This page is a detailed introduction to basic topological homotopy theory. We introduce the fundamental group of topological spaces and the concept of covering spaces. Then we prove the fundamental theorem of covering spaces, saying that they are equivalent to permutation representations of the fundamental group. This is a simple topological version of the general principle of Galois theory and has many applications. As one example application, we use it to prove that the fundamental group of the circle is the integers.

Under construction.

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For introduction to more general and abstract homotopy theory see instead at Introduction to Homotopy Theory.

Basic Homotopy Theory

1. Homotopy
   Fundamental group
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2. Covering spaces
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In order to handle topological spaces, to compute their properties and distinguish them, it turns out to be useful to consider not just continuity within a topological space, but also continuous deformations of continuous functions between
topological spaces. This is the concept of homotopy, and its study is homotopy theory. We introduce the basic concept and consider its most fundamental application: the fundamental group and its relation to the classification of covering spaces.

1. Homotopy

It is clear that for \( n \geq 1 \) the Euclidean space \( \mathbb{R}^n \) or equivalently the open ball \( B_0^1(1) \) in \( \mathbb{R}^n \) is not homeomorphic to the point space \( * = \mathbb{R}^0 \) (simply because there is not even a bijection between the underlying sets). Nevertheless, intuitively the \( n \)-ball is a “continuous deformation” of the point, obtained as the radius of the \( n \)-ball tends to zero.

This intuition is made precise by observing that there is a continuous function out of the product topological space (this example) of the open ball with the closed interval

\[
\eta: [0, 1] \times B_0^1(1) \to B_0^1(1)
\]

which is given by rescaling:

\[
(t, x) \mapsto t \cdot x.
\]

This continuously interpolates between the open ball and the point, in that for \( t = 1 \) it restricts to the identity, while for \( t = 0 \) it restricts to the map constant on the origin.

We may summarize this situation by saying that there is a diagram of continuous functions of the form

\[
\begin{array}{ccc}
B_0^1(1) \times \{0\} & \xrightarrow{\exists !} & * \\
\downarrow & & \downarrow \text{const}_0 \\
[0, 1] \times B_0^1(1) & \xrightarrow{(t, x) \mapsto t \cdot x} & B_0^1(1) \\
\uparrow & & \Uparrow \lambda^m \\
B_0^1(1) \times \{1\} &
\end{array}
\]

Such "continuous deformations" are called homotopies:

In the following we use this terminology:

**Definition 1.1. (topological interval)**

The topological interval is

1. the closed interval \([0, 1] \subset \mathbb{R}^1\) regarded as a topological space in the standard way, as a subspace of the real line with its Euclidean metric topology,

2. equipped with the continuous functions
1. \( \text{const}_0 : * \to [0,1] \)
2. \( \text{const}_1 : * \to [0,1] \)

which include the point space as the two endpoints, respectively

3. equipped with the (unique) continuous function
   \[
   [0,1] \to *
   \]

to the point space (which is the terminal object in \( \text{Top} \))

regarded, in summary, as a factorization

\[
\nabla_* : * \sqcup * \xrightarrow{\text{(\text{const}_0,\text{const}_1)}} [0,1] \to *
\]

of the codiagonal on the point space, namely the unique continuous function \( \nabla_* \) out of the disjoint union space \( * \sqcup * \cong \text{Disc}(\{0,1\}) \) (homeomorphic to the discrete topological space on two elements).

**Definition 1.2. (homotopy)**

Let \( X,Y \in \text{Top} \) be two topological spaces and let

\[
f,g : X \to Y
\]

be two continuous functions between them.

A \((left)\) homotopy from \( f \) to \( g \), to be denoted

\[
\eta : f \Rightarrow g ,
\]

is a continuous function

\[
\eta : X \times [0,1] \to Y
\]

out of the product topological space (this example) of \( X \) the topological interval (def. 1.1) such that this makes the following diagram in \( \text{Top} \) commute:

\[
\begin{array}{ccc}
0 \times X & \xrightarrow{\text{id,\text{const}_0}} & X \times [0,1] \xrightarrow{\eta} Y \\
\downarrow f & & \downarrow g \\
{1} \times X & \xrightarrow{\text{id,\text{const}_1}} & X \times [0,1]
\end{array}
\]

graphics grabbed from J. Tauber [here](https://ncatlab.org/nlab/print/Introduction+to+Topology+--+2)

hence such that

\[
\eta(-,0) = f \quad \text{and} \quad \eta(-,1) = g .
\]
If there is a homotopy \( f \Rightarrow g \) (possibly unspecified) we say that \( f \) is *homotopic* to \( g \), denoted
\[
f \sim_h g.
\]

**Proposition 1.3. (homotopy is an equivalence relation)**

Let \( X, Y \in \text{Top} \) be two *topological spaces*. Write \( \text{Hom}_{\text{Top}}(X, Y) \) for the set of *continuous functions* from \( X \) to \( Y \).

Then the relating of being *homotopic* (def. 1.2) is an *equivalence relation* on this set. The corresponding *quotient set*
\[
[X, Y] := \text{Hom}_{\text{Top}}(X, Y) / \sim_h
\]
is called the set of *homotopy classes* of continuous functions.

Moreover, this equivalence relation is compatible with *composition* of continuous functions:

For \( X, Y, Z \in \text{Top} \) three topological spaces, there is a unique function
\[
[X, Y] \times [Y, Z] \to [X, Z]
\]
such that the following diagram commutes:
\[
\begin{array}{ccc}
\text{Hom}_{\text{Top}}(X, Y) \times \text{Hom}_{\text{Top}}(Y, Z) & \xrightarrow{\text{def. } X,Y,Z} & \text{Hom}_{\text{Top}}(X, Z) \\
\downarrow & & \downarrow \\
[X, Y] \times [Y, Z] & \to & [X, Z]
\end{array}
\]

**Proof.** To see that the relation is *reflexive*: A homotopy \( f \Rightarrow f \) from a function \( f \) to itself is given by the function which is constant on the topological interval:
\[
X \times [0, 1] \xrightarrow{\text{pr}_1} X.
\]
This is continuous because *projections* out of *product topological spaces* are continuous, by the *universal property* of the *Cartesian product*.

To see that the relation is *symmetric*: If \( \eta: f \Rightarrow g \) is a homotopy then
\[
X \times [0, 1] \xrightarrow{id \times (1 - (-))} X \times [0, 1] \xrightarrow{\eta} X
\]
\[
(x, t) \quad \mapsto \quad (x, 1 - t) \quad \mapsto \quad \eta(x, 1 - t)
\]
is a homotopy \( g \Rightarrow f \). This is continuous because \( 1 - (-) \) is a *polynomial* function, and *polynomials are continuous*, and because *Cartesian product* and *composition* of continuous functions is again continuous.

Finally to see that the relation is *transitive*: If \( \eta_1 : f \Rightarrow g \) and \( \eta_2 : g \Rightarrow h \) are two composable homotopies, then consider the "\( X \)-parameterized *path concatenation*"
\[
X \times [0, 1] \xrightarrow{\eta_2 \circ \eta_1} X
\]
\[
(x, t) \mapsto \begin{cases} 
\eta_1(x, 2t) & | t \leq 1/2 \\
\eta_2(x, 2t - 1) & | t \leq 1/2
\end{cases}
\]

To see that this is continuous, observe that \(\{X \times [0, 1/2] \subset X, X \times [1/2, 1] \subset X\}\) is a cover of \(X \times [0, 1]\) by closed subsets (in the product topology) and because \(\eta_1(-,2(-))\) and \(\eta_2(-,2(-) - 1)\) are continuous (being composites of Cartesian products of continuous functions) and agree on the intersection \(X \times \{1/2\}\). Hence the continuity follows by this example.

Finally to see that homotopy respects composition: Let
\[
X \xrightarrow{f_1} Y \xrightarrow{f_2} Z \xrightarrow{f_3} W
\]
be continuous functions, and let
\[
\eta : f_2 \Rightarrow f'_2
\]
be a homotopy. It is sufficient to show that then there is a homotopy of the form
\[
f_3 \circ f_2 \circ f_1 \Rightarrow f_3 \circ f'_2 \circ f_1.
\]

This is exhibited by the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & Y \\
\downarrow & & \downarrow \text{(id}_Y, \text{const}_0) \\
X \times [0, 1] & \xrightarrow{f_1 \times [0,1]} & Y \times [0,1] \\
\uparrow \text{(id}_X, \text{const}_1) & & \uparrow \text{(id}, \text{const}_1) \\
X & \xrightarrow{f_1} & Y \\
\end{array}
\]

\[
\xrightarrow{f_2}
\]

Remark 1.4. **(homotopy category)**

Prop. 1.3 means that homotopy classes of continuous functions are the morphisms in a category whose objects are still the topological spaces.

This category (at least when restricted to spaces that admit the structure of CW-complexes) is called the classical homotopy category, often denoted \(\text{Ho}(\text{Top})\).

Hence for \(X,Y\) topological spaces, then
\[
\text{Hom}_{\text{Ho}(\text{Top})}(X,Y) = [X,Y]
\]
Moreover, sending a continuous function to its homotopy class is a **functor**

\[ \kappa : \text{Top} \to \text{Ho(Top)} \]

from the ordinary category \( \text{Top} \) of topological spaces with actual continuous functions between them.

**Definition 1.5. (homotopy equivalence)**

Let \( X, Y \in \text{Top} \) be two **topological spaces**.

A **continuous function**

\[ f : X \to Y \]

is called a **homotopy equivalence** if there exists

1. a continuous function the other way around,

\[ g : Y \to X \]

2. **homotopies** (def. 1.2) from the two composites to the respective **identity function**:

\[ f \circ g \Rightarrow \text{id}_Y \]

and

\[ g \circ f \Rightarrow \text{id}_X . \]

We indicate that a continuous function is a homotopy equivalence by writing

\[ X \overset{\simeq}{\to} Y . \]

If there exists some (possibly unspecified) homotopy equivalence between topological spaces \( X \) and \( Y \) we write

\[ X \simeq Y . \]

**Remark 1.6. (homotopy equivalences are the isomorphisms in the homotopy category)**

In view of remark 1.4 a continuous function \( f \) is a homotopy equivalence precisely if its image \( \kappa(f) \) in the **homotopy category** is an **isomorphism**.

**Example 1.7. (homeomorphism is homotopy equivalence)**

Every **homeomorphism** is a **homotopy equivalence** (def. 1.5).

**Proposition 1.8. (homotopy equivalence is equivalence relation)**

**Being** homotopy equivalent **is an equivalence relation on the class of topological spaces.**

**Proof.** This is immediate from remark 1.6 by general properties of **categories** and
functors.

But for the record we spell it out. This involves the construction already used in the proof of prop. 1.3:

It is clear that the relation it reflexive and symmetric. To see that it is transitive consider continuous functions

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & Y \\
\downarrow{g_1} & & \downarrow{g_2} \\
Y & \xrightarrow{f_2} & Z
\end{array}
\]

and homotopies

\[
g_1 \circ f_1 \Rightarrow \text{id}_Y \quad f_1 \circ g_1 \Rightarrow \text{id}_X \\
g_2 \circ f_2 \Rightarrow \text{id}_Y \quad f_2 \circ g_2 \Rightarrow \text{id}_Z
\]

We need to produce homotopies of the form

\[
(g_1 \circ g_2) \circ (f_2 \circ f_1) \Rightarrow \text{id}_X
\]

and

\[
(f_2 \circ f_1) \circ (g_1 \circ g_2) \Rightarrow \text{id}_Y
\]

Now the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & Y \\
\downarrow{(\text{id}_X,\text{const}_0)} & & \downarrow{(\text{id}_Y,\text{const}_0)} \\
X \times [0,1] & \xrightarrow{f_1 \times \text{id}_{[0,1]}} & Y \times [0,1] \\
\downarrow{(\text{id}_X,\text{const}_1)} & & \downarrow{(\text{id},\text{const}_1)} \\
X & \xrightarrow{g_1} & X
\end{array}
\]

with \(\eta\) one of the given homotopies, exhibits a homotopy

\[(g_1 \circ g_2) \circ (f_2 \circ f_1) \Rightarrow g_1 \circ f_1\]. Composing this with the given homotopy \(g_1 \circ f_1 \Rightarrow \text{id}_X\) gives the first of the two homotopies required above. The second one follows by the same construction, just with the labels of the functions exchanged. ■

**Definition 1.9. (contractible topological space)**

A topological space \(X\) is called contractible if the unique continuous function to the point space

\[
X \xrightarrow{\approx_h} *
\]

is a homotopy equivalence (def. 1.5).

