible closed subspace corresponds to a point $p \in SX$. By that same lemma, this frame homomorphism $p: \tau_X \rightarrow \tau_*$ takes the value \emptyset on all those opens which are inside U . This means that the topological closure of this point is just $SX \setminus \tilde{U}$.

This shows that there exists at least one point of which $X \setminus \tilde{U}$ is the topological closure. It remains to see that there is no other such point.

So let $p_1 \neq p_2 \in SX$ be two distinct points. This means that there exists $U \in \tau_X$ with $p_1(U) \neq p_2(U)$. Equivalently this says that \tilde{U} contains one of the two points, but not the other. This means that (SX, τ_{SX}) is **T0**. By [this prop.](#) this is equivalent to there being no two points with the same topological closure. ■

Proposition 5.11. *For (X, τ_X) any [topological space](#), for (Y, τ_Y^{sob}) a sober topological space, and for $f: (X, \tau_X) \rightarrow (Y, \tau_Y)$ a [continuous function](#), then it factors uniquely through the soberification $s_X: (X, \tau_X) \rightarrow (SX, \tau_{SX})$ from [def. 5.7](#), [lemma 5.8](#)*

$$\begin{array}{ccc} (X, \tau_X) & \xrightarrow{f} & (Y, \tau_Y^{\text{sob}}) \\ s_X \downarrow & \nearrow_{\exists!} & \\ (SX, \tau_{SX}) & & \end{array} .$$

Proof. By the construction in [def. 5.7](#), we the outer part of the following square [commutes](#):

$$\begin{array}{ccc} (X, \tau_X) & \xrightarrow{f} & (Y, \tau_Y^{\text{sob}}) \\ s_X \downarrow & \nearrow & \downarrow s_{SX} \\ (SX, \tau_{SX}) & \xrightarrow{s_f} & (SSX, \tau_{SSX}) \end{array} .$$

By [lemma 5.10](#) and [lemma 5.9](#), the right vertical morphism s_{SX} is an isomorphism (a [homeomorphism](#)), hence has an [inverse morphism](#). This defines the diagonal morphism, which is the desired factorization.

To see that this factorization is unique, consider two factorizations $\tilde{f}, \bar{f}: (SX, \tau_{SX}) \rightarrow (Y, \tau_Y^{\text{sob}})$ and apply the soberification construction once more to the triangles

$$\begin{array}{ccc} (X, \tau_X) & \xrightarrow{f} & (Y, \tau_Y^{\text{sob}}) & & (SX, \tau_{SX}) & \xrightarrow{s_f} & (Y, \tau_Y^{\text{sob}}) \\ s_X \downarrow & \nearrow_{\tilde{f}, \bar{f}} & & \mapsto & \simeq \downarrow & \nearrow_{\tilde{f}, \bar{f}} & \\ (SX, \tau_{SX}) & & & & (SX, \tau_{SX}) & & \end{array} .$$

Here on the right we used again [lemma 5.9](#) to find that the vertical morphism is an isomorphism, and that \tilde{f} and \bar{f} do not change under soberification, as they already map between sober spaces. But now that the left vertical morphism is an isomorphism, the commutativity of this triangle for both \tilde{f} and \bar{f} implies that $\tilde{f} = \bar{f}$. ■

6. Compact spaces

From the discussion of [compact metric spaces](#) in [def. 1.19](#) and [prop. 1.20](#) it is now immediate how to generalize these concepts to [topological spaces](#).

The most naive version of the definition directly generalizes the concept via converging sequences from def. 1.19:

Definition 6.1. (converging sequence in a topological space)

Let (X, τ) be a [topological space](#) (def. 2.3) and let $(x_n)_{n \in \mathbb{N}}$ be a [sequence](#) of points (x_n) in X (def. 1.15). We say that this sequence [converges](#) in (X, τ) to a point $x_\infty \in X$, denoted

$$x_n \xrightarrow{n \rightarrow \infty} x_\infty$$

if for each open [neighbourhood](#) U_{x_∞} of x_∞ there exists a $k \in \mathbb{N}$ such that for all $n \geq k$ then $x_n \in U_{x_\infty}$:

$$\left(x_n \xrightarrow{n \rightarrow \infty} x_\infty \right) \Leftrightarrow \bigwedge_{\substack{U_{x_\infty} \in \tau_X \\ x_\infty \in U_{x_\infty}}} \left(\exists k \in \mathbb{N} \left(\forall_{n \geq k} x_n \in U_{x_\infty} \right) \right).$$

Definition 6.2. (sequentially compact topological space)

Let (X, τ) be a [topological space](#) (def. 2.3). It is called [sequentially compact](#) if for every [sequence](#) of points (x_n) in X (def. 1.15) there exists a sub-sequence $(x_{n_k})_{k \in \mathbb{N}}$ which [converges](#) according to def. 6.1.

But prop. 1.20 suggests to consider also another definition of compactness for topological spaces:

Definition 6.3. (open cover)

An [open cover](#) of a [topological space](#) X (def. 2.3) is a [set](#) $\{U_i \subset X\}_{i \in I}$ of [open subsets](#) U_i of X , indexed by some [set](#) I , such that their [union](#) is all of X :

$$\bigcup_{i \in I} U_i = X.$$

Definition 6.4. (compact topological space)

A [topological space](#) X (def. 2.3) is called a [compact topological space](#) if every [open cover](#) $\{U_i \subset X\}_{i \in I}$ (def. 6.3) has a [finite subcover](#) in that there is a [finite subset](#) $J \subset I$ such that $\{U_i \subset X\}_{i \in J}$ is still a cover of X in that $\bigcup_{i \in J} U_i = X$.

Remark 6.5. (terminology issue regarding "compact")

Beware that the following terminology issue persists in the literature:

Some authors use "compact" to mean "[Hausdorff and compact](#)". To disambiguate this, some authors (mostly in [algebraic geometry](#)) say "quasi-compact" for what we call "compact" in prop. 6.4.

