nLab

Introduction to Topology -- 1

This page contains a detailed introduction to basic topology. Starting from scratch (required background is just a basic concept of *sets*), and amplifying motivation from analysis, it first develops standard point-set topology (topological spaces). In passing, some basics of category theory make an informal appearance, used to transparently summarize some conceptually important aspects of the theory, such as initial and final topologies and the reflection into Hausdorff and sober topological spaces. We close with discussion of the basics of topological manifolds and differentiable manifolds, laying the foundations for differential geometry. The second part introduces some basics of homotopy theory, mostly the fundamental group, and ends with their first application to the classification of covering spaces.

main page: *Introduction to Topology*

this chapter: *Introduction to Topology 1 – Point-set topology*

next chapter: *Introduction to Topology 2 -- Basic Homotopy Theory*

For introduction to more general and abstract *homotopy theory* see instead at *Introduction to Homotopy Theory*.

Point-set Topology

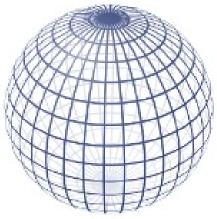
Context

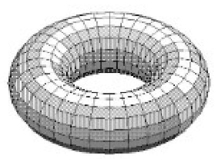
Topology

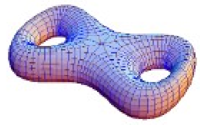
The idea of *topology* is to study "spaces" with "continuous functions" between them. Specifically one considers functions between sets (whence "point-set topology", see below) such that there is a concept for what it means that these functions depend continuously on their arguments, in that their values do not "jump". Such a concept of continuity is familiar from analysis on metric spaces, (recalled below) but the definition in topology generalizes this analytic concept and renders it more foundational, generalizing the concept of metric spaces to that of *topological spaces*. (def. 2.3 below).

Hence, topology is the study of the category whose objects are topological spaces, and whose morphisms are continuous functions (see also remark 3.3 below). This category is much more flexible than that of metric spaces, for example it admits the construction of arbitrary quotients and intersections of spaces. Accordingly, topology underlies or informs many and diverse areas of mathematics, such as functional analysis, operator algebra, manifold/scheme theory, hence algebraic geometry and differential geometry, and the study of topological groups, topological vector spaces, local rings, etc. Not the least, it gives rise to the field of homotopy theory, where one considers also continuous deformations of continuous functions themselves ("homotopies"). Topology itself has many branches, such as *low-dimensional topology* or *topological domain theory*.

A popular imagery for the concept of a continuous function is provided by deformations of elastic physical bodies, which may be deformed by stretching them without tearing. The canonical illustration is a continuous bijective function from the torus to the surface of a coffee mug, which maps half of the torus to the handle of the coffee mug, and continuously deforms parts of the other half in order to form the actual cup. Since the inverse function to this function is itself continuous, the torus and the coffee mug, both regarded as topological spaces, are "the same" for the purposes of topology; one says they are *homeomorphic*.







On the other hand, there is *no* homeomorphism from the torus to, for instance, the sphere, signifying that these represent two topologically distinct spaces. Part of topology is concerned with studying homeomorphism-invariants of topological spaces ("topological properties") which allow to detect by means of algebraic manipulations whether two topological spaces are homeomorphic (or more generally homotopy equivalent) or not. This is called *algebraic topology*. A basic algebraic invariant is the fundamental group of a topological space (discussed below), which measures how many ways there are to wind loops inside a topological space.

Beware the popular imagery of "rubber-sheet geometry", which only captures part of the full scope of topology, in that it invokes spaces that *locally* still look like metric spaces (called *topological manifolds*, see below). But the concept of topological spaces is a good bit more general. Notably, finite topological spaces are either discrete or very much unlike metric spaces (example 4.7 below); the former play a role in categorical logic. Also, in geometry, exotic topological spaces frequently arise when forming non-free quotients. In order to gauge just how many of such "exotic" examples of topological spaces beyond locally metric spaces one wishes to admit in the theory, extra "separation axioms" are imposed on topological spaces (see below), and the flavour of topology as a field depends on this choice.

Among the separation axioms, the *Hausdorff space* axiom is the most popular (see below). But the weaker axiom of sobriety (see below) stands out, because on the one hand it is the weakest axiom that is still naturally satisfied in applications to algebraic geometry (schemes are sober) and computer science (Vickers 89), and on the other, it fully realizes the strong roots that topology has in formal logic: sober topological spaces are entirely characterized by the union-, intersection- and inclusion-relations (logical conjunction, disjunction and implication) among their open subsets (propositions). This leads to a natural and fruitful generalization of topology to more general "purely logic-determined spaces", called *locales*, and in yet more generality, *toposes* and *higher toposes*. While the latter are beyond the scope of this introduction, their rich theory and relation to the foundations of mathematics and geometry provide an outlook on the relevance of the basic ideas of topology.

In this first part we discuss the foundations of the concept of "sets equipped with topology" (topological spaces) and of continuous functions between them.

(classical logic)

The proofs in the following freely use the principle of excluded middle, hence proof by contradiction, and in a few places they also use the axiom of choice/Zorn's lemma.

Hence we discuss topology in its traditional form with classical logic.

We do however highlight the role of frame homomorphisms (def. 2.34 below) and that of sober topological spaces (def. 5.1 below). These concepts pave the way to a constructive formulation of topology in terms not of topological spaces but in terms of *locales*, see remark 5.8 below. The reader interested in questions of intuitionistic mathematics in topology may benefit from looking at (Waaldijk 96).

1. Metric spaces

The concept of continuity was first made precise in analysis, in terms of epsilontic analysis on metric spaces, recalled as def. 1.8 below. Then it was realized that this has a more elegant formulation in terms of the more general concept of *open sets*, this is prop. 1.14 below. Adopting the latter as the definition leads to a more abstract concept of "continuous space", this is the concept of *topological spaces*, def. 2.3 below.

Here we briefly recall the relevant basic concepts from analysis, as a motivation for various definitions in topology. The reader who either already recalls these concepts in analysis or is content with ignoring the motivation coming from analysis should skip right away to the section *Topological spaces*.

Definition 1.1. (**metric space**)

A *metric space* is

1. a set X (the "underlying set");
2. a function $d : X \times X \to [0, \infty)$ (the "distance function") from the Cartesian product of the set with itself to the non-negative real numbers

such that for all $x, y, z \in X$:

1. (symmetry) $d(x, y) = d(y, x)$
2. (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.
3. (non-degeneracy) $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$

Definition 1.2. (**open balls**)

Let (X, d), be a metric space. Then for every element $x \in X$ and every $\epsilon \in \mathbb{R}_+$ a positive real number, we write

$$B_x^\circ(\epsilon) \;\coloneqq\; \{y \in X \mid d(x, y) < \epsilon\}$$

for the open ball of radius ϵ around x. Similarly we write

$$B_x(\epsilon) \;\coloneqq\; \{y \in X \;|\; d(x,y) \leq \epsilon\}$$

for the *closed ball* of radius ϵ around x. Finally we write

$$S_x(\epsilon) \;\coloneqq\; \{y \in X \;|\; d(x,y) = \epsilon\}$$

for the sphere of radius ϵ around x.

For $\epsilon = 1$ we also speak of the *unit open/closed ball* and the *unit sphere*.

Definition 1.3. For (X,d) a metric space (def. 1.1) then a subset $S \subset X$ is called a *bounded subset* if S is contained in some open ball (def. 1.2)

$$S \subset B^\circ_x(r)$$

around some $x \in X$ of some radius $r \in \mathbb{R}$.

A key source of metric spaces are normed *vector spaces*:

Dedfinition 1.4. **(normed vector space)**

A *normed vector space* is

1. a real vector space V;
2. a function (the *norm*)

$$\Vert - \Vert \;:\; V \longrightarrow \mathbb{R}_{\geq 0}$$

   from the underlying set of V to the non-negative real numbers,

such that for all $c \in \mathbb{R}$ with absolute value $|c|$ and all $v,w \in V$ it holds true that

1. (linearity) $\Vert c v\Vert = |c| \Vert v \Vert$;
2. (triangle inequality) $\Vert v + w\Vert \leq \Vert v \Vert + \Vert w \Vert$;
3. (non-degeneracy) if $\Vert v \Vert = 0$ then $v = 0$.

Proposition 1.5. *Every normed vector space $(V, \Vert - \Vert)$ becomes a metric space according to def. 1.1 by setting*

$$d(x,y) \coloneqq \Vert x - y \Vert \;.$$

Examples of normed vector spaces (def. 1.4) and hence, via prop. 1.5, of metric spaces include the following:

Example 1.6. For $n \in \mathbb{N}$, the Cartesian space

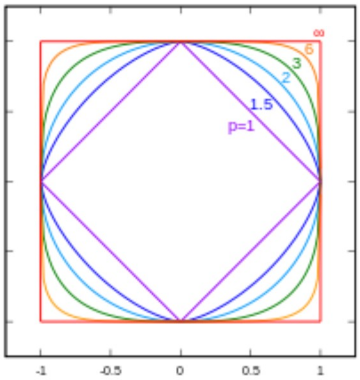
$$\mathbb{R}^n = \{ \vec x = (x_i)_{i = 1}^n \,|\, x_i \in \mathbb{R} \}$$

carries a norm (the *Euclidean norm*) given by the square root of the sum of the squares of the components:

$$\|\vec{x}\| \coloneqq \sqrt{\sum_{i=1}^{n} (x_i)^2}\,.$$

Via prop. 1.5 this gives $\mathbb{R}^n$ the structure of a metric space, and as such it is called the *Euclidean space* of dimension n.

Example 1.7. More generally, for $n \in \mathbb{N}$, and $p \in \mathbb{R}$, $p \geq 1$, then the Cartesian space $\mathbb{R}^n$ carries the *p-norm*



$$\|\vec{x}\|_p \coloneqq \sqrt[p]{\sum_i |x_i|^p}$$

One also sets

$$\|\vec{x}\|_\infty \coloneqq \max_{i \in I} |x_i|$$

and calls this the *supremum norm*.

The graphics on the right (grabbed from Wikipedia) shows unit circles (def. 1.2) in $\mathbb{R}^2$ with respect to various p-norms.

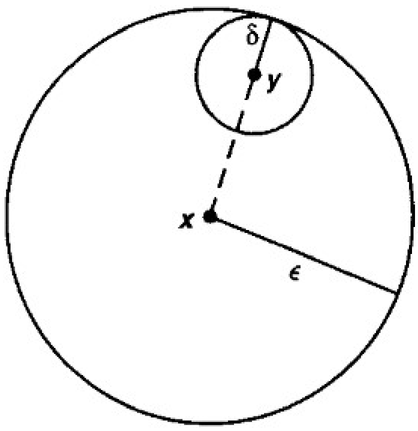
By the Minkowski inequality, the p-norm generalizes to non-finite dimensional vector spaces such as sequence spaces and Lebesgue spaces.

Continuity

The following is now the fairly obvious definition of continuity for functions between metric spaces.

Definition 1.8. **(epsilontic definition of continuity)**

For (X, d_X) and (Y, d_Y) two metric spaces (def. 1.1), then a function



$$f : X \longrightarrow Y$$

is said to be *continuous at a point* $x \in X$ if for every positive real number ϵ there exists a positive real number δ such that for all $x' \in X$ that are a distance smaller than δ from x then their image $f(x')$ is a distance smaller than ϵ from $f(x)$:

$$(f \text{ continuous at } x) \coloneqq \underset{\epsilon > 0}{\underset{\epsilon \in \mathbb{R}}{\forall}} \left(\underset{\delta > 0}{\underset{\delta \in \mathbb{R}}{\exists}} \left((d_X(x, x') < \delta) \;\; \Rightarrow \;\; (d_Y(f(x), f(x')) < \epsilon) \right) \right).$$

The function f is said to be *continuous* if it is continuous at every point $x \in X$.

Example 1.9. **(distance function from a subset is continuous)**

Let (X, d) be a metric space (def. 1.1) and let $S \subset X$ be a subset of the underlying set. Define then the function

$$d(S, -) : X \to \mathbb{R}$$

from the underlying set X to the real numbers by assigning to a point $x \in X$ the infimum of the distances from x to s, as s ranges over the elements of S:

$$d(S, x) \coloneqq \inf\{d(s, x) \mid s \in S\} .$$

This is a continuous function, with $\mathbb{R}$ regarded as a metric space via its Euclidean norm (example 1.6).

In particular the original distance function $d(x, -) = d(\{x\}, -)$ is continuous in both its arguments.

Proof. Let $x \in X$ and let ϵ be a positive real number. We need to find a positive real number δ such that for $y \in X$ with $d(x, y) < \delta$ then $|d(S, x) - d(S, y)| < \epsilon$.

For $s \in S$ and $y \in X$, consider the triangle inequalities

$$\begin{aligned} d(s, x) &\leq d(s, y) + d(y, x) \\ d(s, y) &\leq d(s, x) + d(x, y) \end{aligned} .$$

Forming the infimum over $s \in S$ of all terms appearing here yields

$$\begin{aligned} d(S, x) &\leq d(S, y) + d(y, x) \\ d(S, y) &\leq d(S, x) + d(x, y) \end{aligned}$$

which implies

$$|d(S, x) - d(S, y)| \leq d(x, y) .$$

This means that we may take for instance $\delta \coloneqq \epsilon$. ▮

Example 1.10. **(rational functions are continuous)**

Consider the real line $\mathbb{R}$ regarded as the 1-dimensional Euclidean space $\mathbb{R}$ from example 1.6.

For $P \in \mathbb{R}[X]$ a polynomial, then the function

$$\begin{array}{cccc} f_P & : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & & x & \mapsto & P(x) \end{array}$$

is a continuous function in the sense of def. 1.8. Hence polynomials are continuous functions.

Similarly rational functions are continuous on their domain of definition: for $P, Q \in \mathbb{R}[X]$ two polynomials, then $\frac{f_P}{f_Q} : \mathbb{R}\backslash\{x \vert f_Q(x) = 0\} \to \mathbb{R}$ is a continuous function.

Also for instance forming the square root is a continuous function $\sqrt(-) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

On the other hand, a step function is continuous everywhere except at the finite number of points at which it changes its value, see example 1.15 below.

We now reformulate the analytic concept of continuity from def. 1.8 in terms of the simple but important concept of *open sets*:

Definition 1.11. **(neighbourhood and open set)**

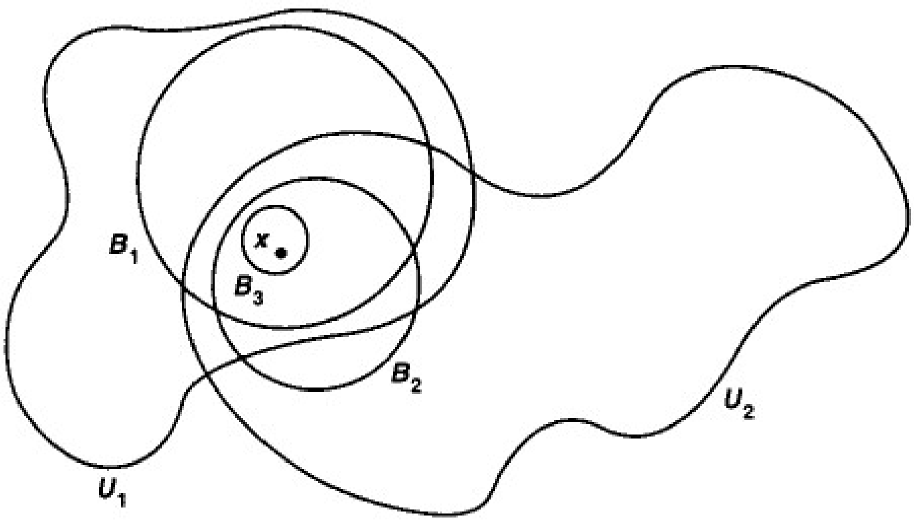
Let (X,d) be a metric space (def. 1.1). Say that:

1. A *neighbourhood* of a point $x \in X$ is a subset $U_x \subset X$ which contains some open ball $B_x^\circ(\epsilon) \subset U_x$ around x (def. 1.2).

2. An *open subset* of X is a subset $U \subset X$ such that for every $x \in U$ it also contains an open ball $B_x^\circ(\epsilon)$ around x (def. 1.2).

3. An *open neighbourhood* of a point $x \in X$ is a neighbourhood U_x of x which is also an open subset, hence equivalently this is any open subset of X that contains x.

The following picture shows a point x, some open balls B_i containing it, and two of its neighbourhoods U_i:



graphics grabbed from Munkres 75

Example 1.12. **(the empty subset is open)**

Notice that for (X, d) a metric space, then the empty subset $\emptyset \subset X$ is always an open subset of (X, d) according to def. 1.11. This is because the clause for open subsets $U \subset X$ says that "for every point $x \in U$ there exists…", but since there is no x in $U = \emptyset$, this clause is always satisfied in this case.

Conversely, the entire set X is always an open subset of (X, d).

Example 1.13. (open/closed intervals)

Regard the real numbers $\mathbb{R}$ as the 1-dimensional Euclidean space (example 1.6).

For $a < b \in \mathbb{R}$ consider the following subsets:

1. $(a, b) \coloneqq \{x \in \mathbb{R} \vert a < x < b\}$ (*open interval*)
2. $(a, b] \coloneqq \{x \in \mathbb{R} \vert a < x \leq b\}$ (*half-open interval*)
3. $[a, b) \coloneqq \{x \in \mathbb{R} \vert a \leq x < b\}$ (*half-open interval*)
4. $[a, b] \coloneqq \{x \in \mathbb{R} \vert a \leq x \leq b\}$ (*closed interval*)

The first of these is an open subset according to def. 1.11, the other three are not. The first one is called an *open interval*, the last one a *closed interval* and the middle two are called *half-open intervals*.

Similarly for $a, b \in \mathbb{R}$ one considers

1. $(-\infty, b) \coloneqq \{x \in \mathbb{R} \vert x < b\}$ (*unbounded open interval*)
2. $(a, \infty) \coloneqq \{x \in \mathbb{R} \vert a < x\}$ (*unbounded open interval*)
3. $(-\infty, b] \coloneqq \{x \in \mathbb{R} \vert x \leq b\}$ (*unbounded half-open interval*)
4. $[a, \infty) \coloneqq \{x \in \mathbb{R} \vert a \leq x\}$ (*unbounded half-open interval*)

The first two of these are open subsets, the last two are not.

For completeness we may also consider

- $(-\infty, \infty) = \mathbb{R}$
- $(a, a) = \emptyset$

which are both open, according to def. 2.3.

We may now rephrase the analytic definition of continuity entirely in terms of open subsets (def. 1.11):

Proposition 1.14. *(rephrasing continuity in terms of open sets)*

Let (X, d_X) and (Y, d_Y) be two metric spaces (def. 1.1). Then a function $f : X \to Y$ is continuous in the epsilontic sense of def. 1.8 precisely if it has the property that its pre-images of open subsets of Y (in the sense of def. 1.11) are open subsets *of X:*

$$(f \text{ continuous}) \;\; \Leftrightarrow \;\; \left((O_Y \subset Y \text{ open}) \Rightarrow \left(f^{-1}(O_Y) \subset X \text{ open} \right) \right) .$$

principle of continuity

Continuous pre-Images of open subsets are open.

Proof. Observe, by direct unwinding the definitions, that the epsilontic definition of continuity (def. 1.8) says equivalently in terms of open balls (def. 1.2) that f is continous at x precisely if for every open ball $B^\circ_{f(x)}(\epsilon)$ around an image point, there exists an open ball $B^\circ_x(\delta)$ around the corresponding pre-image point which maps into it:

$$\begin{aligned} (f \text{ continuous at } x) \qquad & \Leftrightarrow \underset{\epsilon > 0}{\forall} \left(\underset{\delta > 0}{\exists} \left(f(\, B^\circ_x(\delta)\,) \; \subset \; B^\circ_{f(x)}(\epsilon) \right) \right) \\ & \Leftrightarrow \underset{\epsilon > 0}{\forall} \left(\underset{\delta > 0}{\exists} \left(B^\circ_x(\delta) \subset f^{-1}\left(B^\circ_{f(x)}(\epsilon) \right) \right) \right) \end{aligned} .$$

With this observation the proof immediate. For the record, we spell it out:

First assume that f is continuous in the epsilontic sense. Then for $O_Y \subset Y$ any open subset and $x \in f^{-1}(O_Y)$ any point in the pre-image, we need to show that there exists an open neighbourhood of x in $f^{-1}(O_Y)$.

That O_Y is open in Y means by definition that there exists an open ball $B^\circ_{f(x)}(\epsilon)$ in O_Y around $f(x)$ for some radius ϵ. By the assumption that f is continuous and using the above observation, this implies that there exists an open ball $B^\circ_x(\delta)$ in X such that $f(B^\circ_x(\delta)) \subset B^\circ_{f(x)}(\epsilon) \subset Y$, hence such that $B^\circ_x(\delta) \subset f^{-1}\left(B^\circ_{f(x)}(\epsilon)\right) \subset f^{-1}(O_Y)$. Hence this is an open ball of the required kind.

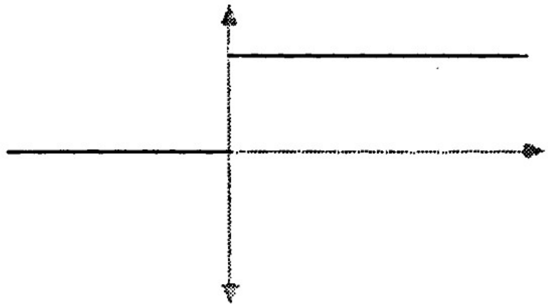
Conversely, assume that the pre-image function f^{-1} takes open subsets to open subsets. Then for every $x \in X$ and $B^\circ_{f(x)}(\epsilon) \subset Y$ an open ball around its image, we need to produce an open ball $B^\circ_x(\delta) \subset X$ around x such that $f(B^\circ_x(\delta)) \subset B^\circ_{f(x)}(\epsilon)$.

But by definition of open subsets, $B^\circ_{f(x)}(\epsilon) \subset Y$ is open, and therefore by assumption on f its pre-image $f^{-1}(B^\circ_{f(x)}(\epsilon)) \subset X$ is also an open subset of X. Again by definition of open subsets, this implies that it contains an open ball as required. ▮

Example 1.15. **(step function)**

Consider $\mathbb{R}$ as the 1-dimensional Euclidean space (example 1.6) and consider the step function

$$\begin{array}{ccc} \mathbb{R} & \overset{H}{\longrightarrow} & \mathbb{R} \\ x & \mapsto & \begin{cases} 0 & \vert\, x \leq 0 \\ 1 & \vert\, x \gt 0 \end{cases} \end{array} \,.$$



graphics grabbed from Vickers 89

Consider then for $a \lt b \in \mathbb{R}$ the open interval $(a,b) \subset \mathbb{R}$, an open subset according to example 1.13. The preimage $H^{-1}(a,b)$ of this open subset is

$$H^{-1} \;:\; (a,b) \mapsto \left\{ \begin{array}{ccc} \emptyset & \vert\, a \geq 1 \;\text{or}\; b \leq 0 \\ \mathbb{R} & \vert\, a \lt 0 \;\text{and}\; b \gt 1 \\ \emptyset & \vert\, a \geq 0 \;\text{and}\; b \leq 1 \\ (0,\infty) & \vert\, 0 \leq a \lt 1 \;\text{and}\; b \gt 1 \\ (-\infty, 0] & \vert\, a \lt 0 \;\text{and}\; b \leq 1 \end{array} \right. \,.$$

By example 1.13, all except the last of these pre-images listed are open subsets.

The failure of the last of the pre-images to be open witnesses that the step function is not continuous at $x = 0$.

Compactness

A key application of metric spaces in analysis is that they allow a formalization of what it means for an infinite sequence of elements in the metric space (def. 1.16 below) to *converge* to a *limit of a sequence* (def. 1.17 below). Of particular interest are therefore those metric spaces for which each sequence has a converging subsequence: the *sequentially compact metric spaces* (def. 1.20).

We now briefly recall these concepts from analysis. Then, in the above spirit, we reformulate their epsilontic definition in terms of open subsets. This gives a useful definition that generalizes to topological spaces, the *compact topological spaces* discussed further below.

Definition 1.16. (sequence)

Given a set X, then a *sequence* of elements in X is a function

$$x_{(-)} \;:\; \mathbb{N} \longrightarrow X$$

from the natural numbers to X.

A *sub-sequence* of such a sequence is a sequence of the form

$$x_{\iota(-)} \;:\; \mathbb{N} \overset{\iota}{\hookrightarrow} \mathbb{N} \overset{x_{(-)}}{\longrightarrow} X$$

for some injection ι.

Definition 1.17. (convergence to limit of a sequence)

Let (X, d) be a metric space (def. 1.1). Then a sequence

$$x_{(-)} : \mathbb{N} \longrightarrow X$$

in the underlying set X (def. 1.16) is said to *converge* to a point $x_\infty \in X$, denoted

$$x_i \overset{i \to \infty}{\longrightarrow} x_\infty$$

if for every positive real number ϵ, there exists a natural number n, such that all elements in the sequence after the nth one have distance less than ϵ from x_∞.

$$\left(x_i \overset{i \to \infty}{\longrightarrow} x_\infty\right) \Leftrightarrow \left(\underset{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}}{\forall}\left(\underset{n \in \mathbb{N}}{\exists}\left(\underset{\substack{i \in \mathbb{N} \\ i > n}}{\forall} d(x_i, x_\infty) \leq \epsilon\right)\right)\right).$$

Here the point x_∞ is called the *limit of the sequence*. Often one writes $\underset{i \to \infty}{\lim} x_i$ for this point.

Definition 1.18. (Cauchy sequence)

Given a metric space (X, d) (def. 1.1), then a sequence of points in X (def. 1.16)

$$x_{(-)} : \mathbb{N} \longrightarrow X$$

is called a *Cauchy sequence* if for every positive real number ϵ there exists a natural number $n \in \mathbb{N}$ such that the distance between any two elements of the sequence beyond the nth one is less than ϵ

$$\left(x_{(-)} \text{ Cauchy}\right) \Leftrightarrow \left(\underset{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}}{\forall}\left(\underset{N \in \mathbb{N}}{\exists}\left(\underset{\substack{i, j \in \mathbb{N} \\ i, j > N}}{\forall} d(x_i, x_j) \leq \epsilon\right)\right)\right).$$

Definition 1.19. (complete metric space)

A metric space (X, d) (def. 1.1), for which every Cauchy sequence (def. 1.18) converges (def. 1.17) is called a *complete metric space*.

A normed vector space, regarded as a metric space via prop. 1.5 that is complete in this sense is called a *Banach space*.

Finally recall the concept of *compactness* of metric spaces via epsilontic analysis:

Definition 1.20. (sequentially compact metric space)

A metric space (X, d) (def. 1.1) is called *sequentially compact* if every sequence in X has a subsequence (def. 1.16) which converges (def. 1.17).

The key fact to translate this epsilontic definition of compactness to a concept that makes sense for general topological spaces (below) is the following:

Proposition 1.21. ***(sequentially compact metric spaces are equivalently compact metric spaces)***

For a metric space (X, d) (def. 1.1) the following are equivalent:

1. *X is sequentially compact;*
2. *for every set $\{U_i \subset X\}_{i \in I}$ of open subsets U_i of X (def. 1.11) which cover X in that $X = \underset{i \in I}{\cup} U_i$, then there exists a finite subset $J \subset I$ of these open subsets which still covers X in that also $X = \underset{i \in J \subset I}{\cup} U_i$.*

The **proof** of prop. 1.21 is most conveniently formulated with some of the terminology of topology in hand, which we introduce now. Therefore we postpone the proof to below.

In **summary** prop. 1.14 and prop. 1.21 show that the purely combinatorial and in particular non-epsilontic concept of *open subsets* captures a substantial part of the nature of metric spaces in analysis. This motivates to reverse the logic and consider more general "spaces" which are *only* characterized by what counts as their open subsets. These are the *topological spaces* which we turn to now in def. 2.3 (or, more generally, these are the "locales", which we briefly consider below in remark 5.8).

2. Topological spaces

Due to prop. 1.14 we should pay attention to open subsets in metric spaces. It turns out that the following closure property, which follow directly from the definitions, is at the heart of the concept:

Proposition 2.1. ***(closure properties of open sets in a metric space)***

The collection of open subsets of a metric space (X, d) as in def. 1.11 has the following properties:

1. *The union of any set of open subsets is again an open subset.*
2. *The intersection of any finite number of open subsets is again an open subset.*

Remark 2.2. **(empty union and empty intersection)**

Notice the degenerate case of unions $\underset{i \in I}{\cup} U_i$ and intersections $\underset{i \in I}{\cap} U_i$ of subsets $U_i \subset X$ for the case that they are indexed by the empty set $I = \emptyset$:

1. the *empty union* is the empty set itself;

2. the *empty intersection* is all of X.

(The second of these may seem less obvious than the first. We discuss the general logic behind these kinds of phenomena below.)

This way prop. 2.1 is indeed compatible with the degenerate cases of examples of open subsets in example 1.12.

Proposition 2.1 motivates the following generalized definition, which abstracts away from the concept of metric space just its system of open subsets:

Definition 2.3. **(topological spaces)**

Given a set X, then a *topology* on X is a collection τ of subsets of X called the *open subsets*, hence a subset of the power set $P(X)$

$$\tau \subset P(X)$$

such that this is closed under forming

1. finite intersections;
2. arbitrary unions.

In particular (by remark 2.2):

- the empty set $\emptyset \subset X$ is in τ (being the union of no subsets)

and

- the whole set $X \subset X$ itself is in τ (being the intersection of no subsets).

A set X equipped with such a topology is called a *topological space*.

Remark 2.4. In the field of topology it is common to eventually simply say "space" as shorthand for "topological space". This is especially so as further qualifiers are added, such as "Hausdorff space" (def. 4.4 below). But beware that there are other kinds of spaces in mathematics.

Remark 2.5. The simple definition of open subsets in def. 2.3 and the simple implementation of the *principle of continuity* below in def. 3.1 gives the field of topology its fundamental and universal flavor. The combinatorial nature of these definitions makes topology be closely related to formal logic. This becomes more manifest still for the "sober topological space" discussed below. For more on this perspective see the remark on *locales* below, remark 5.8. An introductory textbook amplifying this perspective is (Vickers 89).

Before we look at first examples below, here is some common **further terminology** regarding topological spaces:

There is an evident partial ordering on the set of topologies that a given set may carry:

Definition 2.6. (**finer/coarser topologies**)

Let X be a set, and let $\tau_1, \tau_2 \in P(X)$ be two topologies on X, hence two choices of open subsets for X, making it a topological space. If

$$\tau_1 \subset \tau_2$$

hence if every open subset of X with respect to τ_1 is also regarded as open by τ_2, then one says that

- the topology τ_2 is *finer* than the topology τ_2
- the topology τ_1 is *coarser* than the topology τ_1.

With any kind of structure on sets, it is of interest how to "generate" such structures from a small amount of data:

Definition 2.7. (**basis for the topology**)

Let (X, τ) be a topological space, def. 2.3, and let

$$\beta \subset \tau$$

be a subset of its set of open subsets. We say that

1. β is a *basis for the topology* τ if every open subset $O \in \tau$ is a union of elements of β;
2. β is a *sub-basis for the topology* if every open subset $O \in \tau$ is a union of finite intersections of elements of β.

Often it is convenient to *define* topologies by defining some (sub-)basis as in def. 2.7. Examples are the the metric topology below, example 2.9, the binary product topology in def. 2.18 below, and the compact-open topology on mapping spaces below in def. 7.28. To make use of this, we need to recognize sets of open subsets that serve as the basis for some topology:

Lemma 2.8. ***(recognition of topological bases)***

Let X be a set.

1. *A collection $\beta \subset P(X)$ of subsets of X is a basis for some topology $\tau \subset P(X)$ (def. 2.7) precisely if*
   1. *every point of X is contained in at least one element of β;*
   2. *for every two subsets $B_1, B_2 \in \beta$ and for every point $x \in B_1 \cap B_2$ in their intersection, then there exists a $B \in \beta$ that contains x and is contained in the intersection: $x \in B \subset B_1 \cap B_2$.*
2. *A subset $B \subset \tau$ of open subsets is a sub-basis for a topology τ on X precisely if τ is the coarsest topology (def. 2.6) which contains B.*

Examples

We discuss here some basic examples of topological spaces (def. 2.3), to get a feeling for the scope of the concept. But topological spaces are ubiquituous in mathematics, so that there are many more examples and many more classes of examples than could be listed. As we further develop the theory below, we encounter more examples, and more classes of examples. Below in *Universal constructions* we discuss a very general construction principle of new topological space from given ones.

First of all, our motivating example from above now reads as follows:

Example 2.9. (**metric topology**)

Let (X, d) be a metric space (def. 1.1). Then the collection of its open subsets in def. 1.11 constitutes a *topology* on the set X, making it a *topological space* in the sense of def. 2.3. This is called the *metric topology*.

The open balls in a metric space constitute a basis of a topology (def. 2.7) for the metric topology.

While the example of metric space topologies (example 2.9) is the motivating example for the concept of topological spaces, it is important to notice that the concept of topological spaces is considerably more general, as some of the following examples show.

The following simplistic example of a (metric) topological space is important for the theory (for instance in prop. 2.37):

Example 2.10. (**empty space** and **point space**)

On the empty set there exists a unique topology τ making it a topological space according to def. 2.3. We write also

$$\emptyset \;\coloneqq\; \left(\emptyset, \tau_\emptyset = \{\emptyset\}\right)$$

for the resulting topological space, which we call the *empty topological space*.

On a singleton set $\{1\}$ there exists a unique topology τ making it a topological space according to def. 2.3, namely

$$\tau \coloneqq \{\emptyset, \{1\}\} \; .$$

We write

$$* \coloneqq (\{1\}, \tau \coloneqq \{\emptyset, \{1\}\})$$

for this topological space and call it *the point topological space*.

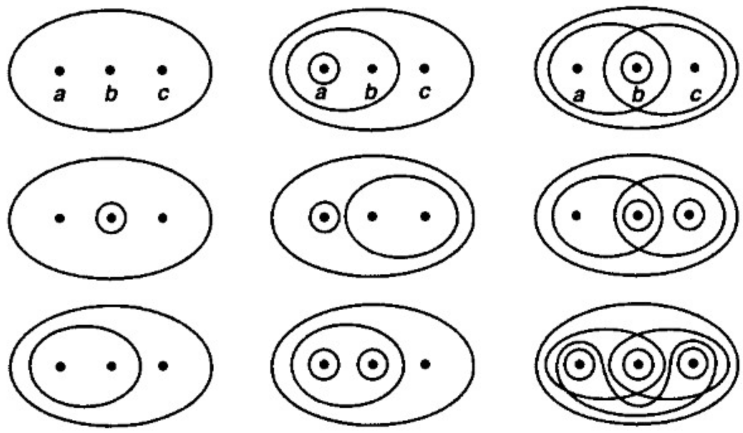
This is equivalently the metric topology (example 2.9) on $\mathbb{R}^0$, regarded as the

0-dimensional Euclidean space (example 1.6).

Example 2.11. On the 2-element set $\{0,1\}$ there are (up to permutation of elements) three distinct topologies:

1. the *codiscrete topology* (def. 2.13) $\tau = \{\emptyset, \{0,1\}\}$;
2. the *discrete topology* (def. 2.13), $\tau = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$;
3. the *Sierpinski space* topology $\tau = \{\emptyset, \{1\}, \{0,1\}\}$.

Example 2.12. The following shows all the topologies on the 3-element set (up to permutation of elements)



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Example 2.13. **(discrete and co-discrete topology)**

Let S be any set. Then there are always the following two extreme possibilities of equipping X with a topology $\tau \subset P(X)$ in the sense of def. 2.3, and hence making it a topological space:

1. $\tau \coloneqq P(S)$ the set of *all* open subsets;

   this is called the *discrete topology* on S, it is the finest topology (def. 2.6) on X,

   we write $\mathrm{Disc}(S)$ for the resulting topological space;

2. $\tau \coloneqq \{\emptyset, S\}$ the set contaning only the empty subset of S and all of S itself;

   this is called the *codiscrete topology* on S, it is the coarsest topology (def. 2.6) on X,

   we write $\mathrm{CoDisc}(S)$ for the resulting topological space.

The reason for this terminology is best seen when considering continuous functions into or out of these (co-)discrete topological spaces, we come to this in example 3.8 below.

Example 2.14. (cofinite topology)

Given a set X, then the *cofinite topology* or *finite complement topology* on X is the topology (def. 2.3) whose open subsets are precisely

1. all cofinite subsets $S \subset X$ (i.e. those such that the complement $X \backslash S$ is a finite set);

2. the empty set.

If X is itself a finite set (but not otherwise) then the cofinite topology on X coincides with the discrete topology on X (example 2.13).

We now consider basic construction principles of new topological spaces from given ones:

1. disjoint union spaces (example 2.15)

2. subspaces (example 2.16),

3. quotient spaces (example 2.17)

4. product spaces (example 2.18).

Below in *Universal constructions* we will recognize these as simple special cases of a general construction principle.

Example 2.15. (disjoint union space)

For $\{(X_i, \tau_i)\}_{i \in I}$ a set of topological spaces, then their *disjoint union*

$$\underset{i \in I}{\sqcup} (X_i, \tau_i)$$

is the topological space whose underlying set is the disjoint union of the underlying sets of the summand spaces, and whose open subsets are precisely the disjoint unions of the open subsets of the summand spaces.

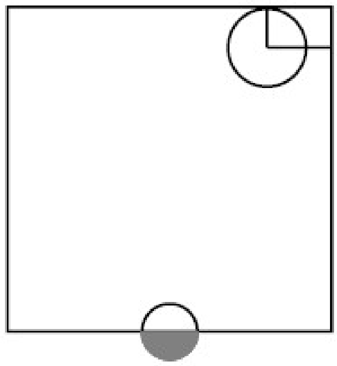
In particular, for I any index set, then the disjoint union of I copies of the point space (example 2.10) is equivalently the discrete topological space (example 2.13) on that index set:

$$\underset{i \in I}{\sqcup} * \;=\; \mathrm{Disc}(I) \,.$$

Example 2.16. (subspace topology)

Let (X, τ_X) be a topological space, and let $S \subset X$ be a subset of the underlying set. Then the corresponding *topological subspace* has S as its underlying set, and its open subsets are those subsets of S which arise as restrictions of open subsets of X.

$$(U_S \subset S \text{ open}) \Leftrightarrow \left(\underset{U_X \in \tau_X}{\exists} (U_S = U_X \cap S) \right) .$$



(This is also called the *initial topology* of the inclusion map. We come back to this below in def. 6.17.)