**Remark 1.10. (contractible topological spaces are the terminal objects in the homotopy category)**
In view of remark 1.4, a topological space $X$ is contractible (def. 1.9) precisely if its image $\kappa(X)$ in the classical homotopy category is a terminal object.

**Example 1.11. (closed ball and Euclidean space are contractible)**

Let $B^n \subset \mathbb{R}^n$ be the unit open ball or closed ball in Euclidean space. This is contractible (def. 1.9):

$$p : B^n \xrightarrow{\cong_h} \ast.$$  

The homotopy inverse function is necessarily constant on a point, we may just as well choose it to go pick the origin:

$$\text{const}_0 : \ast \to B^n.$$  

For one way of composing these functions we have the equality

$$p \circ \text{const}_0 = \text{id}_\ast,$$

with the identity function. This is a homotopy by prop. 1.3.

The other composite is

$$\text{const}_0 \circ p = \text{const}_0 : B^n \to B^n.$$  

Hence we need to produce a homotopy

$$\text{const}_0 \Rightarrow \text{id}_{B^n}.$$  

This is given by the function

$$B^n \times [0,1] \xrightarrow{\eta} B^n,$$

$$(x,t) \mapsto tx,$$

where on the right we use the multiplication with respect to the standard real vector space structure in $\mathbb{R}^n$.

Since the open ball is homeomorphic to the whole Cartesian space $\mathbb{R}^n$ (this example) it follows with example 1.7 and example 1.3 that also $\mathbb{R}^n$ is a contractible topological space:

$$\mathbb{R}^n \xrightarrow{\cong_h} \ast.$$  

In direct generalization of the construction in example 1.11 one finds further examples as follows:

**Example 1.12.** The following three graphs
(i.e. the evident **topological subspaces** of the **plane** $\mathbb{R}^2$ that these pictures indicate) are not **homeomorphic**. But they are **homotopy equivalent**, in fact they are each homotopy equivalent to the **disk** with two points removed, by the homotopies indicated by the following pictures:

![Graphics grabbed from Hatcher](image)

**Fundamental group**

**Definition 1.13. (homotopy relative boundary)**

Let $X$ be a **topological space** and let

$$\gamma_1, \gamma_2 : [0, 1] \to X$$

be two **paths** in $X$, i.e. two **continuous functions** from the **closed interval** to $X$, such that their endpoints agree:

$$\gamma_1(0) = \gamma_2(0) \quad \gamma_1(1) = \gamma_2(1).$$

Then a **homotopy relative boundary** from $\gamma_1$ to $\gamma_2$ is a **homotopy** (def. 1.2)

$$\eta : \gamma_1 \Rightarrow \gamma_2$$

such that it does not move the endpoints:

$$\eta(0, -) = \text{const}_{\gamma_1(0)} = \text{const}_{\gamma_2(0)} \quad \eta(1, -) = \text{const}_{\gamma_1(0)} = \text{const}_{\gamma_2(1)}.$$

**Proposition 1.14. (homotopy relative boundary is equivalence relation on sets of paths)**

Let $X$ be a **topological space** and let $x, y \in X$ be two points. Write

$$P_{x,y} X$$

for the set of **paths** $\gamma$ in $X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Then **homotopy relative boundary** (def. 1.13) is an **equivalence relation** on $P_{x,y} X$.

The corresponding set of **equivalence classes** is denoted

$$\text{Hom}_{\pi_1(X)}(x, y) := (P_{x,y} X)/\sim.$$

Recall the operations on **paths**: **path concatenation** $\gamma_2 \cdot \gamma_1$, **path reversion** $\bar{\gamma}$ and **constant paths**
Proposition 1.15. (concatenation of homotopy relative boundary-classes of paths)

For $X$ a topological space, then the operation of path concatenation descends to homotopy relative boundary equivalence classes, so that for all $x, y, z \in X$ there is a function

$$\Hom_{\pi_1(X)}(x, y) \times \Hom_{\pi_1(X)}(y, z) \rightarrow \Hom_{\pi_1(X)}(x, z) \quad \text{with} \quad ([y_1], [y_2]) \mapsto [y_2] \cdot [y_1] = [y_2 \cdot y_1].$$

Moreover,

1. this composition operation is associative in that for all $x, y, z, w \in X$ and $[y_1] \in \Hom_{\pi_1(X)}(x, y)$, $[y_2] \in \Hom_{\pi_1(X)}(y, z)$ and $[y_3] \in \Hom_{\pi_1(X)}(z, w)$ then

$$[y_3] \cdot ([y_2] \cdot [y_1]) = ([y_3] \cdot [y_2]) \cdot [y_1].$$

2. this composition operation is unital with neutral elements the constant paths in that for all $x, y \in X$ and $[y] \in \Hom_{\pi_1(X)}(x, y)$ we have

$$[\text{const}_y] \cdot [y] = [y] = [y] \cdot [\text{const}_x].$$

3. this composition operation has inverse elements given by path reversal in that for all $x, y \in X$ and $[y] \in \Hom_{\pi_1(X)}(x, y)$ we have

$$[\bar{y}] \cdot [y] = [\text{const}_x] \quad [y] \cdot [\bar{y}] = [\text{const}_y].$$

Definition 1.16. (fundamental groupoid and fundamental groups)

Let $X$ be a topological space. Then set of points of $X$ together with the sets $\Hom_{\pi_1(X)}(x, y)$ of homotopy relative boundary-classes of paths (def. 1.13) for all points of points and equipped with the concatenation operation from prop. 1.15 is called the fundamental groupoid of $X$, denoted

$$\Pi_1(X).$$

Given a choice of point $x \in X$, then one writes

$$\pi_1(X, x) := \Hom_{\Pi_1(X)}(x, x).$$

Prop. 1.15 says that under concatenation of paths, this set is a group. As such it is called the fundamental group of $X$ at $x$.

The following picture indicates the four non-equivalent non-trivial generators of the fundamental group of the oriented surface of genus 2:
Example 1.17. (fundamental group of Euclidean space)

For \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^n \) any point in the \( n \)-dimensional Euclidean space (regarded with its metric topology) we have that the fundamental group (def. 1.16) at that point is trivial:

\[ \pi_1(\mathbb{R}^n, x) = \ast. \]

Remark 1.18. (basepoints)

Definition 1.16 intentionally offers two variants of the definition.

The first, the fundamental groupoid is canonically given, without choosing a basepoint. As a result, it is a structure that is not quite a group but, slightly more generally, a "groupoid" (a "group with many objects"). We discuss the concept of groupoids below.

The second, the fundamental group, is a genuine group, but its definition requires picking a base point \( x \in X \).

In this context it is useful to say that

1. a pointed topological space \((X, x)\) is
   
   1. a topological space \( X \);
   2. a \( x \in X \) in the underlying set.

2. a homomorphism of pointed topological spaces \( f : (X, x) \to (Y, y) \) is a basepoint preserving continuous function, namely
   
   1. a continuous function \( f : X \to Y \)
   2. such that \( f(x) = y \).

Hence there is a category, to be denoted, \( \text{Top}^*/ \), whose objects are the pointed topological spaces, and whose morphisms are the base-point preserving continuous functions.

Similarly, a homotopy between morphisms \( f, f' : (X, x) \to (Y, y) \) in \( \text{Top}^*/ \) is a homotopy \( \eta : f \Rightarrow f' \) of underlying continuous functions, as in def. 1.2, such that
the corresponding function

\[ \eta : X \times [0,1] \to Y \]

preserves the basepoints in that

\[ \forall t \in [0,1], \eta(x, t) = y. \]

These pointed homotopies still form an equivalence relation as in prop. 1.3 and hence quotienting these out yields the pointed analogue of the homotopy category from def. 1.4, now denoted

\[ \kappa : \text{Top}^*/ \to \text{Ho}(\text{Top}^*/). \]

In general it is hard to explicitly compute the fundamental group of a topological space. But often it is already useful to know if two spaces have the same fundamental group or not:

**Definition 1.19. (pushforward of elements of fundamental groups)**

Let \((X, x)\) and \((Y, y)\) be pointed topological space (remark 1.18) and let

\[ f : X \to Y \]

be a continuous function which respects the chosen points, in that \(f(x) = y\).

Then there is an induced homomorphism of fundamental groups (def. 1.16)

\[ \pi_1(X, x) \xrightarrow{f_*} \pi_1(Y, y) \]

\[ [\gamma] \mapsto [f \circ \gamma] \]

given by sending a closed path \(\gamma : [0,1] \to X\) to the composite

\[ f \circ \gamma : [0,1] \xrightarrow{\gamma} X \xrightarrow{f} Y. \]

**Remark 1.20. (fundamental group is functor on pointed topological spaces)**

The pushforward operation in def. 1.19 is functorial, now on the category \(\text{Top}^*/\) of pointed topological spaces (remark 1.18)

\[ \pi_1 : \text{Top}^*/ \to \text{Grp}. \]

**Proposition 1.21. (fundamental group depends only on homotopy classes)**

Let \(X, Y \in \text{Top}^*/\) be pointed topological space and let \(f_1, f_2 : X \to Y\) be two base-point preserving continuous functions. If there is a pointed homotopy (def. 1.2, remark 1.18)

\[ \eta : f_1 \Rightarrow f_2 \]

then the induced homomorphisms on fundamental groups (def. 1.19) agree
\[(f_1)_* = (f_2)_* : \pi_1(X,x) \rightarrow \pi_1(Y,y).\]

In particular if \(f : X \rightarrow Y\) is a homotopy equivalence (def. 1.5) then \(f_* : \pi_1(X,x) \rightarrow \pi_1(Y,y)\) is an isomorphism.

**Proof.** This follows by the fact that homotopy respects composition (prop. 1.3):

If \(\gamma : [0,1] \rightarrow X\) is a closed path representing a given element of \(\pi_1(X,x)\), then the homotopy \(f_1 \Rightarrow f_2\) induces a homotopy

\[f_1 \circ \gamma \Rightarrow f_2 \circ \gamma\]

and therefore these represent the same elements in \(\pi_1(Y,y)\).

If follows that if \(f\) is a homotopy equivalence with homotopy inverse \(g\), then \(g_* : \pi_1(Y,y) \rightarrow \pi_1(X,x)\) is an inverse morphism to \(f_* : \pi_1(X,x) \rightarrow \pi_1(Y,y)\) and hence \(f_*\) is an isomorphism. □

**Remark 1.22.** Prop. 1.21 says that the fundamental group functor from def. 1.19 and remark 1.20 factors through the classical pointed homotopy category from remark 1.18:

\[
\begin{array}{ccc}
\text{Top}^*/ & \xrightarrow{\pi_1} & \text{Grp} \\
\kappa \downarrow & & \nearrow \\
\text{Ho(\text{Top}^*)} & & \\
\end{array}
\]

**Definition 1.23.** *(simply connected topological space)*

A topological space \(X\) for which

1. \(\pi_0(X) \simeq *\) *(path connected)*
2. \(\pi_1(X,x) \simeq 1\) *(the fundamental group is trivial, def. 1.16),

is called simply connected.

We will need also the following local version:

**Definition 1.24.** *(semi-locally simply connected topological space)*

A topological space \(X\) is called semi-locally simply connected if every point \(x \in X\) has a neighbourhood \(U_x \subset X\) such that every loop in \(X\) is contractible as a loop in \(X\), hence such that the induced morphism of fundamental groups (def. 1.19)

\[\pi_1(U,x) \rightarrow \pi_1(X,x)\]

is trivial (i.e. sends everything to the neutral element).