Example 6.6. (finite discrete spaces are compact)

A [discrete topological space](#) (def. 2.13) is [compact](#) (def. 6.4) precisely if its

underlying set is [finite](#).

Example 6.7. (closed interval is compact)

For any $a < b \in \mathbb{R}$ the [closed interval](#) (example 1.12)

$$[a, b] \subset \mathbb{R}$$

regarded with its [subspace topology](#) is a [compact topological space](#) (def. 6.4).

Proof. Since all the closed intervals are [homeomorphic](#) (by example 3.25) it is sufficient to show the statement for $[0, 1]$. Hence let $\{U_i \subset [0, 1]\}_{i \in I}$ be an open cover. We need to show that it has an open subcover.

Say that an element $x \in [0, 1]$ is *admissible* if the closed sub-interval $[0, x]$ is covered by finitely many of the U_i . In this terminology, what we need to show is that 1 is admissible.

Observe from the definition that

1. 0 is admissible,
2. if $y < x \in [0, 1]$ and x is admissible, then also y is admissible.

This means that the set of admissible x forms either an [open interval](#) $[0, g)$ or a [closed interval](#) $[0, g]$, for some $g \in [0, 1]$. We need to show that the latter is true, and for $g = 1$. We do so by observing that the alternatives lead to contradictions:

1. Assume that the set of admissible values were an open interval $[0, g)$. By assumption there would be a finite subset $J \subset I$ such that $\{U_i \subset [0, 1]\}_{i \in J \subset I}$ were a finite open cover of $[0, g)$. Accordingly, since there is some $i_g \in I$ such that $g \in U_{i_g}$, the union $\{U_i\}_{i \in J} \sqcup \{U_{i_g}\}$ were a finite cover of the closed interval $[0, g]$, contradicting the assumption that g itself is not admissible (since it is not contained in $[0, g)$).
2. Assume that the set of admissible values were a closed interval $[0, g]$ for $g < 1$. By assumption there would then be a finite set $J \subset I$ such that $\{U_i \subset [0, 1]\}_{i \in J \subset I}$ were a finite cover of $[0, g]$. Hence there would be an index $i_g \in J$ such that $g \in U_{i_g}$. But then by the nature of open subsets in the Euclidean space \mathbb{R} , this U_{i_g} would also contain an open ball $B_g^\circ(\epsilon) = (g - \epsilon, g + \epsilon)$. This would mean that the set of admissible values includes the open interval $[0, g + \epsilon)$, contradicting the assumption.

This gives a [proof by contradiction](#). ■

Proposition 6.8. (binary [Tychonoff theorem](#))

Let (X, τ_X) and (Y, τ_Y) be two [compact topological spaces](#) (def. 6.4). Then also their [product topological space](#) (def. 2.18) $(X \times Y, \tau_{X \times Y})$ is compact.

Proof. Let $\{U_i \subset X \times Y\}_{i \in I}$ be an [open cover](#) of the product space. We need to show that this has a finite subcover.

By definition of the product space topology, each U_i is the union, indexed by some set K_i , of [Cartesian products](#) of open subsets of X and Y :

$$U_i = \bigcup_{k_i \in K_i} (V_{k_i} \times W_{k_i}) \quad V_{k_i} \in \tau_X \quad \text{and} \quad W_{k_i} \in \tau_Y .$$

Consider then the [disjoint union](#) of all these index sets

$$K := \bigsqcup_{i \in I} K_i .$$

This is such that

$$(\star) \quad \{V_{k_i} \times W_{k_i} \subset X \times Y\}_{k_i \in K}$$

is again an open cover of $X \times Y$.

But by construction, each element $V_{k_i} \times W_{k_i}$ of this new cover is contained in at least one $U_{j(k_i)}$ of the original cover. Therefore it is now sufficient to show that there is a finite subcover of (\star) , consisting of elements indexed by $k_i \in K_{\text{fin}} \subset K$ for some [finite set](#) K_{fin} . Because then the corresponding $U_{j(k_i)}$ for $k_i \in K_{\text{fin}}$ form a finite subcover of the original cover.

In order to see that (\star) has a finite subcover, first fix a point $x \in X$ and write $\{x\} \subset X$ for the corresponding [singleton topological subspace](#). By [example 3.24](#) this is [homeomorphic](#) to the abstract [point](#) space $*$. By [example 3.26](#) there is thus a [homeomorphism](#) of the form

$$\{x\} \times Y \simeq Y .$$

Therefore, since (Y, τ_Y) is assumed to be [compact](#), the open cover

$$\{(V_{k_1} \times W_{k_1}) \cap (\{x\} \times Y)\}_{k_i \in K}$$

has a finite subcover, indexed by a [finite subset](#) $J_x \subset K$.

Here we may assume without restriction of generality that $x \in V_{k_i}$ for all $k_i \in J_x \subset K$, because if not then we may simply remove that index and still have a (finite) subcover.

By finiteness of J_x it now follows that the intersection

$$V_x := \bigcap_{k_i \in J_x} V_{k_i}$$

is still an open subset, and by the previous remark we may assume without restriction that

$$x \in V_x .$$

Now observe that by the nature of the above cover of $\{x\} \times Y$ we have

$$\{x\} \times Y \subset \bigcup_{k_i \in J_x \subset K} V_{k_i} \times W_{k_i}$$

and hence

$$\{x\} \times Y \subset \{x\} \times \bigcup_{k_i \in J_x \subset K} W_{k_i} .$$

Since by construction $V_x \subset V_{k_i}$ for all $k_i \in J_x \subset K$, it follows that we have found a finite cover not just of $\{x\} \times Y$ but of $V_x \times Y$

$$V_x \times Y \subset \bigcup_{k_i \in J_x \subset K} (V_{k_i} \times W_{k_i}) .$$

To conclude, observe that $\{V_x \subset X\}_{x \in X}$ is clearly an open cover of X , so that by the assumption that also X is compact there is a finite set of points $S \subset X$ so that $\{V_x \subset X\}_{x \in S \subset X}$ is still a cover. In summary then

$$\{V_{k_i} \times W_{k_i} \subset X \times Y\}_{\substack{x \in S \subset X \\ k_i \in J_x \subset K}}$$

is a finite subcover as required. ■

In terms of the topological incarnation of the definitions of compactness, the familiar statement about metric spaces from prop. [1.20](#) now equivalently says the following:

Proposition 6.9. ([sequentially compact metric spaces are equivalently compact metric spaces](#))

If (X, d) is a [metric space](#), regarded as a [topological space](#) via its [metric topology](#) (def. [2.9](#)), then the following are equivalent:

1. (X, d) is a [compact topological space](#) (def. [6.4](#)).
2. (X, d) is a [sequentially compact topological space](#) (def. [6.2](#)).

