nLab

# Introduction to Topology -- 2

This page is a detailed introduction to basic topological homotopy theory. We introduce the fundamental group of topological spaces and the concept of covering spaces. Then we prove the fundamental theorem of covering spaces, saying that they are equivalent to permutation representations of the fundamental group. This is a simple topological version of the general principle of *Galois theory* and has many applications. As one example application, we use it to prove that the fundamental group of the circle is the integers.

> *Under construction.*

main page: *Introduction to Topology*

previous chapter: *Introduction to Topology 1 -- Point-set topology*

this chapter: **Introduction to Topology 2 – Basic Homotopy Theory**

For introduction to more general and abstract *homotopy theory* see instead at *Introduction to Homotopy Theory*.

---

# Basic Homotopy Theory



**Context**

**Topology**

**Homotopy theory**

In order to handle topological spaces, to compute their properties and to distinguish them, it turns out to be useful to consider not just continuous variation within a topological space, i.e. continuous functions between topological spaces, but also continuous deformations of continuous functions themselves. This is the concept of *homotopy* (def. 1.2 below), and its study is called *homotopy theory*. If one regards topological spaces with homotopy classes of continuous functions between them then their nature changes, and one speaks of *homotopy types* (remark 1.6 below).

Of particular interest are homotopies between paths in a topological space. If a loop in a topological space is homotopic to the constant loop, this means that it does not "wind around a hole" in the space. Hence the set of homotopy classes of loops in a topological space, which is a group under concatenation of paths, detects crucial information about the global structure of the space, and hence is called the *fundamental group* of the space (def. 1.16).

This same information turns out to be encoded in "continuously varying sets" over a topological space, hence in "bundles of sets", called *covering spaces* (def. 2.1 below). As one moves around a loop, then the parameterized set comes back to itself up to a bijection called the *monodromy* of the loop. This encodes an *action* or *permutation representation* of the fundamental group. The *fundamental theorem of covering spaces* (prop. 2.22 below) says that covering spaces are equivalently characterized by their monodromy representation of the fundamental group. This is an incarnation of the general principle of *Galois theory* in topological homotopy theory. Sometimes this allows to compute fundamental groups from behaviour of covering spaces, for instance it allows to prove that the fundamental group of the circle is the integers (prop. 3.1 below).

In order to formulate and prove these statements, it turns out convenient to do away with the arbitrary choice of basepoint that is involved in the definition of fundamental groups, and instead collect *all* homotopy classes of paths into a single structure, called the *fundamental groupoid* of a topological space (example 1.27 below) an example of a generalization of groups to *groupoids* (discussed below). The fundamental groupoid may be regarded as an algebraic incarnation of the homotopy type presented by a topological space, *up to level 1* (the *homotopy 1-type*).

The algebraic reflection of the full homotopy type of a topological space involves higher dimensional analogs fo the fundamental group called the *higher homotopy groups*. We close with an outlook on these below.

# 1. Homotopy

It is clear that for $n \geq 1$ the Euclidean space $\mathbb{R}^n$ or equivalently the open ball $B_0^\circ(1)$ in $\mathbb{R}^n$ is *not* homeomorphic to the point space $* = \mathbb{R}^0$ (simply because there is not even a bijection between the underlying sets). Nevertheless, intuitively the $n$-ball is a "continuous deformation" of the point, obtained as the radius of the $n$-ball tends to zero.

This intuition is made precise by observing that there is a continuous function out of the product topological space (this example) of the open ball with the closed interval

$$\eta : [0,1] \times B^\circ_0(1) \longrightarrow B^\circ_0(1)$$

which is given by rescaling:

$$(t,x) \mapsto t \cdot x \,.$$

This continuously interpolates between the open ball and the point, in that for $t = 1$ it restricts to the identity, while for $t = 0$ it restricts to the map constant on the origin.