If every \(x\) has a neighbourhood \(U_x\) which is itself simply connected, then \(X\) is called a locally simply connected topological space. This implies semi-local simply-connectedness.
Example 1.25. (Euclidean space is simply connected)

For \( n \in \mathbb{N} \), then the Euclidean space \( \mathbb{R}^n \) is a simply connected topological space (def. 1.23).

Groupoids

In def. 1.16 we extracted the fundamental group at some point \( x \in X \) from a larger algebraic structure, that incorporates all the basepoints, to be called the fundamental groupoid. This larger algebraic structure of groupoids is usefully made explicit for the formulation and proof of the fundamental theorem of covering spaces (theorem 3.1 below) and the development of homotopy theory in general.

Where a group may be thought of as a group of symmetry transformations that isomorphically relates one object to itself (the symmetries of one object, such as the isometries of a polyhedron) a groupoid is a collection of symmetry transformations acting between possibly more than one object.

Hence a groupoid consists of a set of objects \( x, y, z, \ldots \) and for each pair of objects \( (x, y) \) there is a set of transformations, usually denoted by arrows

\[
x \xrightarrow{f} y
\]

which may be composed if they are composable (i.e. if the first ends where the second starts)

\[
\begin{array}{ccc}
  y & \xrightarrow{g} & z \\
  f \swarrow & \swarrow & \searrow \\
  x & \xrightarrow{g \circ f} & y
\end{array}
\]

such that this composition is associative and such that for each object \( x \) there is identity transformation \( x \xrightarrow{\text{id}_x} x \) in that this is a neutral element for the composition of transformations, whenever defined.

So far this structure is what is called a small category. What makes this a (small) groupoid is that all these transformations are to be “symmetries” in that they are invertible morphisms meaning that for each transformation \( x \xrightarrow{f} y \) there is a transformation the other way around \( y \xrightarrow{f^{-1}} x \) such that

\[
f^{-1} \circ f = \text{id}_x \quad f \circ f^{-1} = \text{id}_y.
\]

If there is only a single object \( x \), then this definition reduces to that of a group, and in this sense groupoids are “groups with many objects”. Conversely, given any groupoid \( G \) and a choice of one of its objects \( x \), then the subcollection of
transformations from and to $x$ is a group, sometimes called the **automorphism group** $\text{Aut}_G(x)$ of $x$ in $G$.

Just as for groups, the “transformations” above need not necessarily be given by concrete transformations (say by bijections between objects which are sets). Just as for groups, such a concrete realization is always possible, but is an extra choice (called a representation of the groupoid). Generally one calls these “transformations” morphisms: $x \xymatrix{ f \ar[r] & y}$ is a morphism with “source” $x$ and “domain” $y$.

An archetypical example of a groupoid is the **fundamental groupoid** $\Pi_1(X)$ of a topological space (def. \ref{FundamentalGroupoid} below, for introduction see here): For $X$ a topological space, this is the groupoid whose

- **objects** are the points $x \in X$;
- **morphisms** $x \xymatrix{ y \ar[r] & y}$ are the homotopy relative boundary-equivalence classes $[y]$ of paths $y: [0,1] \to X$ in $X$, with $y(0) = x$ and $y(1) = y$;

and **composition** is given, on representatives, by concatenation of paths. Here the class of the reverse path $\hat{y} : t \mapsto y(1 - t)$ constitutes the inverse morphism, making this a groupoid.

If one chooses a point $x \in X$, then the corresponding group at that point is the **fundamental group** $\pi_1(X, x) \coloneqq \text{Aut}_{\Pi_1(X)}(x)$ of $X$ at that point.

This highlights one of the reasons for being interested in groupoids over groups: Sometimes this allows to avoid unnatural ad-hoc choices and it serves to streamline and simplify the theory.

A **homomorphism** between groupoids is the obvious: a function between their underlying objects together with a function between their morphisms which respects source and target objects as well as composition and identity morphisms. If one thinks of the groupoid as a special case of a category, then this is a functor. Between groupoids with only a single object this is the same as a group homomorphism.

For example if $f : X \to Y$ is a continuous function between topological spaces, then postcomposition of paths with this function induces a groupoid homomorphism $f_* : \Pi_1(X) \to \Pi_1(Y)$ between the fundamental groupoids from above.

Groupoids with groupoid homomorphisms (functors) between them form a category $\text{Grp}$ (def. 1.32 below) which includes the category $\text{Grp}$ of groups as the full subcategory of the groupoids with a single object. This makes precise how groupoid theory is a generalization of group theory.

However, for groupoids more than for groups one is typically interested in “conjugation actions” on homomorphisms. These are richer for groupoids than for groups, because one may conjugate with a different morphism at each object. If we think of groupoids as special cases of categories, then these “conjugation actions”
actions on homomorphisms” are natural transformations between functors.

For examples if \( f, g : X \to Y \) are two continuous functions between topological spaces, and if \( \eta : f \Rightarrow g \) is a homotopy from \( f \) to \( g \), then the homotopy relative boundary classes of the paths \( \eta(x, -) : [0,1] \to Y \) constitute a natural transformation between \( f_* : \Pi_1(X) \to \Pi_Y(Y) \) in that for all paths \( x_1 \overset{[\gamma]}{\longrightarrow} x_2 \) in \( X \) we have the “conjugation relation”

\[
\begin{array}{c}
\eta(x_1, \cdot) \cdot [f \circ \gamma] = [g \circ \gamma] \cdot [\eta(x_2, \cdot)] \\
\end{array}
\]

i.e.

\[
\begin{array}{c}
f(x_1) \overset{[\eta(x_1, \cdot)]}{\longrightarrow} g(x_1) \\
\end{array}
\]

\[
\begin{array}{c}
[f \circ \gamma] \downarrow \\
\downarrow \quad [f \circ \gamma] \\
\end{array}
\]

\[
\begin{array}{c}
f(x_1) \overset{[\eta(x_2, \cdot)]}{\longrightarrow} g(x_2) \\
\end{array}
\]

**Definition 1.26.** (groupoid – dependently typed definition)

A small groupoid \( \mathcal{G} \) is

1. a set \( X \), to be called the set of objects;

2. for all pairs of objects \( (x, y) \in X \times X \) a set \( \text{Hom}(x, y) \), to be called the set of morphisms with domain or source \( x \) and codomain or target \( y \);

3. for all triples of objects \( (x, y, z) \in X \times X \times X \) a function

\[
io_{x,y,z} : \text{Hom}(y, z) \times \text{Hom}(x, y) \to \text{Hom}(x, z)
\]

to be called composition

4. for all objects \( x \in X \) an element

\[
id_x \in \text{Hom}(x, x)
\]

to be called the identity morphism on \( x \);

5. for all pairs \( x, y \in \text{Hom}(x, y) \) of objects a function

\[
(\cdot)^{-1} : \text{Hom}(x, y) \to \text{Hom}(y, x)
\]

to be called the inverse-assigning function

such that

1. (associativity) for all quadruples of objects \( x_1, x_2, x_3, x_4 \in X \) and all triples of morphisms \( f \in \text{Hom}(x_1, x_2), g \in \text{Hom}(x_2, x_3) \) and \( h \in \text{Hom}(x_3, x_4) \) an equality

\[
h \circ (g \circ f) = (h \circ g) \circ f
\]

2. (unitality) for all pairs of objects \( x, y \in X \) and all morphisms \( f \in \text{Hom}(x, y) \) equalities

\[
id_y \circ f = f \quad f \circ id_x = f
\]
3. (invertibility) for all pairs of objects \( x, y \in X \) and every morphism \( f \in \text{Hom}(x, y) \) equalities

\[
f^{-1} \circ f = \text{id}_x \quad f \circ f^{-1} = \text{id}_y.
\]

If \( \mathcal{G}_1, \mathcal{G}_2 \) are two groupoids, then a \textbf{homomorphism} or \textbf{functor} between them, denoted

\[
F : \mathcal{G}_1 \rightarrow \mathcal{G}_2
\]

is

1. a \textbf{function} \( F_0 : X_1 \rightarrow X_2 \) between the respective sets of objects;
2. for each pair \( x, y \in X_1 \) of objects a function

\[
F_{x,y} : \text{Hom}_{\mathcal{G}_1}(x, y) \rightarrow \text{Hom}_{\mathcal{G}_2}(F_0(x), F_0(y))
\]

between sets of morphisms

such that

1. (respect for composition) for all triples \( x, y, z \in X_1 \) and all \( f \in \text{Hom}(x, y) \) and \( g \in \text{Hom}(y, z) \) an equality

\[
F_{y,z}(g) \circ F_{x,y}(f) = F_{x,z}(g \circ f)
\]
2. (respect for identities) for all \( x \in X \) an equality

\[
F_{x,x}(\text{id}_x) = \text{id}_{F_0(x)}.
\]

For \( \mathcal{G}_1, \mathcal{G}_2 \) two groupoids, and for \( F, G : \mathcal{G}_1 \rightarrow \mathcal{G}_2 \) two groupoid homomorphisms/functors, then a \textbf{conjugation} or \textbf{homotopy} or \textbf{natural transformation} (necessarily a \textbf{natural isomorphism})

\[
\eta : F \Rightarrow G
\]

is

- for each object \( x \in X_1 \) of \( \mathcal{G}_1 \) a morphism \( \eta_x \in \text{Hom}_{\mathcal{G}_2}(F(x), G(y)) \)

such that

- for all \( x, y \in X_1 \) and \( f \in \text{Hom}_{\mathcal{G}_1}(x, y) \) an equality

\[
\eta_y \circ F(f) = G(f) \circ \eta_x
\]
\( \eta_1 : \text{Fr} \text{I} \text{harrow} G \) and \( \eta_2 : G \Rightarrow H \) conjugation actions/natural isomorphisms between these, there is the composite

\[ \eta_2 \circ \eta_1 : F \Rightarrow H \]

with components the composite of the components

\[ (\eta_2 \circ \eta_1)(x) = \eta_2(x) \circ \eta_1(x). \]

This yields for any two groupoid a \textit{hom-groupoid}

\[ \text{Hom}_{\text{Grpd}}(G_1, G_2) \]

whose objects are the groupoid homomorphisms / functors, and whose morphisms are the conjugation actions / natural transformations.

**Remark 1.27. (groupoids are special cases of categories)**

A \textit{small groupoid} (def. \ref{GroupoidGlobalDefinition}) is equivalently a \textit{small category} in which all morphisms are \textit{isomorphisms}.

While therefore groupoid theory may be regarded as a special case of \textit{category theory}, it is noteworthy that the two theories are quite different in character. For example \textit{higher groupoid} theory is \textit{homotopy theory} which is rich but quite tractable, for instance via tools such as \textit{simplicial homotopy theory} or \textit{homotopy type theory}, while \textit{higher category theory} is intricate and becomes tractable mostly by making recourse to higher groupoid theory in the guise of \textit{(infinity,1)-category theory} and \textit{(infinity,n)-categories}.

**Example 1.28. (delooping of a group)**

Let \( G \) be a group. Then there is a groupoid, denoted \( BG \), with a single object \( p \), with morphisms

\[ \text{Hom}_{BG}(p, p) := G \]

the elements of \( G \), with composition the multiplication in \( G \), with identity morphism the \textit{neutral element} in \( G \) and with inverse morphisms the inverse elements in \( G \).

This is also called the \textit{delooping} of \( G \) (because the \textit{loop space object} of \( BG \) at the unique point is the given group: \( \Omega BG \simeq G \)).

**Example 1.29. (disjoint union/\textit{coproduct} of groupoids)**

Let \( \{G_i\}_{i \in I} \) be a set of groupoids. Then their \textit{disjoint union} (\textit{coproduct}) is the groupoid

\[ \biguplus_{i \in I} G_i \]

whose set of objects is the disjoint union of the sets of objects of the summand groupoids, and whose sets of morphisms between two objects is that of \( G_i \) if
both objects are form this groupoid, and is empty otherwise.