Proof. of prop. [1.20](#) and prop. [6.9](#)

Assume first that (X, d) is a [compact topological space](#). Let $(x_k)_{k \in \mathbb{N}}$ be a [sequence](#) in X . We need to show that it has a sub-sequence which [converges](#).

Consider the [topological closures](#) of the sub-sequences that omit the first n elements of the sequence

$$F_n := \text{Cl}(\{x_k \mid k \geq n\})$$

and write

$$U_n := X \setminus F_n$$

for their [open complements](#).

Assume now that the [intersection](#) of all the F_n were [empty](#)

$$(*) \quad \bigcap_{n \in \mathbb{N}} F_n = \emptyset$$

or equivalently that the [union](#) of all the U_n were all of X

$$\bigcup_{n \in \mathbb{N}} U_n = X,$$

hence that $\{U_n \rightarrow X\}_{n \in \mathbb{N}}$ were an [open cover](#). By the assumption that X is compact, this would imply that there is a [finite subset](#) $\{i_1 < i_2 < \dots < i_k\} \subset \mathbb{N}$ with

$$\begin{aligned} X &= U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_k} \\ &= U_{i_k} \end{aligned}$$

This in turn would mean that $F_{i_k} = \emptyset$, which contradicts the construction of F_{i_k} . Hence we have a [proof by contradiction](#) that assumption $(*)$ is wrong, and hence that there must exist an element

$$x \in \bigcap_{n \in \mathbb{N}} F_n.$$

By definition of [topological closure](#) this means that for all n the [open ball](#) $B_x^\circ(1/(n+1))$ around x of [radius](#) $1/(n+1)$ must intersect the n th of the above subsequence:

$$B_x^\circ(1/(n+1)) \cap \{x_k \mid k \geq n\} \neq \emptyset.$$

Picking one point (x'_n) in the n th such intersection for all n hence defines a sub-sequence, which converges to x .

This proves that compact implies sequentially compact for metric spaces.

For the converse, assume now that (X, d) is sequentially compact. Let $\{U_i \rightarrow X\}_{i \in I}$ be an [open cover](#) of X . We need to show that there exists a finite sub-cover.

Now by the [Lebesgue number lemma](#), there exists a positive real number $\delta > 0$ such that for each $x \in X$ there is $i_x \in I$ such that $B_x^\circ(\delta) \subset U_{i_x}$. Moreover, since [sequentially compact metric spaces are totally bounded](#), there exists then a [finite set](#) $S \subset X$ such that

$$X = \bigcup_{s \in S} B_s^\circ(\delta).$$

Therefore $\{U_{i_s} \rightarrow X\}_{s \in S}$ is a finite sub-cover as required. ■

Remark 6.10. (neither compactness nor sequential compactness implies the other)

Beware that, in contrast to prop. 6.9, for general topological spaces being [sequentially compact](#) neither implies nor is implied by being [compact](#). The corresponding counter-examples are maybe beyond the scope of this note, but see here:

- an example of a compact topological space which is not sequentially compact is given in ([Steen-Seebach 70, item 105](#)), see at [compact space – Compact spaces which are not sequentially compact](#);
- an example of a sequentially compact space which is not compact is discussed in ([Patty 08, chapter 4, example 13](#)).

In [analysis](#), the [extreme value theorem](#) asserts that a [real-valued continuous function](#) on the [bounded closed interval](#) (def. 1.12) attains its [maximum](#) and [minimum](#). The following is the generalization of this statement to general topological spaces:

Lemma 6.11. ([continuous surjections out of compact spaces have compact codomain](#))

Let $f: (X, \tau_X) \rightarrow (Y, \tau_Y)$ be a [continuous function](#) between [topological spaces](#) such that

1. (X, τ_X) is a [compact topological space](#);
2. $f: X \rightarrow Y$ is a [surjective function](#).

Then also (Y, τ_Y) is [compact](#).

Proof. Let $\{U_i \subset Y\}_{i \in I}$ be an [open cover](#) of Y . We need show that this has a finite sub-cover.

By the continuity of f the [pre-images](#) $f^{-1}(U_i)$ are [open subsets](#) of X , and by the surjectivity of f they form an [open cover](#) $\{f^{-1}(U_i) \subset X\}_{i \in I}$ of X . Hence by compactness of X , there exists a [finite subset](#) $J \subset I$ such that $\{f^{-1}(U_i) \subset X\}_{i \in J \subset I}$ is still an open cover of X . Finally, using again that f is assumed to be surjective, it follows that

$$\begin{aligned} Y &= f(X) \\ &= f\left(\bigcup_{i \in J} f^{-1}(U_i)\right) \\ &= \bigcup_{i \in J} U_i \end{aligned}$$

which means that also $\{U_i \subset Y\}_{i \in J \subset I}$ is still an open cover of Y , and in particular a finite subcover of the original cover. ■

Corollary 6.12. ([continuous images of compact spaces are compact](#))

If $f: X \rightarrow Y$ is a continuous function out of a compact topological space X which is not necessarily surjective, then we may consider its image factorization

$$f : X \rightarrow f(X) \hookrightarrow Y$$

as in example 3.10. Now by construction $X \rightarrow f(X)$ is surjective, and so lemma 6.11 implies that $f(X)$ is compact.