We may summarize this situation by saying that there is a diagram of continuous functions of the form

$$\begin{array}{ccc} B^\circ_0(1) \times \{0\} & \overset{\exists !}{\longrightarrow} & \ast \\ \downarrow && \downarrow^{\mathrlap{const_0}} \\ [0,1] \times B^\circ_0(1) & \overset{(t,x) \mapsto t \cdot x}{\longrightarrow} & B^\circ_0(1) \\ \uparrow & \nearrow_{\simeq} \\ B^\circ_0(1) \times \{1\} \end{array}$$

Such "continuous deformations" are called *homotopies*:

In the following we use this terminlogy:

**Definition 1.1. (topological interval)**

The *topological interval* is

1. the closed interval $[0,1] \subset \mathbb{R}^1$ regarded as a topological space in the standard way, as a subspace of the real line with its Euclidean metric topology,

2. equipped with the continuous functions

    1. $const_0 \;:\; \ast \to [0,1]$

    2. $const_1 \;:\; \ast \to [0,1]$

    which include the point space as the two endpoints, respectively

3. equipped with the (unique) continuous function

    $$[0,1] \longrightarrow \ast$$

    to the point space (which is the terminal object in Top)

regarded, in summary, as a factorization

$$\nabla_* \;:\; * \sqcup * \xrightarrow{(\mathrm{const}_0, \mathrm{const}_1)} [0,1] \longrightarrow *$$

of the codiagonal on the point space, namely the unique continuous function $\nabla_*$ out of the disjoint union space $* \sqcup * \simeq \mathrm{Disc}(\{0,1\})$ (homeomorphic to the discrete topological space on two elements).

**Definition 1.2**. (**homotopy**)

Let $X, Y \in$ Top be two topological spaces and let

$$f, g : X \longrightarrow Y$$

be two continuous functions between them.

A *(left) homotopy* from $f$ to $g$, to be denoted

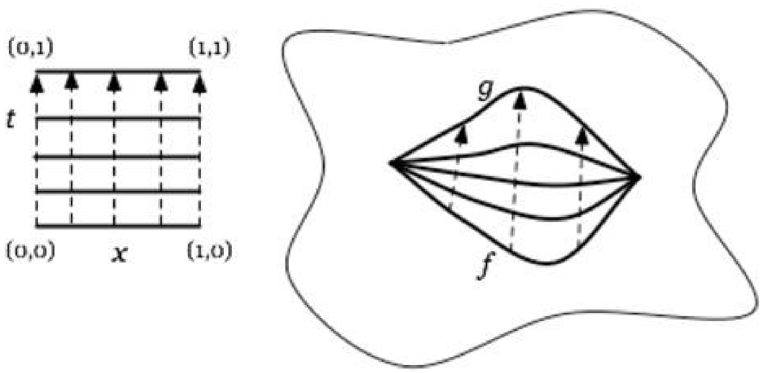
$$\eta \;:\; f \;\Rightarrow\; g\,,$$

is a continuous function

$$\eta \;:\; X \times [0,1] \longrightarrow Y$$

out of the product topological space (this example) of $X$ the topological interval (def. 1.1) such that this makes the following diagram in Top commute:

$$\begin{tikzcd}
0 \times X \arrow[d, "(\mathrm{id}, \mathrm{const}_0)"'] \arrow[dr, "f"] & \\
X \times [0,1] \arrow[r, "\eta"] & Y \\
\{1\} \times X \arrow[u, "(\mathrm{id}, \mathrm{const}_1)"] \arrow[ur, "g"'] &
\end{tikzcd}$$




*graphics grabbed from J. Tauber here*

hence such that

$$\eta(-,0) = f \qquad \text{and} \qquad \eta(-,1) = g\,.$$

If there is a homotopy $f \Rightarrow g$ (possibly unspecified) we say that $f$ is *homotopic* to $g$, denoted

$$f \sim_h g\,.$$

**Proposition 1.3**. *(**homotopy is an equivalence relation**)*

*Let $X, Y \in$ Top be two topological spaces. Write $\mathrm{Hom}_{\mathrm{Top}}(X,Y)$ for the set of continuous functions from $X$ to $Y$.*

*Then the relating of being homotopic (def. 1.2) is an equivalence relation on this set. The correspnding quotient set*

$$[X, Y] \coloneqq \mathrm{Hom}_{\mathrm{Top}}(X, Y)/\sim_h$$

*is called the set of homotopy classes of continuous functions.*

*Moreover, this equivalence relation is compatible with composition of continuous functions:*

*For $X, Y, Z \in$ Top three topological spaces, there is a unique function*

$$[X, Y] \times [Y, Z] \longrightarrow [X, Z]$$

*such that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Top}}(X, Y) \times \mathrm{Hom}_{\mathrm{Top}}(Y, Z) & \overset{\circ_{X,Y,Z}}{\longrightarrow} & \mathrm{Hom}_{\mathrm{Top}}(X, Z) \\ \downarrow & & \downarrow \\ [X, Y] \times [Y, Z] & \longrightarrow & [X, Z] \end{array} .$$

***Proof***. To see that the relation is reflexive: A homotopy $f \Rightarrow f$ from a function $f$ to itself is given by the function which is constant on the topological interval:

$$X \times [0, 1] \overset{\mathrm{pr}_1}{\longrightarrow} X .$$

This is continuous becaue projections out of product topological spaces are continuous, by the universal property of the Cartesian product.