**Example 1.30. (disjoint union of delooping groupoids)**

Let \( \{ G_i \}_{i \in I} \) be a set of groups. Then there is a groupoid \( \bigcup_{i \in I} B G_i \) which is the disjoint union groupoid (example 1.29) of the delooping groupoids \( B G_i \) (example 1.28).

Its set of objects is the index set \( I \), and

\[
\text{Hom}(i, j) = \begin{cases} 
G_i & \text{if } i = j \\
\emptyset & \text{otherwise}
\end{cases}
\]

**Example 1.31. (groupoid core of a category)**

For \( \mathcal{C} \) any (small) category, then there is a maximal groupoid inside

\[ \text{Core}(\mathcal{C}) \hookrightarrow \mathcal{C} \]

sometimes called the core of \( \mathcal{C} \). This is obtained from \( \mathcal{C} \) simply by discarding all those morphisms that are not isomorphisms.

For instance

- For \( \mathcal{C} = \text{Set} \) then \( \text{Core}(\text{Set}) \) is the groupoid of sets and bijections between them.

- For \( \mathcal{C} = \text{FinSet} \) then the skeleton of this groupoid (prop. 1.43) is the disjoint union of deloopings (example 1.30) of all the symmetric groups:

\[ \text{Core}(\text{FinSet}) \simeq \bigsqcup_{n \in \mathbb{N}} \Sigma(n) \]

- For \( \mathcal{C} = \text{Vect} \) then \( \text{Core}(\text{Vect}) \) is the groupoid of vector spaces and linear bijections between them.

- For \( \mathcal{C} = \text{FinVect} \) then the skeleton of this groupoid is the disjoint union of delooping of all the general linear groups:

\[ \text{Core}(\text{FinVect}) \simeq \bigsqcup_{n \in \mathbb{N}} \text{GL}(n) \]

**Remark 1.32. (1-category of groupoids)**

From def. 1.26 we see that there is a category whose

- objects are the small groupoids;
- morphisms are the groupoid homomorphisms (functors).

But since this 1-category does not reflect the existence of homotopies/natural isomorphisms between homomorphisms/functors of groupoids (def. 1.26) this 1-category is not what one is interested in when considering homotopy theory/higher category theory.
In order to obtain the right notion of category of groupoids that does reflect homotopies, we first consider now the \textit{horizontal} composition of homotopies/natural transformations.

**Lemma 1.33.** (\textbf{horizontal composition of homotopies with morphisms})

Let \(G_1, G_2, G_3, G_4\) be groupoid and let

\[
\begin{array}{cccc}
  G_1 & \xrightarrow{F_1} & G_2 & \xrightarrow{F_2} \\
  & \downarrow \eta & \downarrow & \\
  F_2 & \xrightarrow{G} & G_3 & \xrightarrow{F_3} G_3
\end{array}
\]

be morphisms and a homotopy \(\eta\). Then there is a homotopy

\[
\begin{array}{cccc}
  G_1 & \xrightarrow{F_3 \circ F_2 \circ F_1} & G_2 & \xrightarrow{F_2 \cdot \eta \cdot F_1} G_2
\end{array}
\]

between the respective composites, with components given by

\[
(F_2 \cdot \eta \cdot F_1)(x) \coloneqq F_2(\eta(F_1(x))).
\]

This operation constitutes a groupoid homomorphism/\textit{functor}

\[
F_3 \cdot (-) \cdot F_1 : \text{Hom}_{\text{Grp}}(G_2, G_3) \to \text{Hom}_{\text{Grp}}(G_1, G_4).
\]

**Proof.** The respect for identities is clear. To see the respect for composition, let

\[
\begin{array}{cccc}
  & \xrightarrow{F} & \\
  & \downarrow \eta_1 & \\
  G_2 & \xrightarrow{G} & G_3 & \xrightarrow{\eta_2} \\
  & \downarrow & \\
  & \xrightarrow{H}
\end{array}
\]

be two composable homotopies. We need to show that

\[
F_3 \cdot (\eta_2 \circ \eta_1 \cdot F_1) = (F_3 \cdot \eta_2 \cdot F_1) \circ (F_3 \cdot \eta_1 \cdot F_1).
\]

Now for \(x\) any object of \(G_1\) we find

\[
(F_3 \cdot (\eta_2 \circ \eta_1 \cdot F_1))(x) \coloneqq F_2((\eta_2 \circ \eta_1)(F_1(x)))
\]

\[
= F_3(\eta_2(F_1(x)) \circ \eta_1(F_1(x)))
\]

\[
= F_2(\eta_2(F_1(x))) \circ F_2(\eta_1(F_1(X)))
\]

\[
= ((F_3 \cdot \eta_2 \cdot F_1) \circ (F_3 \cdot \eta_1 \cdot F_1))(x).
\]

Here all steps are unwinding of the definition of horizontal and of ordinary (vertical) composition of homotopies, except the third equality, which is the functoriality of \(F_2\). \qed
Lemma 1.34. (horizontal composition of homotopies)

Consider a diagram of groupoids, groupoid homomorphisms (functors) and homotopies (natural transformations) as follows:

\[
\begin{array}{ccc}
G_1 & \xrightarrow{F_1} & G_2 \\
\downarrow \eta_1 & & \downarrow \eta_2 \\
F_1 & & F_2
\end{array}
\]

The horizontal composition of the homotopies to a single homotopy of the form

\[
\begin{array}{ccc}
G_1 & \xrightarrow{F_2 \circ F_1} & G_2 \\
\downarrow \eta_2 \cdot \eta_1 G_3 \\
F_{12} \circ F_{11}
\end{array}
\]

may be defined in terms of the horizontal composition of homotopies with morphisms (lemma 1.33) and the ("vertical") composition of homotopies with themselves, in two different ways, namely by decomposing the above diagram as

\[
\begin{array}{ccc}
G_1 & \xrightarrow{F_1} & G_2 \xrightarrow{F_2} G_3 \\
\downarrow \eta_1 & & \downarrow \eta_2 \\
F_1 & & F_{12}
\end{array}
\]

or as

\[
\begin{array}{ccc}
G_1 & \xrightarrow{F_1} & G_2 \xrightarrow{F_2} G_3 \\
\downarrow \eta_1 & & \downarrow \eta_2 \\
F_1 & & F_{12}
\end{array}
\]

In the first case we get

\[
\eta_2 \cdot \eta_1 := (\eta_2 \cdot F'_1) \circ (F_2 \cdot \eta_1)
\]

while in the second case we get

\[
\eta_2 \cdot \eta_1 := (F'_2 \cdot \eta_1) \circ (\eta_2 \cdot F_1)
\]

These two definitions coincide.

Proof. For \(x\) an object of \(G_1\), then we need that the following square diagram commutes in \(G_3\)
But the ommutativity of the square on the right is the defining compatibility condition on the components of \( \eta_2 \) applied to the morphism \( \eta_1(x) \) in \( G_2 \). ■

**Proposition 1.35. (horizontal composition with homotopy is natural transformation)**

Consider groupoids, homomorphisms and homotopies of the form

\[
\begin{array}{ccc}
G_1 & \xrightarrow{F_1} & G_2 \\
\eta_1 & \downarrow & \eta_2 \\
F_{12} & \xrightarrow{F_2} & F_{12}
\end{array}
\]

Then horizontal composition with the homotopies (lemma 1.34) constitutes a natural transformation between the functors of horizontal composition with morphisms (lemma 1.33)

\[
(\eta_3 \cdot (-) \cdot \eta_1) : (F_3 \cdot (-) \cdot F_1) \Rightarrow (F'_{32} (-) \cdot F'_1) : \text{Hom}_{\text{Grpd}}(G_2, G_2) \to \text{Hom}_{\text{Grpd}}(G_1, G_4).
\]

**Proof.** By lemma 1.34. ■

It first of all follows that the following makes sense

**Definition 1.36. (homotopy category of groupoids)**

There is also the homotopy category \( \text{Ho}(\text{Grpd}) \) whose

- **objects** are small groupoids;
- **morphisms** are equivalence classes of groupoid homomorphisms modulo homotopy (i.e. functors modulo natural transformations).

This is usually denoted \( \text{Ho}(\text{Grpd}) \).

Of course what the above really means is that, without quotienting out homotopies, groupoids form a 2-category, in fact a \((2,1)\)-category, in fact an enriched category which is enriched over the naive 1-category of groupoids from remark 1.32, hece a strict 2-category with hom-groupoids.

**Definition 1.37. (equivalence of groupoids)**

Given two groupoids \( G_1 \) and \( G_2 \), then a homomorphism

\[
F : G_1 \to G_2
\]

is an equivalence if it is an isomorphism in the homotopy category \( \text{Ho}(\text{Grpd}) \) (def. 1.36), hence if there exists a homomorphism the other way around
$G : G_2 \to G_1$

and a homotopy/natural transformations of the form

$G \circ F \simeq \text{id}_{G_1} \quad F \circ G \simeq \text{id}_{G_2}$.

**Definition 1.38. (connected components of a groupoid)**

Given a groupoid $\mathcal{G}$ with set of objects $X$, then the relation “there exists a morphism from $x$ to $y$”, i.e.

$(x \sim y) \equiv (\text{Hom}(x,y) \neq \emptyset)$

is clearly an equivalence relation on $X$. The corresponding set of equivalence classes is denoted

$\pi_0(\mathcal{G})$

and called the set of connected components of $\mathcal{G}$.

**Definition 1.39. (automorphism groups)**

Given a groupoid $\mathcal{G}$ and an object $x$, then under composition the set $\text{Hom}_x(x,x)$ forms a group. This is called the automorphism group $\text{Aut}_\mathcal{G}(x)$ or vertex group or isotropy group of $x$ in $\mathcal{G}$.

**Definition 1.40. (weak homotopy equivalence of groupoids)**

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be groupoids. Then a morphism (functor)

$F : \mathcal{G}_1 \to \mathcal{G}_2$

is called a weak homotopy equivalence if

1. it induces a bijection on connected components (def. 1.38):

$\pi_0(F) : \pi_0(\mathcal{G}_1) \xrightarrow{\simeq} \pi_0(\mathcal{G}_2)$

2. for each object $x$ of $\mathcal{G}_1$ the morphism

$F_{x,x} : \text{Aut}_{\mathcal{G}_1}(x) \xrightarrow{\simeq} \text{Aut}_{\mathcal{G}_2}(F_0(X))$

is an isomorphism of automorphism groups (def. 1.39)

**Lemma 1.41. (automorphism group depends on basepoint only up to conjugation)**

For $\mathcal{G}$ a groupoid, let $x$ and $y$ be two objects in the same connected component (def. 1.38). Then there is a group isomorphism

$\text{Aut}_\mathcal{G}(x) \simeq \text{Aut}_\mathcal{G}(y)$

between their automorphism groups (def. 1.39).
Proof. By assumption, there exists some morphism from $x$ to $y$

$$x \xrightarrow{f} y.$$ 

The operation of conjugation with this morphism

$$\text{Aut}_G(x) \xrightarrow{Ad_f} \text{Aut}_G(y)$$

$$g \mapsto f^{-1} \circ g \circ f$$

is clearly a group isomorphism as required. ▮

Lemma 1.42. (equivalences between disjoint unions of delooping groupoids)

Let $\{G_i\}_{i \in I}$ and $\{H_j\}_{j \in J}$ be sets of groups and consider a homomorphism (functor)

$$F : \sqcup_{i \in I} G_i \rightarrow \sqcup_{j \in J} H_j$$

between the corresponding disjoint unions of delooping groupoids (example 1.28).

Then the following are equivalent:

1. $F$ is an equivalence of groupoids (def. 1.37);
2. $F$ is a weak homotopy equivalence (def. 1.40).

Proof. The implication 2) $\Rightarrow$ 1) is immediate.