The converse to cor. 6.12 does not hold in general: the pre-image of a compact subset under a continuous function need not be compact again. If this is the case, then we speak of proper maps:

Definition 6.13. (proper maps)

A continuous function $f: (X, \tau_X) \rightarrow (Y, \tau_Y)$ is called proper if for $C \in Y$ a compact topological subspace of Y , then also its pre-image $f^{-1}(C)$ is compact in X .

Definition 6.14. (mapping space)

For X a topological space and Y a locally compact topological space (in that for every point, every neighbourhood contains a compact neighbourhood), the mapping space

$$X^Y \in \text{Top}$$

is the topological space

- whose underlying set is the set $\text{Hom}_{\text{Top}}(Y, X)$ of continuous functions $Y \rightarrow X$,
- whose open subsets are unions of finitary intersections of the following subbase elements of standard open subsets:

the standard open subset $U^K \subset \text{Hom}_{\text{Top}}(Y, X)$ for

- $K \hookrightarrow Y$ a compact topological space subset
- $U \hookrightarrow X$ an open subset

is the subset of all those continuous functions f that fit into a commuting diagram of the form

$$\begin{array}{ccc} K & \hookrightarrow & Y \\ \downarrow & & \downarrow f \\ U & \hookrightarrow & X \end{array}$$

Accordingly this is called the compact-open topology on the set of functions.

The construction extends to a functor

$$(-)^{(-)} : \text{Top}_{\text{lc}}^{\text{op}} \times \text{Top} \rightarrow \text{Top} .$$

Relation to Hausdorff spaces

We discuss some important relations between the concepts of compact spaces and of [Hausdorff topological spaces](#).

In [analysis](#) the key recognition principle for compact spaces is the following:

Proposition 6.15. ([Heine-Borel theorem](#))

For $n \in \mathbb{N}$, regard \mathbb{R}^n as the n -dimensional [Euclidean space](#) via [example 1.6](#), regarded as a [topological space](#) via its [metric topology](#) ([def. 2.9](#)).

Then for a [topological subspace](#) $S \subset \mathbb{R}^n$ the following are equivalent:

1. S is [compact](#) ([def. 6.4](#));
2. S is [closed](#) ([def. 2.20](#)) and [bounded](#) ([def. 1.3](#)).

We **prove** this [below](#) as a consequence of the following more general statement for [topological space](#):

Proposition 6.16. ([closed subspaces of compact Hausdorff spaces are equivalently compact subspaces](#))

Let (X, τ) be a [compact Hausdorff topological space](#) ([def. 4.1](#), [def. 6.4](#)) and let $Y \subset X$ be a [topological subspace](#). Then the following are equivalent:

1. $Y \subset X$ is a [closed subspace](#) ([def. 2.20](#));
2. Y is a [compact topological space](#).

Proof. By [lemma 6.17](#) and [lemma 6.19](#) below. ■

Lemma 6.17. ([closed subspaces of compact spaces are compact](#))

Let (X, τ) be a [compact topological space](#) ([def. 6.4](#)), and let $Y \subset X$ be a [closed topological subspace](#). Then also Y is [compact](#).

Proof. Let $\{V_i \subset Y\}_{i \in I}$ be an [open cover](#) of Y . We need to show that this has a finite sub-cover.

By definition of the [subspace topology](#), there exist open subsets U_i of X with

$$V_i = U_i \cap Y.$$

By the assumption that Y is closed, the [complement](#) $X \setminus Y$ is an open subset of X , and therefore

$$\{X \setminus Y \subset X\} \cup \{U_i \subset X\}_{i \in I}$$

is an [open cover](#) of X . Now by the assumption that X is compact, this latter cover has a finite subcover, hence there exists a [finite subset](#) $J \subset I$ such that

$$\{X \setminus Y \subset X\} \cup \{U_i \subset X\}_{i \in J \subset I}$$

is still an open cover of X , hence in particular intersects to a finite open cover of Y . But since $Y \cap (X \setminus Y) = \emptyset$, it follows that indeed

$$\{V_i \subset Y\}_{i \in J \subset I}$$

is a cover of Y , and in indeed a finite subcover of the original one. ■

Lemma 6.18. (separation by neighbourhoods of points from compact subspaces in Hausdorff spaces)

Let

1. (X, τ) be a [Hausdorff topological space](#);
2. $Y \subset X$ a [compact subspace](#).

Then for every $x \in X \setminus Y$ there exists

1. an [open neighbourhood](#) $U_x \supset \{x\}$;
2. an [open neighbourhood](#) $U_Y \supset Y$

such that

- they are still disjoint: $U_x \cap U_Y = \emptyset$.

Proof. By the assumption that (X, τ) is Hausdorff, we find for every point $y \in Y$ disjoint open neighbourhoods $U_{x,y} \supset \{x\}$ and $U_y \supset \{y\}$. By the nature of the [subspace topology](#) of Y , the restriction of all the U_y to Y is an [open cover](#) of Y :

$$\{(U_y \cap Y) \subset Y\}_{y \in Y}.$$

Now by the assumption that Y is compact, there exists a finite subcover, hence a [finite set](#) $S \subset Y$ such that

$$\{(U_y \cap Y) \subset Y\}_{y \in S \subset Y}$$

is still a cover.

But the finite intersection

$$U_x := \bigcap_{s \in S \subset Y} U_{x,s}$$

of the corresponding open neighbourhoods of x is still open, and by construction it is disjoint from all the U_s , hence in particular from their union

$$U_Y := \bigcup_{s \in S \subset Y} U_s.$$

Therefore U_x and U_Y are two open subsets as required. ■

Lemma [6.18](#) immediately implies the following:

Lemma 6.19. ([compact subspaces of Hausdorff spaces are closed](#))

Let (X, τ) be a [Hausdorff topological space](#) (def. [4.1](#)) and let $C \subset X$ be a [compact](#) (def. [6.4](#)) [topological subspace](#) (example [2.16](#)). Then $C \subset X$ is also a [closed subspace](#) (def. [2.20](#)).