The picture on the right shows two open subsets inside the square, regarded as a topological subspace of the plane $\mathbb{R}^2$:

graphics grabbed from Munkres 75

Example 2.17. (**quotient topological space**)

Let (X, τ_X) be a topological space (def. 2.3) and let

$$R_\sim \subset X \times X$$

be an equivalence relation on its underlying set. Then the *quotient topological space* has

- as underlying set the quotient set $X/\sim$, hence the set of equivalence classes,

and

- a subset $O \subset X/\sim$ is declared to be an open subset precisely if its preimage $\pi^{-1}(O)$ under the canonical projection map

  $$\pi : X \to X/\sim$$

  is open in X.

(This is also called the *final topology* of the projection π. We come back to this below in def. 6.17.)

Often one considers this with input datum not the equivalence relation, but any surjection

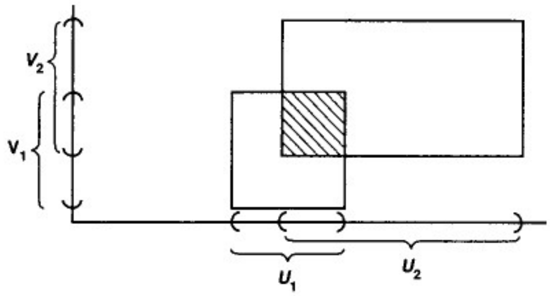
$$\pi : X \longrightarrow Y$$

of sets. Of course this identifies $Y = X/\sim$ with $(x_1 \sim x_2) \Leftrightarrow (\pi(x_1) = \pi(x_2))$. Hence the *quotient topology* on the codomain set of a function out of any topological space has as open subsets those whose pre-images are open.

To see that this indeed does define a topology on $X/\sim$ it is sufficient to observe that taking pre-images commutes with taking unions and with taking intersections.

Example 2.18. (**binary product topological space**)

For (X_1, τ_{X_1}) and (X_2, τ_{X_2}) two topological spaces, then their *binary product topological space* has as underlying set the Cartesian product $X_1 \times X_2$ of the corresponding two underlying sets, and its topology is generated from the basis (def. 2.7) given by the Cartesian products $U_1 \times U_2$ of the opens $U_i \in \tau_i$.



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Beware for non-finite products, the descriptions of the product topology is not as simple. This we turn to below in example 6.25, after introducing the general concept of limits in the category of topological spaces.

The following examples illustrate how all these ingredients and construction principles may be combined.

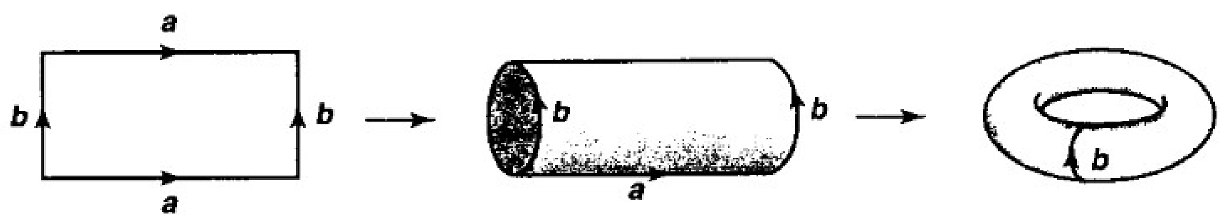
The following example we will examine in more detail below in example 3.30, after we have introduced the concept of *homeomorphisms* below.

Example 2.19. Consider the real numbers $\mathbb{R}$ as the 1-dimensional Euclidean space (example 1.6) and hence as a topological space via the corresponding metric topology (example 2.9). Moreover, consider the closed interval $[0,1] \subset \mathbb{R}$ from example 1.13, regarded as a subspace (def. 2.16) of $\mathbb{R}$.

The product space (example 2.18) of this interval with itself

$$[0,1] \times [0,1]$$

is a topological space modelling the closed square. The quotient space (example 2.17) of that by the relation which identifies a pair of opposite sides is a model for the cylinder. The further quotient by the relation that identifies the remaining pair of sides yields a model for the torus.



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Example 2.20. (**spheres and disks**)

For $n \in \mathbb{N}$ write

- D^n for the n-disk, the closed unit ball (def. 1.2) in the n-dimensional Euclidean space $\mathbb{R}^n$ (example 1.6) and equipped with the induced subspace topology (example 2.16) of the corresponding metric topology (example 2.9);
- S^{n-1} for the (n-1)-sphere (def. 1.2) also equipped with the corresponding

subspace topology;

- $i_n : S^{n-1} \hookrightarrow D^n$ for the continuous function that exhibits this boundary inclusion.

Notice that

- $S^{-1} = \emptyset$ is the empty topological space (example 2.10);
- $S^0 = * \sqcup *$ is the disjoint union space (example 2.15) of the point topological space (example 2.10) with itself, equivalently the discrete topological space on two elements (example 2.11).

The following important class of topological spaces form the foundation of algebraic geometry:

Example 2.21. (Zariski topology on affine space)

Let k be a field, let $n \in \mathbb{N}$, and write $k[X_1, \cdots, X_n]$ for the set of polynomials in n variables over k.

For $\mathcal{F} \subset k[X_1, \cdots, X_n]$ a subset of polynomials, let the subset $V(\mathcal{F}) \subset k^n$ of the n-fold Cartesian product of the underlying set of k (the *vanishing set* of $\mathcal{F}$) be the subset of points on which all these polynomials jointly vanish:

$$V(\mathcal{F}) \coloneqq \left\{ (a_1, \cdots, a_n) \in k^n \mid \underset{f \in \mathcal{F}}{\forall} f(a_1, \cdots, a_n) = 0 \right\}.$$

These subsets are called the *Zariski closed subsets*.

Write

$$\tau_{\mathbb{A}^n_k} \coloneqq \left\{ k^n \backslash V(\mathcal{F}) \subset k^n \mid \mathcal{F} \subset k[X_1, \cdots, X_n] \right\}$$

for the set of complements of the Zariski closed subsets. These are called the *Zariski open subsets* of k^n.

The Zariski open subsets of k^n form a topology (def. 2.3), called the *Zariski topology*. The resulting topological space

$$\mathbb{A}^n_k \coloneqq \left(k^n, \tau_{\mathbb{A}^n_k} \right)$$

is also called the n-dimensional *affine space* over k.

More generally

Example 2.22. (Zariski topology on the prime spectrum of a commutative ring)

Let R be a commutative ring. Write $\mathrm{PrimeIdl}(R)$ for its set of prime ideals. For $\mathcal{F} \subset R$ any subset of elements of the ring, consider the subsets of those prime ideals that contain $\mathcal{F}$:

$$V(\mathcal{F}) \;\coloneqq\; \{ p \in \mathrm{PrimeIdl}(R) \mid \mathcal{F} \subset p \} \,.$$

These are called the *Zariski closed subsets* of $\mathrm{PrimeIdl}(R)$. Their complements are called the *Zariski open subsets*.

Then the collection of Zariski open subsets in its set of prime ideals

$$\tau_{\mathrm{Spec}(R)} \subset P(\mathrm{PrimeIdl}(R))$$

satisfies the axioms of a topology (def. 2.3), the *Zariski topology*.

This topological space

$$\mathrm{Spec}(R) \coloneqq (\mathrm{PrimeIdl}(R), \tau_{\mathrm{Spec}(R)})$$

is called (the space underlying) the *prime spectrum of the commutative ring*.

Closed subsets

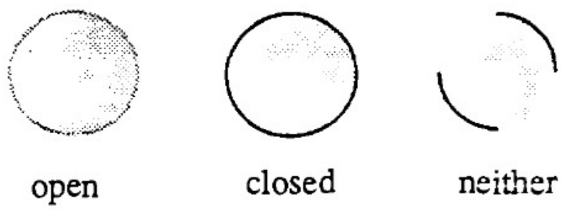
The complements of open subsets in a topological space are called *closed subsets* (def. 2.23 below). This simple definition indeed captures the concept of closure in the analytic sense of convergence of sequences (prop. 2.29 below). Of particular interest for the theory of topological spaces in the discussion of separation axioms below are those closed subsets which are "irreducible" (def. 2.30 below). These happen to be equivalently the "frame homomorphisms" (def. 2.34) to the frame of opens of the point (prop. 2.37 below).

Definition 2.23. (**closed subsets**)

Let (X,τ) be a topological space (def. 2.3).

1. A subset $S \subset X$ is called a *closed subset* if its complement $X \backslash S$ is an *open subset*:

   $$(S \subset X \;\text{is closed}) \qquad \Leftrightarrow \qquad (X \backslash S \;\subset X \;\text{is open}) \,.$$



   graphics grabbed from Vickers 89

2. If a singleton subset $\{x\} \subset X$ is closed, one says that x is a *closed point* of X.

3. Given any subset $S \subset X$, then its *topological closure* $\mathrm{Cl}(S)$ is the smallest closed subset containing S:

   $$\mathrm{Cl}(S) \;\coloneqq\; \underset{\substack{C \subset X \,\text{closed} \\ S \subset C}}{\cap} (C) \,.$$

4. A subset $S \subset X$ such that $\mathrm{Cl}(S) = X$ is called a *dense subset* of (X,τ).

Remark 2.24. (**de Morgan's law**)

In reasoning about closed subsets in topology we are concerned with complements of unions and intersections as well as with unions/intersections of complements. Recall therefore that taking complements of subsets exchanges unions with intersections (de Morgan's law):

Given a set X and a set of subsets

$$\{S_i \subset X\}_{i \in I}$$

then

$$X \backslash \left(\underset{i \in I}{\cup} S_i \right) \;=\; \underset{i \in I}{\cap} (X \backslash S_i)$$

and

$$X \backslash \left(\underset{i \in I}{\cap} S_i \right) \;=\; \underset{i \in I}{\cup} (X \backslash S_i)\,.$$

Also notice that taking complements reverses inclusion relations:

$$(S_1 \subset S_2) \;\Leftrightarrow\; (X \backslash S_2 \subset X \backslash S_1)\,.$$

Often it is useful to reformulate def. 2.23 of closed subsets as follows:

Lemma 2.25. ***(alternative characterization of closed subsets)***

Let (X,τ) be a topological space and let $S \subset X$ be a subset of its underlying set. Then a point $x \in X$ is contained in the topological closure $\mathrm{Cl}(S)$ (def. 2.23) precisely if every open neighbourhood $U_x \subset X$ of x intersects S:

$$(x \in \mathrm{Cl}(S)) \quad \Leftrightarrow \quad \neg \left(\underset{\substack{U \subset X \backslash S \\ U \subset X \,\text{open}}}{\exists} (x \in U) \right).$$

Proof. In view of remark 2.24 we may rephrase the definition of the topological closure as follows:

$$\begin{aligned} \mathrm{Cl}(S) & \coloneqq \underset{\substack{S \subset C \\ C \subset X \,\text{closed}}}{\cap} (C) \\ & = \underset{\substack{U \subset X \backslash S \\ U \subset X \,\text{open}}}{\cap} (X \backslash U) \\ & = X \backslash \left(\underset{\substack{U \subset X \backslash S \\ U \subset X \,\text{open}}}{\cup} U \right) \end{aligned}.$$

■

Definition 2.26. **(topological interior and boundary)**

Let (X,τ) be a topological space (def. 2.3) and let $S \subset X$ be a subset. Then the *topological interior* of S is the largest open subset $\mathrm{Int}(S) \in \tau$ still contained in S,

$\mathrm{Int}(S) \subset S \subset X$:

$$\mathrm{Int}(S) \coloneqq \underset{O \subset X \text{ open}}{\underset{O \subset S}{\cup}} (U) \,.$$

The *boundary* ∂S of S is the complement of its interior inside its topological closure (def. 2.23):

$$\partial S \;\coloneqq\; \mathrm{Cl}(S) \backslash \mathrm{Int}(S) \,.$$

Lemma 2.27. ***(duality between closure and interior)***

Let (X, τ) be a topological space and let $S \subset X$ be a subset. Then the topological interior of S (def. 2.26) is the same as the complement of the topological closure $\mathrm{Cl}(X \backslash S)$ of the complement of S:

$$X \backslash \mathrm{Int}(S) \;=\; \mathrm{Cl}(\, X \backslash S \,)$$

and conversely

$$X \backslash \mathrm{Cl}(S) \;=\; \mathrm{Int}(\, X \backslash S \,) \,.$$

Proof. Using remark 2.24, we compute as follows:

$$\begin{aligned} X \backslash \mathrm{Int}(S) &= X \backslash \left(\underset{U \subset X \text{ open}}{\underset{U \subset S}{\cup}} U \right) \\ &= \underset{U \subset X \text{ open}}{\underset{U \subset S}{\cap}} (X \backslash U) \\ &= \underset{C \text{ closed}}{\underset{C \supset X \backslash S}{\cap}} (C) \\ &= \mathrm{Cl}(X \backslash S) \end{aligned}$$

Similarly for the other case. ▮

Example 2.28. **(topological closure and interior of closed and open intervals)**

Regard the real numbers as the 1-dimensional Euclidean space (example 1.6) and equipped with the corresponding metric topology (example 2.9) . Let $a < b \in \mathbb{R}$. Then the topological interior (def. 2.26) of the closed interval $[a,b] \subset \mathbb{R}$ (example 1.13) is the open interval $(a,b) \subset \mathbb{R}$, moreover the closed interval is its own topological closure (def. 2.23) and the converse holds (by lemma 2.27):

$$\begin{array}{ll} \mathrm{Cl}(\,(a,b)\,) \;=\; [a,b] & \quad \mathrm{Int}(\,(a,b)\,) \;=\; (a,b) \\ \mathrm{Cl}(\,[a,b]\,) \;=\; [a,b] & \quad \mathrm{Int}(\,[a,b]\,) \;=\; (a,b) \end{array} \,.$$

Hence the boundary of the closed interval is its endpoints, while the boundary of the open interval is empty

$$\partial[a,b] = \{a\} \cup \{b\} \qquad \partial(a,b) = \emptyset \,.$$

The terminology "closed" subspace for complements of opens is justified by the following statement, which is a further example of how the combinatorial concept of open subsets captures key phenomena in analysis:

Proposition 2.29. ***(convergence in closed subspaces)***

Let (X,d) be a metric space (def. 1.1), regarded as a topological space via example 2.9, and let $V \subset X$ be a subset. Then the following are equivalent:

1. *$V \subset X$ is a closed subspace according to def. 2.23.*
2. *For every sequence $x_i \in V \subset X$ (def. 1.16) with elements in V, which converges as a sequence in X (def. 1.17) to some $x_\infty \in X$, we have $x_\infty \in V \subset X$.*

Proof. First assume that $V \subset X$ is closed and that $x_i \overset{i\to\infty}{\longrightarrow} x_\infty$ for some $x_\infty \in X$. We need to show that then $x_\infty \in V$. Suppose it were not, hence that $x_\infty \in X \backslash V$. Since, by assumption on V, this complement $X\backslash V \subset X$ is an open subset, it would follow that there exists a real number $\epsilon > 0$ such that the open ball around x of radius ϵ were still contained in the complement: $B^\circ_x(\epsilon) \subset X\backslash V$. But since the sequence is assumed to converge in X, this would mean that there exists N_ϵ such that all $x_{i > N_\epsilon}$ are in $B^\circ_x(\epsilon)$, hence in $X\backslash V$. This contradicts the assumption that all x_i are in V, and hence we have proved by contradiction that $x_\infty \in V$.

Conversely, assume that for all sequences in V that converge to some $x_\infty \in X$ then $x_\infty \in V \subset X$. We need to show that then V is closed, hence that $X\backslash V \subset X$ is an open subset, hence that for every $x \in X\backslash V$ we may find a real number $\epsilon > 0$ such that the open ball $B^\circ_x(\epsilon)$ around x of radius ϵ is still contained in $X\backslash V$. Suppose on the contrary that such ϵ did not exist. This would mean that for each $k \in \mathbb{N}$ with $k \geq 1$ then the intersection $B^\circ_x(1/k) \cap V$ were non-empty. Hence then we could choose points $x_k \in B^\circ_x(1/k) \cap V$ in these intersections. These would form a sequence which clearly converges to the original x, and so by assumption we would conclude that $x \in V$, which violates the assumption that $x \in X\backslash V$. Hence we proved by contradiction $X\backslash V$ is in fact open. ▮

A special role in the theory is played by the "irreducible" closed subspaces:

Definition 2.30. **(irreducible closed subspace)**

A closed subset $S \subset X$ (def. 2.23) of a topological space X is called *irreducible* if it is non-empty and not the union of two closed proper (i.e. smaller) subsets. In other words, a non-empty closed subset $S \subset X$ is irreducible if whenever $S_1, S_2 \subset X$ are two closed subspace such that

$$S = S_1 \cup S_2$$

then $S_1 = S$ or $S_2 = S$.

Example 2.31. **(closures of points are irreducible)**

For $x \in X$ a point inside a topological space, then the closure $\mathrm{Cl}(\{x\})$ of the singleton subset $\{x\} \subset X$ is irreducible (def. 2.30).

Example 2.32. **(no nontrivial closed irreducibles in metric spaces)**

Let (X, d) be a metric space, regarded as a topological space via its metric topology (example 2.9). Then every point $x \in X$ is closed (def 2.23), hence every singleton subset $\{x\} \subset X$ is irreducible according to def. 2.31.

Let $\mathbb{R}$ be the 1-dimensional Euclidean space (example 1.6) with its metric topology (example 2.9). Then for $a < c \subset \mathbb{R}$ the closed interval $[a, c] \subset \mathbb{R}$ (example 1.13) is *not* irreducible, since for any $b \in \mathbb{R}$ with $a < b < c$ it is the union of two smaller closed subintervals:

$$[a, c] \;=\; [a, b] \cup [b, c] \;.$$

In fact we will see below (prop. 5.3) that in a metric space the singleton subsets are precisely the only irreducible closed subsets.

Often it is useful to re-express the condition of irreducibility of closed subspaces in terms of complementary open subsets:

Proposition 2.33. ***(irreducible closed subsets in terms of prime open subsets)***

Let (X, τ) be a topological space, and let $P \in \tau$ be a proper open subset of X, hence so that the complement $F \coloneqq X \backslash P$ is a non-empty closed subspace. Then F is irreducible in the sense of def. 2.30 precisely if whenever $U_1, U_2 \in \tau$ are open subsets with $U_1 \cap U_2 \subset P$ then $U_1 \subset P$ or $U_2 \subset P$:

$$(X \backslash P \text{ irreducible}) \;\Leftrightarrow\; \left(\underset{U_1, U_2 \in \tau}{\forall} \left((U_1 \cap U_2 \subset P) \;\Rightarrow\; (U_1 \subset P \text{ or } U_2 \subset P) \right) \right) .$$

The open subsets $P \subset X$ with this property are also called the prime open subsets in τ_X.

Proof. Observe that every closed subset $F_i \subset F$ may be exhibited as the complement

$$F_i = F \backslash U_i$$

of some open subset $U_i \in \tau$ with respect to F. Observe that under this identification the condition that $U_1 \cap U_2 \subset P$ is equivalent to the condition that $F_1 \cup F_2 = F$, because it is equivalent to the equation labeled $(\star)$ in the following sequence of equations:

$$
\begin{aligned}
F_1 \cup F_2 & = (F \backslash U_1) \cup (F \backslash U_2) \\
& = (X \backslash (P \cup U_1)) \cup (X \backslash P \cup U_2) \\
& = X \backslash ((P \cup U_1) \cap (P \cup U_2)) \\
& = X \backslash (P \cup (U_1 \cap U_2)) \\
& \overset{(\star)}{=} X \backslash P \\
& = F \;.
\end{aligned}
\,.
$$

Similarly, the condition that $U_i \subset P$ is equivalent to the condition that $F_i = F$, because it is equivalent to the equality $(\star)$ in the following sequence of equalities:

$$
\begin{aligned}
F_i & = F \backslash U_i \\
& = X \backslash (P \cup U_i) \\
& \overset{(\star)}{=} X \backslash P \\
& = F
\end{aligned}
\,.
$$

Under these identifications, the two conditions are manifestly the same. ▮

We consider yet another equivalent characterization of irreducible closed subsets, prop. 2.37 below, which will be needed in the discussion of the separation axioms further below. Stating this requires the following concept of "frame" homomorphism, the natural kind of homomorphisms between topological spaces if we were to forget the underlying set of points of a topological space, and only remember the set τ_X with its operations induced by taking finite intersections and arbitrary unions:

Definition 2.34. (frame homomorphisms)

Let (X, τ_X) and (Y, τ_Y) be topological spaces (def. 2.3). Then a function

$$
\tau_X \longleftarrow \tau_Y \;:\; \phi
$$

between their sets of open subsets is called a *frame homomorphism* if it preserves

1. arbitrary unions;

2. finite intersections.

In other words, ϕ is a frame homomorphism precisely if

1. for every set I and every I-indexed set $\{U_i \in \tau_Y\}_{i \in I}$ of elements of τ_Y, then

$$
\phi\Big(\underset{i \in I}{\cup} U_i \Big) \;=\; \underset{i \in I}{\cup} \phi(U_i) \quad \in \tau_X \,,
$$

2. for every finite set J and every J-indexed set $\{U_j \in \tau_Y\}_{j \in J}$ of elements in τ_Y, then

$$\phi\left(\underset{j \in J}{\cap} U_j \right) = \underset{j \in J}{\cap} \phi(U_j) \quad \in \tau_X \,.$$

Remark 2.35. **(frame homomorphisms preserve inclusions)**

A frame homomorphism ϕ as in def. 2.34 necessarily also preserves inclusions in that

- for every inclusion $U_1 \subset U_2$ with $U_1, U_2 \in \tau_Y \subset P(Y)$ then

$$\phi(U_1) \subset \phi(U_2) \qquad \in \tau_X \,.$$

This is because inclusions are witnessed by unions

$$(U_1 \subset U_2) \;\Leftrightarrow\; (U_1 \cup U_2 = U_2)$$

or alternatively because inclusions are witnessed by finite intersections:

$$(U_1 \subset U_2) \;\Leftrightarrow\; (U_1 \cap U_2 = U_1) \,.$$

Example 2.36. **(pre-images of continuous functions are frame homomorphisms)**

Let (X, τ_X) and (Y, τ_Y) be two topological spaces. One way to obtain a function between their sets of open subsets

$$\tau_X \longleftarrow \tau_Y \;:\; \phi$$

is to specifiy a function

$$f : X \longrightarrow Y$$

of their underlying sets, and take $\phi \coloneqq f^{-1}$ to be the pre-image operation. A priori this is a function of the form

$$P(Y) \longleftarrow P(X) \;:\; f^{-1}$$

and hence in order for this to co-restrict to $\tau_X \subset P(X)$ when restricted to $\tau_Y \subset P(Y)$ we need to demand that, under f, pre-images of open subsets of Y are open subsets of Z. Below in def. 3.1 we highlight these as the *continuous functions* between toopological spaces.

$$f \;:\; (X, \tau_X) \longrightarrow (Y, \tau_Y)$$

In this case then

$$\tau_X \longleftarrow \tau_Y \;:\; f^{-1}$$

is a frame homomorphism in the sense of def. 2.34.

For the following recall from example 2.10 the point topological space $* = (\{1\}, \tau_* = \{\emptyset, \{1\}\})$.

Proposition 2.37. ***(irreducible closed subsets are equivalently frame***

homomorphisms to opens of the point)

For (X,τ) a topological space, then there is a natural bijection between the irreducible closed subspaces of (X,τ) (def. 2.30) and the frame homomorphisms from τ_X to τ_*, and this bijection is given by

$$\begin{array}{ccc} \mathrm{FrameHom}(\tau_X, \tau_*) & \overset{\simeq}{\longrightarrow} & \mathrm{IrrClSub}(X) \\ \phi & \mapsto & X \backslash \left(U_\emptyset(\phi) \right) \end{array}$$

where $U_\emptyset(\phi)$ is the union of all elements $U \in \tau_x$ such that $\phi(U) = \emptyset$:

$$U_\emptyset(\phi) \coloneqq \underset{ \underset{\phi(U) = \emptyset}{U \in \tau_X} }{\cup} (U) \,.$$

See also (Johnstone 82, II 1.3).

Proof. First we need to show that the function is well defined in that given a frame homomorphism $\phi : \tau_X \to \tau_*$ then $X \backslash U_\emptyset(\phi)$ is indeed an irreducible closed subspace.

To that end observe that:

$(*)$ *If there are two elements $U_1, U_2 \in \tau_X$ with $U_1 \cap U_2 \subset U_\emptyset(\phi)$ then $U_1 \subset U_\emptyset(\phi)$ or $U_2 \subset U_\emptyset(\phi)$.*

This is because

$$\begin{aligned} \phi(U_1) \cap \phi(U_2) & = \phi(U_1 \cap U_2) \\ & \subset \phi(U_\emptyset(\phi)) \quad , \\ & = \emptyset \end{aligned}$$

where the first equality holds because ϕ preserves finite intersections by def. 2.34, the inclusion holds because ϕ respects inclusions by remark 2.35, and the second equality holds because ϕ preserves arbitrary unions by def. 2.34. But in $\tau_* = \{\emptyset, \{1\}\}$ the intersection of two open subsets is empty precisely if at least one of them is empty, hence $\phi(U_1) = \emptyset$ or $\phi(U_2) = \emptyset$. But this means that $U_1 \subset U_\emptyset(\phi)$ or $U_2 \subset U_\emptyset(\phi)$, as claimed.

Now according to prop. 2.33 the condition $(*)$ identifies the complement $X \backslash U_\emptyset(\phi)$ as an irreducible closed subspace of (X,τ).

Conversely, given an irreducible closed subset $X \backslash U_0$, define ϕ by

$$\phi \;:\; U \mapsto \left\{ \begin{array}{ll} \emptyset & \vert \, \text{if}\, U \subset U_0 \\ \{1\} & \vert \, \text{otherwise} \end{array} \right. \,.$$

This does preserve

1. arbitrary unions

   because $\phi(\underset{i}{\cup} U_i) = \{\emptyset\}$ precisely if $\underset{i}{\cup} U_i \subset U_0$ which is the case precisely if all

$U_i \subset U_0$, which means that all $\phi(U_i) = \emptyset$ and because $\underset{i}{\cup} \emptyset = \emptyset$;

while $\phi(\underset{i}{\cup} U_1) = \{1\}$ as soon as one of the U_i is not contained in U_0, which means that one of the $\phi(U_i) = \{1\}$ which means that $\underset{i}{\cup} \phi(U_i) = \{1\}$;

2. finite intersections

   because if $U_1 \cap U_2 \subset U_0$, then by $(*)$ $U_1 \in U_0$ or $U_2 \in U_0$, whence $\phi(U_1) = \emptyset$ or $\phi(U_2) = \emptyset$, whence with $\phi(U_1 \cap U_2) = \emptyset$ also $\phi(U_1) \cap \phi(U_2) = \emptyset$;

   while if $U_1 \cap U_2$ is not contained in U_0 then neither U_1 nor U_2 is contained in U_0 and hence with $\phi(U_1 \cap U_2) = \{1\}$ also $\phi(U_1) \cap \phi(U_2) = \{1\} \cap \{1\} = \{1\}$.

Hence this is indeed a frame homomorphism $\tau_X \to \tau_*$.

Finally, it is clear that these two operations are inverse to each other. ▮

3. Continuous functions

With the concept of topological spaces in hand (def. 2.3) it is now immediate to formally implement in abstract generality the statement of prop. 1.14:

principle of continuity

Continuous pre-Images of open subsets are open.

Definition 3.1. **(continuous function)**

A *continuous function* between topological spaces (def. 2.3)

$$f : (X, \tau_X) \to (Y, \tau_Y)$$

is a function between the underlying sets,

$$f : X \longrightarrow Y$$

such that pre-images under f of open subsets of Y are open subsets of X.

We may equivalently state this in terms of closed subsets:

Proposition 3.2. *Let (X_1, τ_X) and (Y, τ_Y) be two topological spaces (def. 2.3). Then a function*

$$f \,:\, X \longrightarrow Y$$

between the underlying sets is continuous in the sense of def. 3.1 precisely if pre-images under f of closed subsets of Y (def. 2.23) are closed subsets of X.

Proof. This follows since taking pre-images commutes with taking complements. ▮

Before looking at first examples of continuous functions below we consider now an informal remark on the resulting global structure, the "category of topological spaces", remark 3.3 below. This is a language that serves to make transparent key phenomena in topology which we encounter further below, such as the Tn-reflection (remark 4.24 below), and the universal constructions.

Remark 3.3. **(concrete category of topological spaces)**

For X_1, X_2, X_3 three topological spaces and for

$$X_1 \overset{f}{\longrightarrow} X_2 \quad \text{and} \quad X_2 \overset{g}{\longrightarrow} X_3$$

two continuous functions (def. 3.1) then their composition

$$f_2 \circ f_1 : X_1 \overset{f}{\longrightarrow} X_2 \overset{f_2}{\longrightarrow} X_3$$

is clearly itself again a continuous function from X_1 to X_3. Moreover, this composition operation is clearly *associative*, in that for

$$X_1 \overset{f}{\longrightarrow} X_2 \quad \text{and} \quad X_2 \overset{g}{\longrightarrow} X_3 \quad \text{and} \quad X_3 \overset{h}{\longrightarrow} X_4$$

three continuous functions, then

$$f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1 \ : X_1 \longrightarrow X_3 \ .$$

Finally, the composition operation is also clearly *unital*, in that for each topological space X there exists the identity function $\mathrm{id}_X : X \to X$ and for $f : X_1 \to X_2$ any continuous function then

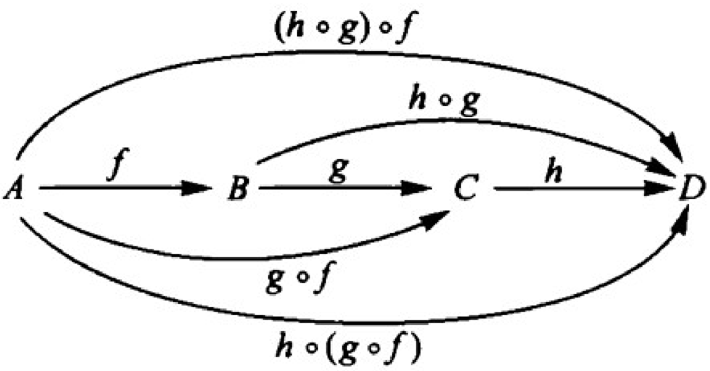
$$\mathrm{id}_{X_2} \circ f \ = \ f \ = \ f \circ \mathrm{id}_{X_1} \ .$$

One summarizes this situation by saying that:

1. topological spaces constitute the *objects*,
2. continuous functions constitute the *morphisms* (homomorphisms)

of a *category*, called the *category of topological spaces* ("Top" for short).

It is useful to depict collections of objects with morphisms between them by diagrams, like this one:



graphics grabbed from Lawvere-Schanuel 09.

There are other categories. For instance there is the category of sets ("Set" for short) whose

1. objects are sets,

2. morphisms are plain functions between these.

The two categories Top and Set are different, but related. After all,

1. an object of Top (hence a topological space) is an object of Set (hence a set) equipped with extra structure (namely with a topology);

2. a morphism in Top (hence a continuous function) is a morphism in Set (hence a plain function) with the extra property that it preserves this extra structure.

Hence we have the *underlying set assigning function*

$$\begin{array}{ccc} \mathrm{Top} & \overset{U}{\longrightarrow} & \mathrm{Set} \\ (X,\tau) & \longmapsto & X \end{array}$$

from the class of topological spaces to the class of sets. But more is true: every continuous function between topological spaces is, by definition, in particular a function on underlying sets:

$$\begin{array}{ccc} \mathrm{Top} & \overset{U}{\longrightarrow} & \mathrm{Set} \\ (X,\tau_X) & \longmapsto & X \\ {}^{\mathllap{f}}\downarrow & \mapsto & \downarrow^{\mathrlap{f}} \\ (Y,\tau_Y) & \longmapsto & Y \end{array}$$

and this assignment (trivially) respects the composition of morphisms and the identity morphisms.

Such a function between classes of objects of categories, which is extended to a function on the sets of homomorphisms between these objects in a way that respects composition and identity morphisms is called a *functor*. If we write an arrow between categories

$$U \;\colon\; \mathrm{Top} \longrightarrow \mathrm{Set}$$

then it is understood that we mean not just a function between their classes of objects, but a functor.

The functor U at hand has the special property that it does not do much except *forgetting extra structure*, namely the extra structure on a set X given by a choice of topology τ_X. One also speaks of a *forgetful functor*.

This is intuitively clear, and we may easily formalize it: The functor U has the special property that as a function between sets of homomorphisms ("hom sets", for short) it is injective. More in detail, given topological spaces (X, τ_X) and (Y, τ_Y) then the component function of U from the set of continuous function between these spaces to the set of plain functions between their underlying sets

$$\left\{ (X, \tau_X) \underset{\text{function}}{\overset{\text{continuous}}{\longrightarrow}} (Y, \tau_Y) \right\} \quad \overset{U}{\longmapsto} \quad \left\{ X \underset{\text{function}}{\longrightarrow} Y \right\}$$

is an injective function, including the continuous functions among all functions of underlying sets.

A functor with this property, that its component functions between all hom-sets are injective, is called a *faithful functor*.

A category equipped with a faithful functor to Set is called a *concrete category*.

Hence Top is canonically a concrete category.

Example 3.4. (**product topological space construction is functorial**)

For $\mathcal{C}$ and $\mathcal{D}$ two categories as in remark 3.3 (for instance Top or Set) then we obtain a new category denoted $\mathcal{C} \times \mathcal{D}$ and called their *product category* whose

1. objects are pairs (c, d) with c an object of $\mathcal{C}$ and d an object of $\mathcal{D}$;

- morphisms are pairs $(f, g) : (c, d) \to (c', d')$ with $f : c \to d$ a morphism of $\mathcal{C}$ and $g : d \to d'$ a morphisms of $\mathcal{D}$,

- composition of morphisms is defined pairwise $(f', g') \circ (f, g) \coloneqq (f' \circ f, g' \circ g)$.

This concept secretly underlies the construction of product topological spaces:

Let (X_1, τ_{X_1}), (X_2, τ_{X_2}), (Y_1, τ_{Y_1}) and (Y_2, τ_{Y_2}) be topological spaces. Then for all pairs of continuous functions

$$f_1 \; : \; (X_1, \tau_{X_1}) \longrightarrow (Y_1, \tau_{Y_1})$$

and

$$f_2 \; : \; (X_2, \tau_{X_2}) \longrightarrow (Y_2, \tau_{Y_2})$$

the canonically induced function on Cartesian products of sets

$$\begin{array}{ccc} X_1 \times X_2 & \overset{f_1 \times f_2}{\longrightarrow} & Y_1 \times Y_2 \\ (x_1, x_2) & \mapsto & (f_1(x_1), f_2(x_2)) \end{array}$$

is a continuous function with respect to the binary product space topologies (def. 2.18)

$$f_1 \times f_2 : (X_1 \times X_2, \tau_{X_1 \times X_2}) \longrightarrow (Y_1, \times Y_2, \tau_{Y_1 \times Y_2}) .$$

Moreover, this construction respects identity functions and composition of functions in both arguments.

In the language of category theory (remark 3.3), this is summarized by saying that the product topological space construction $(-) \times (-)$ extends to a *functor* from the product category of the category Top with itself to itself:

$$(-) \times (-) : \mathrm{Top} \times \mathrm{Top} \longrightarrow \mathrm{Top} .$$

Examples

We discuss here some basic examples of continuous functions (def. 3.1) between topological spaces (def. 2.3) to get a feeling for the nature of the concept. But as with topological spaces themselves, continuous functions between them are ubiquitous in mathematics, and no list will exhaust all classes of examples. Below in the section *Universal constructions* we discuss a general principle that serves to produce examples of continuous functions with prescribed "universal properties".

Example 3.5. (point space is terminal)

For (X, τ) any topological space, then there is a *unique* continuous function

1. from the empty topological space (def. 2.10) X

$$\emptyset \overset{\exists !}{\longrightarrow} X$$

2. from X to the point topological space (def. 2.10).

$$X \overset{\exists !}{\longrightarrow} *$$

In the language of category theory (remark 3.3), this says that

1. the empty topological space is the *initial object*

2. the point space $*$ is the *terminal object*

in the category Top of topological spaces. We come back to this below in example 6.12.

Example 3.6. (constant continuous functions)

For (X, τ) a topological space then for $x \in X$ any element of the underlying set, there is a unique continuous function (which we denote by the same symbol)

$$x : * \longrightarrow X$$

from the point topological space (def. 2.10), whose image in X is that element. Hence there is a natural bijection

$$\left\{ * \overset{f}{\to} X \mid f \text{ continuous} \right\} \simeq X$$

between the continuous functions from the point to any topological space, and the underlying set of that topological space.

More generally, for (X, τ_X) and (Y, τ_Y) two topological spaces, then a continuous function $X \to Y$ between them is called a *constant function* with value some point $y \in Y$ if it factors through the point spaces as

$$\mathrm{const}_y : X \overset{\exists !}{\longrightarrow} * \overset{y}{\longrightarrow} Y .$$

Definition 3.7. (**locally constant function**)

For $(X . \tau_X)$, (Y, τ_Y) two topological spaces, then a a continuous function $f : (X, \tau_X) \to (Y, \tau_Y)$ (def. 3.1) is called *locally constant* if every point $x \in X$ has a neighbourhood on which the function is constant.

Example 3.8. (**continuous functions into and out of discrete and codiscrete spaces**)

Let S be a set and let (X, τ) be a topological space. Recall from example 2.13

1. the discrete topological space $\mathrm{Disc}(S)$;

2. the co-discrete topological space $\mathrm{CoDisc}(S)$

on the underlying set S. Then continuous functions (def. 3.1) into/out of these satisfy:

1. every function (of sets) $\mathrm{Disc}(S) \longrightarrow X$ out of a discrete space is continuous;

2. every function (of sets) $X \longrightarrow \mathrm{CoDisc}(S)$ into a codiscrete space is continuous.

Also:

- every continuous function $(X, \tau) \longrightarrow \mathrm{Disc}(S)$ into a discrete space is locally constant (def. 3.7).

Example 3.9. (**diagonal**)

For X a set, its *diagonal* Δ_X is the function from X to the Cartesian product of X with itsef, given by

$$\begin{array}{ccc} X & \overset{\Delta_X}{\longrightarrow} & X \times X \\ x & \mapsto & (x, x) \end{array}$$

For (X, τ) a topological space, then the diagonal is a continuous function to the product topological space (def. 2.18) of X with itself.

$$\Delta_X : (X, \tau) \longrightarrow (X \times X, \tau_{X \times X}) .$$

To see this, it is sufficient to see that the preimages of basic opens $U_1 \times U_2$ in $\tau_{X \times X}$ are in τ_X. But these pre-images are the intersections $U_1 \cap U_2 \subset X$, which are open by the axioms on the topology τ_X.