To see that the relation is symmetric: If $\eta : f \Rightarrow g$ is a homotopy then

$$\begin{array}{ccccc} X \times [0, 1] & \overset{\mathrm{id}_X \times (1 - (-))}{\longrightarrow} & X \times [0, 1] & \overset{\eta}{\longrightarrow} & X \\ (x, t) & \mapsto & (x, 1 - t) & \mapsto & \eta(x, 1 - t) \end{array}$$

is a homotopy $g \Rightarrow f$. This is continuous because $1 - (-)$ is a polynomial function, and polynomials are continuous, and because Cartesian product and composition of continuous functions is again continuous.

Finally to see that the relation is transitive: If $\eta_1 : f \Rightarrow g$ and $\eta_2 : g \Rightarrow h$ are two composable homotopies, then consider the "$X$-parameterized path concatenation"

$$\begin{array}{ccc} X \times [0, 1] & \overset{\eta_2 \circ \eta_1}{\longrightarrow} & X \\ (x, t) & \mapsto & \left\{ \begin{array}{ccc} \eta_1(x, 2t) & | & t \leq 1/2 \\ \eta_2(x, 2t - 1) & | & t \leq 1/2 \end{array} \right. \end{array} .$$

To see that this is continuous, observe that $\{X \times [0, 1/2] \subset X, X \times [1/2, 1] \subset X\}$ is a cover of $X \times [0, 1]$ by closed subsets (in the product topology) and because $\eta_1(-, 2(-))$ and $\eta_2(-, 2(-) - 1)$ are continuous (being composites of Cartesian products of continuous functions) and agree on the intersection $X \times \{1/2\}$. Hence the continuity follows by this example.

Finally to see that homotopy respects composition: Let

$$X \overset{f_1}{\longrightarrow} Y \underoverset{\underset{f'_2}{\longrightarrow}}{\overset{f_2}{\longrightarrow}}{} Z \overset{f_3}{\longrightarrow} W$$

be continuous functions, and let

$$\eta : f_2 \Rightarrow f'_2$$

be a homotopy. It is sufficient to show that then there is a homotopy of the form

$$f_3 \circ f_2 \circ f_1 \Rightarrow f_3 \circ f'_2 \circ f_1 \,.$$

This is exhibited by the following diagram

$$\begin{array}{ccccccc} X & \overset{f_1}{\longrightarrow} & Y \\ {}^{(\mathrm{id}_X, \mathrm{const}_0)}\downarrow && {}^{(\mathrm{id}_Y, \mathrm{const}_0)}\downarrow & \searrow^{f_2} \\ X \times [0,1] & \overset{f_1 \times \mathrm{id}_{[0,1]}}{\longrightarrow} & Y \times [0,1] & \overset{\eta}{\longrightarrow} & Z & \overset{f_3}{\longrightarrow} & W \,. \\ {}^{(\mathrm{id}_X, \mathrm{const}_1)}\uparrow && {}^{(\mathrm{id}, \mathrm{const}_1)}\uparrow & \nearrow_{f'_2} \\ X & \underset{f_1}{\longrightarrow} & Y \end{array}$$

■

**Remark 1.4**. (**homotopy category**)

Prop. 1.3 means that homotopy classes of continuous functions are the morphisms in a category whose objects are still the topological spaces.

This category (at least when restricted to spaces that admit the structure of CW-complexes) is called the *classical homotopy category*, often denoted

$$\mathrm{Ho}(\mathrm{Top}) \,.$$

Hence for $X, Y$ topological spaces, then

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Top})}(X,Y) = [X,Y]$$

Moreover, sending a continuous function to its homotopy class is a functor

$$\kappa : \mathrm{Top} \longrightarrow \mathrm{Ho}(\mathrm{Top})$$

from the ordinary category Top of topological spaces with actual continuous functions between them.

**Definition 1.5**. (**homotopy equivalence**)

Let $X, Y \in$ Top be two topological spaces.