In the other direction, assume that $F$ is an equivalence of groupoids, and let $G$ be an inverse up to natural isomorphism. It is clear that both induces bijections on connected components. To see that both are isomorphisms of automorphisms groups, observe that the conditions for the natural isomorphisms

$$\alpha : G \circ F \Rightarrow \text{id} \quad \beta : F \circ G \Rightarrow \text{id}$$

are in each separate delooping groupoid $BH_j$ of the form

$$
\begin{array}{ccc}
* & \xrightarrow{\alpha} & *\\
G_{F_0(i),F_0(i)}(F_{i,i}(f)) & \downarrow & \text{id} \\
* & \xrightarrow{\beta} & *
\end{array}
\quad
\begin{array}{ccc}
* & \xrightarrow{\alpha} & *\\
F_{G_0(j),G_0(j)}(G_{j,j}(f)) & \downarrow & \text{id} \\
* & \xrightarrow{\beta} & *
\end{array}
$$

since there is only a single object. But this means $F_{i,i}$ and $F_{j,j}$ are group isomorphisms. ▮

Proposition 1.43. (every groupoid is equivalent to a disjoint union of group deloopings)

Assuming the axiom of choice, then:
For \( \mathcal{G} \) any groupoid, then there exists a set \( \{ G_i \}_{i \in I} \) of groups and an equivalence of groupoids (def. 1.37)

\[
\mathcal{G} \simeq \bigcup_{i \in I} B G_i
\]

between \( \mathcal{G} \) and a disjoint union of delooping groupoids (example 1.30). This is called a skeleton of \( \mathcal{G} \).

Concretely, this exists for \( I = \pi_0(\mathcal{G}) \) the set of connected components of \( \mathcal{G} \) (def. 1.38) and for \( G_i := \text{Aut}_\mathcal{G}(x) \) the automorphism group (def. 1.39) of any object \( x \) in the given connected component.

**Proof.** Using the axiom of choice we may find a set \( \{ x_i \}_{i \in \pi_0(\mathcal{G})} \) of objects of \( \mathcal{G} \), with \( x_i \) being in the connected component \( i \in \pi_0(\mathcal{G}) \).

This choice induces a functor

\[
\text{inc} : \bigcup_{i \in \pi_0(\mathcal{G})} \text{Aut}_\mathcal{G}(x_i) \to \mathcal{G}
\]

which takes each object and morphism “to itself”.

Now using the axiom of choice once more, we choose in each connected component \( i \in \pi_0(\mathcal{G}) \) and for each object \( y \) in that connected component a morphism

\[
x_i \xrightarrow{f_{x_i,y}} y.
\]

Using this we obtain a functor the other way around

\[
p : \mathcal{G} \to \bigcup_{i \in \pi_0(\mathcal{G})} \text{Aut}_\mathcal{G}(x_i)
\]

which sends each object to its connected component, and which for pairs of objects \( y, z \) of \( \mathcal{G} \) is given by conjugation with the morphisms choosen above:

\[
\text{Hom}_\mathcal{G}(y, z) \xrightarrow{p_{y,z}} \text{Aut}_\mathcal{G}(x_i)
\]

It is now sufficient to show that there are conjugations/natural isomorphisms

\[
p \circ \text{inc} \simeq \text{id} \quad \text{inc} \circ p \simeq \text{id}.
\]

For the first this is immediate, since we even have equality

\[
p \circ \text{inc} = \text{id}.
\]

For the second we observe that choosing
\[ \eta(y) := f_{x_i,y} \]
yields a naturality square by the above construction:

\[
x_i \xymatrix{ f_{x_i, y} \ar[r] & y } \quad f_{x_i, z} \circ f \circ f_{x_i, y} \ar[d] \quad \downarrow f. \quad x_i \ar[r]_{f_{x_i, z}} & z
\]

### Proposition 1.44. (weak homotopy equivalence is equivalence of groupoids)

Let \( F : G_1 \to G_2 \) be a homomorphism of groupoids.

Assuming the axiom of choice then the following are equivalent:

1. \( F \) is an equivalence of groupoids (def. 1.37);

2. \( F \) is a weak homotopy equivalence in that

   1. it induces an bijection of sets of connected components (def. 1.38):
   
   \[ \pi_0(F) : \pi_0(G_1) \xrightarrow{\cong} \pi_0(G_0), \]

3. for each object \( x \in G_1 \) it induces an isomorphism of automorphism groups (def. 1.39):

   \[ F_{x,x} : \text{Aut}_{G_1}(x) \xrightarrow{\cong} \text{Aut}_{G_2}(F_0(x)). \]

### Proof. In one direction, if \( F \) has an inverse up to natural isomorphism, then this induces by definition a bijection on connected components, and it induces isomorphism on homotopy groups by lemma 1.41.

In the other direction, choose equivalences to skeleta as in prop. 1.43:

\[
\xymatrix{ G_1 \ar[r]^{\text{inc}_1} \ar[d]_F & \bigsqcup_{i \in \pi_0(g_1)} \text{Aut}_{G_1}(x_i) \ar[d]^{\tilde{F} := p_2 \circ F \circ \text{inc}_1} \ar[r]_{p_2} & G_2 \ar[r]_{\text{inc}_2} & \bigsqcup_{j \in \pi_0(g_2)} \text{Aut}_{G_2}(x_j) }
\]

Here \( \text{inc}_1 \) and \( p_2 \) are equivalences of groupoids by prop. 1.43 and hence are weak homotopy equivalences by the statement above. Since moreover \( F \) is a weak homotopy equivalence by assumption, it follows clearly that also \( \tilde{F} \) is a weak homotopy equivalence.

Since \( \tilde{F} \) is a morphism between disjoint unions of delooping groupoids, the statement follows now with lemma 1.42. ■
2. Covering spaces

Definition 2.1. (covering space)

Let $X$ be a topological space. A covering space of $X$ is a continuous function

$$p: E \rightarrow X$$

such that there exists an open cover $\bigcup_i U_i \rightarrow X$, such that restricted to each $U_i$ then $E \rightarrow X$ is homeomorphic over $U_i$ to the product topological space (this example) of $U_i$ with the discrete topological space (this example) on a set $F_i$.

In summary this says that $p: E \rightarrow X$ is a covering space if there exists a pullback diagram in $\text{Top}$ of the form

$$
\begin{array}{ccc}
\bigcup_i U_i \times \text{Disc}(F_i) & \rightarrow & E \\
\downarrow & & \downarrow (\text{pb}) \\
\bigcup_{i \in I} U_i & \rightarrow & X
\end{array}
$$

For $x \in U_i \subset X$ a point, then the elements in $F_x = F_i$ are called the leaves of the covering at $x$.

Given two covering spaces $p_i: E_i \rightarrow X$, then a homomorphism between them is a continuous function $f: E_1 \rightarrow E_2$ between the total covering spaces, which respects the fibers in that the following diagram commutes

$$
\begin{array}{ccc}
E_1 & \xrightarrow{f} & E_2 \\
\downarrow & & \downarrow \quad \checkmark \\
X & & X
\end{array}
$$

This defines a category $\text{Cov}(X)$ whose

- objects are the covering spaces over $X$;
- morphisms are the homomorphisms between these.

Example 2.2. (covering of circle by circle)

Regard the circle $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ as the topological subspace of elements of unit absolute value in the complex plane. For $k \in \mathbb{N}$, consider the continuous function

$$p := (-)^k : S^1 \rightarrow S^1$$

given by taking a complex number to its $k$th power. This may be thought of as the result of “winding the circle $k$ times around itself”. Precisely, for $k \geq 1$ this is a covering space (def. 2.1) with $k$ leaves at each point.

graphics grabbed from Hatcher
Example 2.3. (covering of circle by real line)

Consider the continuous function
\[ \exp(2\pi i(-)) : \mathbb{R}^1 \rightarrow S^1 \]
from the real line to the circle, which,

1. with the circle regarded as the unit circle in the complex plane \( \mathbb{C} \), is given by
\[ t \mapsto \exp(2\pi t) \]
2. with the circle regarded as the unit circle in \( \mathbb{R}^2 \), is given by
\[ t \mapsto (\cos(2\pi t), \sin(2\pi t)) \]

We may think of this as the result of “winding the line around the circle ad infinitum”. Precisely, this is a covering space (def. 2.1) with the leaves at each point forming the set \( \mathbb{Z} \) of natural numbers.

Definition 2.4. (action of fundamental group on fibers of covering)

Let \( E \xrightarrow{\pi} X \) be a covering space (def. 2.1)

Then for \( x \in X \) any point, and any choice of element \( e \in F_x \) of the leaf space over \( x \), there is, up to homotopy, a unique way to lift a representative path in \( X \) of an element \( \gamma \) of the fundamental group \( \pi_1(X,x) \) (def. 1.16) to a continuous path in \( E \) that starts at \( e \). This path necessarily ends at some (other) point \( \rho_\gamma(e) \in F_x \) in the same fiber. This construction provides a function
\[ \rho : F_x \times \pi_1(X,x) \rightarrow F_x \]
\[ (e, \gamma) \mapsto \rho_\gamma(e) \]

from the Cartesian product of the leaf space with the fundamental group. This function is compatible with the group-structure on \( \pi_1(X,x) \), in that the following diagrams commute:

\[ F_x \times \{\text{const}_x\} \xrightarrow{\text{id}} F_x \times \pi_1(X,x) \xrightarrow{\rho} F_x \]
\[ \text{id, the neutral element,} \quad \text{i.e. the constant loop,} \quad \text{acts trivially} \]

and
\[ F_x \times \pi_1(X,x) \times \pi_1(X,x) \xrightarrow{\rho \times \text{id}} F_x \times \pi_1(X,x) \]
\[ \text{id} \times ((-) \cdot (-)) \downarrow \quad \downarrow \rho \quad \rho \times \text{id} \]
\[ F_x \times \pi_1(X,x) \xrightarrow{\rho} F_x \]
\[ \text{acting with two group elements} \]
\[ \text{is the same as} \]
\[ \text{first multiplying them} \]
\[ \text{and then acting with their product element} \]
One says that $\rho$ is an action or permutation representation of $\pi_1(X,x)$ on $F_x$.

For $G$ any group, then there is a category $G\text{Set}$ whose objects are sets equipped with an action of $G$, and whose morphisms are functions which respect these actions. The above construction is a functor of the form

$$\text{Fib}_x : \text{Cov}(X) \to \pi_1(X,x)\text{Set}.$$  

**Example 2.5. (three-sheeted covers of the circle)**

There are, up to isomorphism, three different 3-sheeted covering spaces of the circle $S^1$.

The one from example 2.2 for $k = 3$. Another one. And the trivial one. Their corresponding permutation actions may be seen from the pictures on the right.

*graphics grabbed from Hatcher*

**Proposition 2.6. (covering projections are open maps)**

If $p : E \to X$ is a covering space projection, then $p$ is an open map.

**Proof.** By definition of covering space there exists an open cover $\{U_i \subset X\}_{i \in I}$ and homeomorphisms $p^{-1}(U_i) \simeq U_i \times \text{Disc}(F_i)$ for all $i \in I$. Since the projections out of a product topological space are open maps (this prop.), it follows that $p$ is an open map when restricted to any of the $p^{-1}(U_i)$. But a general open subset $W \subset E$ is the union of its restrictions to these subspaces:

$$W = \bigcup_{i \in I} (W \cap p^{-1}(U_i)).$$

Since images preserve unions (this prop.) it follows that

$$p(W) = \bigcup_{i \in I} p(W \cap p^{-1}(U_i))$$

is a union of open sets, and hence itself open. □

We discuss left lifting properties satisfied by covering spaces.

1. path-lifting property,
2. homotopy-lifting property,
3. the lifting theorem.

**Lemma 2.7. (path lifting property)**

Let $p : E \to X$ be any covering space. Given

1. $\gamma : [0,1] \to X$ a path in $X$, 

2. \( \hat{x}_0 \in E \) be a lift of its starting point, hence such that \( p(\hat{x}_0) = \gamma(0) \)

then there exists a unique path \( \hat{\gamma} : [0, 1] \to E \) such that

1. it is a lift of the original path: \( p \circ \hat{\gamma} = \gamma \);
2. it starts at the given lifted point: \( \hat{\gamma}(0) = \hat{x}_0 \).