Example 3.10. (**image factorization**)

Let $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ be a continuous function.

Write $f(X) \subset Y$ for the image of f on underlying sets, and consider the resulting factorization of f through $f(X)$ on underlying sets:

$$f : X \xrightarrow{\text{surjective}} f(X) \xrightarrow{\text{injective}} Y \; .$$

There are the following two ways to topologize the image $f(X)$ such as to make this a sequence of two continuous functions:

1. By example 2.16 $f(X)$ inherits a subspace topology from (Y, τ_Y) which evidently makes the inclusion $f(X) \longrightarrow Y$ a continuous function.

   Observe that this also makes $X \to f(X)$ a continuous function: An open subset of $f(X)$ in this case is of the form $U_Y \cap f(X)$ for $U_Y \in \tau_Y$, and $f^{-1}(U_Y \cap f(X)) = f^{-1}(U_Y)$, which is open in X since f is continuous.

2. By example 2.17 $f(X)$ inherits a quotient topology from (X, τ_X) which evidently makes the surjection $X \longrightarrow f(X)$ a continuous function.

   Observe that this also makes $f(X) \longrightarrow Y$ a continuous function: The preimage under this map of an open subset $U_Y \in \tau_Y$ is the restriction $U_Y \cap f(X)$, and the pre-image of that under $X \to f(X)$ is $f^{-1}(U_Y)$, as before, which is open since f is continuous, and therefore $U_Y \cap f(X)$ is open in the quotient topology.

Beware, in general a continuous function itself (as opposed to its pre-image function) neither preserves open subsets, nor closed subsets, as the following examples show:

Example 3.11. Regard the real numbers $\mathbb{R}$ as the 1-dimensional Euclidean space (example 1.6) equipped with the metric topology (example 2.9). For $a \in \mathbb{R}$ the constant function (example 3.6)

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\text{const}_a} & \mathbb{R} \\ x & \mapsto & a \end{array}$$

maps every open subset $U \subset \mathbb{R}$ to the singleton set $\{a\} \subset \mathbb{R}$, which is not open.

Example 3.12. Write $\mathrm{Disc}(\mathbb{R})$ for the set of real numbers equipped with its discrete topology (def. 2.13) and $\mathbb{R}$ for the set of real numbers equipped with its Euclidean metric topology (example 1.6, example 2.9). Then the identity function on the underlying sets

$$\mathrm{id}_{\mathbb{R}} : \mathrm{Disc}(\mathbb{R}) \longrightarrow \mathbb{R}$$

is a continuous function (a special case of example 3.8). A singleton subset $\{a\} \in \mathrm{Disc}(\mathbb{R})$ is open, but regarded as a subset $\{a\} \in \mathbb{R}$ it is not open.

Example 3.13. Consider the set of real numbers $\mathbb{R}$ equipped with its Euclidean metric topology (example 1.6, example 2.9). The exponential function

$$\exp(-) : \mathbb{R} \longrightarrow \mathbb{R}$$

maps all of $\mathbb{R}$ (which is a closed subset, since $\mathbb{R} = \mathbb{R} \backslash \emptyset$) to the open interval $(0, \infty) \subset \mathbb{R}$, which is not closed.

Those continuous functions that do happen to preserve open or closed subsets get a special name:

Definition 3.14. **(open maps and closed maps)**

A continuous function $f : (X, \tau_X) \to (Y, \tau_Y)$ (def. 3.1) is called

- an *open map* if the image under f of an open subset of X is an open subset of Y;
- a *closed map* if the image under f of a closed subset of X (def. 2.23) is a closed subset of Y.

Example 3.15. **(image projections of open/closed maps are themselves open/closed)**

If a continuous function $f : (X, \tau_X) \to (Y, \tau_Y)$ is an open map or closed map (def. 3.14) then so its its image projection $X \to f(X) \subset Y$, respectively, for $f(X) \subset Y$ regarded with its subspace topology (example 3.10).

Proof. If f is an open map, and $O \subset X$ is an open subset, so that $f(O) \subset Y$ is also open in Y, then, since $f(O) = f(O) \cap f(X)$, it is also still open in the subspace topology, hence $X \to f(X)$ is an open map.

If f is a closed map, and $C \subset X$ is a closed subset so that also $f(C) \subset Y$ is a closed subset, then the complement $Y \backslash f(C)$ is open in Y and hence $(Y \backslash f(C)) \cap f(X) = f(X) \backslash f(C)$ is open in the subspace topology, which means that $f(C)$ is closed in the subspace topology. ▮

Example 3.16. **(projections are open continuous functions)**

For (X_1, τ_{X_1}) and (X_2, τ_{X_2}) two topological spaces, then the projection maps

$$\mathrm{pr}_i : (X_1 \times X_2, \tau_{X_1 \times X_2}) \longrightarrow (X_i, \tau_{X_i})$$

out of their product topological space (def. 2.18)

$$
\begin{aligned}
X_1 \times X_2 &\overset{\mathrm{pr}_1}{\longrightarrow} X_1 \\
(x_1, x_2) &\longmapsto x_1 \\
X_1 \times X_2 &\overset{\mathrm{pr}_2}{\longrightarrow} X_2 \\
(x_1, x_2) &\longmapsto x_2
\end{aligned}
$$

are open continuous functions (def. 3.14).

This is because, by definition, every open subset $O \subset X_1 \times X_2$ in the product space topology is a union of products of open subsets $U_i \in X_1$ and $V_i \in X_2$ in the factor spaces

$$
O = \underset{i \in I}{\cup} (U_i \times V_i)
$$

and because taking the image of a function preserves unions of subsets

$$
\begin{aligned}
\mathrm{pr}_1\left(\underset{i \in I}{\cup} (U_i \times V_i) \right) &= \underset{i \in I}{\cup} \mathrm{pr}_1 (U_i \times V_i) \\
&= \underset{i \in I}{\cup} U_i
\end{aligned}
.
$$

Below in prop. 7.39 we find a large supply of closed maps.

Sometimes it is useful to recognize quotient topological space projections via saturated subsets (essentially another term for pre-images of underlying sets):

Definition 3.17. **(saturated subset)**

Let $f : X \longrightarrow Y$ be a function of sets. Then a subset $S \subset X$ is called an f-*saturated subset* (or just *saturated subset*, if f is understood) if S is the pre-image of its image:

$$
(S \subset X\ f\text{-saturated}) \Leftrightarrow \left(S = f^{-1}(f(S)) \right) .
$$

Here $f^{-1}(f(S))$ is also called the f-*saturation* of S.

Example 3.18. **(pre-images are saturated subsets)**

For $f : X \to Y$ any function of sets, and $S_Y \subset Y$ any subset of Y, then the pre-image $f^{-1}(S_Y) \subset X$ is an f-saturated subset of X (def. 3.17).

Observe that:

Lemma 3.19. *Let $f : X \longrightarrow Y$ be a function. Then a subset $S \subset X$ is f-saturated (def. 3.17) precisely if its complement $X \backslash S$ is saturated.*

Proposition 3.20. ***(recognition of quotient topologies)***

A continuous function (def. 3.1)

$$f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$$

whose underlying function $f : X \longrightarrow Y$ is surjective exhibits τ_Y as the corresponding quotient topology (def. 2.17) precisely if f sends open and f-saturated subsets in X (def. 3.17) to open subsets of Y. By lemma 3.19 this is the case precisely if it sends closed and f-saturated subsets to closed subsets.

We record the following technical lemma about saturated subspaces, which we will need below to prove prop. 7.44.

Lemma 3.21. ***(saturated open neighbourhoods of saturated closed subsets under closed maps)***

Let

1. *$f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ be a closed map (def. 3.14);*

2. *$C \subset X$ be a closed subset of X (def. 2.23) which is f-saturated (def. 3.17);*

3. *$U \supset C$ be an open subset containing C;*

then there exists a smaller open subset V still containing C

$$U \supset V \supset C$$

and such that V is still f-saturated.

Proof. We claim that the complement of X by the f-saturation (def. 3.17) of the complement of X by U

$$V \coloneqq X \backslash \left(f^{-1}(f(X \backslash U))\right)$$

has the desired properties. To see this, observe first that

1. the complement $X \backslash U$ is closed, since U is assumed to be open;

2. hence the image $f(X \backslash U)$ is closed, since f is assumed to be a closed map;

3. hence the pre-image $f^{-1}(f(X \backslash U))$ is closed, since f is continuous (using prop. 3.2), therefore its complement V is indeed open;

4. this pre-image $f^{-1}(f(X \backslash U))$ is saturated (by example 3.18) and hence also its complement V is saturated (by lemma 3.19).

Therefore it now only remains to see that $U \supset V \supset C$.

By de Morgan's law (remark 2.24) the inclusion $U \supset V$ is equivalent to the inclusion $f^{-1}(f(X \backslash U)) \supset X \backslash U$, which is clearly the case.

The inclusion $V \supset C$ is equivalent to $f^{-1}(f(X \backslash U)) \cap C = \emptyset$. Since C is saturated by assumption, this is equivalent to $f^{-1}(f(X \backslash U)) \cap f^{-1}(f(C)) = \emptyset$. This in turn holds precisely if $f(X \backslash U) \cap f(C) = \emptyset$. Since C is saturated, this holds precisely if

$X \backslash U \cap C = \emptyset$, and this is true by the assumption that $U \supset C$. ▮

Homeomorphisms

With the objects (topological spaces) and the morphisms (continuous functions) of the category Top thus defined (remark 3.3), we obtain the concept of “sameness” in topology. To make this precise, one says that a morphism

$$X \overset{f}{\to} Y$$

in a category is an *isomorphism* if there exists a morphism going the other way around

$$X \overset{g}{\longleftarrow} Y$$

which is an inverse in the sense that both its compositions with f yield an identity morphism:

$$f \circ g = \mathrm{id}_Y \quad \text{and} \quad g \circ f = \mathrm{id}_X \; .$$

Since such g is unique if it exsist, one often writes “f^{-1}” for this inverse morphism. However, in the context of topology then f^{-1} usually refers to the pre-image function of a given function f, and in these notes we will stick to this usage and never use “$(-)^{-1}$” to denote inverses.

Definition 3.22. **(homeomorphisms)**

An isomorphism in the category Top (remark 3.3) of topological spaces (def. 2.3) with continuous functions between them (def. 3.1) is called a *homeomorphism*.

Hence a *homeomorphism* is a continuous function

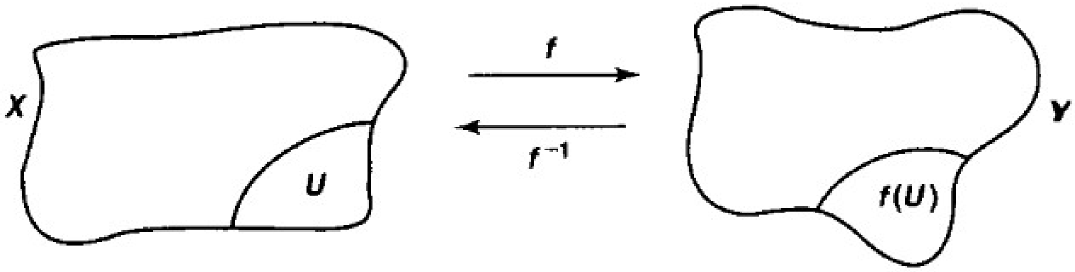
$$f \;:\; (X, \tau_X) \longrightarrow (Y, \tau_Y)$$

between two topological spaces (X, τ_X), (Y, τ_Y) such that there exists another continuous function the other way around

$$(X, \tau_X) \longleftarrow (Y, \tau_Y) \;:\; g$$

such that their composites are the identity functions on X and Y, respectively:

$$f \circ g = \mathrm{id}_Y \quad \text{and} \quad g \circ f = \mathrm{id}_X \; .$$



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We notationally indicate that a continuous function is a homeomorphism by the symbol "≃".

$$f \;:\; (X,\tau_X) \overset{\simeq}{\longrightarrow} (Y,\tau_Y)\;.$$

If there is *some*, possibly unspecified, homeomorphism between topological spaces (X,τ_X) and (Y,τ_Y), then we also write

$$(X,\tau_X) \;\simeq\; (Y,\tau_Y)$$

and say that the two topological spaces *are homeomorphic.*

A property/predicate P of topological spaces which is invariant under homeomorphism in that

$$\left((X,\tau_X) \;\simeq\; (Y,\tau_Y) \right) \;\Rightarrow\; \left(P(X,\tau_X) \;\Leftrightarrow\; P(Y,\tau_Y) \right)$$

is called a *topological property* or *topological invariant*.

Remark 3.23. If $f : (X,\tau_X) \to (Y,\tau_Y)$ is a homeomorphism (def. 3.22) with inverse coninuous function g, then

1. also g is a homeomophism, with inverse continuous function f;

2. the underlying function of sets $f : X \to Y$ of a homeomorphism f is necessarily a bijection, with inverse bijection g.

But beware that not every continuous function which is bijective on underlying sets is a homeomorphism. While an inverse function g will exists on the level of functions of sets, this inverse may fail to be continuous:

Counter Example 3.24. Consider the continuous function

$$\begin{array}{ccc} [0,2\pi) & \longrightarrow & S^1 \subset \mathbb{R}^2 \\ t & \mapsto & (\cos(t), \sin(t)) \end{array}$$

from the half-open interval (def. 1.13) to the unit circle $S^1 \coloneqq S_0(1) \subset \mathbb{R}^2$ (def. 1.2), regarded as a topological subspace (example 2.16) of the Euclidean plane (example 1.6).

The underlying function of sets of f is a bijection. The inverse function of sets however fails to be continuous at $(1,0) \in S^1 \subset \mathbb{R}^2$. Hence this f is *not* a homeomorphism.

Indeed, below we see that the two topological spaces $[0,2\pi)$ and S^1 are distinguished by topological invariants, meaning that they cannot be homeomorphic via *any* (other) choice of homeomorphism. For example S^1 is a compact topological space (def. 7.2) while $[0,2\pi)$ is not, and S^1 has a non-trivial fundamental group, while that of $[0,2\pi)$ is trivial (this prop.).

Below in example 7.45 we discuss a practical criterion under which continuous bijections are homeomorphisms after all. But immediate from the definitions is the following characterization:

Proposition 3.25. ***(homeomorphisms are the continuous and open bijections)***

Let $f : (X,\tau_X) \longrightarrow (Y,\tau_Y)$ be a continuous function between topological spaces (def. 3.1). Then the following are equivalence:

1. *f is a homeomorphism;*

2. *f is a bijection and an open map (def. 3.14);*

3. *f is a bijection and a closed map (def. 3.14).*

Proof. It is clear from the definition that a homeomorphism in particular has to be a bijection. The condition that the inverse function $Y \leftarrow X : g$ be continuous means that the pre-image function of g sends open subsets to open subsets. But by g being the inverse to f, that pre-image function is equal to f, regarded as a function on subsets:

$$g^{-1} = f \,:\, P(X) \to P(Y)\;.$$

Hence g^{-1} sends opens to opens precisely if f does, which is the case precisely if f is an open map, by definition. This shows the equivalence of the first two items. The equivalence between the first and the third follows similarly via prop. 3.2. ▮

Now we consider some actual **examples** of homeomorphisms:

Example 3.26. **(concrete point homeomorphic to abstract point space)**

Let (X,τ_X) be a non-empty topological space, and let $x \in X$ be any point. Regard the corresponding singleton subset $\{x\} \subset X$ as equipped with its subspace topology $\tau_{\{x\}}$ (example 2.16). Then this is homeomorphic (def. 3.22) to the abstract point space from example 2.10:

$$(\{x\}, \tau_{\{x\}}) \;\simeq\; * \;.$$

Example 3.27. **(open interval homeomorphic to the real line)**

Regard the real line as the 1-dimensional Euclidean space (example 1.6) with its metric topology (example 2.9).

Then the open interval $(-1,1) \subset \mathbb{R}$ (def. 1.13) regarded with its subspace topology (example 2.16) is homeomorphic (def.3.22) to all of the real line

$$(-1,1) \simeq \mathbb{R}^1 \,.$$

An inverse pair of continuous functions is for instance given (via example 1.10) by

$$\begin{array}{ccc} f & : & \mathbb{R}^1 & \longrightarrow & (-1,+1) \\ && x & \mapsto & \frac{x}{\sqrt{1+x^2}} \end{array}$$

and

$$\begin{array}{ccc} g & : & (-1,+1) & \longrightarrow & \mathbb{R}^1 \\ && x & \mapsto & \frac{x}{\sqrt{1-x^2}} \end{array} \,.$$

But there are many other choices for f and g that yield a homeomorphism.

Similarly, for all $a < b \in \mathbb{R}$

1. the open intervals $(a,b) \subset \mathbb{R}$ (example 1.13) equipped with their subspace topology are all homeomorphic to each other,

2. the closed intervals $[a,b]$ are all homeomorphic to each other,

3. the half-open intervals of the form $[a,b)$ are all homeomophic to each other;

4. the half-open intervals of the form $(a,b]$ are all homeomophic to each other.

Generally, every open ball in $\mathbb{R}^n$ (def. 1.2) is homeomorphic to all of $\mathbb{R}^n$:

$$\left(B^\circ_0(\epsilon) \subset \mathbb{R}^n\right) \simeq \mathbb{R}^n \,.$$

While mostly the interest in a given homeomorphism is in it being non-obvious from the definitions, many homeomorphisms that appear in practice exhibit “obvious re-identifications” for which it is of interest to leave them *consistently implicit*:

Example 3.28. **(homeomorphisms between iterated product spaces)**

Let (X,τ_X), (Y,τ_Y) and (Z,τ_Z) be topological spaces.

Then:

1. There is an evident homeomorphism between the two ways of bracketing the three factors when forming their product topological space (def. 2.18), called the *associator*:

$$\alpha_{X,Y,Z} \;:\; ((X,\tau_X) \times (Y,\tau_Y)) \times (Z,\tau_Z) \overset{\simeq}{\longrightarrow} (X,\tau_X) \times ((Y,\tau_Y) \times (Z,\tau_Z)) \;.$$

2. There are evident homeomorphism between (X,τ) and its product topological space (def. 2.18) with the point space $*$ (example 2.10), called the left and right *unitors*:

$$\lambda_X \;:\; * \times (X,\tau_X) \overset{\simeq}{\longrightarrow} (X,\tau_X)$$

and

$$\rho_X \;:\; (X,\tau_X) \times * \overset{\simeq}{\longrightarrow} (X,\tau_X) \;.$$

3. There is an evident homeomorphism between the results of the two orders in which to form their product topological spaces (def. 2.18), called the *braiding*:

$$\beta_{X,Y} \;:\; (X,\tau_X) \times (Y,\tau_Y) \overset{\simeq}{\longrightarrow} (Y,\tau_Y) \times (X,\tau_X) \;.$$

Moreover, all these homeomorphisms are compatible with each other, in that they make the following diagrams commute (recall remark 3.3):

1. (triangle identity)

$$\begin{tikzcd} (X \times *) \times Y \arrow[rr, "\alpha_{X,*,Y}"] \arrow[dr, "\rho_X \times \mathrm{id}_Y"'] & & X \times (* \times Y) \arrow[dl, "\mathrm{id}_X \times \lambda_Y"] \\ & X \times Y & \end{tikzcd}$$

2. (pentagon identity)

$$\begin{tikzcd} & (W \times X) \times (Y \times Z) \arrow[dr, "\alpha_{W,X,Y\times Z}"] & \\ ((W \times X) \times Y) \times Z \arrow[ur, "\alpha_{W\times X,Y,Z}"] \arrow[d, "\alpha_{W,X,Y} \times \mathrm{id}_Z"'] & & (W \times (X \times (Y \times Z))) \\ (W \times (X \times Y)) \times Z \arrow[rr, "\alpha_{W,X\times Y,Z}"'] & & W \times ((X \times Y) \times Z) \arrow[u, "\mathrm{id}_W \times \alpha_{X,Y,Z}"'] \end{tikzcd}$$

3. (hexagon identities)

$$\begin{tikzcd} (X \times Y) \times Z \arrow[r, "\alpha_{X,Y,Z}"] \arrow[d, "\beta_{X,Y} \times \mathrm{id}_Z"] & X \times (Y \times Z) \arrow[r, "\beta_{X,Y\times Z}"] & (Y \times Z) \times X \arrow[d, "\alpha_{Y,Z,X}"] \\ (Y \times X) \times Z \arrow[r, "\alpha_{Y,X,Z}"] & Y \times (X \times Z) \arrow[r, "\mathrm{id}_Y \times \beta_{X,Y}"] & Y \times (Z \times X) \end{tikzcd}$$

and

$$\begin{array}{ccccc} X \times (Y \times Z) & \overset{\alpha^{\mathrm{inv}}_{X,Y,Z}}{\longrightarrow} & (X \times Y) \times Z & \overset{\beta_{X \times Y, Z}}{\longrightarrow} & Z \times (X \times Y) \\ \downarrow^{\mathrm{id}_X \times \beta_{Y,Z}} && && \downarrow^{\alpha^{\mathrm{inv}}_{Z,X,Y}} \\ X \times (Z \times Y) & \overset{\alpha^{\mathrm{inv}}_{X,Z,Y}}{\longrightarrow} & (X \times Z) \times Y & \overset{\beta_{X,Z} \times \mathrm{id}}{\longrightarrow} & (Z \times X) \times Y \end{array} ,$$

4. (symmetry)

$$\beta_{Y,X} \circ \beta_{X,Y} \;=\; \mathrm{id} : (X_1 \times X_2 \tau_{X_1 \times X_2}) \to (X_1 \times X_2 \tau_{X_1 \times X_2}) \,.$$

In the language of category theory (remark 3.3), all this is summarized by saying that the the functorial construction $(-) \times (-)$ of product topological spaces (example 3.4) gives the category Top of topological spaces the structure of a *monoidal category* which moreover is *symmetrically braided*.

From this, a basic result of category theory, the *MacLane coherence theorem*, guarantees that there is no essential ambiguity re-backeting arbitrary iterations of the binary product topological space construction, as long as the above homeomorphsims are understood.

Accordingly, we may write

$$(X_1, \tau_1) \times (X_2, \tau_2) \times \cdots \times (X_n, \tau_n)$$

for iterated product topological spaces without putting parenthesis.

The following are a sequence of examples all of the form that an abstractly constructed topological space is homeomorphic to a certain subspace of a Euclidean space. These examples are going to be useful in further developments below, for example in the proof below of the Heine-Borel theorem (prop. 7.37).

- *Products of intervals are homeomorphic to hypercubes* (example 3.29).
- *The closed interval glued at its endpoints is homeomorphic to the circle* (example 3.30).
- *The cylinder, the Möbius strip and the torus are all homeomorphic to quotients of the square* (example 3.31).

Example 3.29. **(product of closed intervals homeomorphic to hypercubes)**

Let $n \in \mathbb{N}$, and let $[a_i, b_i] \subset \mathbb{R}$ for $i \in \{1, \cdots, n\}$ be n closed intervals in the real line (example 1.13), regarded as topological subspaces of the 1-dimensional Euclidean space (example 1.6) with its metric topology (example 2.9). Then the product topological space (def. 2.18, example 3.28) of all these intervals is homeomorphic (def. 3.22) to the corresponding topological subspace of the n-dimensional Euclidean space (example 1.6):

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \;\simeq\; \left\{ \vec{x} \in \mathbb{R}^n \,|\, \underset{i}{\forall} (a_i \leq x_i \leq b_i) \right\} \subset \mathbb{R}^n \,.$$

Proof. There is a canonical bijection between the underlying sets. It remains to see that this, as well and its inverse, are continuous functions. For this it is sufficient to see that under this bijection the defining basis (def. 2.7) for the product topology is also a basis for the subspace topology. But this is immediate from lemma 2.8. ▮

Example 3.30. **(closed interval glued at endpoints homeomorphic circle)**

As topological spaces, the closed interval $[0,1]$ (def. 1.13) with its two endpoints identified is homeomorphic (def. 3.22) to the standard circle:

$$[0,1]_{/(0\sim 1)} \;\simeq\; S^1 \,.$$

More in detail: let

$$S^1 \hookrightarrow \mathbb{R}^2$$

be the unit circle in the plane

$$S^1 = \{(x,y) \in \mathbb{R}^2, x^2 + y^2 = 1\}$$

equipped with the subspace topology (example 2.16) of the plane $\mathbb{R}^2$, which is itself equipped with its standard metric topology (example 2.9).

Moreover, let

$$[0,1]_{/(0\sim 1)}$$

be the quotient topological space (example 2.17) obtained from the interval $[0,1] \subset \mathbb{R}^1$ with its subspace topology by applying the equivalence relation which identifies the two endpoints (and nothing else).

Consider then the function

$$f \,:\, [0,1] \longrightarrow S^1$$

given by

$$t \mapsto (\cos(t), \sin(t)) \,.$$

This has the property that $f(0) = f(1)$, so that it descends to the quotient topological space

$$\begin{array}{ccc} [0,1] & \longrightarrow & [0,1]_{/(0\sim 1)} \\ & {}_{f}\searrow & \downarrow^{\tilde f} \\ && S^1 \end{array} \,.$$

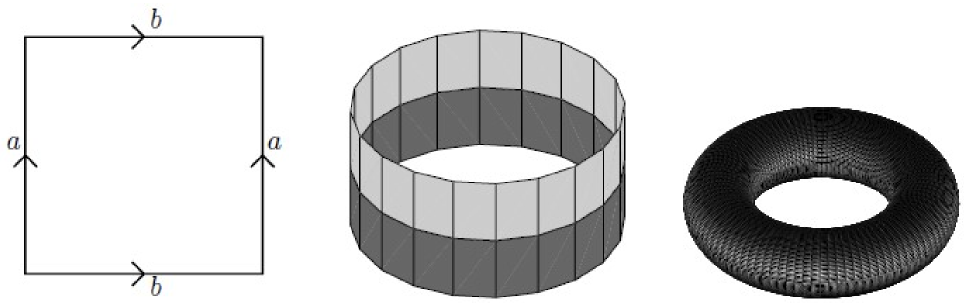
We claim that $\tilde f$ is a homeomorphism (definition 3.22).

First of all it is immediate that $\tilde f$ is a continuous function. This follows immediately from the fact that f is a continuous function and by definition of the quotient topology (example 2.17).

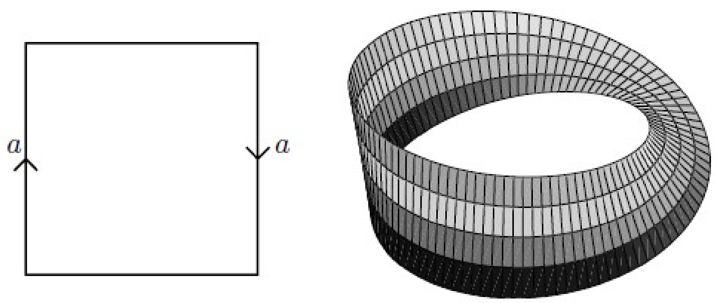
So we need to check that $\tilde f$ has a continuous inverse function. Clearly the restriction of f itself to the open interval $(0,1)$ has a continuous inverse. It fails to have a continuous inverse on $[0,1)$ and on $(0,1]$ and fails to have an inverse at all on [0,1], due to the fact that $f(0) = f(1)$. But the relation quotiented out in $[0,1]_{/(0\sim 1)}$ is exactly such as to fix this failure.

Example 3.31. **(cylinder, Möbius strip and torus homeomorphic to quotients of the square)**

The square $[0,1]^2$ with two of its sides identified is the cylinder, and with also the other two sides identified is the torus:



If the sides are identified with opposite orientation, the result is the *Möbius strip*:



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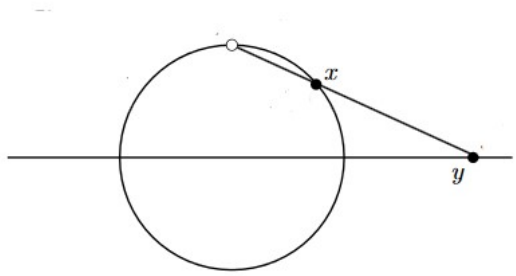
Example 3.32. **(stereographic projection)**

For $n \in \mathbb{N}$ then there is a homeomorphism (def. 3.22) between between the n-sphere S^n (example 2.20) with one point $p \in S^n$ removed and the n-dimensional Euclidean space $\mathbb{R}^n$ (example 1.6) with its metric topology (example 2.9):

$$S^n \backslash \{p\} \xrightarrow{\simeq} \mathbb{R}^n \,.$$

This homeomorphism is given by "stereographic projection": One thinks of both the n-sphere as well as the Euclidean space $\mathbb{R}^n$ as topological subspaces (example 2.16) of $\mathbb{R}^{n+1}$ in the standard way (example 2.20), such that they intersect in the equator of the n-sphere.



For $p \in S^n$ one of the corresponding poles, then the homeomorphism is the function which sends a point $x \in S^n \backslash \{p\}$ along the line connecting it with p to the point y where this line intersects the equatorial plane.

In the canonical ambient coordinates this stereographic projection is given as follows:

$$\begin{array}{ccccc} \mathbb{R}^{n+1} \supset & S^n \backslash (1,0,\cdots,0) & \xrightarrow{\simeq} & \mathbb{R}^n & \subset \mathbb{R}^{n+1} \\ & (x_1, x_2, \cdots, x_{n+1}) & \longmapsto & \frac{1}{1-x_1}(0, x_2, \cdots, x_{n+1}) & \end{array} .$$

Important examples of pairs of spaces that are *not* homeomorphic include the following:

Theorem 3.33. ***(topological invariance of dimension)***

For $n_1, n_2 \in \mathbb{N}$ but $n_1 \neq n_2$, then the Euclidean spaces $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ (example 1.6, example 2.9) are not homeomorphic.

More generally, an open subset in $\mathbb{R}^{n_1}$ is never homeomorphic to an open subset in $\mathbb{R}^{n_2}$ if $n_1 \neq n_2$.

The proofs of theorem 3.33 are not elementary, in contrast to how obvious the statement seems to be intuitively. One approach is to use tools from *algebraic topology*: One assigns topological invariants to topological spaces, notably classes in ordinary cohomology or in topological K-theory), quantities that are invariant under homeomorphism, and then shows that these classes coincide for $\mathbb{R}^{n_1} - \{0\}$ and for $\mathbb{R}^{n_2} - \{0\}$ precisely only if $n_1 = n_2$.

One indication that topological invariance of dimension is not an *elementary* consequence of the axioms of topological spaces is that a related "intuitively obvious" statement is in fact false: One might think that there is no *surjective* continuous function $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ if $n_1 < n_2$. But there are: these are called the *Peano curves*.

Often it is important to know whether a given space is homeomorphism to its *image*, under some continuous function, in some other space:

Definition 3.34. **(embedding of topological spaces)**

Let (X, τ_X) and (Y, τ_Y) be topological spaces. A continuous function $f : X \longrightarrow Y$ is called an *embedding of topological spaces* if in its image factorization (example 3.10)

$$f : X \xrightarrow{\simeq} f(X) \hookrightarrow Y$$

with the image $f(X) \hookrightarrow Y$ equipped with the subspace topology, we have that

$X \to f(X)$ is a homeomorphism.

Proposition 3.35. ***(open/closed continuous injections are embeddings)***

A continuous function $f : (X, \tau_X) \to (Y, \tau_Y)$ which is

1. *an injective function*
2. *an open map or a closed map (def. 3.14)*

is an embedding of topological spaces (def. 3.34).

This is called a closed embedding if the image $f(X) \subset Y$ is a closed subset.

Proof. If f is injective, then the map onto its image $X \to f(X) \subset Y$ is a bijection. Moreover, it is still continuous with respect to the subspace topology on $f(X)$ (example 3.10). Now a bijective continuous function is a homeomorphism precisely if it is an open map or a closed map prop. 3.25 . But the image projection of f has this property, respectively, if f does, by prop 3.15. ▮

4. Separation axioms

The plain definition of *topological space* (above) happens to admit examples where distinct points or distinct subsets of the underlying set appear as more-or-less unseparable as seen by the topology on that set.

The extreme class of examples of topological spaces in which the open subsets do not distinguish distinct underlying points, or in fact any distinct subsets, are the codiscrete spaces (example 2.13). This does occur in practice:

Example 4.1. **(real numbers quotiented by rational numbers)**

Consider the real line $\mathbb{R}$ regarded as the 1-dimensional Euclidean space (example 1.6) with its metric topology (example 2.9) and consider the equivalence relation $\sim$ on $\mathbb{R}$ which identifies two real numbers if they differ by a rational number:

$$(x \sim y) \;\Leftrightarrow\; \left(\underset{p/q \in \mathbb{Q} \subset \mathbb{R}}{\exists} (x = y + p/q) \right) .$$

Then the quotient topological space (def. 2.17)

$$\mathbb{R}/\mathbb{Q} \;\coloneqq\; \mathbb{R}/\sim$$

is a codiscrete topological space (def. 2.13), hence its topology does not distinguish any distinct proper subsets.

Here are some less extreme examples:

Example 4.2. **(open neighbourhoods in the Sierpinski space)**

Consider the Sierpinski space from example 2.11, whose underlying set consists of two points $\{0,1\}$, and whose open subsets form the set $\tau = \{\emptyset, \{1\}, \{0,1\}\}$. This means that the only (open) neighbourhood of the point $\{0\}$ is the entire space.

Incidentally, also the topological closure of $\{0\}$ (def. 2.23) is the entire space.

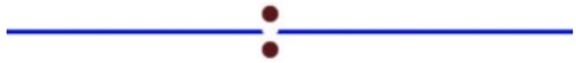
Example 4.3. (line with two origins)

Consider the disjoint union space $\mathbb{R} \sqcup \mathbb{R}$ (example 2.15) of two copies of the real line $\mathbb{R}$ regarded as the 1-dimensional Euclidean space (example 1.6) with its metric topology (example 2.9), which is equivalently the product topological space (example 2.18) of $\mathbb{R}$ with the discrete topological space on the 2-element set (example 2.13):

$$\mathbb{R} \sqcup \mathbb{R} \;\simeq\; \mathbb{R} \times \mathrm{Disc}(\{0,1\})$$

Moreover, consider the equivalence relation on the underlying set which identifies every point x_i in the ith copy of $\mathbb{R}$ with the corresponding point in the other, the $(1-i)$th copy, except when $x = 0$:

$$\left(x_i \sim y_j\right) \;\Leftrightarrow\; \left((x = y) \,\mathrm{and}\, ((x \neq 0) \,\mathrm{or}\, (i = j))\right) \,.$$



The quotient topological space by this equivalence relation (def. 2.17)

$$(\mathbb{R} \sqcup \mathbb{R})/\sim$$

is called the **line with two origins**. These "two origins" are the points 0_0 and 0_1.

We claim that in this space *every neighbourhood of 0_0 intersects every neighbouhood of 0_1.*

Because, by definition of the quotient space topology, the open neighbourhoods of $0_i \in (\mathbb{R} \sqcup \mathbb{R})/\sim$ are precisely those that contain subsets of the form

$$(-\epsilon, \epsilon)_i \;\coloneqq\; (-\epsilon, 0) \cup \{0_i\} \cup (0, \epsilon) \,.$$

But this means that the "two origins" 0_0 and 0_1 may not be separated by neighbourhoods, since the intersection of $(-\epsilon,\epsilon)_0$ with $(-\epsilon,\epsilon)_i$ is always non-empty:

$$(-\epsilon,\epsilon)_0 \cap (-\epsilon,\epsilon)_1 \;=\; (-\epsilon, 0) \cup (0, \epsilon) \,.$$

In many applications one wants to exclude at least some such exotic examples of topologial spaces from the discussion and instead concentrate on those examples for which the topology recognizes the separation of distinct points, or of more general disjoint subsets. The relevant conditions to be imposed on top of the plain axioms of a topological space are hence known as *separation axioms* which we discuss in the following.

These axioms are all of the form of saying that two subsets (of certain kinds) in the topological space are 'separated' from each other in one sense if they are 'separated' in a (generally) weaker sense. For example the weakest axiom (called T_0) demands that if two points are distinct as elements of the underlying set of points, then there exists at least one open subset that contains one but not the other.

In this fashion one may impose a hierarchy of stronger axioms. For example demanding that given two distinct points, then each of them is contained in some open subset not containing the other (T_1) or that such a pair of open subsets around two distinct points may in addition be chosen to be disjoint (T_2). Below in *Tn-spaces* we discuss the following hierarchy:

the **main separation axioms**

number	name	statement	reformulation
T_0	Kolmogorov	given two distinct points, at least one of them has an open neighbourhood not containing the other point	every irreducible closed subset is the closure of at most one point
T_1		given two distinct points, both have an open neighbourhood not containing the other point	all points are closed
T_2	Hausdorff	given two distinct points, they have disjoint open neighbourhoods	the diagonal is a closed map
$T_{>2}$		T_1 and…	all points are closed and…
T_3	regular Hausdorff	…given a point and a closed subset not containing it, they have disjoint open neighbourhoods	…every neighbourhood of a point contains the closure of an open neighbourhood
T_4	normal Hausdorff	…given two disjoint closed subsets, they have disjoint open neighbourhoods	…every neighbourhood of a closed set also contains the closure of an open neighbourhood … every pair of disjoint closed subsets is separated by an Urysohn function

The condition, T_2, also called the *Hausdorff condition* is the most common among all separation axioms. Historically this axiom was originally taken as part of the definition of topological spaces, and it is still often (but by no means always) considered by default.

However, there are respectable areas of mathematics that involve topological spaces where the Hausdorff axiom fails, but a weaker axiom is still satisfied, called sobriety. This is the case notably in algebraic geometry (schemes are sober) and in computer science (Vickers 89). These *sober topological spaces* are singled out by the fact that they are entirely characterized by their sets of open subsets with their union and intersection structure (as in def. 2.34) and may hence be understood independently from their underlying sets of points. This we discuss further below.

hierarchy of separation axioms

hierarchy of separation axioms

$$
\array{
 && \text{metric space} \\
 && \Downarrow \\
 && \vdots \\
 && \Downarrow \\
 && T_4 = \text{normal Hausdorff} \\
 && \Downarrow \\
 && T_3 = \text{regular Hausdorff} \\
 && \Downarrow \\
 && T_2 = \text{Hausdorff} \\
 & \swArrow && \seArrow \\
 T_1 &&&& \text{sober} \\
 & \seArrow && \swArrow \\
 && T_0 = \text{Kolmogorov}
}
$$

All separation axioms are satisfied by metric spaces (example 4.8, example 4.14 below), from whom the concept of topological space was originally abstracted above. Hence imposing some of them may also be understood as gauging just how far one allows topological spaces to generalize away from metric spaces

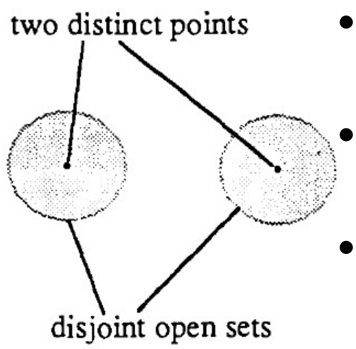
T_n spaces

There are many variants of separation axims. The classical ones are labeled T_n (for German "Trennungsaxiom") with $n \in \{0, 1, 2, 3, 4, 5\}$ or higher. These we now introduce in def. 4.4 and def. 4.13.