A continuous function

$$f : X \longrightarrow Y$$

is called a *homotopy equivalence* if there exists

1. a continuous function the other way around,

$$g : Y \longrightarrow X$$

2. homotopies (def. 1.2) from the two composites to the respective identity function:

$$f \circ g \Rightarrow \mathrm{id}_Y$$

and

$$g \circ f \Rightarrow \mathrm{id}_X \,.$$

We indicate that a continuous function is a homotopy equivalence by writing

$$X \overset{\simeq_h}{\longrightarrow} Y \,.$$

If there exists *some* (possibly unspecified) homotopy equivalence between topological spaces $X$ and $Y$ we write

$$X \simeq_h Y \,.$$

**Remark 1.6**. **(homotopy equivalences are the isomorphisms in the homotopy category)**

In view of remark 1.4 a continuous function $f$ is a homotopy equivalence precisely if its image $\kappa(f)$ in the homotopy category is an isomorphism.

As an object of the homotopy category, a topoogical space is often referred to as a (strong) homotopy type. Homotopy types have a different nature than the topological spaces which *present* them, in that topological spaces that are far from being homeomorphic may still be equivalent as homotopy types.

**Example 1.7**. **(homeomorphism is homotopy equivalence)**

Every homeomorphism is a homotopy equivalence (def. 1.5).

**Proposition 1.8**. ***(homotopy equivalence is equivalence relation)***

*Being homotopy equivalent is an equivalence relation on the class of topological spaces.*

***Proof***. This is immediate from remark 1.6 by general properties of categories and functors.

But for the record we spell it out. This involves the construction already used in the proof of prop. 1.3:

It is clear that the relation it reflexive and symmetric. To see that it is transitive consider continuous functions

$$X \underoverset{\longleftarrow}{\overset{f_1}{\longrightarrow}}{g_1} Y \underoverset{\longleftarrow}{\overset{f_2}{\longrightarrow}}{g_2} Z$$

and homotopies

$$\begin{aligned} g_1 \circ f_1 \Rightarrow \mathrm{id}_X & \qquad f_1 \circ g_1 \Rightarrow \mathrm{id}_Y \\ g_2 \circ f_2 \Rightarrow \mathrm{id}_Y & \qquad f_2 \circ g_2 \Rightarrow \mathrm{id}_Z \;. \end{aligned}$$

We need to produce homotopies of the form

$$(g_1 \circ g_2) \circ (f_2 \circ f_1) \Rightarrow \mathrm{id}_X$$

and

$$(f_2 \circ f_1) \circ (g_1 \circ g_2) \Rightarrow \mathrm{id}_Y \;.$$

Now the diagram

$$\begin{array}{ccccccc} X & \overset{f_1}{\longrightarrow} & Y \\ {}^{\mathllap{(\mathrm{id}_X, \mathrm{const}_0)}}\downarrow && {}^{\mathllap{(\mathrm{id}_Y, \mathrm{const}_0)}}\downarrow & \searrow^{\mathrlap{g_2 \circ f_2}} \\ X \times [0,1] & \overset{f_1 \times \mathrm{id}_{[0,1]}}{\longrightarrow} & Y \times [0,1] & \overset{\eta}{\longrightarrow} & Y & \overset{g_1}{\longrightarrow} & X\,, \\ {}^{\mathllap{(\mathrm{id}_X, \mathrm{const}_1)}}\uparrow && {}^{\mathllap{(\mathrm{id}, \mathrm{const}_1)}}\uparrow & \nearrow_{\mathrlap{\mathrm{id}_Y}} \\ X & \underset{f_1}{\longrightarrow} & Y \end{array}$$

with $\eta$ one of the given homotopies, exhibits a homotopy $(g_1 \circ g_2) \circ (f_2 \circ f_1) \Rightarrow g_1 \circ f_1$. Composing this with the given homotopy $g_1 \circ f_1 \Rightarrow \mathrm{id}_X$ gives the first of the two homotopies required above. The second one follows by the same construction, just with the lables of the functions exchanged. ▮

**Definition 1.9**. (**contractible topological space**)

A topological space $X$ is called *contractible* if the unique continuous function to the point space

$$X \overset{\simeq_h}{\longrightarrow} *$$

is a homotopy equivalence (def. 1.5).

**Remark 1.10**. (**contractible topological spaces are the terminal objects in the homotopy category**)

In view of remark 1.4, a topological space $X$ is contractible (def. 1.9) precisely if its image $\kappa(X)$ in the classical homotopy category is a terminal object.