Proof. Let $x \in X \setminus C$ be any point of X not contained in C . We need to show that there exists an [open neighbourhood](#) of x in X which does not [intersect](#) C . This is implied by lemma [6.18](#). ■

Now we may give the proof of the [Heine-Borel theorem](#):

Proof. of the [Heine-Borel theorem](#) (prop. [6.15](#))

First consider a [subset](#) $S \subset \mathbb{R}^n$ which is closed and bounded. We need to show that regarded as a [topological subspace](#) it is [compact](#).

The assumption that S is bounded by (hence contained in) some [open ball](#) $B_x^\circ(\epsilon)$ in \mathbb{R}^n implies that it is contained in $\{(x_i)_{i=1}^n \in \mathbb{R}^n \mid -\epsilon \leq x_i \leq \epsilon\}$. By example [3.27](#), this topological subspace is homeomorphic to the n -cube $[-\epsilon, \epsilon]^n$. Since the closed interval $[-\epsilon, \epsilon]$ is compact by example [6.7](#), the binary [Tychonoff theorem](#) (prop. [6.8](#)) implies that this n -cube is compact. Since [closed subspaces of compact spaces are compact](#) (lemma [6.17](#)) this implies that S is compact.

Conversely, assume that $S \subset \mathbb{R}^n$ is a compact subspace. We need to show that it is closed and bounded.

The first statement follows since the [Euclidean space](#) \mathbb{R}^n is [Hausdorff](#) (example [4.2](#)) and since [compact subspaces of Hausdorff spaces are closed](#) (prop. [6.19](#)).

Hence what remains is to show that S is bounded.

To that end, choose any [positive real number](#) $\epsilon \in \mathbb{R}_{>0}$ and consider the [open cover](#) of all of \mathbb{R}^n by the open [n-cubes](#)

$$(k_1 - \epsilon, k_1 + 1 + \epsilon) \times (k_2 - \epsilon, k_2 + 1 + \epsilon) \times \cdots \times (k_n - \epsilon, k_n + 1 + \epsilon)$$

for [n-tuples of integers](#) $(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$. The restrictions of these to S hence form an open cover of the subspace S . By the assumption that S is compact, there is then a finite subset of n -tuples of integers such that the corresponding n -cubes still cover S . But the union of any finite number of bounded closed n -cubes in \mathbb{R}^n is clearly a bounded subset, and hence so is S . ■

Proposition 6.20. ([maps from compact spaces to Hausdorff spaces are closed and proper](#))

Let $f: (X, \tau_X) \rightarrow (Y, \tau_Y)$ be a [continuous function](#) between [topological spaces](#) such that

1. (X, τ_X) is a compact topological space;
2. (Y, τ_Y) is a Hausdorff topological space.

Then f is

1. a closed map (def. 3.14);
2. a proper map (def. 6.13).

Proof. For the first statement, we need to show that if $C \subset X$ is a closed subset of X , then also $f(C) \subset Y$ is a closed subset of Y .

Now

1. since closed subsets of compact spaces are compact (lemma 6.17) it follows that $C \subset C$ is also compact;
2. since continuous images of compact spaces are compact (cor. 6.12) it then follows that $f(C) \subset Y$ is compact;
3. since compact subspaces of Hausdorff spaces are closed (prop. 6.19) it finally follow that $f(C)$ is also closed in Y .

For the second statement we need to show that if $C \subset Y$ is a compact subset, then also its pre-image $f^{-1}(C)$ is compact.

Now

1. since compact subspaces of Hausdorff spaces are closed (prop. 6.19) it follows that $C \subset Y$ is closed;
2. since pre-images under continuous of closed subsets are closed (prop. 3.2), also $f^{-1}(C) \subset X$ is closed;
3. since closed subsets of compact spaces are compact (lemma 6.17), it follows that $f^{-1}(C)$ is compact.

■

Proposition 6.21. (continuous bijections from compact spaces to Hausdorff spaces are homeomorphisms)

Let $f: (X, \tau_X) \rightarrow (Y, \tau_Y)$ be a continuous function between topological spaces such that

1. (X, τ_X) is a compact topological space;
2. (Y, τ_Y) is a Hausdorff topological space.
3. $f : X \rightarrow Y$ is a bijection of sets.

Then f is a homeomorphism, i. e. its inverse function $Y \rightarrow X$ is also a continuous function.

In particular then both (X, τ_X) and (Y, τ_Y) are compact Hausdorff spaces.

Proof. Write $g: Y \rightarrow X$ for the inverse function of f .

We need to show that g is continuous, hence that for $U \subset X$ an open subset, then also its pre-image $g^{-1}(U) \subset Y$ is open in Y . By prop. 3.2 this is equivalent to the statement that for $C \subset X$ a closed subset then the pre-image $g^{-1}(C) \subset Y$ is also closed in Y .

But since g is the inverse function to f , its pre-images are the images of f . Hence the last statement above equivalently says that f sends closed subsets to closed subsets. This is true by prop. 6.20. ■

Proposition 6.22. (compact Hausdorff spaces are normal)

Every compact Hausdorff topological space is a normal topological space (def. 4.8).

Proof. First we claim that (X, τ) is regular. To show this, we need to find for each point $x \in X$ and each disjoint closed subset $Y \in X$ disjoint open neighbourhoods $U_x \supset \{x\}$ and $U_Y \supset Y$. But since closed subspaces of compact spaces are compact (lemma 6.17), the subset Y is in fact compact, and hence this is in fact the statement of lemma 6.18.