Definition 4.4. (the first three separation axioms)

Let (X, τ) be a topological space (def. 2.3).

For $x \neq y \in X$ any two points in the underlying set of X which are not equal as elements of this set, consider the following propositions:



- **(T0)** *There exists a neighbourhood of one of the two points which does not contain the other point.*
- **(T1)** *There exist neighbourhoods of both points which do not contain the other point.*
- **(T2)** *There exists neighbourhoods_ of both points which do not intersect each other.*

graphics grabbed from Vickers 89

The topological space X is called a T_n*-topological space* or just T_n*-space*, for short, if it satisfies condition T_n above for all pairs of distinct points.

A T_0-topological space is also called a *Kolmogorov space*.

A T_2-topological space is also called a *Hausdorff topological space*.

For definiteness, we re-state these conditions formally. Write $x, y \in X$ for points in X, write $U_x, U_y \in \tau$ for open neighbourhoods of these points. Then:

- **(T0)** $\underset{x \neq y}{\forall} \left(\left(\underset{U_y}{\exists} \left(\{x\} \cap U_y = \emptyset \right) \right) \text{or} \left(\underset{U_x}{\exists} \left(U_x \cap \{y\} = \emptyset \right) \right) \right)$

- **((T1)** $\underset{x \neq y}{\forall} \left(\underset{U_x, U_y}{\exists} \left(\left(\{x\} \cap U_y = \emptyset \right) \text{and} \left(U_x \cap \{y\} = \emptyset \right) \right) \right)$

- **(T2)** $\underset{x \neq y}{\forall} \left(\underset{U_x, U_y}{\exists} \left(U_x \cap U_y = \emptyset \right) \right)$

The following is evident but important:

Proposition 4.5. ***(T_n are topological properties of increasing strength)***

The separation properties T_n from def. 4.4 are topological properties in that if two topological spaces are homeomorphic (def. 3.22) then one of them satisfies T_n precisely if the other does.

Moreover, these properties imply each other as

$$T2 \Rightarrow T1 \Rightarrow T0 \,.$$

Example 4.6. Examples of topological spaces that are *not* Hausdorff (def. 4.4) include

1. the Sierpinski space (example 4.2),
2. the line with two origins (example 4.3),
3. the quotient topological space $\mathbb{R}/\mathbb{Q}$ (example 4.1).

Example 4.7. **(finite T_1-spaces are discrete)**

For a finite topological space (X, τ), hence one for which the underlying set X is a finite set, the following are equivalent:

1. (X, τ) is T_1 (def. 4.4);
2. (X, τ) is a discrete topological space (def. 2.13).

Example 4.8. **(metric spaces are Hausdorff)**

Every metric space (def 1.1), regarded as a topological space via its metric topology (example 2.9) is a Hausdorff topological space (def. 4.4).

Because for $x \neq y \in X$ two distinct points, then the distance $d(x, y)$ between them is positive number, by the non-degeneracy axiom in def. 1.1. Accordingly the open balls (def. 1.2)

$$B_x^\circ(d(x,y)) \supset \{x\} \quad \text{and} \quad B_y^\circ(d(x,y)) \supset \{y\}$$

are disjoint open neighbourhoods.

Example 4.9. **(subspace of T_n-space is T_n)**

Let (X,τ) be a topological space satisfying the T_n separation axiom for some $n \in \{0,1,2\}$ according to def. 4.4. Then also every topological subspace $S \subset X$ (example 2.16) satisfies T_n.

Separation in terms of topological closures

The conditions T_0, T_1 and T_2 have the following equivalent formulation in terms of topological closures (def. 2.23).

Proposition 4.10. ***(T_0 in terms of topological closures)***

A topological space (X,τ) is T_0 (def. 4.4) precisely if the function $\mathrm{Cl}(\{-\})$ that forms topological closures (def. 2.23) of singleton subsets from the underlying set of X to the set of irreducible closed subsets of X (def. 2.30, which is well defined according to example 2.31), is injective:

$$\mathrm{Cl}(\{-\}) \; : \; X \hookrightarrow \mathrm{IrrClSub}(X)$$

Proof. Assume first that X is T_0. Then we need to show that if $x,y \in X$ are such that $\mathrm{Cl}(\{x\}) = \mathrm{Cl}(\{y\})$ then $x = y$. Hence assume that $\mathrm{Cl}(\{x\}) = \mathrm{Cl}(\{y\})$. Since the closure of a point is the complement of the union of the open subsets not containing the point (lemma 2.25), this means that the union of open subsets that do not contain x is the same as the union of open subsets that do not contain y:

$$\underset{\substack{U \subset X \,\text{open} \\ U \subset X \setminus \{x\}}}{\cup} (U) \;=\; \underset{\substack{U \subset X \,\text{open} \\ U \subset X \setminus \{y\}}}{\cup} (U)$$

But if the two points were distinct, $x \neq y$, then by T_0 one of the above unions would contain x or y, while the other would not, in contradiction to the above equality. Hence we have a proof by contradiction.

Conversely, assume that $(\mathrm{Cl}\{x\} = \mathrm{Cl}\{y\}) \Rightarrow (x = y)$, and assume that $x \neq y$. Hence by contraposition $\mathrm{Cl}(\{x\}) \neq \mathrm{Cl}(\{y\})$. We need to show that there exists an open set which contains one of the two points, but not the other.

Assume there were no such open subset, hence that every open subset containing one of the two points would also contain then other. Then by lemma 2.25 this would mean that $x \in \mathrm{Cl}(\{y\})$ and that $y \in \mathrm{Cl}(\{x\})$. But this would imply that $\mathrm{Cl}(\{x\}) \subset \mathrm{Cl}(\{y\})$ and that $\mathrm{Cl}(\{y\}) \subset \mathrm{Cl}(\{x\})$, hence that $\mathrm{Cl}(\{x\}) = \mathrm{Cl}(\{y\})$. This is a proof by contradiction. ▮

Proposition 4.11. ***(T_1 in terms of topological closures)***

A topological space (X,τ) is T_1 (def. 4.4) precisely if all its points are closed points (def. 2.23).

Proof. We have

$$
\begin{aligned}
\text{all points in } (X,\tau) \text{ are closed} & \coloneqq \underset{x \in X}{\forall} \left(\mathrm{Cl}(\{x\}) = \{x\}\right) \\
& \Leftrightarrow X \backslash \left(\underset{\substack{U \subset X \text{ open} \\ x \notin U}}{\cup} (U) \right) = \{x\} \\
& \Leftrightarrow \left(\underset{\substack{U \subset X \text{ open} \\ x \notin U}}{\cup} (U) \right) = X \backslash \{x\} \\
& \Leftrightarrow \underset{y \in Y}{\forall} \left(\left(\underset{\substack{U \subset X \text{ open} \\ x \notin U}}{\exists} (y \in U) \right) \Leftrightarrow (y \neq x) \right) \\
& \Leftrightarrow (X,\tau) \text{ is } T_1
\end{aligned}
.
$$

Here the first step is the reformulation of closure from lemma 2.25, the second is another application of the de Morgan law (remark 2.24), the third is the definition of union and complement, and the last one is manifestly by definition of T_1. ▮

Proposition 4.12. ***(T_2 in terms of topological closures)***

A topological space (X,τ_X) is T_2=Hausdorff precisely if the image of the diagonal

$$
\begin{aligned}
X & \overset{\Delta_X}{\longrightarrow} X \times X \\
x & \longmapsto (x,x)
\end{aligned}
$$

is a closed subset in the product topological space $(X \times X, \tau_{X \times X})$.

Proof. Observe that the Hausdorff condition is equivalently rephrased in terms of the product topology as: *Every point $(x,y) \in X$ which is not on the diagonal has an open neighbourhood $U_{(x,y)} \times U_{(x,y)}$ which still does not intersect the diagonal*, hence:

$$
\begin{aligned}
& (X,\tau) \text{ Hausdorff} \\
\Leftrightarrow & \underset{(x,y) \in (X \times X) \backslash \Delta_X(X)}{\forall} \left(\underset{\substack{U_{(x,y)} \times V_{(x,y)} \in \tau_{X \times Y} \\ (x,y) \in U_{(x,y)} \times V_{(x,y)}}}{\exists} \left(U_{(x,y)} \times V_{(x,y)} \cap \Delta_X(X) = \emptyset \right) \right)
\end{aligned}
$$

Therefore if X is Hausdorff, then the diagonal $\Delta_X(X) \subset X \times X$ is the complement of a union of such open sets, and hence is closed:

$$
(X,\tau) \text{ Hausdorff} \quad \Rightarrow \quad \Delta_X(X) = X \backslash \left(\underset{(x,y) \in (X \times X) \backslash \Delta_X(X)}{\cup} U_{(x,y)} \times V_{(x,y)} \right).
$$

Conversely, if the diagonal is closed, then (by lemma 2.25) every point $(x,y) \in X \times X$ not on the diagonal, hence with $x \neq y$, has an open neighbourhood $U_{(x,y)} \times V_{(x,y)}$ still not intersecting the diagonal, hence so that $U_{(x,y)} \cap V_{(x,y)} = \emptyset$. Thus (X,τ) is Hausdorff. ▮

Further separation axioms

Clearly one may and does consider further variants of the separation axioms T_0, T_1 and T_2 from def. 4.4. Here we discuss two more:

Definition 4.13. Let (X,τ) be topological space (def. 4.4).

Consider the following conditions

- **(T3)** The space (X,τ) is T_1 (def. 4.4) and for $x \in X$ a point and $C \subset X$ a closed subset (def. 2.23) not containing x, then there exist disjoint open neighbourhoods $U_x \supset \{x\}$ and $U_C \supset C$.
- **(T4)** The space (X,τ) is T_1 (def. 4.4) and for $C_1, C_2 \subset X$ two disjoint closed subsets (def. 2.23) then there exist disjoint open neighbourhoods $U_{C_i} \supset C_i$.

If (X,τ) satisfies T_3 it is said to be a T_3-*space* also called a *regular Hausdorff topological space*.

If (X,τ) satisfies T_4 it is to be a T_4-*space* also called a *normal Hausdorff topological space*.

Example 4.14. **(metric spaces are normal Hausdorff)**

Let (X,d) be a metric space (def. 1.1) regarded as a topological space via its metric topology (example 2.9). Then this is a normal Hausdorff space (def. 4.13).

Proof. By example 4.8 metric spaces are T_2, hence in particular T_1. What we need to show is that given two disjoint closed subsets $C_1, C_2 \subset X$ then their exists disjoint open neighbourhoods $U_{C_1} \subset C_1$ and $U_{C_2} \supset C_2$.

Recall the function

$$d(S,-) : X \to \mathbb{R}$$

computing distances from a subset $S \subset X$ (example 1.9). Then the unions of open balls (def. 1.2)

$$U_{C_1} \coloneqq \underset{x_1 \in C_1}{\cup} B^\circ_{x_1}(d(C_2,x_1)/2)$$

and

$$U_{C_2} \coloneqq \underset{x_2 \in C_2}{\cup} B^\circ_{x_2}(d(C_1,x_2)/2) \,.$$

have the required properties. ▮

Observe that:

Proposition 4.15. ***(T_n are topological properties of increasing strength)***

The separation axioms from def. 4.4, def. 4.13 are topological properties (def. 3.22) which imply each other as

$$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0 \; .$$

Proof. The implications

$$T_2 \Rightarrow T_1 \Rightarrow T_0$$

and

$$T_4 \Rightarrow T_3$$

are immediate from the definitions. The remaining implication $T_3 \Rightarrow T_2$ follows with prop. 4.11: This says that by assumption of T_1 then all points in (X,τ) are closed, and with this the condition T_2 is manifestly a special case of the condition for T_3. ▮

Hence instead of saying "X is T_1 and …" one could just as well phrase the conditions T_3 and T_4 as "X is T_2 and …", which would render the proof of prop. 4.15 even more trivial.

The following shows that not every T_2-space/Hausdorff space is T_3/regular

Example 4.16. **(K-topology)**

Write

$$K \coloneqq \{1/n \mid n \in \mathbb{N}_{\geq 1}\} \subset \mathbb{R}$$

for the subset of natural fractions inside the real numbers.

Define a topological basis $\beta \subset P(\mathbb{R})$ on $\mathbb{R}$ consisting of all the open intervals as well as the complements of K inside them:

$$\beta \;\coloneqq\; \{(a,b), \mid a < b \in \mathbb{R}\} \cup \{(a,b)\backslash K, \mid a < b \in \mathbb{R}\} \; .$$

The topology $\tau_\beta \subset P(\mathbb{R})$ which is generated from this topological basis is called the *K-topology*.

We may denote the resulting topological space by

$$\mathbb{R}_K \;\coloneqq\; (\mathbb{R}, \tau_\beta\} \; .$$

This is a Hausdorff topological space (def. 4.4) which is not a regular Hausdorff space, hence (by prop. 4.15) in particular not a normal Hausdorff space (def. 4.13).

Further separation axioms in terms of topological closures

As before we have equivalent reformulations of the further separation axioms.

Proposition 4.17. ***(T_3 in terms of topological closures)***

A topological space (X,τ) is a regular Hausdorff space (def. 4.13), precisely if all points are closed and for all points $x \in X$ with open neighbourhood $U \supset \{x\}$ there exists a smaller open neighbourhood $V \supset \{x\}$ whose topological closure $\mathrm{Cl}(V)$ is still contained in U:

$$\{x\} \subset V \subset \mathrm{Cl}(V) \subset U \,.$$

The **proof** of prop. 4.17 is the direct specialization of the following proof for prop. 4.18 to the case that $C = \{x\}$ (using that by T_1, which is part of the definition of T_3, the singleton subset is indeed closed, by prop. 4.11).

Proposition 4.18. ***(T_4 in terms of topological closures)***

A topological space (X,τ) is normal Hausdorff space (def. 4.13), precisely if all points are closed and for all closed subsets $C \subset X$ with open neighbourhood $U \supset C$ there exists a smaller open neighbourhood $V \supset C$ whose topological closure $\mathrm{Cl}(V)$ is still contained in U:

$$C \subset V \subset \mathrm{Cl}(V) \subset U \,.$$

Proof. In one direction, assume that (X,τ) is normal, and consider

$$C \subset U \,.$$

It follows that the complement of the open subset U is closed and disjoint from C:

$$C \cap X \backslash U = \emptyset \,.$$

Therefore by assumption of normality of (X,τ), there exist open neighbourhoods with

$$V \supset C \,, \quad W \supset X \backslash U \quad \text{with} \quad V \cap W = \emptyset \,.$$

But this means that

$$V \subset X \backslash W$$

and since the complement $X \backslash W$ of the open set W is closed, it still contains the closure of V, so that we have

$$C \subset V \subset \mathrm{Cl}(V) \subset X \backslash W \subset U$$

as required.

In the other direction, assume that for every open neighbourhood $U \supset C$ of a closed subset C there exists a smaller open neighbourhood V with

$$C \subset V \subset \mathrm{Cl}(V) \subset U \,.$$

Consider disjoint closed subsets

$$C_1, C_2 \subset X \,, \qquad C_1 \cap C_2 = \emptyset \,.$$

We need to produce disjoint open neighbourhoods for them.

From their disjointness it follows that

$$X \backslash C_2 \supset C_1$$

is an open neighbourhood. Hence by assumption there is an open neighbourhood V with

$$C_1 \subset V \subset \mathrm{Cl}(V) \subset X \backslash C_2 \,.$$

Thus

$$V \supset C_1 \,, \qquad X \backslash \mathrm{Cl}(V) \supset C_2$$

are two disjoint open neighbourhoods, as required. ■

But the T_4/normality axiom has yet another equivalent reformulation, which is of a different nature, and will be important when we discuss paracompact topological spaces below:

The following concept of *Urysohn functions* is another approach of thinking about separation of subsets in a topological space, not in terms of their neighbourhoods, but in terms of continuous real-valued "indicator functions" that take different values on the subsets. This perspective will be useful when we consider paracompact topological spaces below.

But the Urysohn lemma (prop. 4.20 below) implies that this concept of separation is in fact equivalent to that of normality of Hausdorff spaces.

Definition 4.19. **(Urysohn function)**

Let (X,τ) be a topological space, and let $A, B \subset X$ be disjoint closed subsets. Then an *Urysohn function separating A from B* is

- a continuous function $f : X \to [0,1]$

to the closed interval equipped with its Euclidean metric topology (example 1.6, example 2.9), such that

- it takes the value 0 on A and the value 1 on B:

$$f(A) = \{0\} \qquad \text{and} \qquad f(B) = \{1\} \,.$$

Proposition 4.20. ***(Urysohn's lemma)***

Let X be a normal Hausdorff topological space (def. 4.13), and let $A, B \subset X$ be two disjoint closed subsets of X. Then there exists an Urysohn function separating A

from B (def. 4.19).

Remark 4.21. Beware, the Urysohn function in prop. 4.20 may take the values 0 or 1 even outside of the two subsets. The condition that the function takes value 0 or 1, respectively, *precisely* on the two subsets corresponds to "perfectly normal spaces".

Proof. of Urysohn's lemma, prop. 4.20

Set

$$C_0 \coloneqq A \qquad U_1 \coloneqq X \backslash B \,.$$

Since by assumption

$$A \cap B = \emptyset \,.$$

we have

$$C_0 \subset U_1 \,.$$

That (X, τ) is normal implies, by lemma 4.18, that every open neighbourhood $U \supset C$ of a closed subset C contains a smaller neighbourhood V together with its topological closure $\mathrm{Cl}(V)$

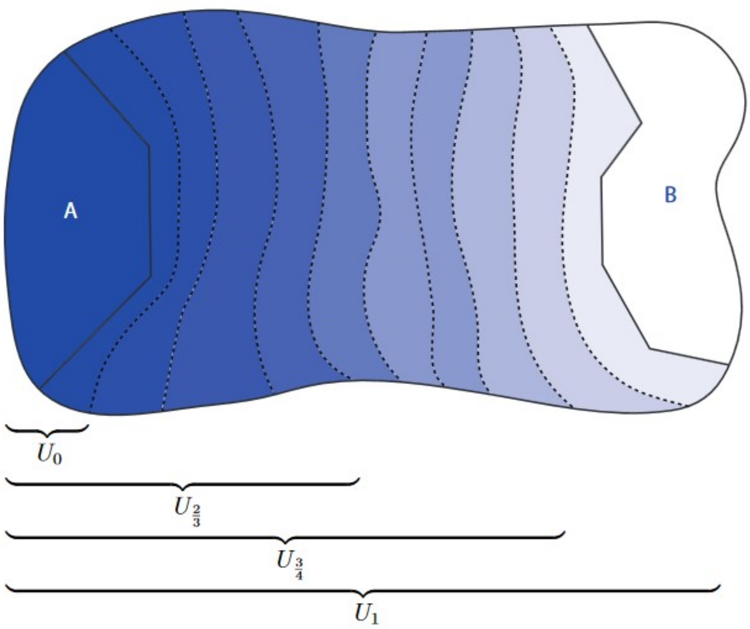
$$U \subset V \subset \mathrm{Cl}(V) \subset C \,.$$

Apply this fact successively to the above situation to obtain the following infinite sequence of nested open subsets U_r and closed subsets C_r

$$\begin{array}{ccccccccccccccc}
C_0 & & & & & & & \subset & & & & & & & U_1 \\
C_0 & & & \subset & & & U_{1/2} & \subset & C_{1/2} & & & \subset & & & U_1 \\
C_0 & \subset & U_{1/4} & \subset & C_{1/4} & \subset & U_{1/2} & \subset & C_{1/2} & \subset & U_{3/4} & \subset & C_{3/4} & \subset & U_1
\end{array}$$

and so on, labeled by the dyadic rational numbers $\mathbb{Q}_{\mathrm{dy}} \subset \mathbb{Q}$ within $(0,1]$

$$\{U_r \subset X\}_{r \in (0,1] \cap \mathbb{Q}_{\mathrm{dy}}}$$



with the property

$$\underset{r_1 < r_2 \in (0,1] \cap \mathbb{Q}_{\mathrm{dy}}}{\forall} \left(U_{r_1} \subset \mathrm{Cl}(U_{r_1}) \subset U_{r_2} \right) \,.$$

Define then the function

$$f : X \longrightarrow [0,1]$$

to assign to a point $x \in X$ the infimum of the labels of those open subsets in this sequence that contain x:

$$f(x) \coloneqq \lim_{U_r \supset \{x\}} r$$

Here the limit is over the directed set of those U_r that contain x, ordered by reverse inclusion.

This function clearly has the property that $f(A) = \{0\}$ and $f(B) = \{1\}$. It only remains to see that it is continuous.

To this end, first observe that

$$\begin{array}{cccc} (\star) & (x \in \mathrm{Cl}(U_r)) & \Rightarrow & (f(x) \leq r) \\ (\star\,\star) & (x \in U_r) & \Leftarrow & (f(x) < r) \end{array} \,.$$

Here it is immediate from the definition that $(x \in U_r) \Rightarrow (f(x) \leq r)$ and that $(f(x) < r) \Rightarrow (x \in U_r \subset \mathrm{Cl}(U_r))$. For the remaining implication, it is sufficient to observe that

$$(x \in \partial U_r) \Rightarrow (f(x) = r)\,,$$

where $\partial U_r \coloneqq \mathrm{Cl}(U_r) \backslash U_r$ is the boundary of U_r.

This holds because the dyadic numbers are dense in $\mathbb{R}$. (And this would fail if we stopped the above decomposition into $U_{a/2^n}$-s at some finite n.) Namely, in one direction, if $x \in \partial U_r$ then for every small positive real number ϵ there exists a dyadic rational number r' with $r < r' < r + \epsilon$, and by construction $U_{r'} \supset \mathrm{Cl}(U_r)$ hence $x \in U_{r'}$. This implies that $\underset{U_r \supset \{x\}}{\lim} = r$.

Now we claim that for all $\alpha \in [0,1]$ then

1. $f^{-1}(\,(\alpha, 1]\,) = \underset{r > \alpha}{\cup} (X \backslash \mathrm{Cl}(U_r))$
2. $f^{-1}(\,[0, \alpha)\,) = \underset{r < \alpha}{\cup} U_r$

Thereby $f^{-1}(\,(\alpha,1]\,)$ and $f^{-1}(\,[0,\alpha)\,)$ are exhibited as unions of open subsets, and hence they are open.

Regarding the first point:

$$\begin{aligned} & x \in f^{-1}(\,(\alpha,1]\,) \\ \Leftrightarrow\; & f(x) > \alpha \\ \Leftrightarrow\; & \underset{r > \alpha}{\exists} (f(x) > r) \\ \overset{(\star)}{\Longrightarrow}\; & \underset{r > \alpha}{\exists} (x \notin \mathrm{Cl}(U_r)) \\ \Leftrightarrow\; & x \in \underset{r > \alpha}{\cup} (X \backslash \mathrm{Cl}(U_r)) \end{aligned}$$

and

$$
\begin{aligned}
& x \in \underset{r > \alpha}{\cup} (X \backslash \mathrm{Cl}(U_r)) \\
\Leftrightarrow & \underset{r > \alpha}{\exists} (x \notin \mathrm{Cl}(U_r)) \\
\Rightarrow & \underset{r > \alpha}{\exists} (x \notin U_r) \\
\overset{(\star\star)}{\Longrightarrow} & \underset{r > \alpha}{\exists} (f(x) \geq r) \\
\Leftrightarrow & f(x) > \alpha \\
\Leftrightarrow & x \in f^{-1}(\,(\alpha, 1]\,)
\end{aligned}
.
$$

Regarding the second point:

$$
\begin{aligned}
& x \in f^{-1}(\,[0, \alpha)\,) \\
\Leftrightarrow & f(x) < \alpha \\
\Leftrightarrow & \underset{r < \alpha}{\exists} (f(x) < r) \\
\overset{(\star\star)}{\Longrightarrow} & \underset{r < \alpha}{\exists} (x \in U_r) \\
\Leftrightarrow & x \in \underset{r < \alpha}{\cup} U_r
\end{aligned}
$$

and

$$
\begin{aligned}
& x \in \underset{r < \alpha}{\cup} U_r \\
\Leftrightarrow & \underset{r < \alpha}{\exists} (x \in U_r) \\
\Rightarrow & \underset{r < \alpha}{\exists} (x \in \mathrm{Cl}(U_r)) \\
\overset{(\star)}{\Longrightarrow} & \underset{r < \alpha}{\exists} (f(x) \leq r) \\
\Leftrightarrow & f(x) < \alpha \\
\Leftrightarrow & x \in f^{-1}(\,[0, \alpha)\,)
\end{aligned}
.
$$

(In these derivations we repeatedly use that $(0,1] \cap \mathbb{Q}_{\mathrm{dy}}$ is dense in $[0,1]$ (def. 2.23), and we use the contrapositions of $(\star)$ and $(\star\star)$.)

Now since the subsets $\{[0,\alpha), (\alpha,1]\}_{\alpha \in [0,1]}$ form a sub-base (def. 2.7) for the Euclidean metric topology on $[0,1]$, it follows that all pre-images of f are open, hence that f is continuous. ▮

As a corollary of Urysohn's lemma we obtain yet another equivalent reformulation of the normality of topological spaces, this one now of a rather different character than the re-formulations in terms of explicit topological closures considered above:

Proposition 4.22. ***(normality equivalent to existence of Urysohn functions)***

A T_1-space (def. 4.4) is normal (def. 4.13) precisely if it admits Urysohn functions (def 4.19) separating every pair of disjoint closed subsets.

Proof. In one direction this is the statement of the Urysohn lemma, prop. 4.20.

In the other direction, assume the existence of Urysohn functions (def. 4.19) separating all disjoint closed subsets. Let $A, B \subset X$ be disjoint closed subsets, then we need to show that these have disjoint open neighbourhoods.

But let $f : X \to [0,1]$ be an Urysohn function with $f(A) = \{0\}$ and $f(B) = \{1\}$ then the pre-images

$$U_A \coloneqq f^{-1}([0,1/3) \qquad U_B \coloneqq f^{-1}((2/3,1])$$

are disjoint open neighbourhoods as required. ▮

T_n reflection

While the topological subspace construction preserves the T_n-property for `n \in \{0,1,2\` (example 4.9) the construction of quotient topological spaces in general does not, as shown by examples 4.1 and 4.3.

Further below we will see that, generally, among all universal constructions in the category Top of all topological spaces those that are limits preserve the T_n property, while those that are colimits in general do not.

But at least for T_0, T_1 and T_2 there is a universal way, called *reflection* (prop. 4.23 below), to approximate any topological space "from the left" by a T_n topological spaces

Hence if one wishes to work within the full subcategory of the T_n-spaces among all topological space, then the correct way to construct quotients and other *colimits* (see below) is to first construct them as usual quotient topological spaces (example 2.17), and then apply the T_n-reflection to the result.

Proposition 4.23. ***(T_n-reflection)***

Let $n \in \{0,1,2\}$. Then for every topological space X there exists

1. *a T_n-topological space $T_n X$*
2. *a continuous function*

$$t_n(X) \,:\, X \longrightarrow T_n X$$

called the T_n-reflection of X,

which is the "closest approximation from the left" to X by a T_n-topological space, in that for Y any T_n-space, then continuous functions of the form

$$f \,:\, X \longrightarrow Y$$

are in bijection with continuous function of the form

$$\tilde f : T_n X \longrightarrow Y$$

and such that the bijection is constituted by

$$f = \tilde f \circ t_n(X) : X \overset{t_n(X)}{\longrightarrow} T_n X \overset{\tilde f}{\longrightarrow} Y \qquad i.e.: \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "t_n(X)"'] & & Y \\ & T_n X \arrow[ur, "\tilde f"'] & \end{tikzcd} .$$

- *For $n = 0$ this is known as the Kolmogorov quotient construction (see prop. 4.26 below).*
- *For $n = 2$ this is known as Hausdorff reflection or Hausdorffication or similar.*

Moreover, the operation $T_n(-)$ extends to continuous functions $f : X \to Y$

$$(X \overset{f}{\to} Y) \;\mapsto\; (T_n X \overset{T_n f}{\longrightarrow} T_n Y)$$

such as to preserve composition of functions as well as identity functions:

$$T_n g \circ T_n f = T_n (g \circ f) \quad , \quad T_n \,\mathrm{id}_X = \mathrm{id}_{T_n X}$$

Finally, the comparison map is compatible with this in that

$$t_n(Y) \circ f = T_n(f) \circ t_n(X) \qquad i.e.: \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t_n(X)"'] & Y \arrow[d, "t_n(Y)"] \\ T_n X \arrow[r, "T_n(f)"'] & T_n Y \end{tikzcd} .$$

We **prove** this via a concrete construction of T_n-reflection in prop. 4.25 below. But first we pause to comment on the bigger picture of the T_n-reflection:

Remark 4.24. (**reflective subcategories**)

In the language of category theory (remark 3.3) the T_n-reflection of prop. 4.23 says that

1. $T_n(-)$ is a *functor* $T_n : \mathrm{Top} \longrightarrow \mathrm{Top}_{T_n}$ from the category Top of topological spaces to the full subcategory $\mathrm{Top}_{T_n} \overset{\iota}{\hookrightarrow} \mathrm{Top}$ of Hausdorff topological spaces;
2. $t_n(X) : X \to T_n X$ is a *natural transformation* from the identity functor on Top to the functor $\iota \circ T_n$
3. T_n-topological spaces form a reflective subcategory of all topological spaces in that T_n is left adjoint to the inclusion functor ι; this situation is denoted as follows:

$$\mathrm{Top}_{T_n} \underoverset{\underset{\iota}{\hookrightarrow}}{\overset{H}{\longleftarrow}}{\bot} \mathrm{Top} \;.$$

Generally, an *adjunction* between two functors

$$L : \mathcal{C} \leftrightarrow \mathcal{D} : R$$

is for all pairs of objects $c \in \mathcal{C}$, $d \in \mathcal{D}$ a bijection between sets of morphisms of the form

$$\{L(c) \longrightarrow d\} \quad \leftrightarrow \quad \{c \longrightarrow R(d)\} \; .$$

i.e.

$$Hom_{\mathcal{D}}(L(c), d) \underset{\simeq}{\overset{\phi_{c,d}}{\longrightarrow}} Hom_{\mathcal{C}}(c, R(d))$$

and such that these bijections are "natural" in that they for all pairs of morphisms $f : c' \to c$ and $g : d \to d'$ then the folowing diagram commutes:

$$\begin{array}{ccc} & Hom_{\mathcal{D}}(L(c), d) & \underset{\simeq}{\overset{\phi_{c,d}}{\longrightarrow}} & Hom_{\mathcal{C}}(c, R(d)) \\ g \circ (-) \circ L(f) \downarrow & & & \downarrow R(g) \circ (-) \circ f \\ & Hom_{\mathcal{C}}(L(c'), d') & \underset{\simeq}{\overset{\phi_{c',d'}}{\longrightarrow}} & Hom_{\mathcal{D}}(c', R(d')) \end{array}$$

One calls the image under $\phi_{c,L(c)}$ of the identity morphism $id_{L(x)}$ the *unit of the adjunction*, written

$$\eta_x : c \longrightarrow R(L(c)) \; .$$

One may show that it follows that the image $\tilde f$ under $\phi_{c,d}$ of a general morphism $f : c \to d$ (called the *adjunct* of f) is given by this composite:

$$\tilde f \; : \; c \overset{\eta_c}{\longrightarrow} R(L(c)) \overset{R(f)}{\longrightarrow} R(d) \; .$$

In the case of the reflective subcategory inclusion $(T_n \dashv \iota)$ of the category of T_n-spaces into the category Top of all topological spaces this adjunction unit is precisely the T_n-reflection $t_n(X) : X \to \iota(T_n(X))$ (only that we originally left the re-embedding ι notationally implicit).

There are various ways to see the existence and to construct the T_n-reflections. The following is the quickest way to see the existence, even though it leaves the actual construction rather implicit.

Proposition 4.25. ***(T_n-reflection via explicit quotients)***

Let $n \in \{0, 1, 2\}$. Let (X, τ) be a topological space and consider the equivalence relation $\sim$ on the underlying set X for which $x_1 \sim x_2$ precisely if for every surjective continuous function $f : X \to Y$ into any T_n-topological space Y (def. 4.4) we have $f(x_1) = f(x_2)$:

$$(x_1 \sim x_2) \coloneqq \underset{X \underset{\text{surjective}}{\overset{f}{\longrightarrow}} Y}{\underset{Y \in \mathrm{Top}_{T_n}}{\forall}} (f(x) = f(y)) \,.$$

Then

1. *the set of equivalence classes*

$$T_n X \coloneqq X/\sim$$

   equipped with the quotient topology (example 2.17) is a T_n-topological space,

2. *the quotient projection*

$$\begin{aligned} X &\overset{t_n(X)}{\longrightarrow} X/\sim \\ x &\longmapsto [x] \end{aligned}$$

   exhibits the T_n-reflection of X, according to prop. 4.23.

Proof. First we observe that every continuous function $f : X \longrightarrow Y$ into a T_n-topological space Y factors uniquely, via $t_n(X)$ through a continuous function $\tilde f$ (this makes use of the "universal property" of the quotient topology, which we dwell on a bit more below in example 6.3):

$$f = \tilde f \circ t_n(X)$$

Clearly this continuous function $\tilde f$ is unique if it exists, because its underlying function of sets must be given by

$$\tilde f : [x] \mapsto f(x) \,.$$

First observe that this is indeed well defined as a function of underlying sets. To that end, factor f through its image $f(X)$

$$f \;:\; X \longrightarrow f(X) \hookrightarrow Y$$

equipped with its subspace topology as a subspace of Y (example 3.10). By prop. 4.9 also the image $f(X)$ is a T_n-topological space, since Y is. This means that if two elements $x_1, x_2 \in X$ have the same equivalence class, then, by definition of the equivalence relation, they have the same image under *all* comntinuous surjective functions into a T_n-space, hence in particular they have the same image under $f : X \overset{\text{surjective}}{\longrightarrow} f(X) \hookrightarrow Y$:

$$\begin{aligned} ([x_1] = [x_2]) &\Leftrightarrow (x_1 \sim x_2) \\ &\Rightarrow (f(x_1) = f(x_2)) \,. \end{aligned}$$

This shows that $\tilde f$ is well defined as a function between sets.

To see that $\tilde f$ is also continuous, consider $U \in Y$ an open subset. We need to show that the pre-image $\tilde f^{-1}(U)$ is open in $X/\sim$. But by definition of the quotient topology (example 2.17), this is open precisely if its pre-image under the quotient projection $t_n(X)$ is open, hence precisely if

$$\begin{aligned} (t_n(X))^{-1}\left(\tilde f^{-1}(U)\right) & = \left(\tilde f \circ t_n(X)\right)^{-1}(U) \\ & = f^{-1}(U) \end{aligned}$$

is open in X. But this is the case by the assumption that f is continuous. Hence $\tilde f$ is indeed the unique continuous function as required.

What remains to be seen is that $T_n X$ as constructed is indeed a T_n-topological space. Hence assume that $[x] \neq [y] \in T_n X$ are two distinct points. Depending on the value of n, need to produce open neighbourhoods around one or both of these points not containing the other point and possibly disjoint to each other.

Now by definition of $T_n X$ the assumption $[x] \neq [y]$ means that there exists a T_n-topological space Y and a surjective continuous function $f : X \xrightarrow{\text{surjective}} Y$ such that $f(x) \neq f(y) \in Y$:

$$([x_1] \neq [x_2]) \;\Leftrightarrow\; \underset{\underset{X \underset{\text{surjective}}{\overset{f}{\longrightarrow}} Y}{Y \in \mathrm{Top}_{T_m}}}{\exists} \left(f(x_1) \neq f(x_2)\right) .$$

Accordingly, since Y is T_n, there exist the respective kinds of neighbourhoods around $f(x_1)$ and $f(x_2)$ in Y. Moreover, by the previous statement there exists the continuous function $\tilde f : T_n X \to Y$ with $\tilde f([x_1]) = f(x_1)$ and $\tilde f([x_2]) = f(x_2)$. By the nature of continuous functions, the pre-images of these open neighbourhoods in Y are still open in X and still satisfy the required disjunction properties. Therefore $T_n X$ is a T_n-space. ▮

Here are alternative constructions of the reflections:

Proposition 4.26. ***(Kolmogorov quotient)***

Let (X,τ) be a topological space. Consider the relation on the underlying set by which $x_1 \sim x_2$ precisely if neither x_i has an open neighbourhood not containing the other. This is an equivalence relation. The quotient topological space $X \to X/\sim$ by this equivalence relation (def. 2.17) exhibits the T_0-reflection of X according to prop. 4.23.

A more explicit construction of the Hausdorff quotient than given by prop. 4.25 is rather more involved. The issue is that the relation "x and y are not separated by disjoint open neighbourhoods" is not transitive;

Proposition 4.27. ***(more explicit Hausdorff reflection)***

For (Y,τ_Y) a topological space, write $r_Y \subset Y \times Y$ for the transitive closure of the relation given by the topological closure $\mathrm{Cl}(\Delta_Y)$ of the image of the diagonal $\Delta_Y : Y \hookrightarrow Y \times Y$.

$$r_Y \coloneqq \mathrm{Trans}(\mathrm{Cl}(\mathrm{Delta}_Y)) .$$

Now for (X, τ_X) a topological space, define by induction for each ordinal number α an equivalence relation r^α on X as follows, where we write $q^\alpha : X \to H^\alpha(X)$ for the corresponding quotient topological space projection:

We start the induction with the trivial equivalence relation:

- $r_X^0 \coloneqq \Delta_X$;

For a successor ordinal we set

- $r_X^{\alpha+1} \coloneqq \left\{ (a,b) \in X \times X \mid (q^\alpha(a), q^\alpha(b)) \in r_{H^\alpha(X)} \right\}$

and for a limit ordinal α we set

- $r_X^\alpha \coloneqq \underset{\beta < \alpha}{\cup} r_X^\beta$.

Then:

1. *there exists an ordinal α such that $r_X^\alpha = r_X^{\alpha+1}$*

2. *for this α then $H^\alpha(X) = H(X)$ is the Hausdorff reflection from prop. 4.25.*

A detailed **proof** is spelled out in (vanMunster 14, section 4).