In other words, every commuting diagram in \( \text{Top} \) of the form

\[
\begin{array}{c}
\{0\} \xrightarrow{\hat{x}_0} E \\
\downarrow \quad \downarrow \Pr \quad \Pr \!
\end{array}
\begin{array}{c}
\{0, 1\} \xrightarrow{\gamma} X
\end{array}
\]

has a unique lift:

\[
\begin{array}{c}
\{0\} \xrightarrow{\hat{x}_0} E \\
\downarrow \hat{\gamma} \quad \downarrow \Pr \!
\end{array}
\begin{array}{c}
\{0, 1\} \xrightarrow{\gamma} X
\end{array}
\]

##### Proof

First consider the case that the covering space is trivial, hence of the Cartesian product form

\( \Pr_1 : X \times \text{Disc}(S) \to X \).

By the universal property of the product topological spaces in this case a lift \( \hat{\gamma} : [0, 1] \to X \times \text{Disc}(S) \) is equivalently a pair of continuous functions

\[
\Pr_1(\hat{\gamma}) : [0, 1] \to X \quad \Pr_2(\hat{\gamma}) : [0, 1] \to \text{Disc}(S),
\]

Now the lifting condition explicitly fixes \( \Pr_1(\hat{\gamma}) = \gamma \). Moreover, a continuous function into a discrete topological space \( \text{Disc}(S) \) is locally constant, and since \( [0, 1] \) is a connected topological space this means that \( \Pr_2(\hat{\gamma}) \) is in fact a constant function (this example), hence uniquely fixed to be \( \Pr_2(\hat{\gamma}) = \hat{x}_0 \).

This shows the statement for the case of trivial covering spaces.

Now consider any covering space \( p : E \to X \). By definition of covering spaces, there exists for every point \( x \in X \) an open neighbourhood \( U_x \subset X \) such that the restriction of \( E \) to \( U_x \) becomes a trivial covering space:

\[
p^{-1}(U_x) \cong U_x \times \text{Disc}(p^{-1}(x)).
\]

Consider such a choice

\[
\{U_x \subset X\}_{x \in X}.
\]
This is an open cover of \( X \). Accordingly, the pre-images
\[
\{ y^{-1}(U_x) \subseteq [0,1] \}_{x \in X}
\]
constitute an open cover of the topological interval \([0,1]\).

Now \([0,1]\) is a compact metric space and therefore the Lebesgue number lemma implies that there is a positive number \( \varepsilon \in (0,\infty) \) and cover of \([0,1]\) by open intervals of the form \((-\varepsilon + x, x + \varepsilon) \cap [0,1] \subseteq [0,1]\) which refines this cover. Again since \([0,1]\) is a compact topological space there is a finite set of such intervals which covers \([0,1]\). This means that we find a finite number of points
\[
t_0 < t_1 < \cdots < t_{n+1} \in [0,1]
\]
with \( t_0 = 0 \) and \( t_{n+1} = 1 \) such that for all \( 0 < j \in \mathbb{N} \) there is \( x_j \in X \) such that the corresponding path segment
\[
\gamma([t_j, t_{j+1}]) \subseteq X
\]
is contained in \( U_{x_j} \) from above.

Now assume that \( \hat{\gamma}|_{[t_0,t_j]} \) has been found. Then by the triviality of the covering space over \( U_{x_j} \) and the first argument above, there is a unique lift of \( \gamma|_{[t_j,t_{j+1}]} \) to a continuous function \( \hat{\gamma}|_{[t_j,t_{j+1}]} \) with starting point \( \hat{\gamma}(t_j) \). Since \([0,t_{j+1}]\) is the pushout \([0,t_j] \sqcup [t_j,t_{j+1}]\) (this example), it follows that \( \hat{\gamma}|_{[0,t_j]} \) and \( \hat{\gamma}|_{[t_j,t_{j+1}]} \) uniquely glue to a continuous function \( \hat{\gamma}|_{[0,t_{j+1}]} \) which lifts \( \gamma|_{[0,t_{j+1}]} \).

By induction over \( j \), this yields the required lift \( \hat{\gamma} \).

Conversely, given any lift, \( \hat{\gamma} \), then its restrictions \( \hat{\gamma}|_{[t_j,t_{j+1}]} \) are uniquely fixed by the above inductive argument. Therefore also the total lift is unique. \( \square \)

**Proposition 2.8. (homotopy lifting property of covering spaces)**

Let \( p:E \to X \) be a covering space. Then given a homotopy relative the starting point between two paths in \( X \), there is for every lift of these two paths to paths in \( E \) with the same starting point a unique homotopy between the lifted paths that lifts the given homotopy:

For commuting squares of the form

\[
\begin{array}{ccc}
\{0\} \times [0,1] & \longrightarrow & * \\
\downarrow & & \downarrow \\
[0,1] \times \{0,1\} & \longrightarrow & E \\
\downarrow & \hat{\eta} \nearrow & \downarrow \hat{p} \\
[0,1] \times [0,1] & \longrightarrow & X
\end{array}
\]
there is a unique diagonal lift in the lower diagram, as shown.

Moreover if the homotopy \( \eta \) also fixes the endpoint, then so does the lifted homotopy \( \hat{\eta} \).

**Proof.** The proof is analogous to that of lemma 2.7: The Lebesgue number lemma gives a partition of \([0,1] \times [0,1]\) into a finite number of squares such that the image of each under \( \gamma \) lands in an open subset over which the covering space trivializes. Then there is inductively a unique appropriate lift over each of these squares.

Finally, if the homotopy in \( X \) is constant also at the endpoint, hence on \([1] \times [0,1]\), then the function constant on \( \{1\} \times [0,1] \), then the function constant on \( \{1\} \times [0,1] \) is clearly a lift of the path \( \eta_{|[1]} \) and by uniqueness of the path lifting (lemma 2.7) this means that also \( \hat{\eta} \) is constant on \([1] \times [0,1] \).

**Example 2.9.** Let \((E,e) \overset{p}{\rightarrow} (X,x)\) be a pointed covering space and let \( f :(Y,y) \rightarrow (X,x) \) be a point-preserving continuous function such that the image of the fundamental group of \((Y,y)\) is contained within the image of the fundamental group of \((E,e)\) in that of \((X,x)\):

\[
f_*(\pi_1(Y,y)) \subset p_*(\pi_1(E,e)) \subset \pi_1(X,x)
\]

Then for \( \ell_Y \) a path in \((Y,y)\) that happens to be a loop, every lift of its image path \( f \circ \ell \) in \((X,x)\) to a path \( \widehat{f \circ \ell_Y} \) in \((E,e)\) is also a loop there.

**Proof.** By assumption, there is a loop \( \ell_E \) in \((E,e)\) and a homotopy fixing the endpoints of the form

\[
\eta_X : p \circ \ell_E \Rightarrow f \circ \ell_Y
\]

Then by the homotopy lifting property (lemma 2.8), there is a homotopy in \((E,e)\) fixing the starting point, of the form

\[
\eta_E : \ell_E \Rightarrow \widehat{f \circ \ell_Y}
\]

and lifting the homotopy \( \eta_X \). Since \( \eta_X \) in addition fixes the endpoint, the uniqueness of the path lifting (lemma 2.7) implies that also \( \eta_E \) fixes the endpoint. Therefore \( \eta_E \) is in fact a homotopy between loops, and so \( f \circ \ell_Y \) is indeed a loop. ■

**Proposition 2.10. (lifting theorem)**

Let

1. \( p : E \rightarrow X \) be a covering space;
2. \( e \in E \) a point, with \( x := p(e) \) denoting its image,
3. \( Y \) be a connected and locally path-connected topological space;
4. \( y \in Y \) a point

5. \( f: (Y, y) \to (X, x) \) a **continuous function** such that \( f(y) = x \).

Then the following are equivalent:

1. There exists a lift \( \hat{f} \) in the diagram

\[
\begin{array}{ccc}
(E, e) & \xrightarrow{(E, e)} & (Y, y) \\
\downarrow{p} & \nearrow{\hat{f}} & \downarrow{f} \\
(X, x) & \xrightarrow{} & (X, x)
\end{array}
\]

of **pointed topological spaces**.

2. The **image** of the **fundamental group** of \( Y \) under \( f \) in that of \( X \) is contained in the image of the fundamental group of \( E \) under \( p \):

\[ f_*(\pi_1(Y, y)) \subset p_*(\pi_1(E, e)) \]

Moreover, if \( Y \) is path-connected, then the lift in the first item is unique.

**Proof.** The implication \( 1 \Rightarrow 2 \) is immediate. We need to show that the second statement already implies the first.

Since \( Y \) is connected and locally path-connected, it is also a **path-connected topological space** (this prop.). Hence for every point \( y' \in Y \) there exists a path \( \gamma \) connecting \( y \) with \( y' \) and hence a path \( f \circ \gamma \) connecting \( x \) with \( f(y') \). By the path-lifting property (lemma 2.7) this has a unique lift

\[
\begin{array}{ccc}
\{0\} & \xrightarrow{e} & E \\
\downarrow{f \circ \gamma} & \nearrow{\hat{f}} & \downarrow{p} \\
[0, 1] & \xrightarrow{f \circ \gamma} & X
\end{array}
\]

Therefore

\[ \hat{f}(y') \coloneqq \hat{f} \circ \gamma \]

if a lift of \( f(y') \).

We claim now that this pointwise construction is independent of the choice \( \gamma \), and that as a function of \( y' \) it is indeed continuous. This will prove the claim.

Now by the path lifting lemma 2.7 the lift \( \hat{f} \circ \gamma \) is unique given \( f \circ \gamma \), and hence \( \hat{f}(y') \) depends at most on the choice of \( \gamma \).

Hence let \( y': [0, 1] \to Y \) be another path in \( Y \) that connects \( y \) with \( y' \). We need to show that then \( \hat{f} \circ y' = \hat{f} \circ \gamma \).
First observe that if $y'$ is related to $y$ by a \textit{homotopy}, so that then also $f \circ y'$ is related to $f \circ y$ by a homotopy, then this is the statement of the homotopy lifting property of lemma 2.8.

Next write $\hat{y}' \cdot y$ for the \textit{path concatenation} of the path $y$ with the \textit{reverse path} of the path $y'$, hence a loop in $Y$, so that $f \circ (\hat{y}' \cdot y)$ is a loop in $X$. The assumption that $f_\ast(\pi_1(Y,y)) \subset p_\ast(\pi_1(E,e))$ implies (example 2.9) that the path $f \circ (\hat{y}' \cdot y)$ which lifts this loop to $E$ is itself a loop in $E$.

By uniqueness of path lifting, this means that the lift of $f \circ (y' \cdot (\hat{y}' \cdot y))$ coincides with that of $f \circ y'$. But $\hat{y}' \cdot (y' \cdot y)$ is homotopic (via reparameterization) to just $y$. Hence it follows now with the first statement that the lift of $f \circ y'$ indeed coincides with that of $f \circ y$.

This shows that the above prescription for $\hat{f}$ is well defined.

It only remains to show that the function $\hat{f}$ obtained this way is continuous.

Let $y' \in Y$ be a point and $W_{\hat{f}(y')}$ be an open neighbourhood of its image in $E$. It is sufficient to see that there is an open neighbourhood $V_{y'} \subset Y$ such that $\hat{f}(V_{y'}) \subset W_{\hat{f}(y')}$.