**Example 1.11**. (**closed ball and Euclidean space are contractible**)

Let $B^n \subset \mathbb{R}^n$ be the unit open ball or closed ball in Euclidean space. This is contractible (def. 1.9):

$$p : B^n \overset{\simeq_h}{\longrightarrow} * \,.$$

The homotopy inverse function is necessarily constant on a point, we may just as well choose it to go pick the origin:

$$\mathrm{const}_0 : * \longrightarrow B^n \,.$$

For one way of composing these functions we have the equality

$$p \circ \mathrm{const}_0 = \mathrm{id}_*$$

with the identity function. This is a homotopy by prop. 1.3.

The other composite is

$$\mathrm{const}_0 \circ p = \mathrm{const}_0 : B^n \longrightarrow B^n \,.$$

Hence we need to produce a homotopy

$$\mathrm{const}_0 \Rightarrow \mathrm{id}_{B^n}$$

This is given by the function

$$\begin{array}{ccc} B^n \times [0,1] & \overset{\eta}{\longrightarrow} & B^n \\ (x,t) & \mapsto & t x \end{array} \,,$$

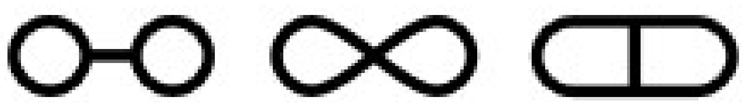
where on the right we use the multiplication with respect to the standard real vector space structure in $\mathbb{R}^n$.

Since the open ball is homeomorphic to the whole Cartesian space $\mathbb{R}^n$ (this example) it follows with example 1.7 and example 1.3 that also $\mathbb{R}^n$ is a contractible topological space:

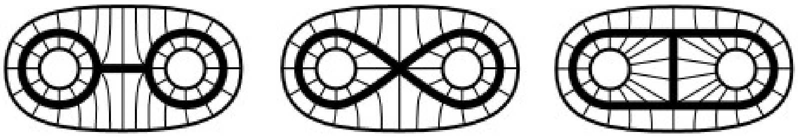
$$\mathbb{R}^n \overset{\simeq_h}{\longrightarrow} * \,.$$

In direct generalization of the construction in example 1.11 one finds further examples as follows:

**Example 1.12**. The following three graphs



(i.e. the evident topological subspaces of the plane $\mathbb{R}^2$ that these pictures indicate) are not homeomorphic. But they are homotopy equivalent, in fact they are each homotopy equivalent to the disk with two points removed, by the homotopies indicated by the following pictures:



*graphics grabbed from Hatcher*

# Fundamental group

**Definition 1.13. (homotopy relative boundary)**

Let $X$ be a topological space and let

$$\gamma_1, \gamma_2 \,:\, [0,1] \longrightarrow X$$

be two paths in $X$, i.e. two continuous functions from the closed interval to $X$, such that their endpoints agree:

$$\gamma_1(0) = \gamma_2(0) \qquad \gamma_1(1) = \gamma_2(1)\,.$$

Then a homotopy relative boundary from $\gamma_1$ to $\gamma_2$ is a homotopy (def. 1.2)

$$\eta \,:\, \gamma_1 \Rightarrow \gamma_2$$

such that it does not move the endpoints:

$$\eta(0,-) = \mathrm{const}_{\gamma_1(0)} = \mathrm{const}_{\gamma_2(0)} \qquad \eta(1,-) = \mathrm{const}_{\gamma_1(0)} = \mathrm{const}_{\gamma_2(1)}\,.$$

**Proposition 1.14.** ***(homotopy relative boundary is equivalence relation on sets of paths)***

*Let $X$ be a topological space and let $x, y \in X$ be two points. Write*

$$P_{x,y} X$$

*for the set of paths $\gamma$ in $X$ with $\gamma(0) = x$ and $\gamma(1) = y$.*

*Then homotopy relative boundary (def. 1.13) is an equivalence relation on $P_{x,y}X$.*

*The corresponding set of equivalence classes is denoted*

$$\mathrm{Hom}_{\Pi_1(X)}(x,y) \;\coloneqq\; (P_{x,y}X)/\sim\,.$$

Recall the operations on paths: path concatenation $\gamma_2 \cdot \gamma_1$, path reversion $\overline{\gamma}$ and constant paths