Example 4.28. (**Hausdorff reflection of the line with two origins**)

The Hausdorff reflection (T_2-reflection, prop. 4.23)

$$T_2 \;:\; \mathrm{Top} \longrightarrow \mathrm{Top}_{\mathrm{Haus}}$$

of the line with two origins from example 4.3 is the real line itself:

$$T_2((\mathbb{R} \sqcup \mathbb{R})/\sim) \;\simeq\; \mathbb{R} .$$

5. Sober spaces

While the original formulation of the separation axioms T_n from def. 4.4 and def. 4.13 clearly does follow some kind of pattern, its equivalent reformulation in terms of closure conditions in prop. 4.10, prop. 4.11, prop 4.12, prop. 4.17 and prop. 4.18 suggests rather different patterns. Therefore it is worthwhile to also consider separation-like axioms that are not among the original list.

In particular, the alternative characterization of the T_0-condition in prop. 4.10 immediately suggests the following strengthening, different from the T_1-condition (see example 5.5 below):

Definition 5.1. (**sober topological space**)

A topological space (X, τ) is called a *sober topological space* precisely if every irreducible closed subspace (def. 2.31) is the topological closure (def. 2.23) of a unique point, hence precisely if the function

$$\mathrm{Cl}(\{-\}) \;:\; X \longrightarrow \mathrm{IrrClSub}(X)$$

from the underlying set of X to the set of irreducible closed subsets of X (def. 2.30, well defined according to example 2.31) is bijective.

Proposition 5.2. ***(sober implies T_0)***

Every sober topological space (def. 5.1) is T_0 (def. 4.4).

Proof. By prop. 4.10. ▮

Proposition 5.3. ***(Hausdorff spaces are sober)***

Every Hausdorff topological space (def. 4.4) is a sober topological space (def. 5.1).

More specifically, in a Hausdorff topological space the irreducible closed subspaces (def. 2.30) are precisely the singleton subspaces (def. 2.16).

Hence, by example 4.8, in particular every metric space with its metric topology (example 2.9) is sober.

Proof. The second statement clearly implies the first. To see the second statement, suppose that F is an irreducible closed subspace which contained two distinct points $x \neq y$. Then by the Hausdorff property there would be disjoint neighbourhoods U_x, U_y, and hence it would follow that the relative complements $F \backslash U_x$ and $F \backslash U_y$ were distinct closed proper subsets of F with

$$F = (F \backslash U_x) \cup (F \backslash U_y)$$

in contradiction to the assumption that F is irreducible.

This proves by contradiction that every irreducible closed subset is a singleton. Conversely, generally the topological closure of every singleton is irreducible closed, by example 2.31. ▮

By prop. 5.2 and prop. 5.3 we have the implications on the right of the following diagram:

separation axioms

$$\begin{array}{ccccc}
 & & T_2 = \text{Hausdorff} & & \\
 & \swArrow & & \seArrow & \\
T_1 & & & & \text{sober} \\
 & \seArrow & & \swArrow & \\
 & & T_0 = \text{Kolmogorov} & &
\end{array}$$

But there there is no implication betwee T_1 and sobriety:

Proposition 5.4. The intersection of the classes of sober topological spaces (def. 5.1) and T_1-topological spaces (def. 4.4) is not empty, but neither class is contained within the other.

That the intersection is not empty follows from prop. 5.3. That neither class is contained in the other is shown by the following counter-examples:

Example 5.5. **(T_1 neither implies nor is implied by sobriety)**

- The Sierpinski space (def. 2.11) is sober, but not T_1.
- The cofinite topology (example 2.14) on a non-finite set is T_1 but not sober.

Finally, sobriety is indeed strictly weaker that Hausdorffness:

Example 5.6. **(schemes are sober but in general not Hausdorff)**

The Zariski topology on an affine space (example 2.21) or more generally on the prime spectrum of a commutative ring (example 2.22) is

1. sober (def 5.1);
2. in general not Hausdorff (def. 4.4).

For details see at *Zariski topology* this prop and this example.

Frames of opens

What makes the concept of sober topological spaces special is that for them the concept of continuous functions may be expressed entirely in terms of the relations between their open subsets, disregarding the underlying set of points of which these opens are in fact subsets.

Recall from example 2.36 that for every continuous function $f:(X,\tau_X) \to (Y,\tau_Y)$ the pre-image function $f^{-1}:\tau_Y \to \tau_X$ is a frame homomorphism (def. 2.34).

For sober topological spaces the converse holds:

Proposition 5.7. *If (X,τ_X) and (Y,τ_Y) are sober topological spaces (def. 5.1), then for every frame homomorphism (def. 2.34)*

$$\tau_X \longleftarrow \tau_Y \,:\, \phi$$

there is a unique continuous function $f:X \to Y$ such that ϕ is the function of forming pre-images under f:

$$\phi = f^{-1}\,.$$

Proof. We first consider the special case of frame homomorphisms of the form

$$\tau_* \longleftarrow \tau_X \,:\, \phi$$

and show that these are in bijection to the underlying set X, identified with the continuous functions $* \to (X, \tau)$ via example 3.6.

By prop. 2.37, the frame homomorphisms $\phi : \tau_X \to \tau_*$ are identified with the irreducible closed subspaces $X \backslash U_\emptyset(\phi)$ of (X, τ_X). Therefore by assumption of sobriety of (X, τ) there is a unique point $x \in X$ with $X \backslash U_\emptyset = \mathrm{Cl}(\{x\})$. In particular this means that for U_x an open neighbourhood of x, then U_x is not a subset of $U_\emptyset(\phi)$, and so it follows that $\phi(U_x) = \{1\}$. In conclusion we have found a unique $x \in X$ such that

$$\phi \;:\; U \mapsto \begin{cases} \{1\} & \vert \, \text{if}\, x \in U \\ \emptyset & \vert \, \text{otherwise} \end{cases} .$$

This is precisely the inverse image function of the continuous function $* \to X$ which sends $1 \mapsto x$.

Hence this establishes the bijection between frame homomorphisms of the form $\tau_* \longleftarrow \tau_X$ and continuous functions of the form $* \to (X, \tau)$.

With this it follows that a general frame homomorphism of the form $\tau_X \overset{\phi}{\longleftarrow} \tau_Y$ defines a function of sets $X \overset{f}{\longrightarrow} Y$ by composition:

$$\begin{array}{ccc} X & \overset{f}{\longrightarrow} & Y \\ (\tau_* \leftarrow \tau_X) & \mapsto & (\tau_* \leftarrow \tau_X \overset{\phi}{\longleftarrow} \tau_Y) \end{array} .$$

By the previous analysis, an element $U_Y \in \tau_Y$ is sent to $\{1\}$ under this composite precisely if the corresponding point $* \to X \overset{f}{\longrightarrow} Y$ is in U_Y, and similarly for an element $U_X \in \tau_X$. It follows that $\phi(U_Y) \in \tau_X$ is precisely that subset of points in X which are sent by f to elements of U_Y, hence that $\phi = f^{-1}$ is the pre-image function of f. Since ϕ by definition sends open subsets of Y to open subsets of X, it follows that f is indeed a continuous function. This proves the claim in generality. ▮

Remark 5.8. **(locales)**

Proposition 5.7 is often stated as saying that sober topological spaces are equivalently the "locales with enough points" (Johnstone 82, II 1.). Here "locale" refers to a concept akin to topological spaces where one considers *just* a "frame of open subsets" τ_X, without requiring that its elements be actual subsets of some ambient set. The natural notion of homomorphism between such generalized topological spaces are clearly the frame homomorphisms $\tau_X \leftarrow \tau_Y$ from def. 2.34.

From this perspective, prop. 5.7 says that sober topological spaces (X, τ_X) are entirely characterized by their frames of opens τ_X and just so happen to "have enough points" such that these are actual open subsets of some ambient set, namely of X.

Sober reflection

We saw above in prop. 4.23 that every T_n-toopological space for $n \in \{0,1,2\}$ has a "best approximation from the left" by a T_n-topological space (for $n = 2$: "Hausdorff reflection"). We now discuss the analogous statement for sober topological spaces.

Recall again the point topological space $* \coloneqq (\{1\}, \tau_* = \{\emptyset, \{1\}\})$ (example 2.10).

Definition 5.9. **(sober reflection)**

Let (X,τ) be a topological space.

Define SX to be the set

$$SX \coloneqq \mathrm{FrameHom}(\tau_X, \tau_*)$$

of frame homomorphisms (def. 2.34) from the frame of opens of X to that of the point. Define a topology $\tau_{SX} \subset P(SX)$ on this set by declaring it to have one element $\tilde U$ for each element $U \in \tau_X$ and given by

$$\tilde U \; \coloneqq \; \{\phi \in SX \mid \phi(U) = \{1\}\} \,.$$

Consider the function

$$\begin{array}{ccc} X & \overset{s_X}{\longrightarrow} & SX \\ x & \mapsto & (\mathrm{const}_x)^{-1} \end{array}$$

which sends an element $x \in X$ to the function which assigns inverse images of the constant function $\mathrm{const}_x \colon \{1\} \to X$ on that element.

We are going to call this function the *sober reflection* of X.

Lemma 5.10. ***(sober reflection is well defined)***

The construction (SX, τ_{SX}) in def. 5.9 is a topological space, and the function $s_X \colon X \to SX$ is a continuous function

$$s_X \colon (X, \tau_X) \longrightarrow (SX, \tau_{SX})$$

Proof. To see that $\tau_{SX} \subset P(SX)$ is closed under arbitrary unions and finite intersections, observe that the function

$$\begin{array}{ccc} \tau_X & \overset{\widetilde{(-)}}{\longrightarrow} & \tau_{SX} \\ U & \mapsto & \tilde U \end{array}$$

in fact preserves arbitrary unions and finite intersections. Whith this the statement follows by the fact that τ_X is closed under these operations.

To see that $\widetilde{(-)}$ indeed preserves unions, observe that (e.g. Johnstone 82, II 1.3 Lemma)

$$
\begin{aligned}
p \in \underset{i \in I}{\cup} \widetilde{U_i} & \Leftrightarrow \underset{i \in I}{\exists} p(U_i) = \{1\} \\
& \Leftrightarrow \underset{i \in I}{\cup} p(U_i) = \{1\} \\
& \Leftrightarrow p\left(\underset{i \in I}{\cup} U_i \right) = \{1\} \\
& \Leftrightarrow p \in \widetilde{\underset{i \in I}{\cup} U_i}
\end{aligned}
,
$$

where we used that the frame homomorphism $p : \tau_X \to \tau_*$ preserves unions. Similarly for intersections, now with I a finite set:

$$
\begin{aligned}
p \in \underset{i \in I}{\cap} \widetilde{U_i} & \Leftrightarrow \underset{i \in I}{\forall} p(U_i) = \{1\} \\
& \Leftrightarrow \underset{i \in I}{\cap} p(U_i) = \{1\} \\
& \Leftrightarrow p\left(\underset{i \in I}{\cap} U_i \right) = \{1\} \\
& \Leftrightarrow p \in \widetilde{\underset{i \in I}{\cap} U_i}
\end{aligned}
,
$$

where we used that the frame homomorphism p preserves finite intersections.

To see that s_X is continuous, observe that $s_X^{-1}(\tilde{U}) = U$, by construction. ▮

Lemma 5.11*. (**sober reflection** detects T_0 and sobriety)*

For (X, τ_X) a topological space, the function $s_X : X \to SX$ from def. 5.9 is

1. *an injection precisely if (X, τ_X) is T_0 (def. 4.4);*
2. *a bijection precisely if (X, τ_Y) is sober (def. 5.1), in which case s_X is in fact a homeomorphism (def. 3.22).*

Proof. By lemma 2.37 there is an identification $SX \simeq \mathrm{IrrClSub}(X)$ and via this s_X is identified with the map $x \mapsto \mathrm{Cl}(\{x\})$.

Hence the second statement follows by definition, and the first statement by prop. 4.10.

That in the second case s_X is in fact a homeomorphism follows from the definition of the opens $\tilde{U}$: they are identified with the opens U in this case (…expand…). ▮

Lemma 5.12*. (**soberification** lands in **sober spaces**, e.g. Johnstone 82, lemma II 1.7)*

For (X, τ) a topological space, then the topological space (SX, τ_{SX}) from def. 5.9, lemma 5.10 is sober.

Proof. Let $SX \backslash \tilde{U}$ be an irreducible closed subspace of (SX, τ_{SX}). We need to show that it is the topological closure of a unique element $\phi \in SX$.

Observe first that also $X \backslash U$ is irreducible.

To see this use prop. 2.33, saying that irreducibility of $X \backslash U$ is equivalent to $U_1 \cap U_2 \subset U \Rightarrow (U_1 \subset U) \text{or} (U_2 \subset U)$. But if $U_1 \cap U_2 \subset U$ then also $\tilde U_1 \cap \tilde U_2 \subset \tilde U$ (as in the proof of lemma 5.10) and hence by assumption on $\tilde U$ it follows that $\tilde U_1 \subset \tilde U$ or $\tilde U_2 \subset \tilde U$. By lemma 2.37 this in turn implies $U_1 \subset U$ or $U_2 \subset U$. In conclusion, this shows that also $X \backslash U$ is irreducible .

By lemma 2.37 this irreducible closed subspace corresponds to a point $p \in SX$. By that same lemma, this frame homomorphism $p : \tau_X \to \tau_*$ takes the value $\emptyset$ on all those opens which are inside U. This means that the topological closure of this point is just $SX \backslash \tilde U$.

This shows that there exists at least one point of which $X \backslash \tilde U$ is the topological closure. It remains to see that there is no other such point.

So let $p_1 \neq p_2 \in SX$ be two distinct points. This means that there exists $U \in \tau_X$ with $p_1(U) \neq p_2(U)$. Equivalently this says that $\tilde U$ contains one of the two points, but not the other. This means that (SX, τ_{SX}) is T0. By prop. 4.10 this is equivalent to there being no two points with the same topological closure. ▮

Proposition 5.13. ***(unique factorization through soberification)***

For (X, τ_X) any topological space, for (Y, τ_Y^{sob}) a sober topological space, and for $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ a continuous function, then it factors uniquely through the soberification $s_X : (X, \tau_X) \longrightarrow (SX, \tau_{SX})$ from def. 5.9, lemma 5.10

$$\begin{array}{ccc} (X, \tau_X) & \overset{f}{\longrightarrow} & (Y, \tau_Y^{\text{sob}}) \\ {}^{\mathllap{s_X}}\downarrow & \nearrow_{\exists !} & \\ (SX, \tau_{SX}) & & \end{array} \,.$$

Proof. By the construction in def. 5.9, we find that the outer part of the following square commutes:

$$\begin{array}{ccc} (X, \tau_X) & \overset{f}{\longrightarrow} & (Y, \tau_Y^{\text{sob}}) \\ {}^{\mathllap{s_X}}\downarrow & \nearrow & \downarrow^{\mathrlap{s_{SX}}} \\ (SX, \tau_{SX}) & \underset{Sf}{\longrightarrow} & (SSX, \tau_{SSX}) \end{array} \,.$$

By lemma 5.12 and lemma 5.11, the right vertical morphism s_{SX} is an isomorphism (a homeomorphism), hence has an inverse morphism. This defines the diagonal morphism, which is the desired factorization.

To see that this factorization is unique, consider two factorizations $\tilde f, \overline{f} : : (SX, \tau_{SX}) \to (Y, \tau_Y^{\text{sob}})$ and apply the soberification construction once more to the triangles

$$
\begin{array}{ccc}
(X, \tau_X) & \overset{f}{\longrightarrow} & (Y, \tau_Y^{\mathrm{sob}}) \\
{}^{s_X}\downarrow & \nearrow_{\tilde f, \overline{f}} & \\
(SX, \tau_{SX}) & &
\end{array}
\quad \mapsto \quad
\begin{array}{ccc}
(SX, \tau_{SX}) & \overset{Sf}{\longrightarrow} & (Y, \tau_Y^{\mathrm{sob}}) \\
{}^{\simeq}\downarrow & \nearrow_{\tilde f, \overline{f}} & \\
(SX, \tau_{SX}) & &
\end{array}
\,.
$$

Here on the right we used again lemma 5.11 to find that the vertical morphism is an isomorphism, and that $\tilde f$ and $\overline{f}$ do not change under soberification, as they already map between sober spaces. But now that the left vertical morphism is an isomorphism, the commutativity of this triangle for both $\tilde f$ and $\overline{f}$ implies that $\tilde f = \overline{f}$. ▮

In **summary** we have found

Proposition 5.14. ***(sober reflection)***

For every topological space X there exists

1. *a sober topological spaces SX;*
2. *a continuous function $s_n : X \longrightarrow SX$*

such that …

As before for the T_n-reflection in remark 4.24, the statement of prop. 5.14 may neatly be re-packaged:

Remark 5.15. (**sober topological spaces are a reflective subcategory**)

In the language of category theory (remark 3.3) and in terms of the concept of adjoint functors (remark 4.24), proposition 5.14 simply says that sober topological spaces form a reflective subcategory $\mathrm{Top}_{\mathrm{sob}}$ of the category Top of all topological spaces

$$
\mathrm{Top}_{\mathrm{sob}} \underset{\hookrightarrow}{\overset{s}{\underset{\bot}{\longleftarrow}}} \mathrm{Top} \,.
$$

6. Universal constructions

We have seen above various construction principles for topological spaces above, such as topological subspaces and topological quotient spaces. It turns out that these constructions enjoy certain "universal properties" which allow us to find continuous functions into or out of these spaces, respectively (examples 6.1, example 6.2 and 6.3 below).

Since this is useful for handling topological spaces (we secretly used the universal property of the quotient space construction already in the proof of prop. 4.25), we next consider, in def. 6.11 below, more general "universal constructions" of topological spaces, called *limits* and *colimits* of topological spaces (and to be distinguished from limits *in* topological spaces, in the sense of convergence of sequences as in def. 1.17).

Moreover, we have seen above that the quotient space construction in general does not preserve the T_n-separation property or sobriety property of topological spaces, while the topological subspace construction does. The same turns out to be true for the more general "colimiting" and "limiting" universal constructions. But we have also seen that we may universally "reflect" any topological space to becomes a T_n-space or sober space. The remaining question then is whether this reflection breaks the desired universal property. We discuss that this is not the case, that instead the universal construction in all topological spaces followed by these reflections gives the correct universal constructions in T_n-separated and sober topological spaces, respectively (remark 6.22 below).

After these general considerations, we finally discuss a list of examples of universal constructions in topological spaces.

To motivate the following generalizations, first observe the universal properties enjoyed by the basic construction principles of topological spaces from above

Example 6.1. (universal property of binary product topological space)

Let X_1, X_2 be topological spaces. Consider their product topological space $X_1 \times X_2$ from example 2.18. By example 3.16 the two projections out of the product space are continuous functions

$$X_1 \overset{\mathrm{pr}_1}{\longleftarrow} X_1 \times X_2 \overset{\mathrm{pr}_2}{\longrightarrow} X_2 \,.$$

Now let Y be any other topological space. Then, by composition, every continuous function $Y \to X_1 \times X_2$ into the product space yields two continuous component functions f_1 and f_2:

$$\begin{array}{ccccc} && Y \\ & {}^{\mathllap{f_1}}\swarrow & \downarrow & \searrow^{\mathrlap{f_2}} \\ X_1 & \underset{\mathrm{pr}_1}{\longleftarrow} & X_1 \times X_2 & \underset{\mathrm{pr}_2}{\longrightarrow} & X_2 \end{array} \,.$$

But in fact these two components completely characterize the function into the product: There is a (natural) bijection between continuous functions into the product space and pairs of continuous functions into the two factor spaces:

$$\{Y \longrightarrow X_1 \times X_2\} \quad \simeq \quad \left\{ \left(\begin{array}{c} Y \longrightarrow X_1, \\ Y \longrightarrow X_2 \end{array} \right) \right\} \,.$$

i.e.:

$$\mathrm{Hom}(Y, X_1 \times X_2) \;\simeq\; \mathrm{Hom}(Y, X_1) \times \mathrm{Hom}(Y, X_2)$$

Example 6.2. (universal property of disjoint union spaces)

Let X_1, X_2 be topological spaces. Consider their disjoint union space $X_1 \sqcup X_2$ from example 2.15. By definition, the two inclusions into the disjoint union space are clearly continuous functions

$$X_1 \overset{i_1}{\longrightarrow} X_1 \sqcup X_2 \overset{i_2}{\longleftarrow} X_2 \,.$$

Now let Y be any other topological space. Then by composition a continuous function $X_1 \sqcup X_2 \longrightarrow Y$ out of the disjoint union space yields two continuous component functions f_1 and f_2:

$$\begin{array}{ccccc} X_1 & \overset{i_1}{\longleftarrow} & X_1 \sqcup X_2 & \overset{i_2}{\longrightarrow} & X_2 \\ & {}_{f_1}\searrow & \downarrow & \swarrow_{f_2} \\ && Y \end{array} \,.$$

But in fact these two components completely characterize the function out of the disjoint union: There is a (natural) bijection between continuous functions out of disjoint union spaces and pairs of continuous functions out of the two summand spaces:

$$\{X_1 \sqcup X_2 \longrightarrow Y\} \quad \simeq \quad \left\{ \begin{pmatrix} X_1 \longrightarrow Y, \\ X_2 \longrightarrow Y \end{pmatrix} \right\} \,.$$

i.e.:

$$\mathrm{Hom}(X_1 \times X_2, Y) \;\simeq\; \mathrm{Hom}(X_1, Y) \times \mathrm{Hom}(X_2, Y)$$

Example 6.3. (**universal property of quotient topological spaces**)

Let X be a topological space, and let $\sim$ be an equivalence relation on its underlying set. Then the corresponding quotient topological space $X/\sim$ together with the corresponding qutient continuous function $p : X \to X/\sim$ has the following universal property:

Given $f : X \longrightarrow Y$ any continuous function out of X with the property that it respects the given equivalence relation, in that

$$(x_1 \sim x_2) \;\Rightarrow\; (f(x_1) = f(x_2))$$

then there is a unique continuous function $\tilde f : X/\sim \longrightarrow Y$ such that

$$f = \tilde f \circ p \qquad i.e. \qquad \begin{array}{ccc} X & \overset{f}{\longrightarrow} & Y \\ {}^{p}\downarrow & \nearrow_{\exists ! \tilde f} \\ X/\sim \end{array} \,.$$

(We already made use of this universal property in the construction of the T_n-reflection in the proof of prop. 4.25.)

Proof. First observe that there is a unique function $\tilde f$ as claimed on the level of functions of the underlying sets: In order for $f = \tilde f \circ p$ to hold, $\tilde f$ must send an equivalence class in $X/\sim$ to one of its members

$$\tilde{f} : [x] \mapsto x$$

and that this is well defined and independent of the choice of representative x is guaranteed by the condition on f above.

Hence it only remains to see that $\tilde{f}$ defined this way is continuous, hence that for $U \subset Y$ an open subset, then its pre-image $\tilde{f}^{-1}(U) \subset X/\sim$ is open in the quotient topology. By definition of the quotient topology (example 2.17), this is the case precisely if its further pre-image under p is open in X. But by the fact that $f = \tilde{f} \circ p$, this is the case by the continuity of f:

$$\begin{aligned} p^{-1}\left(\tilde{f}^{-1}(U)\right) & = \left(\tilde{f} \circ p\right)^{-1}(U) \\ & = f^{-1}(U) \end{aligned} .$$

∎

This kind of example we now generalize.

Limits and colimits

We consider now the general definition of free diagrams of topological spaces (def. 6.4 below), their cones and co-cones (def. 6.9) as well as limiting cones and colimiting cocones (def. 6.11 below).

Then we use these concepts to see generally (remark 6.22 below) why limits (such as product spaces and subspaces) of $T_{n\leq 2}$-spaces and of sober spaces are again T_n or sober, respectively, and to see that the correct colimits (such as disjoint union spaces and quotient spaces) of T_n- or sober spaces are instead the T_n*-reflection* (prop. 4.23) or sober reflection (prop. 5.14), respectively, of these colimit constructions performed in the context of unconstrained topological spaces.

Definition 6.4. **(free diagram of sets/topological spaces)**

A *free diagram* $X_\bullet$ of sets or of topological spaces is

1. a set $\{X_i\}_{i \in I}$ of sets or of topological spaces, respectively;

2. for every pair $(i, j) \in I \times I$ of labels, a set $\{X_i \overset{f_\alpha}{\longrightarrow} X_j\}_{\alpha \in I_{i,j}}$ of functions of of continuous functions, respectively, between these.

Here is a list of basic and important examples of free diagrams

- *discrete diagrams and the empty diagram* (example 6.5);

- *pairs of parallel morphisms* (example 6.6);
- *span and cospan diagram* (example 6.7);
- *tower and cotower diagram* (example 6.8).

Example 6.5. **(discrete diagram and empty diagram)**

Let I be any set, and for each $(i,j) \in I \times I$ let $I_{i,j} = \emptyset$ be the empty set.

The corresponding free diagrams (def. 6.4) are simply a set of sets/topological spaces with no specified (continuous) functions between them. This is called a *discrete diagram*.

For example for $I = \{1,2,3\}$ the set with 3-elements, then such a diagram looks like this:

$$X_1 \qquad X_2 \qquad X_3 \;.$$

Notice that here the index set may be empty set, $I = \emptyset$, in which case the corresponding diagram consists of no data. This is also called the *empty diagram*.

Definition 6.6. **(parallel morphisms diagram)**

Let $I = \{a,b\}$ be the set with two elements, and consider the sets

$$I_{i,j} \;\coloneqq\; \left\{ \begin{array}{ccc} \{1,2\} & | & (i = a) \,\text{and}\, (j = b) \\ \emptyset & | & \text{otherwise} \end{array} \right\} \,.$$

The corresponding free diagrams (def. 6.4) are called *pairs of parallel morphisms*. They may be depicted like so:

$$X_a \underoverset{f_2}{f_1}{\rightrightarrows} X_b \;.$$

Example 6.7. **(span and cospan diagram)**

Let $I = \{a,b,c\}$ the set with three elements, and set

$$I_{i,j} = \left\{ \begin{array}{ccc} \{f_1\} & | & (i = c) \,\text{and}\, (j = a) \\ \{f_2\} & | & (i = c) \,\text{and}\, (j = b) \\ \emptyset & | & \text{otherwise} \end{array} \right.$$

The corresponding free diagrams (def. 6.4) look like so:

$$\begin{array}{ccccc} && X_c \\ & {}^{f_1}\swarrow && \searrow^{f_2} \\ X_a &&&& X_b \end{array} \;.$$

These are called *span diagrams*.

Similary, there is the *cospan* diagram of the form

$$
\begin{array}{ccccc}
 & & X_c & & \\
 & {}^{f_1}\nearrow & & \nwarrow^{f_2} & \\
X_a & & & & X_b
\end{array}
\,.
$$

Example 6.8. (**tower diagram**)

Let $I = \mathbb{N}$ be the set of natural numbers and consider

$$
I_{i,j} \coloneqq \left\{ \begin{array}{ccc} \{f_{i,j}\} & | & j = i+1 \\ \emptyset & | & \text{otherwise} \end{array} \right.
$$

The corresponding free diagrams (def. 6.4) are called *tower diagrams*. They look as follows:

$$
X_0 \overset{f_{0,1}}{\longrightarrow} X_1 \overset{f_{1,2}}{\longrightarrow} X_2 \overset{f_{2,3}}{\longrightarrow} X_3 \longrightarrow \cdots \,.
$$

Similarly there are co-tower diagram

$$
X_0 \overset{f_{0,1}}{\longleftarrow} X_1 \overset{f_{1,2}}{\longleftarrow} X_2 \overset{f_{2,3}}{\longleftarrow} X_3 \longleftarrow \cdots \,.
$$

Definition 6.9. (**cone** over a **free diagram**)

Consider a free diagram of sets or of topological spaces (def. 6.4)

$$
X_\bullet = \left\{ X_i \overset{f_\alpha}{\longrightarrow} X_j \right\}_{i,j \in I, \alpha \in I_{i,j}} \,.
$$

Then

1. a *cone* over this diagram is

   1. a set or topological space $\tilde X$ (called the *tip* of the cone);

   2. for each $i \in I$ a function or continuous function $\tilde X \overset{p_i}{\longrightarrow} X_i$

   such that

   - for all $(i,j) \in I \times I$ and all $\alpha \in I_{i,j}$ then the condition

     $$
     f_\alpha \circ p_i = p_j
     $$

     holds, which we depict as follows:

     $$
     \begin{array}{ccccc}
      & & \tilde X & & \\
      & {}^{p_i}\swarrow & & \searrow^{p_j} & \\
     X_i & & \underset{f_\alpha}{\longrightarrow} & & X_j
     \end{array}
     $$

2. a *co-cone* over this diagram is

   1. a set or topological space $\tilde{X}$ (called the *tip* of the co-cone);

   2. for each $i \in I$ a function or continuous function $q_i : X_i \longrightarrow \tilde{X}$;

   such that

   - for all $(i,j) \in I \times I$ and all $\alpha \in I_{i,j}$ then the condition

     $$q_j \circ f_\alpha = q_i$$

     holds, which we depict as follows:

     $$\begin{array}{ccc} X_i & \overset{f_\alpha}{\longrightarrow} & X_j \\ & {}_{\mathllap{q_i}}\searrow & \swarrow_{\mathrlap{q_j}} \\ & \tilde{X} \end{array} \,.$$

Example 6.10. (solutions to equations are cones)

Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions from the real numbers to themselves, and consider the corresponding parallel morphism diagram of sets (example 6.6):

$$\mathbb{R} \underoverset{\underset{f_2}{\longrightarrow}}{\overset{f_1}{\longrightarrow}}{} \mathbb{R} \,.$$

Then a cone (def. 6.9) over this free diagram with tip the singleton set $*$ is a *solution* to the equation $f(x) = g(x)$

$$\begin{array}{ccc} & * \\ {}^{\mathllap{const_x}}\swarrow && \searrow^{\mathrlap{const_y}} \\ \mathbb{R} & \underoverset{\underset{f_2}{\longrightarrow}}{\overset{f_1}{\longrightarrow}}{} & \mathbb{R} \end{array} \,.$$

Namely the components of the cone are two functions of the form

$$cont_x, const_y \;:\; * \to \mathbb{R}$$

hence equivalently two real numbers, and the conditions on these are

$$f_1 \circ const_x = const_y \qquad f_2 \circ const_x = const_y \,.$$

Definition 6.11. (limiting cone over a diagram)

Consider a free diagram of sets or of topological spaces (def. 6.4):

$$\left\{ X_i \overset{f_\alpha}{\longrightarrow} X_j \right\}_{i,j \in I, \alpha \in I_{i,j}} \,.$$

Then

1. its *limiting cone* (or just *limit* for short, also "inverse limit", for historical reasons) is the cone

$$\left\{ \array{ && \underset{\longleftarrow}{\lim}_k X_k \\ & {}^{\mathllap{p_i}}\swarrow && \searrow^{\mathrlap{p_j}} \\ X_i && \underset{f_\alpha}{\longrightarrow} && X_j } \right\}$$

   over this diagram (def. 6.9) which is *universal* among all possible cones, in that for

$$\left\{ \array{ && \tilde X \\ & {}^{\mathllap{p'_i}}\swarrow && \searrow^{\mathrlap{p'_j}} \\ X_i && \underset{f_\alpha}{\longrightarrow} && X_j } \right\}$$

   any other cone, then there is a unique function or continuous function, respectively

$$\phi \;\colon\; \tilde X \longrightarrow \underset{\longrightarrow}{\lim}_i X_i$$

   that factors the given cone through the limiting cone, in that for all $i \in I$ then

$$p'_i = p_i \circ \phi$$

   which we depict as follows:

$$\array{ \tilde X \\ {}^{\mathllap{\exists ! \phi}}\downarrow & \searrow^{\mathrlap{p'_i}} \\ \underset{\longrightarrow}{\lim}_i X_i &\underset{p_i}{\longrightarrow}& X_i }$$

2. its *colimiting cocone* (or just *colimit* for short, also "direct limit", for historical reasons) is the cocone

$$\left\{ \array{ X_i && \overset{f_\alpha}{\longrightarrow} && X_j \\ & {}_{\mathllap{q_i}}\searrow && \swarrow_{\mathrlap{q_j}} \\ && \underset{\longrightarrow}{\lim}_i X_i } \right\}$$

   under this diagram (def. 6.9) which is *universal* among all possible co-cones, in that it has the property that for

$$\left\{ \array{ X_i && \overset{f_\alpha}{\longrightarrow} && X_j \\ & {}_{\mathllap{q'_i}}\searrow && \swarrow_{\mathrlap{q'_j}} \\ && \tilde X } \right\}$$

   any other cocone, then there is a unique function or continuous function, respectively

$$\phi : \underset{\longrightarrow_i}{\lim} X_i \longrightarrow \tilde{X}$$

that factors the given co-cone through the co-limiting cocone, in that for all $i \in I$ then

$$q'_i = \phi \circ q_i$$

which we depict as follows:

$$\begin{tikzcd} X_i \arrow[r, "q_i"] \arrow[dr, "q'_i"'] & \underset{\longrightarrow_i}{\lim} X_i \arrow[d, "\exists ! \phi"] \\ & \tilde{X} \end{tikzcd}$$

We now briefly mention the names and comment on the general nature of the limits and colimits over the free diagrams from the list of examples above. Further below we discuss examples in more detail.

shapes of free diagrams and the names of their limits/colimits

free diagram	limit/colimit
empty diagram	terminal object/initial object
discrete diagram	product/coproduct
parallel morphisms	equalizer/coequalizer
span/cospan	pullback,fiber product/pushout
tower/cotower	sequential limit/sequential colimit

Example 6.12. **(initial object and terminal object)**

Consider the empty diagram (def. 6.5).

1. A cone over the empty diagram is just an object X, with no further structure or condition. The universal property of the limit "$*$" over the empty diagram is hence that for every object X, there is a unique map of the form $X \to *$, with no further condition. Such an object $*$ is called a *terminal object*.

2. A co-cone over the empty diagram is just an object X, with no further structure or condition. The universal property of the colimit "$\emptyset$" over the empty diagram is hence that for every object X, there is a unique map of the form $\emptyset \to X$. Such an object $\emptyset$ is called an *initial object*.

Example 6.13. **(Cartesian product and coproduct)**

Let $\{X_i\}_{i \in I}$ be a discrete diagram (example 6.5), i.e. just a set of objects.

1. The limit over this diagram is called the *Cartesian product*, denoted $\prod_{i \in I} X_i$;

2. The colimit over this diagram is called the *coproduct*, denoted $\coprod_{i \in I} X_i$.

Example 6.14. (**equalizer**)

Let

$$X_1 \underoverset{f_2}{f_1}{\rightrightarrows} X_2$$

be a free diagram of the shape "pair of parallel morphisms" (example 6.6).

A limit over this diagram according to def. 6.11 is also called the *equalizer* of the maps f_1 and f_2. This is a set or topological space $\mathrm{eq}(f_1, f_2)$ equipped with a map $\mathrm{eq}(f_1, f_2) \overset{p_1}{\longrightarrow} X_1$, so that $f_1 \circ p_1 = f_2 \circ p_1$ and such that if $Y \to X_1$ is any other map with this property

$$\begin{tikzcd} & Y \arrow[d] \arrow[dr] & \\ \mathrm{eq}(f_1,f_2) \arrow[r, "p_1"] & X_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & X_2 \end{tikzcd}$$

then there is a unique factorization through the equalizer:

$$\begin{tikzcd} & Y \arrow[dl, "\exists !"'] \arrow[d] \arrow[dr] & \\ \mathrm{eq}(f_1,f_2) \arrow[r, "p_1"] & X_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & X_2 \end{tikzcd} .$$

In example 6.10 we have seen that a cone over such a pair of parallel morphisms is a *solution* to the equation $f_1(x) = f_2(x)$.

The equalizer above is the *space of all solutions* of this equation.

Example 6.15. (**pullback**/**fiber product** and **coproduct**)

Consider a cospan diagram (example 6.7)

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ X \arrow[r, "g"'] & Z \end{tikzcd} .$$

The limit over this diagram is also called the *fiber product* of X with Y over Z, and denoted $X \underset{Z}{\times} Y$. Thought of as equipped with the projection map to X, this is also called the *pullback* of f along g

$$
\begin{array}{ccc}
X \underset{X}{\times} Z & \longrightarrow & Y \\
\downarrow & (\text{pb}) & \downarrow^{f} \\
X & \underset{g}{\longrightarrow} & Z
\end{array}.
$$

Dually, consider a span diagram (example 6.7)

$$
\begin{array}{ccc}
Z & \overset{g}{\longrightarrow} & Y \\
{}^{f}\downarrow & & \\
X & &
\end{array}
$$

The colimit over this diagram is also called the pushout of f along g, denoted $X \underset{Z}{\sqcup} Y$:

$$
\begin{array}{ccc}
Z & \overset{g}{\longrightarrow} & Y \\
{}^{f}\downarrow & (\text{po}) & \downarrow \\
X & \longrightarrow & X \underset{Z}{\sqcup} Y
\end{array}
$$

Often the defining universal property of a limit/colimit construction is all that one wants to know. But sometimes it is useful to have an explicit description of the limits/colimits, not the least because this proves that these actually exist. Here is the explicit description of the (co-)limiting cone over a diagram of sets:

Proposition 6.16. ***(limits and colimits of sets)***

Let

$$
\left\{ X_i \overset{f_\alpha}{\longrightarrow} X_j \right\}_{i,j \in I, \alpha \in I_{i,j}}
$$

be a free diagram of sets (def. 6.4). Then

1. *its limit cone (def. 6.11) is given by the following subset of the Cartesian product $\prod_{i \in I} X_i$ of all the sets X_i appearing in the diagram*

$$
\underset{\longleftarrow i}{\lim} X_i \hookrightarrow \prod_{i \in I} X_i
$$

on those tuples of elements which match the graphs of the functions appearing in the diagram:

$$
\underset{\longleftarrow i}{\lim} X_i \;\simeq\; \left\{ (x_i)_{i \in I} \;\vert\; \underset{\alpha \in I_{i,j}}{\underset{i,j \in I}{\forall}} \left(f_\alpha(x_i) = x_j \right) \right\}
$$

and the projection functions are $p_i \colon (x_j)_{j \in I} \mapsto x_i$.

2. *its colimiting co-cone (def. 6.11) is given by the quotient set of the disjoint union $\underset{i \in I}{\sqcup} X_i$ of all the sets X_i appearing in the diagram*

$$\underset{i \in I}{\sqcup} X_i \longrightarrow \underset{\longrightarrow i \in I}{\lim} X_i$$

with respect to the equivalence relation which is generated from the graphs of the functions in the diagram:

$$\underset{\longrightarrow i}{\lim} X_i \;\simeq\; \left(\underset{i \in I}{\sqcup} X_i \right) / \left((x \sim x') \Leftrightarrow \left(\underset{\substack{i,j \in I \\ \alpha \in I_{i,j}}}{\exists} \left(f_\alpha(x) = x' \right) \right) \right)$$

and the injection functions are the evident maps to equivalence classes:

$$q_i \,:\, x_i \mapsto [x_i] \,.$$

Proof. We dicuss the proof of the first case. The second is directly analogous.