Next to show that (X, τ) is indeed normal, we apply the idea of the proof of lemma 6.18 once more:

Let $Y_1, Y_2 \subset X$ be two disjoint closed subspaces. By the previous statement then for every point $y_1 \in Y_1$ we find disjoint open neighbourhoods $U_{y_1} \subset \{y_1\}$ and $U_{Y_2, y_1} \supset Y_2$. The union of the U_{y_1} is a cover of Y_1 , and by compactness of Y_1 there is a finite subset $S \subset Y_1$ such that

$$U_{Y_1} := \bigcup_{s \in S \subset Y_1} U_{y_1}$$

is an open neighbourhood of Y_1 and

$$U_{Y_2} := \bigcap_{s \in S \subset Y_1} U_{Y_2, s}$$

is an open neighbourhood of Y_2 , and both are disjoint. ■

Relation to quotient spaces

Proposition 6.23. (continuous surjections from compact spaces to Hausdorff spaces are quotient projections)

Let

$$\pi : (X, \tau_X) \rightarrow (Y, \tau_Y)$$

be a [continuous function](#) between [topological spaces](#) such that

1. (X, τ_X) is a [compact topological space](#) (def. 6.4);
2. (Y, τ_Y) is a [Hausdorff topological space](#) (def. 4.1);
3. $\pi : X \rightarrow Y$ is a [surjective function](#).

Then τ_X is the [quotient topology](#) inherited from τ_X via the surjection f (def. 2.17).

Proof. We need to show that an subset $U \subset Y$ is an [open subset](#) (Y, τ_Y) precisely if its [pre-image](#) $\pi^{-1}(U) \subset X$ is an open subset in (X, τ_X) . Equivalently, as in prop. 3.2, we need to show that U is a [closed subset](#) precisely if $\pi^{-1}(U)$ is a closed subset. The implication

$$(U \text{ closed}) \Rightarrow (f^{-1}(U) \text{ closed})$$

follows via prop. 3.2 from the continuity of π . The implication

$$(f^{-1}(U) \text{ closed}) \Rightarrow (U \text{ closed})$$

follows since π is a [closed map](#) by prop. 6.20. ■

The following proposition allows to recognize when a [quotient space](#) of a compact Hausdorff space is itself still Hausdorff.

Proposition 6.24. (quotient projections out of compact Hausdorff spaces are closed precisely if the codomain is Hausdorff)

Let

$$\pi : (X, \tau_X) \rightarrow (Y, \tau_Y)$$

be a [continuous function](#) between [topological spaces](#) such that

1. (X, τ) is a [compact Hausdorff topological space](#) (def. 6.4, def. 4.1);
2. π is a [surjection](#) and τ_Y is the corresponding [quotient topology](#) (def. 2.17).

Then the following are equivalent

1. (Y, τ_Y) is itself a [Hausdorff topological space](#) (def. 4.1);
2. π is a [closed map](#) (def. 3.14).

Proof. The implicaton $((Y, \tau_Y) \text{ Hausdorff}) \Rightarrow (\pi \text{ closed})$ is given by prop. 6.20. We need to show the converse.

Hence assume that π is a closed map. We need to show that for every pair of

distinct point $y_1 \neq y_2 \in Y$) there exist [open neighbourhoods](#) $U_{y_1}, U_{y_2} \in \tau_Y$ which are disjoint, $U_{y_1} \cap U_{y_2} = \text{empty}$.

Therefore consider the [pre-images](#)

$$C_1 := \pi^{-1}(\{y_1\}) \quad C_2 := \pi^{-1}(\{y_2\}) .$$

Observe that these are [closed subsets](#), because in the Hausdorff space (Y, τ_Y) the singleton subsets $\{y_i\}$ are closed by prop. [4.6](#), and since pre-images under continuous functions preserves closed subsets by prop. [3.2](#).

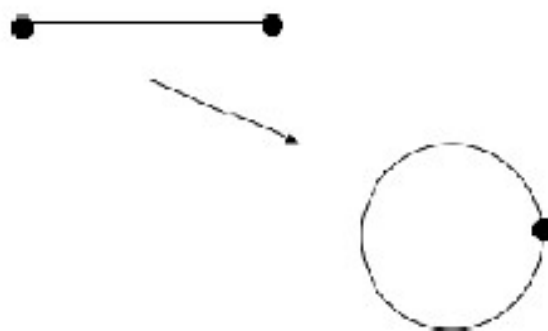
Now since [compact Hausdorff spaces are normal](#) it follows (by def. [4.8](#)) that we may find disjoint open subset $U_1, U_2 \in \tau_X$ such that

$$C_1 \subset U_1 \quad C_2 \subset U_2 .$$

Moreover, by lemma [3.20](#) we may find these U_i such that they are both [saturated subsets](#) (def. [3.16](#)). Therefore finally lemma [3.20](#) says that the images $\pi(U_i)$ are open in (Y, τ_Y) . These are now clearly disjoint open neighbourhoods of y_1 and y_2 . ■

Example 6.25. Consider the function

$$\begin{aligned} [0, 2\pi] / \sim &\rightarrow S^1 \subset \mathbb{R}^2 \\ t &\mapsto (\cos(t), \sin(t)) \end{aligned}$$



- from the [quotient topological space](#) (def. [2.17](#)) of the [closed interval](#) (def. [1.12](#)) by the [equivalence relation](#) which identifies the two endpoints

$$(x \sim y) \Leftrightarrow ((x = y) \text{ or } ((x \in \{0, 2\pi\} \text{ and } (y \in \{0, 2\pi\})))$$

- to the unit [circle](#) $S^1 = S_0(1) \subset \mathbb{R}^2$ (def. [1.2](#)) regarded as a [topological subspace](#) of the 2-dimensional [Euclidean space](#) (def. [1.6](#)) equipped with its [metric topology](#) (def. [2.9](#)).

This is clearly a [continuous function](#) and a [bijection](#) on the underlying sets. Moreover, since [continuous images of compact spaces are compact](#) (cor. [6.12](#)) and since the closed interval $[0, 1]$ is compact (example [6.7](#)) we also obtain another proof that the [circle](#) is compact.

Hence by prop. [6.21](#) the above map is in fact a [homeomorphism](#)

$$[0, 2\pi] / \sim \simeq S^1 .$$

Compare this to the counter-example [3.23](#), which observed that the analogous function

$$\begin{aligned} [0, 2\pi) &\rightarrow S^1 \subset \mathbb{R}^2 \\ t &\mapsto (\cos(t), \sin(t)) \end{aligned}$$

is *not* a homeomorphism, even though this, too, is a bijection on the the underlying sets. But the [half-open interval](#) $[0, 2\pi)$ is not compact, and hence [prop. 6.21](#) does not apply.