**Proposition 1.15**. ***(concatenation of homotopy relative boundary-classes of paths)***

*For $X$ a topological space, then the operation of path concatenation descends to homotopy relative boundary equivalence classes, so that for all $x, y, z \in X$ there is a function*

$$\begin{array}{ccc} \mathrm{Hom}_{\Pi_1(X)}(x,y) \times \mathrm{Hom}_{\Pi_1(X)}(y,z) & \longrightarrow & \mathrm{Hom}_{\Pi_1(X)}(x,z) \\ ([\gamma_1],[\gamma_2]) & \mapsto & [\gamma_2]\cdot[\gamma_1] \coloneqq [\gamma_2 \cdot \gamma_1] \end{array} .$$

*Moreover,*

1. *this composition operation is associative in that for all $x, y, z, w \in X$ and $[\gamma_1] \in \mathrm{Hom}_{\Pi_1(X)}(x,y)$, $[\gamma_2] \in \mathrm{Hom}_{\Pi_1(X)}(y,z)$ and $[\gamma_3] \in \mathrm{Hom}_{\Pi_1(X)})(z,w)$ then*

   $$[\gamma_3]\cdot([\gamma_2]\cdot[\gamma_1]) \;=\; ([\gamma_3]\cdot[\gamma_2])\cdot[\gamma_1]$$

2. *this composition operation is unital with neutral elements the constant paths in that for all $x, y \in X$ and $[\gamma] \in \mathrm{Hom}_{\Pi_1(X)}(x,y)$ we have*

   $$[\mathrm{const}_y]\cdot[\gamma] = [\gamma] = [\gamma]\cdot[\mathrm{const}_x] \;.$$

3. *this composition operation has inverse elements given by path reversal in that for all $x, y \in X$ and $[\gamma] \in \mathrm{Hom}_{\Pi_1(X)}(x,y)$ we have*

   $$[\overline{\gamma}]\cdot[\gamma] = [\mathrm{const}_x] \qquad [\gamma]\cdot[\overline{\gamma}] = [\mathrm{const}_y] \;.$$

**Definition 1.16**. **(fundamental groupoid and fundamental groups)**

Let $X$ be a topological space. Then set of points of $X$ together with the sets $\mathrm{Hom}_{\Pi_1(X)}(x,y)$ of homotopy relative boundary-classes of paths (def. 1.13) for all points of points and equipped with the concatenation operation from prop. 1.15 is called the *fundamental groupoid* of $X$, denoted

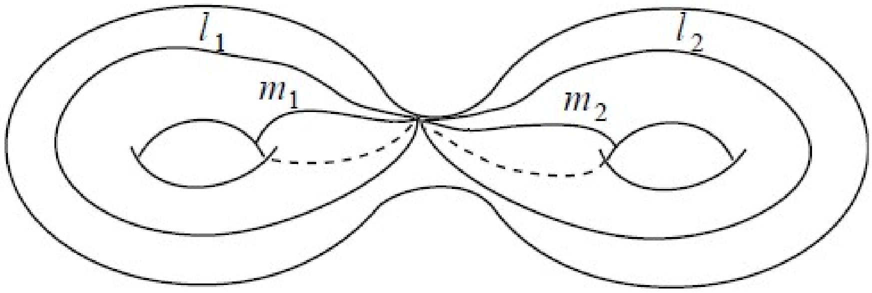
$$\Pi_1(X) \;.$$

Given a choice of point $x \in X$, then one writes

$$\pi_1(X,x) \;\coloneqq\; \mathrm{Hom}_{\Pi_1(X)}(x,x) \;.$$

Prop. 1.15 says that under concatenation of paths, this set is a group. As such it is called the *fundamental group* of $X$ at $x$.

The following picture indicates the four non-equivalent non-trivial generators of the fundamental group of the oriented surface of genus 2:




*graphics grabbed from Lawson 03*

**Example 1.17**. **(fundamental group of Euclidean space)**

For $n \in \mathbb{N}$ and $x \in \mathbb{R}^n$ any point in the $n$-dimensional Euclidean space (regarded with its metric topology) we have that the fundamental group (def. 1.16) at that point is trivial:

$$\pi_1(\mathbb{R}^n, x) = * \;.$$

**Remark 1.18**. **(basepoints)**

Definition 1.16 intentionally offers two variants of the defintion.

The first, the *fundamental groupoid* is canonically given, without choosing a basepoint. As a result, it is a structure that is not quite a group but, slightly more generally, a "groupoid" (a "group with many objects"). We discuss the concept of groupoids below.

The second, the fundamental group, is a genuine group, but its definition requires picking a base point $x \in X$.