Let $U_{f(y')} \subset X$ be an open neighbourhood over which $p$ trivializes. Then the restriction

$$p^{-1}(U_{f(y')}) \cap W_{\hat{f}(y')} \subset U_{f(y')} \times \operatorname{Disc}(p^{-1}(f(y')))$$

is an open subset of the product space. Consider its further restriction

$$\left(U_{f(y')} \times \{\hat{f}(y')\}\right) \cap \left(p^{-1}(U_{f(y')}) \cap W_{\hat{f}(y')}\right)$$

to the \textit{leaf}

$$U_{f(y')} \times \{\hat{f}(y')\} \subset U_{f(y')} \times p^{-1}(f(y'))$$

which is itself an open subset. Since $p$ is an \textit{open map} (this prop.), the subset

$$p\left((U_{f(y')} \times \{\hat{f}(y')\}) \cap \left(p^{-1}(U_{f(y')}) \cap W_{\hat{f}(y')}\right)\right)$$

is open, hence so is its pre-image

$$f^{-1}\left(p\left((U_{f(y')} \times \{\hat{f}(y')\}) \cap \left(p^{-1}(U_{f(y')}) \cap W_{\hat{f}(y')}\right)\right)\right) \subset Y.\text{ }$$

Since $Y$ is assumed to be \textit{locally path-connected}, there exists a path-connected open neighbourhood

$$V_{y'} \subset f^{-1}\left(p\left((U_{f(y')} \times \{\hat{f}(y')\}) \cap \left(p^{-1}(U_{f(y')}) \cap W_{\hat{f}(y')}\right)\right)\right).$$
By the uniqueness of pah lifting, the image of that under $\hat{f}$ is

$$\hat{f}(V_{y'}) = f(V_{y'}) \times \{\hat{f}(y')\}$$

$$\subseteq p\left(\left(U_f(y') \times \{\hat{f}(y')\}\right) \cap \left(p^{-1}(U_{f(y')}) \cap W_{\hat{f}(y')}\right)\right) \times \{\hat{f}(y')\}$$

$$\cong \left(U_f(y') \times \{\hat{f}(y')\}\right) \cap \left(p^{-1}(U_{f(y')}) \cap W_{\hat{f}(y')}\right)$$

$$\subseteq W_{\hat{f}(y')}$$

It remains to show that this lift is unique if $Y$ is path-connected. (...)

### Monodromy

We now extract from a covering space is *monodromy*, which is a *groupoid representation* of the *fundamental groupoid* of the base space.

#### Definition 2.11. (groupoid representation)

Let $\mathcal{G}$ be a *groupoid*. Then:

A *linear representation* of $\mathcal{G}$ is a groupoid homomorphism (*functor*)

$$\rho : \mathcal{G} \to \text{Core}(\text{Vect})$$

to the groupoid core of the category $\text{Vect}$ of *vector spaces* (example 1.31). Hence this is

1. For each object $x$ of $\mathcal{G}$ a *vector space* $V_x$;
2. For each morphism $x \xrightarrow{f} y$ of $\mathcal{G}$ a *linear map* $\rho(f) : V_x \to V_y$

such that

1. (respect for composition) for all composable morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in the groupoid we have an *equality*

$$\rho(g) \circ \rho(f) = \rho(g \circ f)$$

2. (respect for identities) for each object $x$ of the groupoid we have an equality

$$\rho(\text{id}_x) = \text{id}_{V_x}.$$

Similarly a *permutation representation* of $\mathcal{G}$ is a groupoid homomorphism (*functor*)

$$\rho : \mathcal{G} \to \text{Core}(\text{Set})$$

to the groupoid core of $\text{Set}$. Hence this is

1. For each object $x$ of $\mathcal{G}$ a *set* $S_x$;
for each morphism $x \xrightarrow{f} y$ of $\mathcal{G}$ a function $\rho(f) : S_x \to S_y$

such that composition and identities are respected, as above.

For $\rho_1$ and $\rho_2$ two such representations, then a homomorphism of representations

$$\phi : \rho_1 \to \rho_2$$

is a natural transformation between these functors, hence is

- for each object $x$ of the groupoid a (linear) function

$$\phi(x) : (V_1)_x \to (V_2)_x$$

- such that for all morphisms $x \xrightarrow{f} y$ we have

$$\phi(y) \circ \rho_1(f) = \rho_2(x) \circ \phi(x)$$

Representations of $\mathcal{G}$ and homomorphisms between them constitute a category, called the representation category $\text{Rep}_{\text{Grpd}}(\mathcal{G})$.

**Definition 2.12. (monodromy of a covering space)**

Let $X$ be a topological space and $E \xrightarrow{p} X$ a covering space. Write $\Pi_1(X)$ for the fundamental groupoid of $X$.

Define a functor

$$\text{Fib}_E : \Pi_1(X) \to \text{Set}$$

to the category $\text{Set}$ of sets, hence a permutation groupoid representation, as follows:

1. to a point $x \in X$ assign the fiber $p^{-1}(\{x\}) \in \text{Set}$;

2. to the homotopy class of a path $\gamma$ connecting $x := \gamma(0)$ with $y := \gamma(1)$ in $X$ assign the function $p^{-1}(\{x\}) \to p^{-1}(\{y\})$ which takes $\hat{x} \in p^{-1}(\{x\})$ to the endpoint of a path $\hat{\gamma}$ in $E$ which lifts $\gamma$ through $p$ with starting point $\hat{\gamma}(0) = \hat{x}$

$$p^{-1}(x) \quad \rightarrow \quad p^{-1}(y)$$

$$(\hat{x} = \hat{\gamma}(0)) \quad \mapsto \quad \hat{\gamma}(1)$$

This construction is well defined for a given representative $\gamma$ due to the unique path-lifting property of covering spaces (this lemma) and it is independent of the choice of $\gamma$ in the given homotopy class of paths due to the homotopy-lifting
property (this lemma). Similarly, these two lifting properties give that this construction respects composition in \( \Pi_1(X) \) and hence is indeed a functor.

We may also express this in terms of group representations of fundamental groups:

**Proposition 2.13.** *(groupoid representations are products of group representations)*

Assuming the **axiom of choice** then the following holds:

Let \( G \) be a groupoid. Then its category of groupoid representations is equivalent to the product category indexed by the set of connected components \( \pi_0(G) \) (def. 1.38) of group representations of the automorphism group \( G_i := \text{Aut}_g(x_i) \) (def. 1.39) for \( x_i \) any object in the \( i \)th connected component:

\[
\text{Rep}(G) \simeq \prod_{i \in \pi_0(G)} \text{Rep}(G_i).
\]

**Proof.** Let \( \mathcal{C} \) be the category that the representation is on. Then by definition

\[
\text{Rep}(G) = \text{Hom}(G, \mathcal{C}).
\]

Consider the injection functor of the **skeleton** (from lemma 1.42)

\[
\text{inc} : \bigcup_{i \in \pi_0(G)} BG_i \to G.
\]

By lemma 1.33 the pre-composition with this constitutes a functor

\[
\text{inc}^* : \text{Hom}(G, \mathcal{C}) \to \text{Hom}(\bigcup_{i \in \pi_0(G)} BG_i, \mathcal{C})
\]

and by combining lemma 1.42 with lemma 1.35 this is an **equivalence of categories**. Finally, by example \ref{GroupoidRepresentationOfDeloopingGroupoid} the category on the right is the product of group representation categories as claimed. ■

**Proposition 2.14.** Given a **homomorphism** between two covering spaces \( E_1 \overset{p_1}{\to} X \), hence a **continuous function** \( f : E_1 \to E_2 \) which respects fibers in that the diagram

\[
\begin{array}{ccc}
E_1 & \overset{f}{\rightarrow} & E_2 \\
\downarrow_{p_1} & & \downarrow_{p_2} \\
X
\end{array}
\]

**commutes**, then the component functions

\[
f \big|_{(x)} : p_1^{-1}((x)) \to p_2^{-1}((x))
\]

are compatible with the monodromy \( \text{Fib}_E \) (def. 2.12) along any path \( \gamma \) between points \( x \) and \( y \) from def. 2.12 in that the following diagrams of sets commute.
This means that \( f \) induces a **natural transformation** between the monodromy functors of \( E_1 \) and \( E_2 \), respectively, and hence that constructing monodromy is itself a functor from the **category of covering spaces** of \( X \) to that of **permutation representations** of the **fundamental groupoid** of \( X \):

\[
\text{Fib} : \text{Cov}(X) \to \text{Set}^{\pi_1(X)}.
\]

**Example 2.15. (fundamental groupoid of covering space)**

Let \( E \to X \) be a covering space.

Then the **fundamental groupoid** \( \Pi_1(E) \) of the total space \( E \) is **equivalently** the Grothendieck construction of the **monodromy** functor \( \text{Fib}_E : \Pi_1(X) \to \text{Set} \)

\[
\Pi_1(E) \cong \int_{\Pi_1(X)} \text{Fib}_E
\]

whose

- **objects** are pairs \((x, \hat{x})\) consisting of a point \( x \in X \) and an element \( \hat{x} \in \text{Fib}_E(x) \);

- **morphisms** \([\hat{y}] : (x, \hat{x}) \to (x', \hat{x}')\) are morphisms \([y] : x \to x'\) in \( \Pi_1(X) \) such that \( \text{Fib}_E([y])(\hat{x}) = \hat{x}' \).

**Proof.** By the uniqueness of the path-lifting, lemma 2.7 and the very definition of the **monodromy** functor. □

**Proposition 2.16.** Let \( X \) be a **path-connected topological space** and let \( E \to X \) be a **covering space**. Then the total space \( E \) is

1. **path-connected** precisely if the **monodromy** \( \text{Fib}_E \) is a **transitive action**;

2. **simply connected** precisely if the **monodromy** \( \text{Fib}_E \) is a **free action**.

**Proof.** By example 2.15. □

**Reconstruction**

The following is a description of the reconstruction in terms of elementary **point-set topology**.

**Definition 2.17. (reconstruction of covering spaces from monodromy)**

Let...
1. \((X, \tau)\) be a **locally path-connected** *semi-locally simply connected* topological space,

2. \(\rho \in \text{Set}^{\Pi_1(X)}\) a **permutation representation** of its **fundamental groupoid**.

Consider the **disjoint union** set of all the sets appearing in this representation

\[ E(\rho) := \bigcup_{x \in X} \rho(x) \]

For an **open subset** \(U \subset X\) which is **path-connected** and for which every element of the **fundamental group** \(\pi_1(U, x)\) becomes trivial under \(\pi_1(U, x) \to \pi_1(X, x)\), and for \(\hat{x} \in \rho(x)\) with \(x \in U\) consider the subset

\[ V_{U, \hat{x}} := \{\rho(\gamma)(\hat{x}) \mid x' \in U, \ \gamma \text{ path from } x \text{ to } x'\} \subset E(\rho) .\]

The collection of these defines a **base for a topology** (prop. 2.18 below). Write \(\tau_\rho\) for the corresponding topology. Then

\[ (E(\rho), \tau_\rho) \]

is a topological space. It canonically comes with the function

\[ E(\rho) \xrightarrow{p} X \]

\[ \hat{x} \in \rho(x) \mapsto x .\]

Finally, for

\[ f : \rho_1 \to \rho_2 \]

a **homomorphism** of permutation representations, there is the evident induced function

\[ E(\rho_1) \xrightarrow{\text{Rec}(f)} E(\rho_2) \]

\[ (\hat{x} \in \rho_1(x)) \mapsto (f_\times(\hat{x}) \in \rho_2(x)) .\]

**Proposition 2.18.** The construction \(\rho \mapsto E(\rho)\) in def. 2.17 is well defined and yields a **covering space** of \(X\).

Moreover, the construction \(f \mapsto \text{Rec}(f)\) yields a homomorphism of covering spaces.

**Proof.** First to see that we indeed have a topology, we need to check (by this prop.) that every point is contained in some base element, and that every point in the intersection of two base elements has a base neighbourhood that is still contained in that intersection.

So let \(x \in X\) be a point. By the assumption that \(X\) is semi-locally simply connected there exists an open neighbourhood \(U_x \subset X\) such that every loop in \(U_x\) on \(x\) is contractible in \(X\). Moreover by the assumption that \(X\) is locally path-connected
topological space, this contains a possibly smaller open neighbourhood $U'_x \subset U_x$ which is path connected. Moreover, as every subset of $U_x$, it still has the property that every loop in $U'_x$ based on $x$ is contractible as a loop in $X$. Now let $\hat{x} \in E$ be any point over $x$, then it is contained in the base open $V_{U'_x(x)}$.