7. Universal constructions

One point of the general definition of [topological space above](#) is that it admits constructions which intuitively should exist on “continuous spaces”, but which do not in general exist on [metric spaces](#).

Examples include the construction of [quotient topological spaces](#) of metric spaces, which are not [Hausdorff](#) anymore (e.g. [example 4.10](#)), and hence in particular are not metric spaces anymore (by [example 4.2](#)).

Now from a more abstract point of view, a [quotient topological space](#) is a special case of a “[colimit](#)” of topological spaces. This we explain now.

Generally, for every [diagram](#) in the [category Top](#) of topological space ([remark \ref{TopCat}](#)), hence for every collection of topological spaces with a system of [continuous functions](#) between them, then there exists a further topological space, called the [colimiting space](#) of the diagram, which may be thought of as the result of “gluing” all the spaces in the diagram together, while using the maps between them in order to identify those parts “along which” the spaces are to be glued.

One may formalize this intuition by saying that the colimiting space has the property that it receives compatible continuous functions from all the spaces in the diagram, and that it is characterized by the fact that it is [universal with this property](#): every compatible system of maps to another space uniquely factors through the colimiting one.

Therefore forming colimits of topological spaces is a convenient means to construct new spaces which have prescribed properties for continuous functions out of them. We implicitly used a simple special case of this phenomenon in the proof of the [Hausdorff reflection](#) in [prop. 4.12](#), when we concluded the existence of certain unique factorizing maps out of the Hausdorff quotient of a topological space.

[Dual](#) to the concept of [colimits](#) of topological space is that of “[limits](#)” of [diagrams](#) of topological spaces (not to be confused with [limits of sequences](#) in a topological space). Here one considers topological spaces with the [universal property](#) of having compatible continuous functions into a given [diagram](#) of spaces.

Most constructions of new topological spaces that one builds from given spaces are obtained by forming limits and/or colimits of diagrams of the original spaces.

Definition 7.1. Let $\{X_i = (S_i, \tau_i) \in \text{Top}\}_{i \in I}$ be a [class](#) of [topological spaces](#), and let $S \in \text{Set}$ be a bare [set](#). Then

- For $\{S \xrightarrow{f_i} S_i\}_{i \in I}$ a set of **functions** out of S , the **initial topology** $\tau_{\text{initial}}(\{f_i\}_{i \in I})$ is the topology on S with the **minimum** collection of **open subsets** such that all $f_i: (S, \tau_{\text{initial}}(\{f_i\}_{i \in I})) \rightarrow S_i$ are **continuous**.
- For $\{S_i \xrightarrow{f_i} S\}_{i \in I}$ a set of **functions** into S , the **final topology** $\tau_{\text{final}}(\{f_i\}_{i \in I})$ is the topology on S with the **maximum** collection of **open subsets** such that all $f_i: S_i \rightarrow (S, \tau_{\text{final}}(\{f_i\}_{i \in I}))$ are **continuous**.

Example 7.2. For X a single topological space, and $\iota_S: S \hookrightarrow U(X)$ a subset of its underlying set, then the initial topology $\tau_{\text{initial}}(\iota_S)$, def. 7.1, is the **subspace topology**, making

$$\iota_S : (S, \tau_{\text{initial}}(\iota_S)) \hookrightarrow X$$

a **topological subspace** inclusion.

Example 7.3. Conversely, for $p_S: U(X) \rightarrow S$ an **epimorphism**, then the final topology $\tau_{\text{final}}(p_S)$ on S is the **quotient topology**.

Proposition 7.4. Let I be a **small category** and let $X_\bullet: I \rightarrow \text{Top}$ be an **I -diagram** in **Top** (a **functor** from I to **Top**), with components denoted $X_i = (S_i, \tau_i)$, where $S_i \in \text{Set}$ and τ_i a topology on S_i . Then:

1. The **limit** of X_\bullet exists and is given by **the** topological space whose underlying set is **the** limit in **Set** of the underlying sets in the diagram, and whose topology is the **initial topology**, def. 7.1, for the functions p_i which are the limiting **cone** components:

$$\begin{array}{ccc} & \lim_{\longleftarrow i \in I} S_i & \\ p_i \swarrow & & \searrow p_j \\ S_i & \longrightarrow & S_j \end{array}$$

Hence

$$\lim_{\longleftarrow i \in I} X_i \simeq \left(\lim_{\longleftarrow i \in I} S_i, \tau_{\text{initial}}(\{p_i\}_{i \in I}) \right)$$

2. The **colimit** of X_\bullet exists and is the topological space whose underlying set is the colimit in **Set** of the underlying diagram of sets, and whose topology is the **final topology**, def. 7.1 for the component maps ι_i of the colimiting **cocone**

$$\begin{array}{ccc} S_i & \longrightarrow & S_j \\ \iota_i \searrow & & \swarrow \iota_j \\ & \lim_{\longrightarrow i \in I} S_i & \end{array}$$

Hence

$$\lim_{\rightarrow i \in I} X_i \simeq \left(\lim_{\rightarrow i \in I} S_i, \tau_{\text{final}}(\{t_i\}_{i \in I}) \right)$$

(e.g. [Bourbaki 71](#), section I.4)

Proof. The required [universal property](#) of $\left(\lim_{\leftarrow i \in I} S_i, \tau_{\text{initial}}(\{p_i\}_{i \in I}) \right)$ is immediate: for

$$\begin{array}{ccc} & (S, \tau) & \\ f_i \swarrow & & \searrow f_j \\ X_i & \longrightarrow & X_i \end{array}$$

any [cone](#) over the diagram, then by construction there is a unique function of underlying sets $S \rightarrow \lim_{\leftarrow i \in I} S_i$ making the required diagrams commute, and so all that is required is that this unique function is always [continuous](#). But this is precisely what the initial topology ensures.