In this context it is useful to say that

1. a *pointed topological space* $(X, x)$ is
   1. a topological space $X$;
   2. a $x \in X$ in the underlying set.
2. a homomorphism of pointed topological spaces $f : (X, x) \longrightarrow (Y, y)$ is a base-point preserving continuous function, namely
   1. a continuous function $f : X \longrightarrow Y$
   2. such that $f(x) = y$.

Hence there is a category, to be denoted, $\mathrm{Top}^{*/}$, whose objects are the pointed topological spaces, and whose morphisms are tbe base-point preserving continuous functions.

Similarly, a homotopy between morphisms $f, f' : (X, x) \to (Y, y)$ in $\mathrm{Top}^{*/}$ is a homotopy $\eta : f \Rightarrow f'$ of underlying continuous functions, as in def. 1.2, such that the corresponding function

$$\eta \,:\, X \times [0,1] \longrightarrow Y$$

preserves the basepoints in that

$$\underset{t \in [0,1]}{\forall} \eta(x,t) = y \;.$$

These pointed homotopies still form an equivalence relation as in prop. 1.3 and hence quotienting these out yields the pointed analogue of the homotopy category from def. 1.4, now denoted

$$\kappa \,:\, Top^{*/} \longrightarrow Ho(Top^{*/}) \;.$$

In general it is hard to explicitly compute the fundamental group of a topological space. But often it is already useful to know if two spaces have the same fundamental group or not:

**Definition 1.19. (pushforward of elements of fundamental groups)**

Let $(X,x)$ and $(Y,y)$ be pointed topological space (remark 1.18) and let

$$f \,:\, X \longrightarrow Y$$

be a continuous function which respects the chosen points, in that $f(x) = y$.

Then there is an induced homomorphism of fundamental groups (def. 1.16)

$$\begin{array}{ccc} \pi_1(X,x) & \overset{f_*}{\longrightarrow} & \pi_1(Y,y) \\ [\gamma] & \mapsto & [f \circ \gamma] \end{array}$$

given by sending a closed path $\gamma : [0,1] \to X$ to the composite

$$f \circ \gamma \,:\, [0,1] \overset{\gamma}{\longrightarrow} X \overset{f}{\longrightarrow} Y \;.$$

**Remark 1.20. (fundamental group is functor on pointed topological spaces)**

The pushforward operation in def. 1.19 is functorial, now on the category $Top^{*/}$ of pointed topological spaces (remark 1.18)

$$\pi_1 \,:\, Top^{*/} \longrightarrow Grp \;.$$

***Proposition 1.21. (fundamental group depends only on homotopy classes)***

*Let $X, Y \in Top^{*/}$ be pointed topological space and let $f_1, f_2 \,:\, X \longrightarrow Y$ be two base-point preserving continuous functions. If there is a pointed homotopy (def. 1.2, remark 1.18)*

$$\eta : f_1 \Rightarrow f_2$$

*then the induced homomorphisms on fundamental groups (def. 1.19) agree*

$$(f_1)_* = (f_2)_* : \pi_1(X,x) \to \pi_1(Y,y) \,.$$

*In particular if $f :; X \longrightarrow Y$ is a homotopy equivalence (def. 1.5) then $f_* : \pi_1(X,x) \to \pi_1(Y,y)$ is an isomorphism.*

***Proof***. This follows by the fact that homotopy respects composition (prop. 1.3):

If $\gamma : [0,1] \longrightarrow X$ is a closed path representing a given element of $\pi_1(X,x)$, then the homotopy $f_1 \Rightarrow f_2$ induces a homotopy

$$f_1 \circ \gamma \Rightarrow f_2 \circ \gamma$$

and therefore these represent the same elements in $\pi_1(Y,y)$.

If follows that if $f$ is a homotopy equivalence with homotopy inverse $g$, then $g_* : \pi_1(Y,y) \to \pi_1(X,x)$ is an inverse morphism to $f_* : \pi_1(X,x) \to \pi_1(Y,y)$ and hence $f_*$ is an isomorphism. ▮

**Remark 1.22**. Prop. 1.21 says that the fundamental group functor from def. 1.19 and remark 1.20 factors through the classical pointed homotopy category from remark 1.18:

$$\begin{array}{ccc} \mathrm{Top}^{*/} & \overset{\pi_1}{\longrightarrow} & \mathrm{Grp} \\ {}^{\mathllap{\kappa}}\downarrow & \nearrow & \\ \mathrm{Ho}(\mathrm{Top}^{*/}) & & \end{array} \,.$$

**Definition 1.23**. **(simply connected topological space)**

A topological space $X$ for which

1. $\pi_0(X) \simeq *$ (path connected)

2. $\pi_1(X,x) \simeq 1$ (the fundamental group is trivial, def. 1.16),

is called *simply connected*.