First observe that indeed, by construction, the projection maps p_i as given do make a cone over the free diagram, by the very nature of the relation that is imposed on the tuples:

$$\begin{array}{ccccc} & & \left\{ (x_k)_{k \in I} \mid \underset{\substack{i,j \in I \\ \alpha \in I_{i,j}}}{\forall} \left(f_\alpha(x_i) = x_j \right) \right\} & & \\ & {}^{p_i}\swarrow & & \searrow^{p_j} & \\ X_i & & \underset{f_\alpha}{\longrightarrow} & & X_j \end{array} \,.$$

We need to show that this is universal, in that every other cone over the free diagram factors universally through this one. First consider the case that the tip of a given cone is a singleton:

$$\begin{array}{ccccc} & & * & & \\ & {}^{p'_i}\swarrow & & \searrow^{p'_j} & \\ X_i & & \underset{f_\alpha}{\longrightarrow} & & X_j \end{array} \quad = \quad \begin{array}{ccccc} & & * & & \\ & {}^{\mathrm{const}_{x'_i}}\swarrow & & \searrow^{\mathrm{const}_{x'_j}} & \\ X_i & & \underset{f_\alpha}{\longrightarrow} & & X_j \end{array} \,.$$

As shown on the right, the data in such a cone is equivantly: for each $i \in I$ an element $x'_i \in X_i$, such that for all $i, j \in I$ and $\alpha \in I_{i,j}$ then $f_\alpha(x'_i) = x'_j$. But this is precisely the relation used in the construction of the limit above and hence there is a unique map

$$* \overset{(x'_i)_{i \in I}}{\longrightarrow} \left\{ (x_k)_{k \in I} \mid \underset{\substack{i,j \in I \\ \alpha \in I_{i,j}}}{\forall} \left(f_\alpha(x_i) = x_j \right) \right\}$$

such that for all $i \in I$ we have

$$
\begin{array}{ccc}
* & & \\
\downarrow & & \searrow^{p'_i} \\
\left\{ (x_k)_{k \in I} \;\vert\; \underset{\alpha \in I_{i,j}}{\underset{i,j \in I}{\forall}} \left(f_\alpha(x_i) = x_j \right) \right\} & \underset{p_i}{\longrightarrow} & X_i
\end{array}
$$

namely that map is the one that picks the element $(x'_i)_{i \in I}$.

This shows that every cone with tip a singleton factors uniquely through the claimed limiting cone. But then for a cone with tip an arbitrary set Y, this same argument applies to all the single elements of Y. ▮

It will turn out below in prop. 6.20 that limits and colimits of diagrams of topological spaces are computed by first applying prop. 6.16 to the underlying diagram of underlying sets, and then equipping the result with a topology as follows:

Definition 6.17. **(initial topology and final topology)**

Let $\{(X_i, \tau_i)\}_{i \in I}$ be a set of topological spaces, and let S be a bare set. Then

- For

$$\{S \overset{p_i}{\longrightarrow} X_i\}_{i \in I}$$

a set of functions out of S, the *initial topology* $\tau_{\mathrm{initial}}(\{p_i\}_{i \in I})$ is the coarsest topology on S (def. 6.17) such that all $f_i : (S, \tau_{\mathrm{initial}}(\{p_i\}_{i \in I})) \longrightarrow X_i$ are continuous.

By lemma 2.8 this is equivalently the topology whose open subsets are the unions of finite intersections of the preimages of the open subsets of the component spaces under the projection maps, hence the topology generated from the sub-base

$$\beta_{\mathrm{ini}}(\{p_i\}) = \left\{ p_i^{-1}(U_i) \;\vert\; i \in I,\, U_i \subset X_i \,\mathrm{open} \right\} .$$

- For

$$\{X_i \overset{f_i}{\longrightarrow} S\}_{i \in I}$$

a set of functions into S, the *final topology* $\tau_{\mathrm{final}}(\{f_i\}_{i \in I})$ is the finest topology on S (def. 6.17) such that all $q_i : X_i \longrightarrow (S, \tau_{\mathrm{final}}(\{f_i\}_{i \in I}))$ are continuous.

Hence a subset $U \subset S$ is open in the final topology precisely if for all $i \in I$ then the pre-image $q_i^{-1}(U) \subset X_i$ is open.

Beware a variation of synonyms that is in use:

limit topology	**colimit topology**

limit topology	**colimit topology**
initial topology	final topology
weak topology	strong topology
coarse topology	fine topology

We have already seen above simple examples of initial and final topologies:

Example 6.18. **(subspace topology as an initial topology)**

For (X,τ) a single topological space, and $q : S \hookrightarrow X$ a subset of its underlying set, then the initial topology $\tau_{\mathrm{intial}}(p)$, def. 6.17, is the subspace topology from example 2.16, making

$$p \;:\; (S, \tau_{\mathrm{initial}}(p)) \hookrightarrow X$$

a topological subspace inclusion.

Example 6.19. **(quotient topology as a final topology)**

Conversely, for (X,τ) a topological space and for $q : X \longrightarrow S$ a surjective function out of its underlying set, then the final topology $\tau_{\mathrm{final}}(q)$ on S, from def. 6.17, is the quotient topology from example 2.17, making q a continuous function:

$$q \;:\; (X,\tau) \longrightarrow (S, \tau_{\mathrm{final}}(q)) \,.$$

Now we have all the ingredients to explicitly construct limits and colimits of diagrams of topological spaces:

Proposition 6.20. ***(limits and colimits of topological spaces)***

Let

$$\left\{ (X_i, \tau_i) \overset{f_\alpha}{\longrightarrow} (X_j, \tau_j) \right\}_{i,j \in I, \alpha \in I_{i,j}}$$

be a free diagram of topological spaces (def. 6.4).

1. *The limit over this free diagram (def. 6.11) is given by the topological space*

    1. *whose underlying set is the limit of the underlying sets according to prop. 6.16;*

    2. *whose topology is the initial topology, def. 6.17, for the functions p_i which are the limiting cone components:*

$$\begin{array}{ccccc} && \underset{\longleftarrow}{\lim}_{k \in I} X_k \\ & {}^{\mathllap{p_i}}\swarrow && \searrow^{\mathrlap{p_j}} \\ X_i && \longrightarrow && X_j \end{array} \,.$$

    Hence

$$\underset{\longleftarrow}{\lim}_{i \in I} (X_i, \tau_i) \simeq \left(\underset{\longleftarrow}{\lim}_{i \in I} X_i,\ \tau_{\text{initial}}\left(\{p_i\}_{i \in I}\right)\right)$$

2. *The colimit over the free diagram (def. 6.11) is the topological space*

   1. *whose underlying set is the colimit of sets of the underlying diagram of sets according to prop. 6.16,*

   2. *whose topology is the final topology, def. 6.17 for the component maps ι_i of the colimiting cocone*

$$\begin{tikzcd} X_i \arrow[rr] \arrow[dr, "q_i"'] & & X_j \arrow[dl, "q_j"] \\ & \underset{\longrightarrow}{\lim}_{k \in I} X_k & \end{tikzcd} .$$

   Hence

$$\underset{\longrightarrow}{\lim}_{i \in I} (X_i, \tau_i) \simeq \left(\underset{\longrightarrow}{\lim}_{i \in I} X_i,\ \tau_{\text{final}}(\{q_i\}_{i \in I})\right)$$

(e.g. Bourbaki 71, section I.4)

Proof. We discuss the first case, the second is directly analogous:

Consider any cone over the given free diagram:

$$\begin{tikzcd} & (\tilde{X}, \tau_{\tilde{X}}) \arrow[dl, "p'_i"'] \arrow[dr, "p'_j"] & \\ (X_i, \tau_i) \arrow[rr] & & (X_j, \tau_j) \end{tikzcd}$$

By the nature of the limiting cone of the underlying diagram of underlying sets, which always exists by prop. 6.16, there is a unique function of underlying sets of the form

$$\phi : \tilde{X} \longrightarrow \underset{\longleftarrow}{\lim}_{i \in I} S_i$$

satisfying the required conditions $p_i \circ \phi = p'_i$. Since this is already unique on the underlying sets, it is sufficient to show that this function is always continuous with respect to the initial topology.

Hence let $U \subset \underset{\longleftarrow}{\lim}_i X_i$ be in $\tau_{\text{initial}}(\{p_i\})$. By def. 6.17, this means that U is a union of finite intersections of subsets of the form $p_i^{-1}(U_i)$ with $U_i \subset X_i$ open. But since taking pre-images preserves unions and intersections, and since unions and intersections of opens in $(\tilde{X}, \tau_{\tilde{X}})$ are again open, it is sufficient to consider U of the form $U = p_i^{-1}(U_i)$. But then by the condition that $p_i \circ \phi = p'_i$ we find

$$\begin{aligned} \phi^{-1}\left(p_i^{-1}(U_i)\right) &= \left(p_i \circ \phi\right)^{-1}(U_i) \\ &= (p'_i)^{-1}(U_i)\,, \end{aligned}$$

and this is open by the assumption that p'_i is continuous. ■

We discuss a list of examples of (co-)limits of topological spaces in a moment below, but first we conclude with the main theoretical impact of the concept of topological (co-)limits for our our purposes.

Here is a key property of (co-)limits:

Proposition 6.21. ***(functions into a limit cone are the limit of the functions into the diagram)***

Let $\{X_i \overset{f_\alpha}{\longrightarrow} X_j\}_{i,j \in I, \alpha \in I_{i,j}}$ be a free diagram (def. 6.4) of sets or of topological spaces.

1. *If the limit $\underset{\longleftarrow i}{\lim} X_i \in \mathcal{C}$ exists (def. 6.11), then the set of (continuous) function into this limiting object is the limit over the sets $\mathrm{Hom}(-,-)$ of (continuous) functions ("homomorphisms") into the components X_i:*

   $$\mathrm{Hom}\left(Y, \underset{\longleftarrow i}{\lim} X_i\right) \simeq \underset{\longleftarrow i}{\lim} (\mathrm{Hom}(Y, X_i)) .$$

   Here on the right we have the limit over the free diagram of sets given by the operations $f_\alpha \circ (-)$ of post-composition with the maps in the original diagram:

   $$\left\{\mathrm{Hom}(Y, X_i) \xrightarrow{f_\alpha \circ (-)} \mathrm{Hom}(Y, X_j)\right\}_{i,j \in I, \alpha \in I_{i,j}} .$$

2. *If the colimit $\underset{\longrightarrow i}{\lim} X_i \in \mathcal{C}$ exists, then the set of (continuous) functions out of this colimiting object is the limit over the sets of morphisms out of the components of X_i:*

   $$\mathrm{Hom}\left(\underset{\longrightarrow i}{\lim} X_i, Y\right) \simeq \underset{\longleftarrow}{\lim} (\mathrm{Hom}(X_i, Y)) .$$

   Here on the right we have the colimit over the free diagram of sets given by the operations $(-) \circ f_\alpha$ of pre-composition with the original maps:

   $$\left\{\mathrm{Hom}(X_i, Y) \xrightarrow{(-) \circ f_\alpha} \mathrm{Hom}(X_j, Y)\right\}_{i,j \in I, \alpha \in I_{i,j}} .$$

Proof. We give the proof of the first statement. The proof of the second statement is directly analogous (just reverse the direction of all maps).

First observe that, by the very definition of limiting cones, maps out of some Y into them are in natural bijection with the set $\mathrm{Cones}\left(Y, \{X_i \overset{f_\alpha}{\to} X_j\}\right)$ of cones over the corresponding diagram with tip Y:

$$\mathrm{Hom}\left(Y, \underset{\longleftarrow i}{\lim} X_i\right) \simeq \mathrm{Cones}\left(Y, \{X_i \overset{f_\alpha}{\to} X_j\}\right) .$$

Hence it remains to show that there is also a natural bijection like so:

$$\mathrm{Cones}\left(Y, \{X_i \overset{f_\alpha}{\to} X_j\}\right) \simeq \underset{\longleftarrow i}{\lim} (\mathrm{Hom}(Y, X_i)) .$$

Now, again by the very definition of limiting cones, a single element in the limit on the right is equivalently a cone of the form

$$\left\{ \begin{array}{ccc} & * & \\ {}^{\mathrm{const}_{p_i}}\swarrow & & \searrow^{\mathrm{const}_{p_j}} \\ \mathrm{Hom}(Y, X_i) & \underset{f_\alpha \circ (-)}{\longrightarrow} & \mathrm{Hom}(Y, X_j) \end{array} \right\} .$$

This is equivalently for each $i \in I$ a choice of map $p_i : Y \to X_i$, such that for each $i, j \in I$ and $\alpha \in I_{i,j}$ we have $f_\alpha \circ p_i = p_j$. And indeed, this is precisely the characterization of an element in the set $\mathrm{Cones}\left(Y, \{X_i \overset{f_\alpha}{\to} X_j\}\right)$. ▮

Using this, we find the following:

Remark 6.22. **(limits and colimits in categories of nice topological spaces)**

Recall from remark 4.24 the concept of adjoint functors

$$\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{D}$$

witnessed by natural isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(L(c), d) \simeq \mathrm{Hom}_{\mathcal{C}}(c, R(d)) .$$

Then:

1. the left adjoint functor L preserve colimits (def. 6.11)

   in that for every diagram $\{X_i \overset{f_\alpha}{\to} X_j\}$ in $\mathcal{D}$ there is a natural isomorphism of the form

   $$L\left(\underset{\longrightarrow i}{\lim} X_i\right) \simeq \underset{\longrightarrow i}{\lim} L(X_i)$$

2. the right adjoint functor R preserve limits (def. 6.11)

   in that for every diagram $\{X_i \overset{f_\alpha}{\to} X_j\}$ in $\mathcal{C}$ there is a natural isomorphism of the form

   $$R\left(\underset{\longleftarrow i}{\lim} X_i\right) \simeq \underset{\longleftarrow i}{\lim} R(X_i) .$$

This implies that if we have a reflective subcategory of topological spaces

$$\mathrm{Top}_{\mathrm{nice}} \underoverset{\underset{\iota}{\hookrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathrm{Top}$$

(such as with $T_{n \le 2}$-spaces according to remark 4.24 or with sober spaces according to remark 5.15)

then

1. limits in $\mathrm{Top}_{\mathrm{nice}}$ are computed as limits in Top;

2. colimits in $\mathrm{Top}_{\mathrm{nice}}$ are computed as the reflection L of the colimit in Top.

For example let $\{(X_i, \tau_i) \overset{f_\alpha}{\to} (X_j, \tau_j)\}$ be a diagram of Hausdorff spaces, regarded as a diagram of general topological spaces. Then

1. not only is the limit of topological spaces $\underset{\longleftarrow}{\lim}_i (X_i, \tau_i)$ according to prop. 6.20 again a Hausdorff space, but it also satisfies its universal property with respect to the category of Hausdorff spaces;

2. not only is the reflection $T_2\left(\underset{\longrightarrow}{\lim}_i X_i\right)$ of the colimit as topological spaces a Hausdorff space (while the colimit as topological spaces in general is not), but this reflection does satisfy the universal property of a colimit with respect to the category of Hausdorff spaces.

Proof. First to see that right/left adjoint functors preserve limits/colimits: We discuss the case of the right adjoint functor preserving limits. The other case is directly anlogous (just reverse the direction of all arrows).

So let $\underset{\longleftarrow}{\lim}_i X_i$ be the limit over some diagram $\left\{X_i \overset{f_\alpha}{\to} X_j\right\}_{i,j \in I, \alpha \in I_{i,j}}$. To test what a right adjoint functor does to this, we may map any object Y into it. Using prop. 6.21 this yields

$$\begin{aligned} \mathrm{Hom}(Y, R(\underset{\longleftarrow}{\lim}_i X_i)) & \simeq \mathrm{Hom}(L(Y), \underset{\longleftarrow}{\lim}_i X_i) \\ & \simeq \underset{\longleftarrow}{\lim}_i \mathrm{Hom}(L(Y), X_i) \\ & \simeq \underset{\longleftarrow}{\lim}_i \mathrm{Hom}(Y, R(X_i)) \\ & \simeq \mathrm{Hom}(Y, \underset{\longleftarrow}{\lim}_i R(Y_i)) \,. \end{aligned}$$

Since this is true for all Y, it follows that

$$R(\underset{\longleftarrow}{\lim}_i X_i) \simeq \underset{\longleftarrow}{\lim}_i R(X_i) \,.$$

Now to see that limits/colimits in the reflective subcategory are computed as claimed;

(…) ∎

Examples

We now discuss a list of examples of universal constructions of topological spaces as introduced in generality above.

examples of universal constructions of topological spaces:

limits	colimits
point space	empty space
product topological space	disjoint union topological space
topological subspace	quotient topological space
fiber space	space attachment
mapping cocylinder, mapping cocone	mapping cylinder, mapping cone, mapping telescope
	cell complex, CW-complex

Example 6.23. (empty space and point space as empty colimit and limit)

Consider the empty diagram (example 6.5) as a diagram of topological spaces. By example 6.12 the limit and colimit (def. 6.11) over this type of diagram are the *terminal object* and *initial object*, respectively. Applied to topological spaces we find:

1. The limit of topological spaces over the empty diagram is the point space $*$ (example 2.10).

2. The colimit of topological spaces over the empty diagram is the empty topological space $\emptyset$ (example 2.10).

This is because for an empty diagram, the a (co-)cone is just a topological space, without any further data or properties, and it is universal precisely if there is a unique continuous function to (respectively from) this space to any other space X. This is the case for the point space (respectively empty space) by example 3.5:

$$\emptyset \xrightarrow{\exists !} (X, \tau) \xrightarrow{\exists !} * \,.$$

Example 6.24. (binary product topological space and disjoint union space as limit and colimit)

Consider a discrete diagram consisting of two topological spaces $(X, \tau_X), (Y, \tau_Y)$ (example 6.5). Generally, it limit and colimit is the *product* $X \times Y$ and *coproduct* $X \sqcup Y$, respectively (example 6.13).

1. In topological space this product is the binary product topological space from example 2.18, by the universal property observed in example 6.1:

$$(X, \tau_X) \times (Y, \tau_Y) \simeq (X \times Y, \tau_{X \times Y}) \,.$$

2. In topological spaces, this coproduct is the disjoint union space from example 2.15, by the universal property observed in example 6.2:

$$(X, \tau_X) \sqcup (Y, \tau_Y) \simeq (X \sqcup Y, \tau_{X \sqcup Y}) \,.$$

So far these examples just reproduces simple constructions which we already considered. Now the first important application of the general concept of limits of diagrams of topological spaces is the following example 6.25 of product spaces with an non-finite set of factors. It turns out that the correct topology on the underlying infinite Cartesian product of sets is *not* the naive generalization of the binary product topology, but instead is the corresponding weak topology, here called the *Tychonoff topology*.

Example 6.25. (general product topological spaces with Tychonoff topology)

Consider an arbitrary discrete diagram of topological spaces (def. 6.5), hence a set $\{(X_i, \tau_i)\}_{i \in I}$ of topological spaces, indexed by any set I, not necessarily a finite set.

The limit over this diagram (a *Cartesian product*, example 6.13) is called the *product topological space* of the spaces in the diagram, and denoted

$$\underset{i \in I}{\prod} (X_i, \tau_i) \,.$$

By prop. 6.16 and prop. 6.18, the underlying set of this product space is just the Cartesian product of the underlying sets, hence the set of tuples $(x_i \in X_i)_{i \in I}$. This comes for each $i \in I$ with the projection map

$$\begin{array}{ccc} \prod_{j \in I} X_j & \overset{pr_i}{\longrightarrow} & X_i \\ (x_j)_{j \in I} & \longmapsto & x_i \end{array} \,.$$

By prop. 6.18 and def. 6.17, the topology on this set is the coarsest topology such that the pre-images $pr_i(U)$ of open subsets $U \subset X_i$ under these projection maps are open. Now one such pre-image is a Cartesian product of open subsets of the form

$$p_i^{-1}(U_i) \;=\; U_i \times \left(\underset{j \in I \backslash \{i\}}{\prod} X_j \right) \;\subset\; \underset{j \in I}{\prod} X_j \,.$$

The coarsest topology that contains these open subsets ist that generated by these subsets regarded as a sub-basis for the topology (def. 2.7), hence the arbitrary unions of finite intersections of subsets of the above form.

Observe that a binary intersection of these generating open is (for $i \neq j$):

$$p_i^{-1}(U_i) \cap p_j^{-1}(U_j) \;\simeq\; U_i \times U_j \times \left(\underset{k \in I \backslash \{i.j\}}{\prod} X_k \right)$$

and generally for a finite subset $J \subset I$ then

$$\underset{j \in J \subset I}{\cap} p_i^{-1}(U_i) \;=\; \left(\prod_{j \in J \subset I} U_j \right) \times \left(\prod_{i \in I \backslash J} X_i \right).$$

Therefore the open subsets of the product topology are unions of those of this form. Hence the product topology is equivalently that generated by these subsets when regarded as a basis for the topology (def. 2.7).

This is also known as the *Tychonoff topology*.

Notice the subtlety: Naively we could have considered as open subsets the unions of products $\prod_{i \in I} U_i$ of open subsets of the factors, without the constraint that only finitely many of them differ from the corresponding total space. This also defines a topology, called the *box topology*. For a finite index set I the box topology coincides with the product space (Tychinoff) topology, but for non-finite I it is strictly finer (def. 2.6).

Example 6.26. **(equalizer of continuous functions)**

The equalizer (example 6.14) of two continuous functions $f, g : (X, \tau_X) \rightrightarrows (Y, \tau_Y)$ is the equalizer of the underlying functions of sets

$$\mathrm{eq}(f,g) \hookrightarrow X \;\underoverset{g}{f}{\rightrightarrows}\; Y$$

(hence the largest subset of Y on which both functions coincide) and equipped with the subspace topology from example 2.16.

Example 6.27. **(coequalizer of continuous functions)**

The coequalizer of two continuous functions $f, g : (X, \tau_X) \rightrightarrows (Y, \tau_Y)$ is the coequalizer of the underlying functions of sets

$$X \;\underoverset{g}{f}{\rightrightarrows}\; Y \longrightarrow \mathrm{coeq}(f,g)$$

(hence the quotient set by the equivalence relation generated by the relation $f(x) \sim g(x)$ for all $x \in X$) and equipped with the quotient topology, example 2.17.

Example 6.28. **(space attachments)**

Consisder a cospan diagram (example 6.7) of continuous functions

$$\begin{array}{ccc} (A, \tau_A) & \overset{g}{\longrightarrow} & (Y, \tau_Y) \\ {}^{\mathllap{f}}\downarrow & & \\ (X, \tau_X) & & \end{array}$$

The colimit under this diagram called the *pushout* (example 6.15)

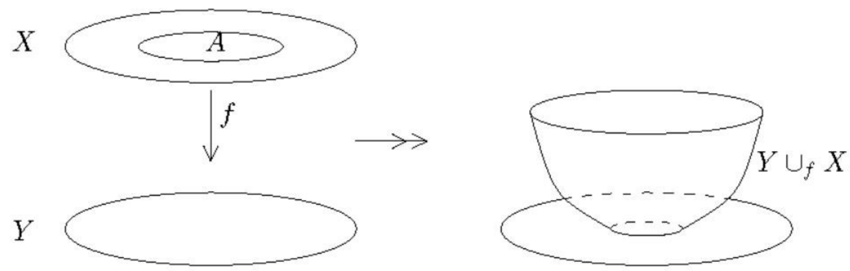
$$\begin{array}{ccc} (A,\tau_A) & \overset{g}{\longrightarrow} & (Y,\tau_Y) \\ {}^{f}\downarrow & (\mathrm{po}) & \downarrow^{g_* f} \\ (X,\tau_X) & \longrightarrow & (X,\tau_X) \underset{(A,\tau_A)}{\sqcup} (Y,\tau_Y) \,. \end{array}$$

Consider on the disjoint union set $X \sqcup Y$ the equivalence relation generated by the relation

$$(x \sim y) \Leftrightarrow \left(\underset{a \in A}{\exists} (x = f(a) \,\text{and}\, y = g(a)) \right) .$$

Then prop. 6.20 implies that the pushout is equivalently the quotient topological space (example 2.17) by this equivalence relation of the disjoint union space (example 2.15) of X and Y:

$$(X,\tau_X) \underset{(A,\tau_A)}{\sqcup} (Y,\tau_Y) \;\simeq\; ((X \sqcup Y, \tau_{X \sqcup Y}))/\sim \,.$$



If g is an topological subspace inclusion $A \subset X$, then in topology its pushout along f is traditionally written as

$$X \cup_f Y \;\coloneqq\; (X,\tau_X) \underset{(A,\tau_A)}{\sqcup} (Y,\tau_Y)$$

and called the *space attachment* (sometimes: *attaching space* or *adjunction space*) of $A \subset X$ along f.

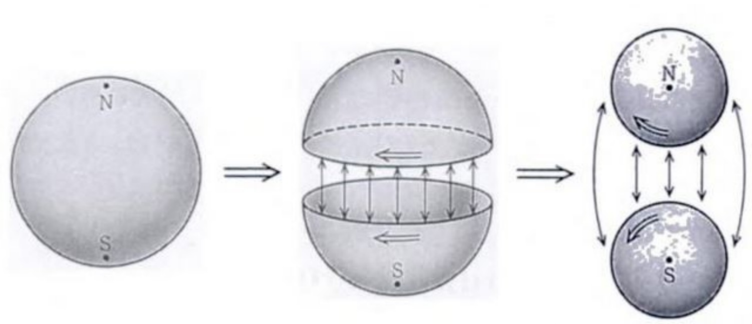
(graphics from Aguilar-Gitler-Prieto 02)

Example 6.29. **(n-sphere as pushout of the equator inclusions into its hemispheres)**

As an important special case of example 6.28, let

$$i_n \,:\, S^{n-1} \longrightarrow D^n$$

be the canonical inclusion of the standard (n-1)-sphere as the boundary of the standard n-disk (example 2.20).



Then the colimit of topological spaces under the span diagram,

$$D^n \overset{i_n}{\longleftarrow} S^{n-1} \overset{i_n}{\longrightarrow} D^n ,$$

is the topological n-sphere S^n (example 2.20):

$$
\begin{array}{ccc}
S^{n-1} & \overset{i_n}{\longrightarrow} & D^n \\
{}^{i_n}\downarrow & (\mathrm{po}) & \downarrow \\
D^n & \longrightarrow & S^n
\end{array}
.
$$

(graphics from Ueno-Shiga-Morita 95)

In generalization of this example, we have the following important concept:

Definition 6.30. (single cell attachment)

For X any topological space and for $n \in \mathbb{N}$, then an n-*cell attachment* to X is the result of gluing an n-disk to X, along a prescribed image of its bounding (n-1)-sphere (def. 2.20):

Let

$$
\phi : S^{n-1} \longrightarrow X
$$

be a continuous function, then the space attachment (example 6.28)

$$
X \cup_\phi D^n \;\in \mathrm{Top}
$$

is the topological space which is the pushout of the boundary inclusion of the n-sphere along ϕ, hence the universal space that makes the following diagram commute:

$$
\begin{array}{ccc}
S^{n-1} & \overset{\phi}{\longrightarrow} & X \\
{}^{\iota_n}\downarrow & (\mathrm{po}) & \downarrow \\
D^n & \longrightarrow & X \cup_\phi D^n
\end{array}
.
$$

Example 6.31. (discrete topological spaces from 0-cell attachment to the empty space)

A single cell attachment of a 0-cell, according to example 6.30 is the same as forming the disjoint union space $X \sqcup *$ with the point space $*$:

$$
\begin{array}{ccc}
(S^{-1} = \emptyset) & \overset{\exists !}{\longrightarrow} & X \\
\downarrow & (\mathrm{po}) & \downarrow \\
(D^0 = *) & \longrightarrow & X \sqcup *
\end{array}
.
$$

In particular if we start with the empty topological space $X = \emptyset$ itself (example 2.10), then by attaching 0-cells we obtain a discrete topological space. To this then we may attach higher dimensional cells.

Definition 6.32. (attaching many cells at once)

If we have a set of attaching maps $\{S^{n_i - 1} \overset{\phi_i}{\longrightarrow} X\}_{i \in I}$ (as in def. 6.30), all to the same space X, we may think of these as one single continuous function out of the disjoint union space of their domain spheres

$$(\phi_i)_{i \in I} : \underset{i \in I}{\sqcup} S^{n_i - 1} \longrightarrow X \,.$$

Then the result of attaching *all* the corresponding n-cells to X is the pushout of the corresponding disjoint union of boundary inclusions:

$$\begin{array}{ccc} \underset{i \in I}{\sqcup} S^{n_i - 1} & \overset{(\phi_i)_{i \in I}}{\longrightarrow} & X \\ \downarrow & (po) & \downarrow \\ \underset{i \in I}{\sqcup} D^n & \longrightarrow & X \cup_{(\phi_i)_{i \in I}} \left(\underset{i \in I}{\sqcup} D^n \right) \end{array} \,.$$

Apart from attaching a set of cells all at once to a fixed base space, we may "attach cells to cells" in that after forming a given cell attachment, then we further attach cells to the resulting attaching space, and ever so on:

Definition 6.33. (**relative cell complexes** and **CW-complexes**)

Let X be a topological space, then A *topological relative cell complex* of countable height based on X is a continuous function

$$f : X \longrightarrow Y$$

and a sequential diagram of topological space of the form

$$X = X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots$$

such that

1. each $X_k \hookrightarrow X_{k+1}$ is exhibited as a cell attachment according to def. 6.32, hence presented by a pushout diagram of the form

$$\begin{array}{ccc} \underset{i \in I}{\sqcup} S^{n_i - 1} & \overset{(\phi_i)_{i \in I}}{\longrightarrow} & X_k \\ \downarrow & (po) & \downarrow \\ \underset{i \in I}{\sqcup} D^{n_i} & \longrightarrow & X_{k+1} \end{array} \,.$$

2. $Y = \underset{k \in \mathbb{N}}{\cup} X_k$ is the union of all these cell attachments, and $f : X \to Y$ is the canonical inclusion; or stated more abstractly: the map $f : X \to Y$ is the inclusion of the first component of the diagram into its colimiting cocone $\underset{\longrightarrow_k}{\lim} X_k$:

$$\begin{array}{ccccccc} X = X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ & {}_{f}\searrow & \downarrow & \swarrow & & \cdots & \\ & & Y = \underset{\longrightarrow}{\lim} X_\bullet & & & & \end{array}$$

If here $X = \emptyset$ is the empty space then the result is a map $\emptyset \hookrightarrow Y$, which is equivalently just a space Y built form "attaching cells to nothing". This is then called just a *topological cell complex* of countable hight.

Finally, a topological (relative) cell complex of countable hight is called a

CW-complex is the $(k+1)$-st cell attachment $X_k \to X_{k+1}$ is entirely by $(k+1)$-cells, hence exhibited specifically by a pushout of the following form:

$$\begin{array}{ccc} \underset{i \in I}{\sqcup} S^k & \overset{(\phi_i)_{i \in I}}{\longrightarrow} & X_k \\ \downarrow & (\text{po}) & \downarrow \\ \underset{i \in I}{\sqcup} D^{k+1} & \longrightarrow & X_{k+1} \end{array} \,.$$

A *finite CW-complex* is one which admits a presentation in which there are only finitely many attaching maps, and similarly a *countable CW-complex* is one which admits a presentation with countably many attaching maps.

Given a CW-complex, then X_n is also called its n-skeleton.

7. Compact spaces

We discuss *compact topological spaces* (def 7.2 below), the generalization of compact metric spaces above. Compact spaces are in some sense the "small" objects among topological spaces, analogous in topology to what finite sets are in set theory, or what finite-dimensional vector spaces are in linear algebra, and equally important in the theory.

Prop. 1.21 suggests the following simple definition 7.2:

Definition 7.1. **(open cover)**

An *open cover* of a topological space (X,τ) (def. 2.3) is a set $\{U_i \subset X\}_{i \in I}$ of open subsets U_i of X, indexed by some set I, such that their union is all of X: $\underset{i \in I}{\cup} U_i \;=\; X$.

A *subcover* of a cover is a subset $J \subset I$ such that $\{U_i \subset X\}_{i \in J \subset I}$ is still a cover.

Definition 7.2. **(compact topological space)**

A topological space X (def. 2.3) is called a *compact topological space* if every open cover $\{U_i \subset X\}_{i \in I}$ (def. 7.1) has a *finite subcover* in that there is a finite subset $J \subset I$ such that $\{U_i \subset X\}_{i \in J}$ is still a cover of X in that also $\underset{i \in J}{\cup} U_i = X$.

Remark 7.3. **(terminology issue regarding "compact")**

Beware the following terminology issue which persists in the literature:

Some authors use "compact" to mean "Hausdorff and compact". To disambiguate this, some authors (mostly in algebraic geometry, but also for instance Waldhausen) say "quasi-compact" for what we call "compact" in def. 7.2.

There are several equivalent reformulations of the compactness condition. An immediate reformulation is prop. 7.4, a more subtle one is prop. 7.13 further below.

Proposition 7.4. ***(compactness in terms of closed subsets)***

Let (X,τ) be a topological space. Then the following are equivalent:

1. *(X,τ) is compact in the sense of def. 7.2.*
2. *Let $\{C_i \subset X\}_{i \in I}$ be a set of closed subsets (def. 2.23) such that their intersection is empty $\underset{i \in I}{\cap} C_i = \emptyset$, then there is a finite subset $J \subset I$ such that the corresponding finite intersection is still empty $\underset{i \in J \subset i}{\cap} C_i = \emptyset$.*
3. *Let $\{C_i \subset X\}_{i \in I}$ be a set of closed subsets (def. 2.23) such that it enjoys the finite intersection property, meaning that for every finite subset $J \subset I$ then the corresponding finite intersection is non-empty $\underset{i \in J \subset I}{\cap} C_i \neq \emptyset$. Then also the total intersection is non-empty, $\underset{i \in I}{\cap} C_i \neq \emptyset$.*

Proof. The equivalence between the first and the second statement is immediate from the definitions after expressing open subsets as complements of closed subsets $U_i = X \backslash C_i$ and applying de Morgan's law (remark 2.24).

We discuss the equivalence between the first and the third statement:

In one direction, assume that (X,τ) is compact in the sense of def. 7.2, and that $\{C_i \subset X\}_{i \in I}$ satisfies the finite intersection property. We need to show that then $\underset{i \in I}{\cap} C_i \neq \emptyset$.

Assume that this were not the case, hence assume that $\underset{i \in I}{\cap} C_i = \emptyset$. This would imply that the open complements were an open cover of X (def. 7.1)

$$\{U_i \coloneqq X \backslash C_i\}_{i \in I},$$

because (using de Morgan's law, remark 2.24)

$$\begin{aligned} \underset{i \in I}{\cup} U_i &\coloneqq \underset{i \in I}{\cup} X \backslash C_i \\ &= X \backslash \left(\underset{i \in I}{\cap} C_i \right) \\ &= X \backslash \emptyset \\ &= X \end{aligned}.$$

But then by compactness of (X,τ) there were a finite subset $J \subset I$ such that $\{U_i \subset X\}_{i \in J \subset I}$ were still an open cover, hence that $\underset{i \in J \subset I}{\cup} U_i = X$. Translating this back through the de Morgan's law again this would mean that

$$
\begin{aligned}
\emptyset &= X \backslash \left(\underset{i \in J \subset I}{\cup} U_i \right) \\
&\coloneqq X \backslash \left(\underset{i \in J \subset I}{\cup} X \backslash C_i \right) \\
&= \underset{i \in J \subset I}{\cap} X \backslash (X \backslash C_i) \\
&= \underset{i \in J \subset I}{\cap} C_i \; .
\end{aligned}
$$

This would be in contradiction with the finite intersection property of $\{C_i \subset X\}_{i \in I}$, and hence we have proof by contradiction.

Conversely, assume that every set of closed subsets in X with the finite intersection property has non-empty total intersection. We need to show that the every open cover $\{U_i \subset X\}_{i \in I}$ of X has a finite subcover.

Write $C_i \coloneqq X \backslash U_i$ for the closed complements of these open subsets.

Assume on the contrary that there were no finite subset $J \subset I$ such that $\underset{i \in J \subset I}{\cup} U_i = X$, hence no finite subset such that $\underset{i \in J \subset I}{\cap} C_i = \emptyset$. This would mean that $\{C_i \subset X\}_{i \in I}$ satisfied the finite intersection property.

But by assumption this would imply that $\underset{i \in I}{\cap} C_i \neq \emptyset$, which, again by de Morgan, would mean that $\underset{i \in I}{\cup} U_i \neq X$. But this contradicts the assumption that the $\{U_i \subset X\}_{i \in I}$ are a cover. Hence we have a proof by contradiction. ▮

Example 7.5. **(finite discrete spaces are compact)**

A discrete topological space (def. 2.13) is compact (def. 7.2) precisely if its underlying set is a finite set.

Example 7.6. **(closed intervals are compact)**

For any $a < b \in \mathbb{R}$ the closed interval (example 1.13)

$$[a, b] \subset \mathbb{R}$$

regarded with its subspace topology of Euclidean space (example 1.6) with its metric topology (example 2.9) is a compact topological space (def. 7.2).

Proof. Since all the closed intervals are homeomorphic (by example 3.27) it is sufficient to show the statement for $[0,1]$. Hence let $\{U_i \subset [0,1]\}_{i \in I}$ be an open cover (def. 7.1). We need to show that it has an open subcover.

Say that an element $x \in [0,1]$ is *admissible* if the closed sub-interval $[0, x]$ is covered by finitely many of the U_i. In this terminology, what we need to show is that 1 is admissible.

Observe from the definition that

1. 0 is admissible,

2. if $y < x \in [0,1]$ and x is admissible, then also y is admissible.

This means that the set of admissible x forms either

1. an open interval $[0,g)$
2. or a closed interval $[0,g]$,

for some $g \in [0,1]$. We need to show that the latter is true, and for $g = 1$. We do so by observing that the alternatives lead to contradictions:

1. Assume that the set of admissible values were an open interval $[0,g)$. Pick an $i_0 \in I$ such that $g \in U_{i_0}$ (this exists because of the covering property). Since such U_{i_0} is an open neighbourhood of g, there is a positive real number ϵ such that the open ball $B^\circ_g(\epsilon) \subset U_{i_0}$ is still contained in the patch. It follows that there is an element $x \in B^\circ_g(\epsilon) \cap [0,g) \subset U_{i_0} \cap [0,g)$ and such that there is a finite subset $J \subset I$ with $\{U_i \subset [0,1]\}_{i \in J \subset I}$ a finite open cover of $[0,x)$. It follows that $\{U_i \subset [0,1]\}_{i \in J \subset I} \sqcup \{U_{i_0}\}$ were a finite open cover of $[0,g]$, hence that g itself were still admissible, in contradiction to the assumption.