The argument for the base open neighbourhoods contained in intersections is similar.

Then we need to see that $p : E(\rho) \to X$ is a continuous function. Since taking pre-images preserves unions (this prop.), and since by semi-local simply-connectedness every neighbourhood contains an open $U \subset X$ that labels a base open, it is sufficient to see that $p^{-1}(U)$ is a base open. But by the very assumption on $U$, there is a unique morphism in $\Pi_1(X)$ from any point $x \in U$ to any other point in $U$, so that $\rho$ applied to these paths establishes a bijection of sets

$$p^{-1}(U) \cong \bigsqcup_{\hat{x} \in \rho(x)} V_{U,\hat{x}} \cong U \times \rho(x),$$

thus exhibiting $p^{-1}(U)$ as a union of base opens.

Finally we need to see that this continuous function $p$ is a covering projection, hence that every point $x \in X$ has a neighbourhood $U$ such that $p^{-1}(U) \cong U \times \rho(x)$. But this is again the case for those $U$ all whose loops are contractible in $X$, by the above identification via $\rho$, and these exist around every point by semi-local simply-connectedness of $X$.

This shows that $p : E(\rho) \to X$ is a covering space. It remains to see that $\text{Ref}(f) : E(\rho_1) \to E(\rho_2)$ is a homomorphism of covering spaces. Now by construction it is immediate that this is a function over $X$, in that this diagram commutes:

$$
\begin{array}{ccc}
E(\rho_1) & \xrightarrow{\text{Rec}(f)} & E(\rho_2) \\
\searrow & & \swarrow \\
& X & 
\end{array}
$$

So it only remains to see that $\text{Ref}(f)$ is a continuous function. So consider $V_{U,y_2 \in \rho_2(x)}$ a base open of $E(\rho_2)$. By naturality of $f$ its pre-image under $\text{Rec}(f)$ is

$$\text{Rec}(f)^{-1}(V_{U,y_2 \in \rho_2(x)}) = \bigsqcup_{y_1 \in f^{-1}(y_2)} V_{U,y_1}$$

and hence a union of base opens.

3. Topological Galois theory

Fundamental theorem of covering spaces

Theorem 3.1. (fundamental theorem of covering spaces)

Let $X$ be a locally path-connected and semi-locally simply-connected topological


space. Then the operations on

1. extracting the monodromy \( \text{Fib}_E \) of a covering space \( E \) over \( X \)

2. reconstructing a covering space from monodromy \( \text{Rec}(\rho) \)

constitute an equivalence of categories

\[
\text{Cov}(X) \xrightarrow{\text{Rec}} \text{Set}^{\Pi_1(X)} \xleftarrow{\text{Fib}} \text{Set}^{\Pi_1(X)}.
\]

**Proof.** Given \( \rho \in \text{Set}^{\Pi_1(X)} \) a permutation representation, we need to exhibit a natural isomorphism of permutation representations.

\[
\eta_\rho : \rho \to \text{Fib}(\text{Rec}(\rho))
\]

First consider what the right hand side is like: By this def. of \( \text{Rec} \) and this def. of \( \text{Fib} \) we have for every \( x \in X \) an actual equality

\[
\text{Fib}(\text{Rec}(\rho))(x) = \rho(x).
\]

To similarly understand the value of \( \text{Fib}(\text{Rec}(\rho)) \) on morphisms \( [\gamma] \in \Pi_1(X) \), let \( \gamma : [0,1] \to X \) be a representing path in \( X \). We find, by the Lebesgue number lemma as in the proof of this lemma (CoveringSpacePathLifting), a finite number of paths \( \{\gamma_i\}_{i \in \{1,n\}} \) such that

1. regarded as morphisms \( [\gamma_i] \) in \( \Pi_1(X) \) they compose to \( [\gamma] \):

\[
[\gamma] = [\gamma_n] \circ \cdots \circ [\gamma_2] \circ [\gamma_1]
\]

2. each \( \gamma_i \) factors through an open subset \( U_i \subset X \) over which \( \text{Rec}(\rho) \) trivializes.

Hence by functoriality of \( \text{Fib}(\text{Rec}(\rho)) \) it is sufficient to understand its value on these paths \( \gamma_i \). But on these we have again by direct unwinding of the definitions that

\[
\text{Fib}(\text{Rec}(\rho))( [\gamma_i] ) = \rho([\gamma_i]).
\]

This means that if we take

\[
\eta_\rho (x): \rho(x) \xrightarrow{\sim} \text{Fib}(\text{Rec}(\rho))
\]

to be the above identification, then this is a natural transformation and hence in a particular a natural isomorphism, as required.

Conversely, given \( E \in \text{Cov}(X) \) a covering space, we need to exhibit a natural isomorphism of covering spaces of the form

\[
\epsilon_E : \text{Rec}(\text{Fib}(E)) \to E.
\]

Again by this def. of \( \text{Rec} \) and this def. of \( \text{Fib} \) the underlying set of \( \text{Rec}(\text{Fib}(E)) \) is actually equal to that of \( E \), hence it is sufficient to check that this identity function
on underlying sets is a **homeomorphism** of **topological spaces**.

By the assumption that \( X \) is **locally path-connected** and **semi-locally simply connected**, it is sufficient to check for \( U \subset X \) an open path-connected subset and \( x \in X \) a point with the property that \( \pi_1(U, x) \to \pi_1(X, x) \) lands is constant on the trivial element, that the open subsets of \( E \) of the form \( U \times \{ \hat{x} \} \subset p^{-1}(U) \) form a basis for the topology of \( \text{Rec}(\text{Fib}(E)) \). But this is the case by definition of \( \text{Rec} \).

This proves the equivalence.

(Notice that the assumption of local path-connectedness and semi-local simply-connectedness of \( X \) is used only to guarantee that the functor \( \text{Rec} \) exists in the first place.) ■

**Applications**

**Proposition 3.2. (fundamental group of the circle is the integers)**

The **fundamental group** \( \pi_1 \) of the **circle** \( S^1 \) is the additive group of **integers**:

\[
\pi_1(S^1) \cong \mathbb{Z}
\]

and the isomorphism is given by assigning **winding number**.

Here in the context of **topological homotopy theory** the **circle** \( S^1 \) is the **topological subspace** \( S^1 = \{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1 \} \subset \mathbb{R}^2 \) of the **Euclidean plane** with its **metric topology**, or any **topological space** of the same **homotopy type**. More generally, the circle in question is, as a **homotopy type**, the **homotopy pushout**

\[
S^1 \cong \ast \bigcup_{\ast} \ast,
\]

hence the **homotopy type** with the **universal property** that it makes a homotopy commuting diagram of the form

\[
\begin{array}{ccc}
\ast & \bigcup & \ast \\
\downarrow & \emptyset & \downarrow \\
\ast & \to & S^1
\end{array}
\]

**Proof.** The **universal covering space** \( \hat{S}^1 \) of \( S^1 \) is the **real line** (by this example):

\[
p := (\cos(2\pi(-)), \sin(2\pi(-))) : \mathbb{R} \to S^1.
\]

Since the **circle** is **locally path-connected** (this example) and **semi-locally simply connected** (this example) the **fundamental theorem of covering spaces** applies and gives that the **automorphism group** of \( \mathbb{R} \) over \( S^1 \) equals the automorphism group of its **monodromy permutation representation**:

\[
\text{Aut}_{\text{CoV}(S^1)}(\mathbb{R}) \cong \text{Aut}_{\pi_1(S^1)\text{Set}}(\text{Fib}_{S^1}).
\]
Moreover, as a corollary of the fundamental theorem of covering spaces we have that the monodromy representation of a universal covering space is given by the action of the fundamental group $\pi_1(S)$ on itself (this prop.).

But the automorphism group of any group regarded as an action on itself by left multiplication is canonically isomorphic to that group itself (by this example), hence we have

$$\text{Aut}_{\pi_1(S)\text{Set}}(\text{Fib}_{S^1}) \cong \text{Aut}_{\pi_1(S)^\text{Set}}(\pi_1(S^1)) \cong \pi_1(S^1).$$

Therefore to conclude the proof it is now sufficient to show that

$$\text{Aut}_{\text{Cov}(S^1)}(\mathbb{R}^1) \cong \mathbb{Z}.$$

To that end, consider a homeomorphism of the form

$$\mathbb{R}^1 \xrightarrow{f} \mathbb{R}^1,$$

$$p \downarrow \nu_p \quad S^1.$$

Let $s \in S^1$ be any point, and consider the restriction of $f$ to the fibers over the complement:

$$p^{-1}(S^1 \setminus \{s\}) \xrightarrow{f} p^{-1}(S^1 \setminus \{s\}),$$

$$p \downarrow \nu_p \quad S^1 \setminus \{s\}.$$

By the covering space property we have (via this example) a homeomorphism

$$p^{-1}(S^1 \setminus \{s\}) \cong (0,1) \times \text{Disc}(\mathbb{Z}).$$

Therefore, up to homeomorphism, the restricted function is of the form

$$(0,1) \times \text{Disc}(\mathbb{Z}) \xrightarrow{f} (0,1) \times \text{Disc}(\mathbb{Z}),$$

$$\text{pr}_1 \downarrow \nu_{\text{pr}_1} \quad (0,1).$$

By the universal property of the product topological space this means that $f$ is equivalently given by its two components

$$(0,1) \times \text{Disc}(\mathbb{Z}) \xrightarrow{\text{pr}_1 \circ f} (0,1) \quad (0,1) \times \text{Disc}(\mathbb{Z}) \xrightarrow{\text{pr}_2 \circ f} \text{Disc}(\mathbb{Z}).$$

By the commutativity of the above diagram, the first component is fixed to be $\text{pr}_1$. Moreover, by the fact that $\text{Disc}(\mathbb{Z})$ is a discrete space it follows that the second component is a locally constant function (by this example). Therefore, since the
**product space** with a **disjoint union space** (via this example)

$$(0, 1) \times \text{Disc}(\mathbb{Z}) \simeq \bigsqcup_{n \in \mathbb{Z}} (0, 1)$$

and since the disjoint summands $(0, 1)$ are **connected topological spaces** (this example), it follows that the second component is a **constant function** on each of these summands (by this example).

Finally, since every function out of a **discrete topological space** is continuous, it follows in conclusion that the second component is a **constant function** on each of these summands (by this example).

Now let $s' \in S^1$ be another point, distinct from $s$. The same analysis as above applies now to the restriction of $f$ to $S^1 \setminus \{s\}$ and yields a function

$$(t, k) \mapsto (t, \phi(k))$$

Since

$$\{p^{-1}(S^1 \setminus \{s\}) \subset \mathbb{R}^1, p^{-1}(S^1 \setminus \{s'\}) \subset \mathbb{R}^1\}$$

is an **open cover** of $\mathbb{R}^1$, it follows that $f$ is uniquely fixed by its restrictions to these two subsets.

Now unwinding the definition of $p$ shows that the condition that the two restrictions coincide on the intersection $S^1 \setminus \{s, s'\}$ implies that there is $n \in \mathbb{Z}$ such that $\phi(k) = k + n$ and $\phi'(k) = k + n$.

This shows that $\text{Aut}_{\text{Cov}(S^1)}(\mathbb{R}^1) \simeq \mathbb{Z}$. ■

This concludes the introduction to basic homotopy theory.

For introduction to more general and abstract homotopy theory see at **Introduction to Homotopy Theory**.

An incarnation of **homotopy theory** in **linear algebra** is **homological algebra**. For introduction to that see at **Introduction to Homological Algebra**.
4. References

A textbook account is in

- Tammo tom Dieck, sections 2 an 3 of *Algebraic Topology*, EMS 2006 (*pdf*)

Lecture notes include


*Revised on July 11, 2017 10:39:02 by Urs Schreiber*