The case of the colimit is [formally dual](#). ■

Examples of (co-)limits of topological spaces

Example 7.5. The limit over the empty diagram in Top is the [point](#) $*$ with its unique topology.

Example 7.6. For $\{X_i\}_{i \in I}$ a set of topological spaces, their [coproduct](#) $\coprod_{i \in I} X_i \in \text{Top}$ is their [disjoint union](#) (example [2.15](#)).

Example 7.7. For $\{X_i\}_{i \in I}$ a set of topological spaces, their [product](#) $\prod_{i \in I} X_i \in \text{Top}$ is the [Cartesian product](#) of the underlying sets equipped with the [product topology](#), also called the [Tychonoff product](#).

In the case that S is a [finite set](#), such as for binary product spaces $X \times Y$, then a [sub-basis](#) for the product topology is given by the [Cartesian products](#) of the open subsets of (a basis for) each factor space.

Example 7.8. The [equalizer](#) of two [continuous functions](#) $f, g: X \rightrightarrows Y$ in Top is the equalizer of the underlying functions of sets

$$\text{eq}(f, g) \hookrightarrow S_X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} S_Y$$

(hence the targets subset of S_X on which both functions coincide) and equipped with the [subspace topology](#), example [7.2](#).

Example 7.9. The [coequalizer](#) of two [continuous functions](#) $f, g: X \rightrightarrows Y$ in Top is the coequalizer of the underlying functions of sets

$$S_X \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} S_Y \rightarrow \text{coeq}(f, g)$$

(hence the quotient set by the equivalence relation generated by $f(x) \sim g(x)$ for all $x \in X$) and equipped with the quotient topology, example 7.3.

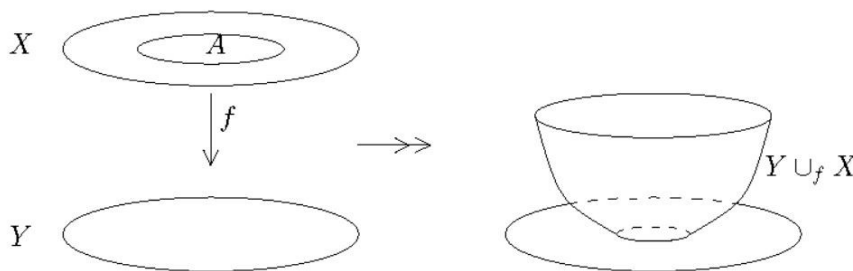
Example 7.10. For

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ f \downarrow & & \\ X & & \end{array}$$

two continuous functions out of the same domain, then the colimit under this diagram is also called the pushout, denoted

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ f \downarrow & & \downarrow g_* f \\ X & \rightarrow & X \sqcup_A Y \end{array}$$

(Here $g_* f$ is also called the pushout of f , or the cobase change of f along g .) If g is an inclusion, one also write $X \cup_f Y$ and calls this the attaching space.



By example 7.9 the pushout/attaching space is the quotient topological space

$$X \sqcup_A Y \simeq (X \sqcup Y) / \sim$$

of the disjoint union of X and Y subject to the

equivalence relation which identifies a point in X with a point in Y if they have the same pre-image in A .

(graphics from Aguilar-Gitler-Prieto 02)

Example 7.11. As an important special case of example 7.10, let

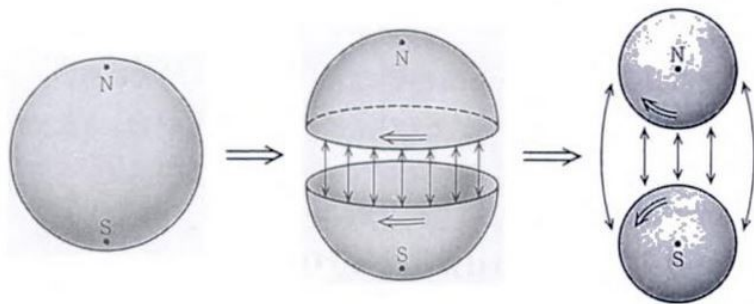
$$i_n : S^{n-1} \rightarrow D^n$$

be the canonical inclusion of the standard (n-1)-sphere as the boundary of the standard n-disk (both regarded as topological spaces with their subspace topology as subspaces of the Cartesian space \mathbb{R}^n).

Then the colimit in Top under the diagram, i.e. the pushout of i_n along itself,

$$\left\{ D^n \xleftarrow{i_n} S^{n-1} \xrightarrow{i_n} D^n \right\},$$

is the n-sphere S^n :



$$\begin{array}{ccc}
 S^{n-1} & \xrightarrow{i_n} & D^n \\
 i_n \downarrow & (\text{po}) & \downarrow \cdot \\
 D^n & \rightarrow & S^n
 \end{array}$$

(graphics from Ueno-Shiga-Morita 95)

(...)

This concludes *Section 1 [Point-set topology](#)*.

For the next section see *[Secton 2 -- Basic homotopy theory](#)*.

8. References

Introductory textbooks to topology include

- [James Munkres](#), *Topology*, Prentice Hall 1975 (2000)
- Terry Lawson, *Topology: A Geometric Approach*, Oxford University Press (2003) ([pdf](#))
- [Steven Vickers](#), *Topology via Logic*, Cambridge University Press (1989)

See also

- [Alan Hatcher](#), *Algebraic Topology*

and see also the references at [algebraic topology](#).

Lecture notes include

- [Friedhelm Waldhausen](#), *Topologie* ([pdf](#))
- Alex Kuronya, *Introduction to topology*, 2010 ([pdf](#))
- Anatole Katok, Alexey Sossinsky, *Introduction to modern topology and geometry* ([pdf](#))

Discussion of [sober topological spaces](#) is in

- [Peter Johnstone](#), section II 1. of *Stone Spaces*, Cambridge Studies in Advanced Mathematics **3**, Cambridge University Press 1982. xxi+370 pp. [MR85f:54002](#), reprinted 1986.

See also

- [Topospaces](#), a Wiki with basic material on topology.

Revised on April 20, 2017 03:36:42 by [Urs Schreiber](#)