2. Assume that the set of admissible values were a closed interval $[0,g]$ for $g < 1$. By assumption there would then be a finite set $J \subset I$ such that $\{U_i \subset [0,1]\}_{i \in J \subset I}$ were a finite cover of $[0,g]$. Hence there would be an index $i_g \in J$ such that $g \in U_{i_g}$. But then by the nature of open subsets in the Euclidean space $\mathbb{R}$, this U_{i_g} would also contain an open ball $B^\circ_g(\epsilon) = (g-\epsilon, g+\epsilon)$. This would mean that the set of admissible values includes the open interval $[0, g+\epsilon)$, contradicting the assumption.

This gives a proof by contradiction. ▮

In contrast:

Nonexample 7.7. **(Euclidean space is non-compact)**

For all $n \in \mathbb{N}$, $n > 0$, the Euclidean space $\mathbb{R}^n$ (example 1.6), regarded with its metric topology (example 2.9), is *not* a compact topological space (def. 7.2).

Proof. Pick any $\epsilon \in (0, 1/2)$. Consider the open cover of $\mathbb{R}^n$ given by

$$\left\{ U_n \coloneqq (n-\epsilon, n+1+\epsilon) \times \mathbb{R}^{n-1} \;\subset\; \mathbb{R}^{n+1} \right\}_{n \in \mathbb{Z}} \,.$$

This is not a finite cover, and removing any one of its patches U_n, it ceases to be a cover, since the points of the form $(n+\epsilon, x_2, x_3, \cdots, x_n)$ are contained only in U_n and in no other patch. ▮

Below we prove the Heine-Borel theorem (prop. 7.37) which generalizes example 7.6 and example 7.7.

In analysis, the extreme value theorem (example 7.11 below) asserts that a real-valued continuous function on the bounded closed interval (def. 1.13) attains its maximum and minimum. The following is the generalization of this statement to general topological spaces, cast in terms of the more abstract concept of compactness from def. 7.2:

Lemma 7.8. ***(continuous surjections out of compact spaces have compact codomain)***

Let $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ be a continuous function between topological spaces such that

1. *(X, τ_X) is a compact topological space (def. 7.2);*
2. *$f : X \to Y$ is a surjective function.*

Then also (Y, τ_Y) is compact.

Proof. Let $\{U_i \subset Y\}_{i \in I}$ be an open cover of Y (def. 7.1). We need show that this has a finite sub-cover.

By continuity of f, the pre-images $f^{-1}(U_i)$ are open subsets of X, and by the surjectivity of f they form an open cover $\{f^{-1}(U_i) \subset X\}_{i \in I}$ of X. Hence by compactness of X, there exists a finite subset $J \subset I$ such that $\{f^{-1}(U_i) \subset X\}_{i \in J \subset I}$ is still an open cover of X. Finally, using again that f is assumed to be surjective, it follows that

$$\begin{aligned} Y & = f(X) \\ & = f\left(\underset{i \in J}{\cup} f^{-1}(U_i) \right) \\ & = \underset{i \in J}{\cup} U_i \end{aligned}$$

which means that also $\{U_i \subset Y\}_{i \in J \subset I}$ is still an open cover of Y, and in particular a finite subcover of the original cover. ▮

As a direct corollary of lemma 7.8 we obtain:

Proposition 7.9. ***(continuous images of compact spaces are compact)***

If $f : X \longrightarrow Y$ is a continuous function out of a compact topological space X (def. 7.2) which is not necessarily surjective, then we may consider its image factorization

$$f \;:\; X \longrightarrow f(X) \hookrightarrow Y$$

as in example 3.10. Now by construction $X \to f(X)$ is surjective, and so lemma 7.8 implies that $f(X)$ is compact.

The converse to cor. 7.9 does not hold in general: the pre-image of a compact subset under a continuous function need not be compact again. If this is the case, then we speak of *proper maps*:

Definition 7.10. (**proper maps**)

A continuous function $f : (X, \tau_X) \to (Y, \tau_Y)$ is called *proper* if for $C \in Y$ a compact topological subspace of Y, then also its pre-image $f^{-1}(C)$ is compact in X.

As a first useful application of the concept of compactness we have:

Proposition 7.11. ***(extreme value theorem)***

Let C be a compact topological space (def. 7.2), and let

$$f : C \longrightarrow \mathbb{R}$$

be a continuous function to the real numbers equipped with their Euclidean metric topology.

Then f attains is maximum and its minimum, i.e. there exist $x_{\min}, x_{\max} \in C$ such that for all $x \in C$ it is true that

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}) \,.$$

Proof. Since continuous images of compact spaces are compact (prop. 7.9) the image $f([a, b]) \subset \mathbb{R}$ is a compact subspace.

Suppose this image did not contain its maximum. Then $\{(-\infty, x)\}_{x \in f([a,b])}$ were an open cover of the image, and hence, by its compactness, there would be a finite subcover, hence a finite set $(x_1 < x_2 < \cdots < x_n)$ of points $x_i \in f([a, b])$, such that the union of the $(-\infty, x_i)$ and hence the single set $(-\infty, x_n)$ alone would cover the image. This were in contradiction to the assumption that $x_n \in f([a, b])$ and hence we have a proof by contradiction.

Similarly for the minimum. ▮

And as a special case:

Example 7.12. (**traditional extreme value theorem**)

Let

$$f : [a, b] \longrightarrow \mathbb{R}$$

be a continuous function from a bounded closed interval ($a < b \in \mathbb{R}$) (def. 1.13) regarded as a topological subspace (example 2.16) of real numbers to the real numbers, with the latter regarded with their Euclidean metric topology (example 1.6, example 2.9).

Then f attains its maximum and minimum: there exists $x_{\max}, x_{\min} \in [a, b]$ such that for all $x \in [a, b]$ we have

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}) \,.$$

Proof. Since continuous images of compact spaces are compact (prop. 7.9) the image $f([a,b]) \subset \mathbb{R}$ is a compact subspace (def. 7.2, example 2.16). By the Heine-Borel theorem this is a bounded closed subset (def. 1.3, def. 2.23). By the nature of the Euclidean metric topology, the image is hence a union of closed intervals. Finally by continuity of f it needs to be a single closed interval, hence (being bounded) of the form

$$f([a,b]) = [f(x_{\min}), f(x_{\max})] \;\subset\; \mathbb{R} \,.$$

■

There is also the following more subtle equivalent reformulation of compactness:

Proposition 7.13. ***(closed-projection characterization of compactness)***

Let (X,τ_X) be a topological space. The following are equivalent:

1. *(X,τ_X) is a compact topological space according to def. 7.2;*

2. *For every topological space (Y,τ_Y) then the projection map out of the product topological space (example 2.18, example 6.25)*

$$\pi_Y \;:\; (Y,\tau_Y) \times (X,\tau_X) \longrightarrow (Y,\tau_Y)$$

is a closed map.

Proof. (due to Todd Trimble)

In one direction, assume that (X,τ_X) is compact and let $C \subset Y \times X$ be a closed subset. We need to show that $\pi_Y(C) \subset Y$ is closed.

By lemma 2.25 this is equivalent to showing that every point $y \in Y \backslash \pi_Y(C)$ in the complement of $\pi_Y(C)$ has an open neighbourhood $V_y \supset \{y\}$ which does not intersect $\pi_Y(C)$:

$$V_y \cap \pi_Y(C) = \emptyset \,.$$

This is clearly equivalent to

$$(V_y \times X) \cap C = \emptyset$$

and this is what we will show.

To this end, consider the set

$$\left\{ U \subset X \,\text{open} \;\vert\; \underset{\substack{V \subset Y \,\text{open} \\ V \supset \{y\}}}{\exists} ((V \times U) \cap C = \emptyset) \right\}$$

Observe that this is an open cover of X: For every $x \in X$ then $(y,x) \notin C$ by assumption of Y, and by closure of C this means that there exists an open neighbourhood of (y,x) in $Y \times X$ not intersecting C, and by nature of the product topology this contains an open neighbourhood of the form $V \times U$.

Hence by compactness of X, there exists a finite subcover $\{U_j \subset X\}_{j \in J}$ of X and a corresponding set $\{V_j \subset Y\}_{j \in J}$ with $V_j \times U_j \cap C = \emptyset$.

The resulting open neighbourhood

$$V \coloneqq \underset{j \in J}{\cap} V_j$$

of y has the required property:

$$\begin{aligned} V \times X & = V \times \left(\underset{j \in J}{\cup} U_j \right) \\ & = \underset{j \in J}{\cup} \left(V \times U_j \right) \\ & \subset \underset{j \in J}{\cup} \left(V_j \times U_j \right) \\ & \subset (Y \times X) \backslash C \;. \end{aligned}$$

Conversely, assume that $\pi_Y : Y \times X \to X$ is a closed map for all Y. We need to show that X is compact. By prop. 7.4 this means equivalently that for every set $\{C_i \subset X\}_{i \in I}$ of closed subsets and satisfying the finite intersection property, then $\underset{i \in I}{\cap} C_i \neq \emptyset$.

Construct a new topological space (Y, τ_Y) by

1. $Y \coloneqq X \sqcup \{\infty\}$;
2. $\beta_Y \coloneqq P(X) \sqcup \{(C_i \cup \{\infty\}) \subset Y\}$ a sub-base for τ_Y (def. 2.7).

Then consider the topological closure of the "diagonal" Δ in $Y \times X$

$$K \coloneqq \mathrm{Cl}(\Delta) \quad \text{with} \quad \Delta \coloneqq \{(x,x) \in Y \times X \mid x \in X\} \;.$$

We claim that there exists $x \in X$ such that

$$(\infty, x) \in K \;.$$

This is because

$$\pi_Y(K) \subset Y \text{ is closed}$$

by assumption and

$$X \subset \pi_Y(K)$$

by construction. So if ∞ were not in $\pi_Y(K)$, then, by lemma 2.25, it would have an open neighbourhood not intersecting X. But by definition of τ_Y, the open neighbourhoods of ∞ are the finite intersections of $C_i \cup \{\infty\}$, and by the assumed finite intersection property all their finite intersections do still intersect X.

Since thus $(\infty, x) \in K$, lemma 2.25 gives again that all of its open neighbourhoods intersect the diagonal, hence that for all $i \in I$ and $U_x \supset \{x\}$ open then

$$((C_i \cup \{\infty\}) \times U_x) \ \cap \ \Delta \ \neq \ \emptyset \ .$$

This means equivalently that

$$C_i \cap U_x \neq \emptyset$$

for all open neighbourhoods $U_x \supset \{x\}$.

But by closure of C_i and using lemma 2.25, this means that

$$x \in C_i$$

for all i, hence that

$$\underset{i \in I}{\cap} C_i \neq \emptyset$$

as required. ▮

The closed-projection characterization of compactness (prop. 7.13) yields direct proof of important facts in topology:

- The *tube lemma*, prop. 7.14 below,
- The *Tychonoff theorem*, prop. 7.15 below.

Lemma 7.14. ***(tube lemma)***

Let

1. (X, τ_X) *be a topological space,*
2. (Y, τ_Y) *a compact topological space,*
3. $x \in X$ *a point,*
4. $W \underset{\text{open}}{\subset} X \times Y$ *an open subset in the product topology (example 2.18, example 7.15),*

such that the Y*-fiber over* x *is contained in* W*:*

$$\{x\} \times Y \ \subseteq \ W \ .$$

Then there exists an open neighborhood U_x *of* x *such that the "tube"* $U_x \times Y$ *around the fiber* $\{x\} \times Y$ *is still contained:*

$$U_x \times Y \subseteq W \ .$$

Proof. Let

$$C \coloneqq (X \times Y) \backslash W$$

be the complement of W. Since this is closed, by prop. 7.13 also its projection $p_X(C) \subset X$ is closed.

Now

$$\{x\} \times Y \subset W \;\Leftrightarrow\; \{x\} \times Y \;\cap\; C = \emptyset$$
$$\Rightarrow\; \{x\} \cap p_X(C) = \emptyset$$

and hence by the closure of $p_X(C)$ there is (by this lemma) an open neighbourhood $U_x \supset \{x\}$ with

$$U_x \cap p_X(C) = \emptyset \,.$$

This means equivalently that $U_x \times Y \cap C = \emptyset$, hence that $U_x \times Y \subset W$. ■

Proposition 7.15. ***(Tychonoff theorem – the product space of compact spaces is compact)***

Let $\{(X_i, \tau_i)\}_{i \in I}$ be a set of compact topological spaces (def. 7.2). Then also their product space $\prod_{i \in I} (X_i, \tau_i)$ (example 6.25) is compact.

We give a proof of the finitary case of the Tychonoff theorem using the closed-projection characterization of compactness from prop. 7.13. This elementary proof generalizes fairly directly to an elementary proof of the general case: see here.

Proof of the finitary case. By prop. 7.13 it is sufficient to show that for every topological space (Y, τ_Y) then the projection

$$\pi_Y \;:\; (Y, \tau_Y) \times \left(\underset{i \in \{1, \cdots, n\}}{\prod} (X_i, \tau_i) \right) \longrightarrow (Y, \tau_Y)$$

is a closed map. We proceed by induction. For $n = 0$ the statement is obvious. Suppose it has been proven for some $n \in \mathbb{N}$. Then the projection for $n+1$ factors is the composite of two consecutive projections

$$\pi_Y \;:\; Y \times \left(\underset{i \in \{1, \cdots, n+1\}}{\prod} X_i \right) = Y \times \left(\underset{i \in \{1, \cdots, n\}}{\prod} X_i \right) \times X_{n+1} \longrightarrow Y \times \left(\underset{i \in \{1, \cdots,, n\}}{\prod} X_i \right) \longrightarrow Y \,.$$

By prop. 7.13, the first map here is closed since (X_{n+1}, τ_{n+1}) is compact by the assumption of the proposition, and similarly the second is closed by induction assumtion. Hence the composite is a closed map. ■

Of course we also want to claim that sequentially compact metric spaces (def. 1.20) are compact as topological spaces when rgarded with their metric topology (example 2.9):

Definition 7.16. **(converging sequence in a topological space)**

Let (X, τ) be a topological space (def. 2.3) and let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points (x_n) in X (def. 1.16). We say that this sequence *converges* in (X, τ) to a point $x_\infty \in X$, denoted

$$x_n \overset{n \to \infty}{\longrightarrow} x_\infty$$

if for each open neighbourhood U_{x_∞} of x_∞ there exists a $k \in \mathbb{N}$ such that for all $n \geq k$ then $x_n \in U_{x_\infty}$:

$$\left(x_n \overset{n \to \infty}{\longrightarrow} x_\infty \right) \;\Leftrightarrow\; \left(\underset{\substack{U_{x_\infty} \in \tau_X \\ x_\infty \in U_{X_\infty}}}{\forall} \left(\underset{k \in \mathbb{N}}{\exists} \left(\underset{n \geq k}{\forall} x_n \in U_{x_\infty} \right)\right)\right).$$

Accordingly it makes sense to consider the following:

Definition 7.17. (**sequentially compact topological space**)

Let (X, τ) be a topological space (def. 2.3). It is called *sequentially compact* if for every sequence of points (x_n) in X (def. 1.16) there exists a sub-sequence $(x_{n_k})_{k \in \mathbb{N}}$ which converges acording to def. 7.16.

Proposition 7.18. ***(sequentially compact metric spaces are equivalently compact metric spaces)***

If (X, d) is a metric space (def. 1.1), regarded as a topological space via its metric topology (example 2.9), then the following are equivalent:

1. *(X, d) is a compact topological space (def. 7.2).*
2. *(X, d) is a sequentially compact metric space (def. 1.20) hence a sequentially compact topological space (def. 7.17).*

Proof. of prop. 1.21 and prop. 7.18

Assume first that (X, d) is a compact topological space. Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in X. We need to show that it has a sub-sequence which converges.

Consider the topological closures of the sub-sequences that omit the first n elements of the sequence

$$F_n \;\coloneqq\; \mathrm{Cl}(\{x_k \mid k \geq n\})$$

and write

$$U_n \coloneqq X \backslash F_n$$

for their open complements.

Assume now that the intersection of all the F_n were empty

$$(\star) \qquad \underset{n \in \mathbb{N}}{\cap} F_n \;=\; \emptyset$$

or equivalently that the union of all the U_n were all of X

$$\underset{n \in \mathbb{N}}{\cup} U_n \;=\; X\,,$$

hence that $\{U_n \subset X\}_{n \in \mathbb{N}}$ were an open cover. By the assumption that X is compact, this would imply that there were a finite subset $\{i_1 < i_2 < \cdots < i_k\} \subset \mathbb{N}$ with

$$\begin{aligned} X &= U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_k} \\ &= U_{i_k} \end{aligned} .$$

This in turn would mean that $F_{i_k} = \emptyset$, which contradicts the construction of F_{i_k}. Hence we have a proof by contradiction that assumption $(*)$ is wrong, and hence that there must exist an element

$$x \in \underset{n \in \mathbb{N}}{\cap} F_n \,.$$

By definition of topological closure this means that for all n the open ball $B^\circ_x(1/(n+1))$ around x of radius $1/(n+1)$ must intersect the nth of the above subsequences:

$$B^\circ_x(1/(n+1)) \cap \{x_k \,\vert\, k \geq n\} \;\neq\; \emptyset \,.$$

If we choose one point (x'_n) in the nth such intersection for all n this defines a sub-sequence, which converges to x.

In summary this proves that *compact* implies *sequentially compact* for metric spaces.

For the converse, assume now that (X,d) is sequentially compact. Let $\{U_i \subset X\}_{i \in I}$ be an open cover of X. We need to show that there exists a finite sub-cover.

Now by the Lebesgue number lemma, there exists a positive real number $\delta > 0$ such that for each $x \in X$ there is $i_x \in I$ such that $B^\circ_x(\delta) \subset U_{i_x}$. Moreover, since sequentially compact metric spaces are totally bounded, there exists then a finite set $S \subset X$ such that

$$X \;=\; \underset{s \in S}{\cup} B^\circ_s(\delta) \,.$$

Therefore $\{U_{i_s} \to X\}_{s \in S}$ is a finite sub-cover as required. ▮

Remark 7.19. **(neither compactness nor sequential compactness implies the other)**

Beware, in contrast to prop. 7.18, general topological spaces being sequentially compact neither implies nor is implied by being compact.

1. The product topological space (example 6.25) $\prod_{r \in [0,1)} \mathrm{Disc}(\{0,1\})$ of copies of the discrete topological space (example 2.13) indexed by the elements of the half-open interval is compact by the Tychonoff theorem (prop. 7.15), but the sequence x_n with

$$\pi_r(x_n) = n\text{th digit of the binary expansion of } r$$

has no convergent subsequence.

2. conversely, there are spaces that are sequentially compact, but not compact, see for instance Vermeeren 10, prop. 18.

Remark 7.20. **(nets fix the shortcomings of sequences)**

That compactness of topological spaces is not detected by convergence of sequences (remark 7.19) may be regarded as a shortcoming of the concept of *sequence*. While a sequence is indexed over the natural numbers, the concept of convergence of sequnces only invokes that the natural numbers form a *directed set*. Hence the concept of convergence immediately generalizes to sets of points in a space which are indexed over an arbitrary directed set. This is called a *net*.

And with these the expected statement does become true (for a proof see here):

A topological space (X,τ) is compact precisely if every net in X has a converging subnet.

In fact convergence of nets also detects closed subsets in topological spaces (hence their topology as such), and it detects the continuity of functions between topological spaces. It also detects for instance the Hausdorff property. (For detailed statements and proofs see *here*.) Hence when analysis is cast in terms of nets instead of just sequences, then it raises to the same level of generality as topology.

There are various variants of the concept of compact spaces. We discuss the following two:

- locally compact topological spaces (def. 7.21);
- paracompact topological spaces (def. 8.3).

Definition 7.21. **(locally compact topological space)**

A topological space X is called *locally compact* if for every point $x \in X$ and every open neighbourhood $U_x \supset \{x\}$ there exists a smaller open neighbourhood $V_x \subset U_x$ whose topological closure is compact (def. 7.2) and still contained in U:

$$\{x\} \subset V_x \subset \underset{\text{compact}}{\mathrm{Cl}(V_x)} \subset U_x \;.$$

Remark 7.22. **(terminology issue regarding "locally compact")**

On top of the terminology issue inherited from that of "compact", remark 7.3 (regarding whether or not to require "Hausdorff" with "compact"; we do not), the definition of "locally compact" is subject to further ambiguity in the literature. There are various definitions of locally compact spaces alternative to def. 7.21. For Hausdorff topological spaces all these definitions happen to be equivalent, but in general they are not. The version we state in def. 7.21 is the one that gives the universal property of the mapping space, prop. 7.29 below, without_ requiring the Hausdorff property.

Example 7.23. **(discrete spaces are locally compact)**

Every discrete topological space (example 2.13) is locally compact (def. 7.21).

Example 7.24. **(metric spaces are locally compact)**

Every metric space (def. 1.1), regarded as a topological space via its metric topology (def. 2.9), are locally compact (def. 7.21).

Example 7.25. **(open subspaces of compact Hausdorff spaces are locally compact)**

Every open topological subspace $X \underset{\text{open}}{\subset} K$ of a compact (def. 7.2) Hausdorff space (def. 4.4) is a locally compact topological space (def. 7.21).

In particular compact Hausdorff spaces themselves are locally compact.

We **prove** this below as prop. 7.42, after having established a list of convenient general facts about compact Hausdorff spaces.

Example 7.26. **(finite product space of locally compact spaces is locally compact)**

The product topological space (example 6.25) $\prod_{i \in J} (X_i, \tau_i)$ of a a *finite set* $\{(X_i, \tau_i)\}_{i \in I}$ of locally compact topological spaces (X_i, τ_i) (def. 7.21) it itself locally compact.

Nonexample 7.27. **(countably infinite products of non-compact spaces are not locally compact)**

Let X be a topological space which is *not* compact (def. 7.2). Then the product topological space (example 6.25) of a countably infinite set of copies of X

$$\prod_{n \in \mathbb{N}} X$$

is *not* a locally compact space (def. 7.21).

Proof. Since the continuous image of a compact space is compact (prop. 7.9), and since the projection maps $p_i : \prod_{\mathbb{N}} X \longrightarrow X$ are continuous (by nature of the initial topology/Tychonoff topology), it follows that every compact subspace of the product space is contained in one of the form

$$\prod_{i \in \mathbb{N}} K_i$$

for $K_i \subset X$ compact.

But by the nature of the Tychonoff topology, a base for the topology on $\prod_{\mathbb{N}} X$ is given by subsets of the form

$$\left(\prod_{i \in \{1, \cdots, n\}} U_i \right) \times \left(\prod_{j \in \mathbb{N}_{>n}} X \right)$$

with $U_i \subset X$ open. Hence every compact neighbourhood in $\prod_{\mathbb{N}} X$ contains a subset of this kind, but if X itself is non-compact, then none of these is contained in a product of compact subsets. ▮

A key application of locally compact spaces is that the space of maps out of them into any given topological space (example 7.28 below) satisfies the expected universal property of a mapping space (prop. 7.29 below).

Example 7.28. (topological mapping space with compact-open topology)

For

1. (X, τ_X) a locally compact topological space (def. 7.21)

2. (Y, τ_Y) any topological space then the *mapping space*

their *mapping space*

$$\mathrm{Maps}((X, \tau_X), (Y, \tau_Y)) \coloneqq \left(\mathrm{Hom}_{\mathrm{Top}}(X, Y), \tau_{\text{cpt-op}} \right)$$

is the topological space

* whose underlying set $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$ is the set of continuous functions $X \to Y$;

* whose topology $\tau_{\text{cpt-op}}$ is generated from the sub-basis for the topology (def. 2.7) which is given by subsets are denoted

  $U^K \subset \mathrm{Hom}_{\mathrm{Top}}(X, Y)$ for labels

  * $K \subset Y$ a compact subset,

  * $U \subset X$ an open subset

  and defined to be those subsets of all those continuous functions f that take K to U:

$$U^K \coloneqq \left\{ f : X \overset{\text{continuous}}{\longrightarrow} Y \;\middle|\; \begin{array}{ccc} K & \hookrightarrow & X \\ \downarrow && \downarrow^{f} \\ U & \hookrightarrow & Y \end{array} \right\} .$$

Accordingly this topology $\tau_{\text{cpt-op}}$ is called the *compact-open topology* on the set of functions.

Proposition 7.29. (universal property of the mapping space)

Let (X, τ_X), (Y, τ_Y), (Z, τ_Z) be topological spaces, with X locally compact (def. 7.21).

Then

1. *The evaluation function*

$$\begin{array}{ccc} (X,\tau_X) \times \mathrm{Maps}((X,\tau_X),(Y,\tau_Y)) & \overset{\mathrm{ev}}{\longrightarrow} & (Y,\tau_Y) \\ (x,f) & \longmapsto & f(x) \end{array}$$

   is a continuous function.

2. *The natural bijection of function sets*

$$\begin{array}{ccc} \underbrace{\{X \times Y \longrightarrow Y\}}_{\mathrm{Hom}_{\mathrm{Set}}(X \times Z, Y)} & \overset{\simeq}{\longrightarrow} & \underbrace{\{Z \longrightarrow \mathrm{Hom}_{\mathrm{Set}}(X,Y)\}}_{\mathrm{Hom}_{\mathrm{Set}}(Z, \mathrm{Hom}_{\mathrm{Set}}(X,Y))} \\ (f : (x,z) \mapsto f(x,z)) & \longmapsto & \tilde f : z \mapsto (x \mapsto f(x,z)) \end{array}$$

   restricts to a natural bijection between sets of continuous functions

$$\underbrace{\left\{(X,\tau_X) \times (Z,\tau_Z) \overset{\mathrm{cts}}{\longrightarrow} (Y,\tau_Y)\right\}}_{\mathrm{Hom}_{\mathrm{Top}}((X,\tau_X)\times(Z,\tau_Z),(Y,\tau_Y))} \overset{\simeq}{\longrightarrow} \underbrace{\left\{(Z,\tau_Z) \overset{\mathrm{cts}}{\longrightarrow} \mathrm{Maps}((X,\tau_X),(Y,\tau_Y))\right\}}_{\mathrm{Hom}_{\mathrm{Top}}((Z,\tau_Z),\mathrm{Maps}((X,\tau_X),(Y,\tau_Y)))} \,.$$

Here $\mathrm{Maps}((X,\tau_X),(Y,\tau_Y))$ is the mapping space with compact-open topology from example 7.28 and $(-)\times(-)$ denotes forming the product topological space (example 2.18, example 6.25).

Proof. To see the continuity of the evaluation map:

Let $V \subset Y$ be an open subset. We need to show that $\mathrm{ev}^{-1}(V) = \{(x,f) \,|\, f(x) \in V\}$ is a union of products of the form $U \times V^K$ with $U \subset X$ open and $U^K \subset \mathrm{Hom}_{\mathrm{Set}}(K,U)$ a basic open according to def. 7.28.

For $(x,f) \in \mathrm{ev}^{-1}(V)$, the preimage $f^{-1}(V) \subset X$ is an open neighbourhood of x in X, by continuity of f. By local compactness of X, there is a compact subset $K \subset f^{-1}(V)$ which is still a neighbourhood of x. Since f also still takes that into V, we have found an open neighbourhood

$$(x,f) \in K \times V^K \underset{\mathrm{open}}{\subset} \mathrm{ev}^{-1}(V)$$

with respect to the product topology. Since this is still contained in $\mathrm{ev}^{-1}(V)$, for all (x,f) as above, $\mathrm{ev}^{-1}(V)$ is exhibited as a union of opens, and is hence itself open.

Regarding the second point:

In one direction, let $f : (X,\tau_X) \times (Y,\tau_Y) \to (Z\,\tau_Z)$ be a continuous function, and let $U^K \subset \mathrm{Maps}(X,Y)$ be a sub-basic open. We need to show that the set

$$\tilde f^{-1}(U) = \{z \in Z \mid f(K,z) \subset U\} \;\subset\; Z$$

is open. To that end, observe that $f(K,z) \subset U$ means that $K \times \{z\} \subset f^{-1}(U)$, where $f^{-1}(U) \subset X \times Y$ is open by the continuity of f. Hence in the topological subspace

$K \times Z \subset X \times Y$ the inclusion

$$K \times \{z\} \subset \left(f^{-1}(U) \cap (K \times Z)\right)$$

is an open neighbourhood. Since K is compact, the tube lemma (prop. 7.14) gives an open neighbourhood $V_z \supset \{z\}$ in Y, hence an open neighbourhood $K \times V_z \subset K \times Y$, which is still contained in the original pre-image:

$$K \times V_z \;\subset\; f^{-1}(U) \cap (K \times Z) \;\subset\; f^{-1}(U)\;.$$

This shows that with every point $z \in \tilde f^{-1}(U^K)$ also an open neighbourhood of z is contained in $\tilde f^{-1}(U^K)$, hence that the latter is a union of open subsets, and hence itself open.

In the other direction, assume that $\tilde f : Z \to \mathrm{Maps}((X,\tau_X),(Y,\tau_Y))$ is continuous: We need to show that f is continuous. But observe that f is the composite

$$f = (X,\tau_X) \times (Z,\tau_Z) \xrightarrow{\mathrm{id}(X,\tau_X)\times \tilde f} (X,\tau_X) \times \mathrm{Maps}((X,\tau_X),(Y,\tau_Y)) \overset{\mathrm{ev}}{\longrightarrow} (X,\tau_X)\;.$$

Here the first function $\mathrm{id} \times \tilde f$ is continuous since $\tilde f$ is by assumption since the product of two continuous functions is again continuous (example 3.4). The second function ev is continuous by the first point above. hence f is continuous. ▮

Remark 7.30. **(topological mapping space is exponential object)**

In the language of category theory (remark 3.3), prop. 7.29 says that the mapping space construction with its compact-open topology from def. 7.28 is an *exponential object* or *internal hom*. This just means that it beahves in all abstract ways just as a function set does for plain functions, but it does so for continuous functions and being itself equipped with a topology.

Moreover, the construction of topological mapping spaces in example 7.28 extends to a functor (remark 3.3)

$$(-)^{(-)} \;:\; \mathrm{Top}_{\mathrm{lcpt}}^{\mathrm{op}} \times \mathrm{Top} \longrightarrow \mathrm{Top}$$

from the product category of the category Top of all topological spaces (remark 3.3) with the opposite category of the subcategory of locally compact topological spaces.

Example 7.31. **(topological mapping space construction out of the point space is the identity)**

The point space $*$ (example 2.10) is clearly a locally compact topological space. Hence for every topological space (X,τ) the mapping space $\mathrm{Maps}(*,(X,\tau))$ (exmaple 7.28) exists. This is homeomorphic (def. 3.22) to the space (x,τ) itself:

$$\mathrm{Maps}(*,(X,\tau)) \simeq (X,\tau)\;.$$

Example 7.32. **(loop space and path space)**

Let (X, τ) be any topological space.

1. The circle S^1 (example 2.20) is a compact Hausdorff space (example 7.38) hence, by prop. 7.25, a locally compact topological space (def. 7.21). Accordingly the mapping space

$$\mathcal{L}X \coloneqq \mathrm{Maps}(S^1, (X,\tau))$$

exists (def. 7.28). This is called the *free loop space* of (X, τ).

If both S^1 and X are equipped with a choice of point ("basepoint") $s_0 \in S^1$, $x_0 \in X$, then the topological subspace

$$\Omega X \subset \mathcal{L}X$$

on those functions which take the basepoint of S^1 to that of X, is called the *loop space* of X, or sometimes *based loop space*, for emphasis.

2. Similarly the closed interval is a compact Hausdorff space (example 7.38) hence, by prop. 7.25, a locally compact topological space (def. 7.21). Accordingly the mapping space

$$\mathrm{Maps}([0,1], (X,\tau))$$

exists (def. 7.28). Again if X is equipped with a choice of basepoint $x_0 \in X$, then the topological subspace of those functions that take $0 \in [0,1]$ to that chosen basepoint is called the *path space* of $(X\tau)$:

$$PX \subset \mathrm{Maps}([0,1], (X,\tau)) \,.$$

Notice that we may encode these subspaces more abstractly in terms of universal properties:

The path space and the loop space are characterized, up to homeomorphisms, as being the limiting cones in the following pullback diagrams of topological spaces (example 6.15):

1. loop space:

$$\begin{array}{ccc} \Omega X & \longrightarrow & \mathrm{Maps}(S^1, (X,\tau)) \\ \downarrow & (pb) & \downarrow^{\mathrm{Maps}(\mathrm{const}_{s_0}, \mathrm{id}_{(X,\tau)})} \\ \ast & \underset{\mathrm{const}_{x_0}}{\longrightarrow} & X \simeq \mathrm{Maps}(\ast, (X,\tau)) \end{array}$$

2. path space:

$$\begin{array}{ccc} PX & \longrightarrow & \mathrm{Maps}([0,1], (X,\tau)) \\ \downarrow & (pb) & \downarrow^{\mathrm{Maps}(\mathrm{const}_{x}, \mathrm{id}_{(X,\tau)})} \\ \ast & \underset{\mathrm{const}_{x_0}}{\longrightarrow} & X \simeq \mathrm{Maps}(\ast, (X,\tau)) \end{array}$$

Here on the right we are using that the mapping space construction is a functor as shown in remark 7.30, and we are using example 7.31 in the identification on the bottom right mapping space out of the point space.

Relation to Hausdorff spaces

We discuss some important relations between the concepts of compact topological spaces (def. 7.2) and of Hausdorff topological spaces (def. 4.4).

Proposition 7.33. ***(closed subspaces of compact Hausdorff spaces are equivalently compact subspaces)***

Let (X, τ) be a compact Hausdorff topological space (def. 4.4, def. 7.2) and let $Y \subset X$ be a topological subspace. Then the following are equivalent:

1. *$Y \subset X$ is a closed subspace (def. 2.23);*
2. *Y is a compact topological space.*

Proof. By lemma 7.34 and lemma 7.36 below. ∎

Lemma 7.34. ***(closed subspaces of compact spaces are compact)***

Let (X, τ) be a compact topological space (def. 7.2), and let $Y \subset X$ be a closed topological subspace. Then also Y is compact.

Proof. Let $\{V_i \subset Y\}_{i \in I}$ be an open cover of Y. We need to show that this has a finite sub-cover.

By definition of the subspace topology, there exist open subsets U_i of X with

$$V_i = U_i \cap Y \,.$$

By the assumption that Y is closed, the complement $X \backslash Y$ is an open subset of X, and therefore

$$\{X \backslash Y \subset X\} \cup \{U_i \subset X\}_{i \in I}$$

is an open cover of X. Now by the assumption that X is compact, this latter cover has a finite subcover, hence there exists a finite subset $J \subset I$ such that

$$\{X \backslash Y \subset X\} \cup \{U_i \subset X\}_{i \in J \subset I}$$

is still an open cover of X, hence in particular intersects to a finite open cover of Y. But since $Y \cap (X \mathrm{backslah} Y) = \emptyset$, it follows that indeed

$$\{V_i \subset Y\}_{i \in J \subset I}$$

is a cover of Y, and in indeed a finite subcover of the original one. ∎

Lemma 7.35. ***(separation by neighbourhoods of points from compact subspaces in Hausdorff spaces)***

Let

1. (X, τ) *be a Hausdorff topological space;*
2. $Y \subset X$ *a compact subspace.*

Then for every $x \in X \backslash Y$ *there exists*

1. *an open neighbourhood* $U_x \supset \{x\}$*;*
2. *an open neighbourhood* $U_Y \supset Y$

such that

- *they are still disjoint:* $U_x \cap U_Y = \emptyset$.

Proof. By the assumption that (X, τ) is Hausdorff, we find for every point $y \in Y$ disjoint open neighbourhoods $U_{x,y} \supset \{x\}$ and $U_y \supset \{y\}$. By the nature of the subspace topology of Y, the restriction of all the U_y to Y is an open cover of Y:

$$\left\{ (U_y \cap Y) \subset Y \right\}_{y \in Y} .$$

Now by the assumption that Y is compact, there exists a finite subcover, hence a finite set $S \subset Y$ such that

$$\left\{ (U_y \cap Y) \subset Y \right\}_{y \in S \subset Y}$$

is still a cover.

But the finite intersection

$$U_x \coloneqq \underset{s \in S \subset Y}{\cap} U_{x,s}$$

of the corresponding open neighbourhoods of x is still open, and by construction it is disjoint from all the U_s, hence in particular from their union

$$U_Y \coloneqq \underset{s \in S \subset Y}{\cup} U_s .$$

Therefore U_x and U_Y are two open subsets as required. ▮

Lemma 7.35 immediately implies the following:

Lemma 7.36. ***(compact subspaces of Hausdorff spaces are closed)***

Let (X, τ) *be a Hausdorff topological space (def. 4.4) and let* $C \subset X$ *be a compact (def. 7.2) topological subspace (example 2.16). Then* $C \subset X$ *is also a closed subspace (def. 2.23).*

Proof. Let $x \in X \backslash C$ be any point of X not contained in C. We need to show that there exists an open neighbourhood of x in X which does not intersect C. This is implied by lemma 7.35. ▮

Proposition 7.37. ***(Heine-Borel theorem)***

For $n \in \mathbb{N}$, regard $\mathbb{R}^n$ as the n-dimensional Euclidean space via example 1.6, regarded as a topological space via its metric topology (example 2.9).

Then for a topological subspace $S \subset \mathbb{R}^n$ the following are equivalent:

1. *S is compact (def. 7.2);*
2. *S is closed (def. 2.23) and bounded (def. 1.3).*

Proof. First consider a subset $S \subset \mathbb{R}^n$ which is closed and bounded. We need to show that regarded as a topological subspace it is compact.

The assumption that S is bounded by (hence contained in) some open ball $B_x^\circ(\epsilon)$ in $\mathbb{R}^n$ implies that it is contained in $\{(x_i)_{i=1}^n \in \mathbb{R}^n \mid -\epsilon \leq x_i \leq \epsilon\}$. By example 3.29, this topological subspace is homeomorphic to the n-cube $[-\epsilon, \epsilon]^n$. Since the closed interval $[-\epsilon, \epsilon]$ is compact by example 7.6, the binary Tychonoff theorem (prop. 7.15) implies that this n-cube is compact. Since closed subspaces of compact spaces are compact (lemma 7.34) this implies that S is compact.

Conversely, assume that $S \subset \mathbb{R}^n$ is a compact subspace. We need to show that it is closed and bounded.

The first statement follows since the Euclidean space $\mathbb{R}^n$ is Hausdorff (example 4.8) and since compact subspaces of Hausdorff spaces are closed (prop. 7.36).

Hence what remains is to show that S is bounded.

To that end, choose any positive real number $\epsilon \in \mathbb{R}_{>0}$ and consider the open cover of all of $\mathbb{R}^n$ by the open n-cubes

$$(k_1 - \epsilon, k_1 + 1 + \epsilon) \times (k_2 - \epsilon, k_2 + 1 + \epsilon) \times \cdots \times (k_n - \epsilon, k_n + 1 + \epsilon)$$

for n-tuples of integers $(k_1, k_2, \cdots, k_n) \in \mathbb{Z}^n$. The restrictions of these to S hence form an open cover of the subspace S. By the assumption that S is compact, there is then a finite subset of n-tuples of integers such that the corresponding n-cubes still cover S. But the union of any finite number of bounded closed n-cubes in $\mathbb{R}^n$ is clearly a bounded subset, and hence so is S. ▮

For the record, we list some examples of compact Hausdorff spaces that are immediately identified by the Heine-Borel theorem:

Example 7.38. **(examples of compact Hausdorff spaces)**

We list some basic examples of compact Hausdorff spaces (def. 4.4, def. 7.2)

1. For $n \in \mathbb{N}$, the n-sphere S^n may canonically be regarded as a topological subspace of Euclidean space $\mathbb{R}^{n+1}$ (example 2.20).

These are clearly closed and bounded subspaces of Euclidean space, hence they are compact topological space, by the Heine-Borel theorem, prop. 7.37.

Proposition 7.39. *(maps from compact spaces to Hausdorff spaces are closed and proper)*

Let $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ be a continuous function between topological spaces such that

1. *(X, τ_X) is a compact topological space;*
2. *(Y, τ_Y) is a Hausdorff topological space.*

Then f is

1. *a closed map (def. 3.14);*
2. *a proper map (def. 7.10))*

Proof. For the first statement, we need to show that if $C \subset X$ is a closed subset of X, then also $f(C) \subset Y$ is a closed subset of Y.

Now

1. since closed subsets of compact spaces are compact (lemma 7.34) it follows that $C \subset X$ is also compact;
2. since continuous images of compact spaces are compact (cor. 7.9) it then follows that $f(C) \subset Y$ is compact;
3. since compact subspaces of Hausdorff spaces are closed (prop. 7.36) it finally follow that $f(C)$ is also closed in Y.

For the second statement we need to show that if $C \subset Y$ is a compact subset, then also its pre-image $f^{-1}(C)$ is compact.

Now

1. since compact subspaces of Hausdorff spaces are closed (prop. 7.36) it follows that $C \subset Y$ is closed;
2. since pre-images under continuous functions of closed subsets are closed (prop. 3.2), also $f^{-1}(C) \subset X$ is closed;
3. since closed subsets of compact spaces are compact (lemma 7.34), it follows that $f^{-1}(C)$ is compact.

∎

Proposition 7.40. *(continuous bijections from compact spaces to Hausdorff spaces are homeomorphisms)*

Let $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ be a continuous function between topological spaces such *that*

1. (X, τ_X) *is a compact topological space;*
2. (Y, τ_Y) *is a Hausdorff topological space.*
3. $f : X \longrightarrow Y$ *is a bijection of sets.*

Then f *is a homeomorphism, i. e. its inverse function* $Y \to X$ *is also a continuous function.*

In particular then both (X, τ_X) *and* (Y, τ_Y) *are compact Hausdorff spaces.*

Proof. Write $g : Y \to X$ for the inverse function of f.

We need to show that g is continuous, hence that for $U \subset X$ an open subset, then also its pre-image $g^{-1}(U) \subset Y$ is open in Y. By prop. 3.2 this is equivalent to the statement that for $C \subset X$ a closed subset then the pre-image $g^{-1}(C) \subset Y$ is also closed in Y.

But since g is the inverse function to f, its pre-images are the images of f. Hence the last statement above equivalently says that f sends closed subsets to closed subsets. This is true by prop. 7.39. ▮

Proposition 7.41. **(compact Hausdorff spaces are normal)**

Every compact Hausdorff topological space is a normal topological space (def. 4.13).

Proof. First we claim that (X, τ) is regular. To show this, we need to find for each point $x \in X$ and each disjoint closed subset $Y \in X$ dijoint open neighbourhoods $U_x \supset \{x\}$ and $U_Y \supset Y$. But since closed subspaces of compact spaces are compact (lemma 7.34), the subset Y is in fact compact, and hence this is the statement of lemma 7.35.

Next to show that (X, τ) is indeed normal, we apply the idea of the proof of lemma 7.35 once more:

Let $Y_1, Y_2 \subset X$ be two disjoint closed subspaces. By the previous statement then for every point $y_1 \in Y$ we find disjoint open neighbourhoods $U_{y_1} \supset \{y_1\}$ and $U_{Y_2, y_1} \supset Y_2$. The union of the U_{y_1} is a cover of Y_1, and by compactness of Y_1 there is a finite subset $S \subset Y$ such that

$$U_{Y_1} \coloneqq \underset{s \in S \subset Y_1}{\cup} U_{y_1}$$

is an open neighbourhood of Y_1 and

$$U_{Y_2} \coloneqq \underset{s \in S \subset Y}{\cap} U_{Y_2, s}$$

is an open neighbourhood of Y_2, and both are disjoint. ▮

With these statements in hand, the remaining proof of example 7.25 is immediate:

Proposition 7.42. ***(open subspaces of compact Hausdorff spaces are locally compact)***

Every open topological subspace $X \underset{\text{open}}{\subset} K$ of a compact (def. 7.2) Hausdorff space (def. 4.4) is a locally compact topological space (def. 7.21).

Proof. Let X be a topological space such that it arises as a topological subspace $X \subset K$ of a compact Hausdorff space. We need to show that X is a locally compact topological space (def. 7.21).

Let $x \in X$ be a point and let $U_x \subset X$ an open neighbourhood. We need to produce a small open neighbourhood whose closure is compact and still contained in U_x.

By the nature of the subspace topology there exists an open subset $V_x \subset K$ such that $U_x = X \cap V_x$. Since X is assumed to be open, it follows that U is also open as a subset of K. Since compact Hausdorff spaces are normal (prop. 7.41) it follows by prop. 4.18 that there exists a smaller open neighbourhood $W_x \subset K$ whose topological closure is still contained in U_x, and since closed subspaces of compact spaces are compact (prop. 7.34):

$$\{x\} \subset W_x \subset \underset{\text{cpt}}{\text{Cl}(W_x)} \subset V_x \subset K \,.$$

The intersection of this situation with X is the required smaller compact neighbourhood $\text{Cl}(W_x) \cap X$:

$$\{x\} \subset W_x \cap X \subset \underset{\text{cpt}}{\text{Cl}(W_x) \cap X} \subset U_x \subset X \,.$$

▮

Relation to quotient spaces

We discuss some important relations between the concept of compact topological spaces and that of quotient topological spaces.

Proposition 7.43. ***(continuous surjections from compact spaces to Hausdorff spaces are quotient projections)***

Let

$$\pi : (X, \tau_X) \longrightarrow (Y, \tau_Y)$$

be a continuous function between topological spaces such that

1. (X, τ_X) *is a compact topological space (def. 7.2);*

2. *(Y, τ_Y) is a Hausdorff topological space (def. 4.4);*

3. *$\pi : X \longrightarrow Y$ is a surjective function.*

Then τ_X is the quotient topology inherited from τ_X via the surjection f (def. 2.17).

Proof. We need to show that a subset $U \subset Y$ is an open subset (Y, τ_Y) precisely if its pre-image $\pi^{-1}(U) \subset X$ is an open subset in (X, τ_X). Equivalenty, as in prop. 3.2, we need to show that U is a closed subset precisely if $\pi^{-1}(U)$ is a closed subset. The implication

$$(U \, \text{closed}) \Rightarrow \left(f^{-1}(U) \, \text{closed}\right)$$

follows via prop. 3.2 from the continuity of π. The implication

$$\left(f^{-1}(U) \, \text{closed}\right) \Rightarrow (U \, \text{closed})$$

follows since π is a closed map by prop. 7.39. ▮

The following proposition allows to recognize when a quotient space of a compact Hausdorff space is itself still Hausdorff.

Proposition 7.44. ***(quotient projections out of compact Hausdorff spaces are closed precisely if the codomain is Hausdorff)***

Let

$$\pi : (X, \tau_X) \longrightarrow (Y, \tau_Y)$$

be a continuous function between topological spaces such that

1. *(X, τ) is a compact Hausdorff topological space (def. 7.2, def. 4.4);*

2. *π is a surjection and τ_Y is the corresponding quotient topology (def. 2.17).*

Then the following are equivalent

1. *(Y, τ_Y) is itself a Hausdorff topological space (def. 4.4);*

2. *π is a closed map (def. 3.14).*

Proof. The implicaton $((Y, \tau_Y) \, \text{Hausdorff}) \Rightarrow (\pi \, \text{closed})$ is given by prop. 7.39. We need to show the converse.

Hence assume that π is a closed map. We need to show that for every pair of distinct point $y_1 \neq y_2 \in Y)$ there exist open neighbourhoods $U_{y_1}, U_{y_2} \in \tau_Y$ which are disjoint, $U_{y_1} \cap U_{y_2} = \emptyset$.

Therefore consider the pre-images

$$C_1 \coloneqq \pi^{-1}(\{y_1\}) \qquad C_2 \coloneqq \pi^{-1}(\{y_2\}) \,.$$

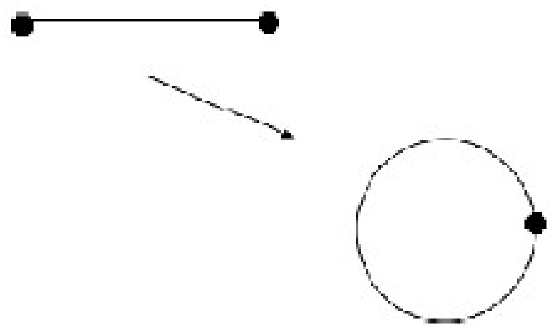
Observe that these are closed subsets, because in the Hausdorff space (Y, τ_Y) (which is hence in particular T_1) the singleton subsets $\{y_i\}$ are closed by prop. 4.11, and since pre-images under continuous functions preserves closed subsets by prop. 3.2.

Now since compact Hausdorff spaces are normal (prop. 7.41) it follows (by def. 4.13) that we may find disjoint open subset $U_1, U_2 \in \tau_X$ such that

$$C_1 \subset U_1 \qquad C_2 \subset U_2 \;.$$

Moreover, by lemma 3.21 we may find these U_i such that they are both saturated subsets (def. 3.17). Therefore finally lemma 3.21 says that the images $\pi(U_i)$ are open in (Y, τ_Y). These are now clearly disjoint open neighbourhoods of y_1 and y_2. ■

Example 7.45. Consider the function



$$\begin{array}{ccc} [0,2\pi]/\sim & \longrightarrow & S^1 \subset \mathbb{R}^2 \\ t & \mapsto & (\cos(t), \sin(t)) \end{array}$$

- from the quotient topological space (def. 2.17) of the closed interval (def. 1.13) by the equivalence relation which identifies the two endpoints

$$(x \sim y) \;\Leftrightarrow\; \left((x = y) \,\text{or}\, \left((x \in \{0, 2\pi\} \,\text{and}\, \left(y \in \{0, 2\text{pi}\} \right) \right) \right) \right)$$

- to the unit circle $S^1 = S_0(1) \subset \mathbb{R}^2$ (def. 1.2) regarded as a topological subspace of the 2-dimensional Euclidean space (example 1.6) equipped with its metric topology (example 2.9).

This is clearly a continuous function and a bijection on the underlying sets. Moreover, since continuous images of compact spaces are compact (cor. 7.9) and since the closed interval $[0,1]$ is compact (example 7.6) we also obtain another proof that the circle is compact.

Hence by prop. 7.40 the above map is in fact a homeomorphism

$$[0,2\pi]/\sim \;\; \simeq \; S^1 \;.$$

Compare this to the counter-example 3.24, which observed that the analogous function

$$\begin{array}{ccc} [0,2\pi) & \longrightarrow & S^1 \subset \mathbb{R}^2 \\ t & \mapsto & (\cos(t), \sin(t)) \end{array}$$

is *not* a homeomorphism, even though this, too, is a bijection on the the underlying sets. But the half-open interval $[0, 2\pi)$ is not compact, and hence prop. 7.40 does not apply.

8. Paracompact spaces

The concept of compactness in topology (above) has several evident weakenings of interest. One is that of *paracompactness* (def. 8.3 below).

A key property is that paracompact Hausdorff spaces are equivalently those (prop. 8.12) all whose open covers admit a subordinate partition of unity (def. 8.10 below), namely a set of real-valued continuous functions each of which is supported in only one patch of the cover, but whose sum is the unit function. Existence of such partitions imply that structures on topological spaces which are glued together via linear maps (such as vector bundles) are well behaved.

In algebraic topology paracompact spaces are important as for them abelian sheaf cohomology may be computed in terms of Cech cohomology.

Definition 8.1. **(locally finite cover)**

Let (X,τ) be a topological space.

An open cover $\{U_i \subset X\}_{i \in I}$ of X is called *locally finite* if for all point $x \in X$, there exists a neighbourhood $U_x \supset \{x\}$ such that it intersects only finitely many elements of the cover, hence such that $U_x \cap U_i \neq \emptyset$ for only a finite number of $i \in I$.

Definition 8.2. **(refinement of open covers)**

Let (X,τ) be a topological space, and let $\{U_i \subset X\}_{i \in I}$ be a open cover.

Then a *refinement* of this open cover is a set of open subsets $\{V_j \subset X\}_{j \in J}$ which is still an open cover in itself and such that for each $j \in J$ there exists an $i \in I$ with $V_j \subset U_i$.

Definition 8.3. **(paracompact topological space)**

A topological space (X,τ) is called *paracompact* if every open cover of X has a refinement (def. 8.2) by a locally finite open cover (def. 8.1).

We consider a couple of technical lemmas related to locally finite covers which will be needed in the proof of prop. 8.12 below:

1. every locally finite refinement induces one with the original index set
2. every locally finite cover of a normal space contains the closure of one with smaller patches ("shrinking lemma")

Lemma 8.4. ***(every locally finite refinement induces one with the original index set)***

Let (X,τ) be a topological space, let $\{U_i \subset X\}_{i \in I}$ be an open cover, and let $(\phi : J \to I, \{V_j \subset X\}_{j \in J})$, be a refinement to a locally finite cover.

Then $\{W_i \subset X\}_{i \in I}$ with

$$W_i \coloneqq \left\{ \underset{j \in \phi^{-1}(\{i\})}{\cup} V_j \right\}$$

is still a refinement of $\{U_i \subset X\}_{i \in I}$ to a locally finite cover.

Proof. It is clear by construction that $W_i \subset U_i$, hence that we have a refinement. We need to show local finiteness.

Hence consider $x \in X$. By the assumption that $\{V_j \subset X\}_{j \in J}$ is locally finite, it follows that there exists an open neighbourhood $U_x \supset \{x\}$ and a finitee subset $K \subset J$ such that

$$\underset{j \in J \backslash K}{\forall} \left(U_x \cap V_j = \emptyset\right) .$$

Hence by construction

$$\underset{I \in I \backslash \phi(K)}{\forall} \left(U_x \cap W_i = \emptyset\right) .$$

Since the image $\phi(K) \subset I$ is still a finite set, this shows that $\{W_i \subset X\}_{i \in I}$ is locally finite. ▮

Lemma 8.5. ***(shrinking lemma for locally finite covers)***

Let X be a topological space which is normal and let $\{U_i \subset X\}_{i \in I}$ be a locally finite open cover.

Then there exists another open cover $\{V_i \subset X\}_{i \in I}$ such that the topological closure $\mathrm{Cl}(V_i)$ of its elements is cotained in the original patches:

$$\underset{i \in I}{\forall} \left(V_i \subset \mathrm{Cl}(V_i) \subset U_i\right) .$$

We now prove this in increasing generality, for binary open covers (lemma 8.6 below), then for finite covers (lemma 8.7), then for locally finite countable covers (lemma 8.9), and finally for general locally finite covers (lemma 8.5, proof below). The last statement needs the axiom of choice.

Lemma 8.6. ***(shrinking lemma for binary covers)***

Let (X, τ) be a normal topological space and let $\{U \subset X\}_{i \in \{1,2\}}$ an open cover by two open subsets.

Then there exists an open set $V_1 \subset X$ whose topological closure is contained in U_1

$$V_1 \subset \mathrm{Cl}(V_1) \subset U_1$$

and such that $\{V_1, U_2\}$ is still an open cover of X.

Proof. Since $U_1 \cup U_2 = X$ it follows (by de Morgan's law) that their complements

$X \backslash U_i$ are disjoint closed subsets. Hence by normality of (X, τ) there exist disjoint open subsets

$$V_1 \supset X \backslash U_2 \qquad V_2 \supset X \backslash U_1 \,.$$

By their disjointness, we have the following inclusions:

$$V_1 \subset X \backslash V_2 \subset U_1 \,.$$

In particular, since $X \backslash V_2$ is closed, this means that $\mathrm{Cl}(V_1) \subset X \backslash (V_2)$.

Hence it only remains to observe that $V_1 \cup U_2 = X$, by definition of V_1. ▮

Lemma 8.7. ***(shrinking lemma for finite covers)***

Let (X, τ) be a normal topological space, and let $\{U_i \subset X\}_{i \in \{1, \cdots, n\}}$ be an open cover with a finite number $n \in \mathbb{N}$ of patches. Then there exists another open cover $\{V_i \subset X\}_{i \in I}$ such that $\mathrm{Cl}(V_i) \subset U_i$ for all $i \in I$.

Proof. By induction using lemma 8.6.

To begin with, consider $\{U_1, \underset{i=2}{\overset{n}{\cup}} U_i\}$. This is a binary open cover, and hence lemma 8.6 gives an open subset $V_1 \subset X$ with $V_1 \subset \mathrm{Cl}(V_1) \subset U_1$ such that $\{V_1, \underset{i=2}{\overset{n}{\cup}} U_i\}$ is still an open cover, and accordingly so is

$$\{V_1\} \cup \{U_i\}_{i \in \{2, \cdots, n\}} \,.$$

Similarly we next find an open subset $V_2 \subset X$ with $V_2 \subset \mathrm{Cl}(V_2) \subset U_2$ and such that

$$\{V_1, , V_2\} \cup \{U_i\}_{i \in \{3, \cdots, n\}}$$

is an open cover. After n such steps we are left with an open cover $\{V_i \subset X\}_{i \in \{1, \cdots, n\}}$ as required. ▮

Remark 8.8. Beware the induction in lemma 8.7 does *not* give the statement for infinite countable covers. The issue is that it is not guaranteed that $\underset{i \in \mathbb{N}}{\cup} V_i$ is a cover.

And in fact, assuming the axiom of choice, then there exists a counter-example of a countable cover on a normal spaces for which the shrinking lemma fails (a Dowker space due to Beslagic 85).

This issue is evaded if we consider locally finite covers:

Lemma 8.9. ***([shrinking lemma]] for locally finite countable covers)***

Let (X, τ) be a normal topological space and $\{U_i \subset X\}_{i \in \mathbb{N}}$ a locally finite countable cover. Then there exists open subsets $V_i \subset X$ for $i \in \mathbb{N}$ such that $V_i \subset \mathrm{Cl}(V_i) \subset U_i$ and such that $\{V_i \subset X\}_{i \in \mathbb{N}}$ is still a cover.

Proof. As in the proof of lemma 8.7, there exist V_i for $i \in \mathbb{N}$ such that $V_i \subset \mathrm{Cl}(V_i) \subset U_i$ and such that for every finite number, hence every $n \in \mathbb{N}$, then

$$\underset{i=0}{\overset{n}{\cup}} V_i \;=\; \underset{i=0}{\overset{n}{\cup}} U_i \,.$$

Now the extra assumption that $\{U_i \subset X\}_{i \in I}$ is locally finite implies that every $x \in X$ is contained in only finitely many of the U_i, hence that for every $x \in X$ there exists $n_x \in \mathbb{N}$ such that

$$x \in \underset{i=0}{\overset{n_x}{\cup}} U_i \,.$$

This implies that for every x then

$$x \in \underset{i=0}{\overset{n_x}{\cup}} V_i \subset \underset{i \in \mathbb{N}}{\cup} V_i$$

hence that $\{V_i \subset X\}_{i \in \mathbb{N}}$ is indeed a cover of X. ▮

We now invoke Zorn's lemma to generalize the shrinking lemma for finitely many patches (lemma 8.7) to arbitrary sets of patches:

Proof. of the general shrinking lemma 8.5

Let $\{U_i \subset X\}_{i \in I}$ be the given locally finite cover of the normal space (X,τ). Consider the set S of pairs $(J, \mathcal{V})$ consisting of

1. a subset $J \subset I$;

2. an I-indexed set of open subsets $\mathcal{V} = \{V_i \subset X\}_{i \in I}$

with the property that

1. $(i \in J \subset I) \Rightarrow (\mathrm{Cl}(V_i) \subset U_i)$;

2. $(i \in I \backslash J) \Rightarrow (V_i = U_i)$.

3. $\{V_i \subset X\}_{i \in I}$ is an open cover of X.

Equip the set S with a partial order by setting

$$\big((J_1, \mathcal{V}) \leq (J_2, \mathcal{V})\big) \Leftrightarrow \left(\big(J_1 \subset J_2\big) \,\text{and} \left(\underset{i \in J_1}{\forall} (V_i = W_i)\right)\right).$$

By definition, an element of S with $J = I$ is an open cover of the required form.

We claim now that a maximal element $(J, \mathcal{V})$ of $(S, \leq)$ has $J = I$.

For assume on the contrary that there were $i \in I \backslash J$. Then we could apply the construction in lemma 8.6 to replace that single V_i with a smaller open subset V'_i to obtain $\mathcal{V}'$ such that $\mathrm{Cl}(V'_i) \subset V_i$ and such $\mathcal{V}'$ is still an open cover. But that would mean that $(J, \mathcal{V}) < (J \cup \{i\}, \mathcal{V}')$, contradicting the assumption that $(J, \mathcal{V})$ is maximal.

This proves by contradiction that a maximal element of $(S, \leq)$ has $J = I$ and hence is an open cover as required.

We are reduced now to showing that a maximal element of $(S, \leq)$ exists. To achieve this we invoke Zorn's lemma. Hence we have to check that every chain in $(S, \leq)$, hence every totally ordered subset has an upper bound.

So let $T \subset S$ be a totally ordered subset. Consider the union of all the index sets appearing in pairs in this subset:

$$K \coloneqq \underset{(J,\mathcal{V}) \in T}{\cup} J \,.$$

Now define open subsets $\mathcal{W}_i$ for $i \in K$ picking any $(J, \mathcal{V})$ in T with $i \in J$ and setting

$$W_i \coloneqq V_i \qquad i \in K \,.$$

This is independent of the choice of $(J, \mathcal{V})$, hence well defined, by the assumption that $(T, \leq)$ is totally ordered.

Moreover, for $i \in I \backslash K$ define

$$W_i \coloneqq U_i \qquad i \in I \backslash K \,.$$

We claim now that $\{W_i \subset X\}_{i \in I}$ thus defined is a cover of X. Because by assumption that $\{U_i \subset X\}_{i \in I}$ is locally finite, also all the $\{V_i \subset X\}_{i \in I}$ are locally finite, hence for every point $x \in X$ there exists a finite set $J_x \subset I$ such that $(i \in I \backslash J_x) \Rightarrow (i \notin U_i)$. Since $(T, \leq)$ is a total order, it must contain an element $(J, \mathcal{V})$ such that $J_x \cap K \subset J$. Since that $\mathcal{V}$ is a cover, it follows that $x \in \underset{i \in I}{\cup} V_i$, hence in $\underset{i \in I}{\cup} W_i$.

This shows that $(K, \mathcal{W})$ is indeed an element of S. It is clear by construction that it is an upper bound for $(T, \leq)$. Hence we have shown that every chain in $(S, \leq)$ has an upper bound, and so Zorn's lemma implies the claim. ▮

Partitions of unity

Definition 8.10. (**partition of unity**)

Let (X, τ) be a topological space, and let $\{U_i \subset X\}_{i \in I}$ be an open cover. Then a *partition of unity subordinate to the cover* is

- a set $\{f_i\}_{i \in I}$ of continuous functions

$$f_i \colon U_i \longrightarrow [0,1]$$

  (where $U_i \subset X$ and $[0,1] \subset \mathbb{R}$ are equipped with their subspace topology, the real numbers $\mathbb{R}$ is regarded as the 1-dimensional Euclidean space equipped with its metric topology);

such that with

$$\mathrm{Supp}(f_i) \coloneqq \mathrm{Cl}\left(f_i^{-1}((0,1])\right)$$

denoting the support of f_i (the topological closure of the subset of points on which it does not vanish) then

1. $\underset{i \in I}{\forall} \left(\mathrm{Supp}(f_i) \subset U_i\right)$;

2. $\left\{\mathrm{Supp}(f_i) \subset X\right\}_{i \in I}$ is a locally finite cover (def. 8.1);

3. $\underset{x \in X}{\forall} \left(\sum_{i \in I} f_i(x) = 1\right)$.

Remark 8.11. Due to the second clause in def. 8.10, the sum in the third clause involves only a finite number of elements not equal to zero, and therefore is well defined.

Proposition 8.12. ***(paracompact Hausdorff spaces equivalently admit subordinate partitions of unity)***

Let (X, τ) be a topological space. Then the following are equivalent:

1. *(X, τ) is a paracompact Hausdorff space (def. 4.4, def. 8.3).*

2. *Every open cover of (X, τ) admits a subordinate partition of unity (def. 8.10).*

Proof. One direction is immediate: Assume that every open cover $\{U_i \subset X\}_{i \in I}$ admits a subordinate partition of unity $\{f_i\}_{i \in I}$. Then by definition (def. 8.10) $\{\mathrm{Int}(\mathrm{Supp}(f)_i) \subset X\}_{i \in I}$ is a locally finite open cover refining the original one.

We need to show the converse: If (X, τ) is a paracompact topological space, then for every open cover $\{U_i \subset X\}_{i \in I}$ there is a subordinate partition of unity (def. 8.10).

To that end, first apply the shrinking lemma 8.5 to the given locally finite open cover $\{U_i \subset X\}$, to obtain a smaller locally finite open cover $\{V_i \subset X\}_{i \in I}$, and then apply the lemma once more to that result to get a yet smaller open cover $\{W_i \subset X\}_{i \in I}$, so that now

$$\underset{i \in I}{\forall} \left(W_i \subset \mathrm{Cl}(W_i) \subset V_i \subset \mathrm{Cl}(V_i) \subset U_i\right) .$$

It follows that for each $i \in I$ we have two disjoint closed subsets, namely the topological closure $\mathrm{Cl}(W_i)$ and the complement $X \backslash V_i$

$$\mathrm{Cl}(W_i) \cap X \backslash V_i = \emptyset .$$

Now since paracompact Hausdorff spaces are normal, Urysohn's lemma says that there exist continuous functions

$$h_i : X \longrightarrow [0,1]$$

with the property that

$$h_i(\mathrm{Cl}(W_i)) = \{1\}\,, \qquad h_i(X \backslash V_i) = \{0\}\;.$$

This means in particular that $h_i^{-1}((0,1]) \subset V_i$ and hence that

$$\mathrm{Supp}(h_i) = \mathrm{Cl}(h_i^{-1}((0,1])) \subset \mathrm{Cl}(V_i) \subset U_i\;.$$

By construction, the set of function $\{h_i\}_{i \in I}$ already satisfies two of the three conditions on a partition of unity subordinate to $\{U_i \subset X\}_{i \in I}$ from def. 8.10. It just remains to normalize these functions so that they indeed sum to unity. To that end, consider the continuous function

$$h : X \longrightarrow [0,1]$$

defined on $x \in X$

$$h(x) \coloneqq \sum_{i \in I} h_i(x)\;.$$

Notice that the sum on the right has only a finite number of non-zero summands, due to the local finiteness of the cover, so that this is well-defined.

Then set

$$f_i \;\coloneqq\; g_i / g\;.$$

This is now manifestly such that $\sum_{i \in I} f_i = 1$, and so

$$\{f_i\}_{i \in I}$$

is a partition of unity as required. ▮

Manifolds

A *topological manifold* is a topological space which is *locally* homeomorphic to a Euclidean space (def. 8.13 below), but which may globally look very different. These are the kinds of topological spaces that are really meant when people advertise topology as "rubber-sheet geometry".

If the gluing functions which relate the Euclidean local charts of topological manifolds to each other are differentiable functions, for a fixed degree of differentiability, then one speaks of *differentiable manifolds* (def 8.16 below) or of_smooth manifolds_ if the gluing functions are arbitrarily differentiable.

Accordingly, a differentiable manifold is a space to which the tools of (infinitesimal9 analysis may be applied *locally*. Notably we may ask whether a continuous function between differentiable manifolds is differentiable by computing its derivatives pointwise in any of the Euclidean coordinate charts. This way differential and smooth manifolds are the basis for much of differential geometry. They are the analogs in differential geometry of what schemes are in algebraic geometry.

Definition 8.13. (topological manifold)

Let $n \in \mathbb{N}$ be a natural number.

A *topological manifold* of *dimension* n (also "n-fold") is

- a paracompact Hausdorff topological space X

such that

- every point $x \in X$ has an open neighbourhood $U_x \supset \{x\}$ which is homeomorphic to the Euclidean space $\mathbb{R}^n$ with its metric topology.

Remark 8.14. (varying terminology)

There is some variance in the choice of regularity condition in def. 8.13. Often it is required in addition to being a paracompact Hausdorff space that a manifold have a countable set of connected components, which then means that it is sigma-compact.

This is the relevant condition for the Whitney embedding theorem to apply.

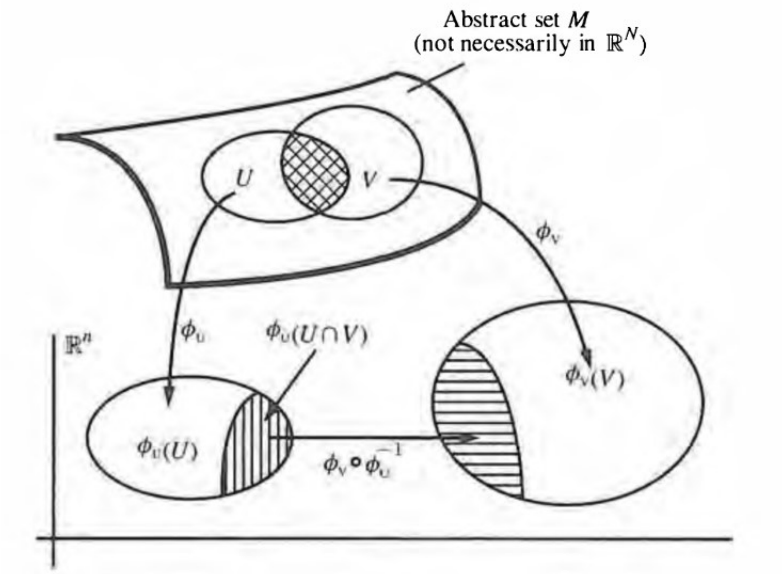
Very rarely one considers non-Hausdorff topological spaces as manifolds.

Definition 8.15. (local chart, atlas and gluing function)

Given an n-dimensional topological manifold X (def. 8.13), then

1. an open subset $U \subset X$ and a homeomorphism $\phi : \mathbb{R}^n \overset{\simeq}{\longrightarrow} U$ is also called a *local coordinate chart* of X.

2. an open cover of X by local charts $\left\{ \mathbb{R}^n \overset{\phi_i}{\to} U \subset X \right\}_{i \in I}$ is called an *atlas* of the topological manifold.



3. denoting for each $i, j \in I$ the intersection of the ith chart with the jth chart in such an atlas by

$$U_{ij} \coloneqq U_i \cap U_j$$

then the induced homeomorphism

$$\mathbb{R}^n \supset \quad \phi_i^{-1}(U_{ij}) \xrightarrow{\phi_i} U_{ij} \xrightarrow{\phi_j^{-1}} \phi_j^{-1}(U_{ij}) \quad \subset \mathbb{R}^n$$

is called the *gluing function* from chart i to chart j.

graphics grabbed from Frankel

Definition 8.16. (differentiable manifold)

For $p \in \mathbb{N} \cup \{\infty\}$ then a p-fold *differentiable manifold* or C^p*-manifold* for short is

1. a topological manifold X (def. 8.13);
2. an atlas $\{\mathbb{R}^n \overset{\phi_i}{\to} X\}$ (def. 8.15) all whose gluing functions are p times continuously differentiable.

A p-fold differentiable function between p-fold differentiable manifolds

$$\left(X, \{\mathbb{R}^n \overset{\phi_i}{\to} U_i \subset X\}_{i \in I}\right) \overset{f}{\longrightarrow} \left(Y, \{\mathbb{R}^{n'} \overset{\psi_j}{\to} V_j \subset Y\}_{j \in J}\right)$$

is

- a continuous function $f : X \to Y$

such that

- for all $i \in I$ and $j \in J$ then

$$\mathbb{R}^n \supset \quad (f \circ \phi_i)^{-1}(V_j) \overset{\phi_i}{\to} f^{-1}(V_j) \overset{f}{\longrightarrow} V_j \overset{\psi_j^{-1}}{\longrightarrow} \mathbb{R}^{n'}$$

  is a p-fold differentiable function between open subsets of Euclidean space.

Notice that this in in general a non-trivial condition even if $X = Y$ and f is the identity function. In this case the above exhibits a passage to a different, but equivalent, differentiable atlas.

Remark 8.17. (category Diff of differentiable manifolds)

In analogy to remark 3.3 there is a category Diff_p whose objects are C^p-differentiable manifolds and whose morphisms are C^p-differentiable functions.

Example 8.18. (Cartesian space as a smooth manifold)

For $n \in \mathbb{N}$ then Cartesian space $\mathbb{R}^n$ equipped with the atlas consisting of the single chart $\mathbb{R}^n \overset{\mathrm{id}}{\to} \mathbb{R}^n$ is a smooth manifold, in particularly a p-fold differentiable manifold for every $p \in \mathbb{N}$ according to def. 8.16.

Similarly the open disk D^n becomes a smooth manifold when equipped with the atlas whose single chart is the homeomorphism $\mathbb{R}^n \to D^n$.

Example 8.19. (n-sphere as a smooth manifold)

For all $n \in \mathbb{N}$, the n-sphere S^n becomes a smooth manfold, with atlas consisting of the two local charts that are given by the inverse functions of the stereographic projection from the two poles of the sphere onto the equatorial hyperplane

$$\left\{ \mathbb{R}^n \underset{\simeq}{\overset{\sigma_i^{-1}}{\longrightarrow}} S^n \right\}_{i \in \{+,-\}} \,.$$

By the formulas given in this prop. the induced gluing function $\mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ is smooth.

- embedding of smooth manifolds
- tangent bundle

(…)

(…)

This concludes *Section 1 Point-set topology*.

For the next section see *Secton 2 -- Basic homotopy theory*.

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9. References

General

A canonical compendium is

- Nicolas Bourbaki, chapter 1 *Topological Structures* in *Elements of Mathematics III: General topology*, Springer (1971, 1990)

Introductory textbooks include

- John Kelley *General Topology*, Graduate Texts in Mathematics, Springer (1955)
- James Munkres, *Topology*, Prentice Hall (1975, 2000)

Lecture notes include

- Friedhelm Waldhausen, *Topologie* (pdf)

See also the references at *algebraic topology*.

Special topics

The standard literature typically omits the following important topics:

Discussion of sober topological spaces is briefly in

- Peter Johnstone, section II 1. of *Stone Spaces*, Cambridge Studies in Advanced Mathematics **3**, Cambridge University Press 1982. xxi+370 pp. MR85f:54002, reprinted 1986.

An introductory textbook that takes sober spaces, and their relation to logic, as the starting point for toplogy is

- Steven Vickers, *Topology via Logic*, Cambridge University Press (1989)

Detailed discussion of the Hausdorff reflection is in

- Bart van Munster, *The Hausdorff quotient*, 2014 (pdf)

10. Index

Basic concepts

- open subset, closed subset
- topological space (see also *locale*)
- basis for the topology, finer/coarser topology
- closure, interior, boundary
- separation axiom
- continuous function, homeomorphism
- embedding
- open map, closed map
- sequence, net, sub-net, filter
- convergence
- category Top
  - convenient category of topological spaces

Universal constructions

- initial topology, final topology

- subspace, quotient space,
- fiber space, attaching space
- product space, disjoint union space
- mapping cylinder, mapping cocylinder
- mapping cone, mapping cocone
- mapping telescope

Extra stuff, structure, properties

- nice topological space
- metric space
- Kolmogorov space, Hausdorff space, regular space, normal space
- sober space
- compact space (sequentially compact, countably compact, paracompact, countably paracompact, locally compact, strongly compact)
- compactly generated space
- second-countable space, first-countable space
- contractible space, locally contractible space
- connected space, locally connected space
- simply-connected space, locally simply-connected space
- topological vector space, Banach space, Hilbert space
- topological manifold
- CW-complex

Examples

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- discrete space, codiscrete space
- order topology, specialization topology, Scott topology
- Euclidean space
  - real line, plane
- sphere, ball,
- circle, torus, annulus

- polytope, polyhedron
- projective space (real, complex)
- classifying space
- mapping space, loop space, path space
- Zariski topology
- Cantor space, Sierpinski space
- long line, line with two origins
- K-topology, Dowker space
- Warsaw circle
- Peano curve

Basic statements

- Hausdorff spaces are sober
- CW-complexes are Hausdorff
- (para-)compact Hausdorff spaces are normal
- continuous image of a compact space is compact
- closed subspaces of compact Hausdorff spaces are equivalently compact subspaces
- open subspaces of compact Hausdorff spaces are locally compact
- quotient projections out of compact Hausdorff spaces are closed precisely if the codomain is Hausdorff
- compact spaces equivalently have converging subnet of every net
    - Lebesgue number lemma
    - sequentially compact metric spaces are equivalently compact metric spaces
    - compact spaces equivalently have converging subnet of every net
    - sequentially compact metric spaces are totally bounded
- paracompact Hausdorff spaces equivalently admit subordinate partitions of unity

Theorems

- Urysohn's lemma

- Tietze extension theorem
- tube lemma
- Tychonoff theorem
- Heine-Borel theorem
- Brouwer's fixed point theorem
- topological invariance of dimension
- Jordan curve theorem

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