We will need also the following local version:

**Definition 1.24**. **(semi-locally simply connected topological space)**

A topological space $X$ is called *semi-locally simply connected* if every point $x \in X$ has a neighbourhood $U_x \subset X$ such that every loop in $X$ is contractible as a loop in $X$, hence such that the induced morphism of fundamental groups (def. 1.19)

$$\pi_1(U,x) \to \pi_1(X,x)$$

is trivial (i.e. sends everything to the neutral element).

If every $x$ has a neighbourhood $U_x$ which is itself simyply connected, then $X$ is called a *locally simply connected topological space*. This implies semi-local simply-connectedness.

**Example 1.25**. **(Euclidean space is simply connected)**

For $n \in \mathbb{N}$, then the Euclidean space $\mathbb{R}^n$ is a simply connected topological space (def. 1.23).

# Groupoids

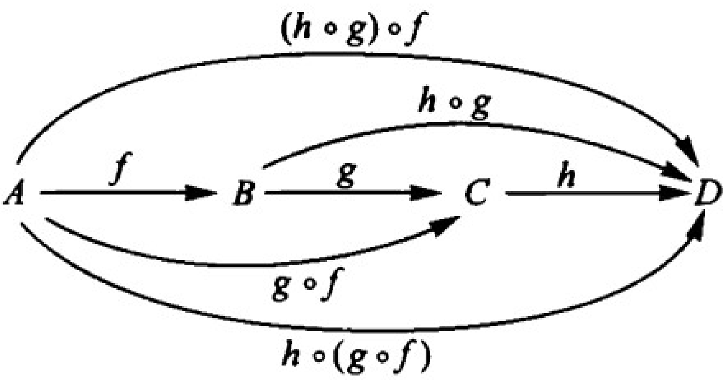
In def. 1.16 we extracted the fundamental group at some point $x \in X$ from a larger algebraic structure, that incorporates all the basepoints, to be called the *fundamental groupoid*. This larger algebraic structure of *groupoids* is usefully made explicit for the formulation and proof of the fundamental theorem of covering spaces (theorem 2.22 below) and the development of homotopy theory in general.

Where a *group* may be thought of as a *group of symmetry transformations* that isomorphically relates one object to itself (the *symmetries* of one object, such as the isometries of a polyhedron) a *groupoid* is a collection of symmetry transformations acting between possibly more than one object.

Hence a groupoid consists of a set of objects $x, y, z, \cdots$ and for each pair of objects $(x, y)$ there is a set of transformations, usually denoted by arrows

$$x \overset{f}{\longrightarrow} y$$

which may be composed if they are composable (i.e. if the first ends where the second starts)




$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & z \end{tikzcd}$$

such that this composition is associative and such that for each object $x$ there is identity transformation $x \overset{\mathrm{id}_X}{\longrightarrow} x$ in that this is a neutral element for the composition of transformations, whenever defined.

So far this structure is what is called a *small category*. What makes this a *(small) groupoid* is that all these transformations are to be "symmetries" in that they are invertible morphisms meaning that for each transformation $x \overset{f}{\longrightarrow} y$ there is a transformation the other way around $y \overset{f^{-1}}{\longrightarrow} x$ such that

$$f^{-1} \circ f = \mathrm{id}_x \qquad f \circ f^{-1} = \mathrm{id}_y \,.$$

If there is only a single object $x$, then this definition reduces to that of a group, and in this sense groupoids are "groups with many objects". Conversely, given any groupoid $\mathcal{G}$ and a choice of one of its objects $x$, then the subcollection of transformations from and to $x$ is a group, sometimes called the automorphism group $\mathrm{Aut}_{\mathcal{G}}(x)$ of $x$ in $\mathcal{G}$.

Just as for groups, the "transformations" above need not necessarily be given by concrete transformations (say by bijections between objects which are sets). Just as for groups, such a concrete realization is always possible, but is an extra choice (called a representation of the groupoid). Generally one calls these "transformations" *morphisms*: $x \overset{f}{\longrightarrow} y$ is a morphism with "source" $x$ and "domain" $y$.

An archetypical example of a groupoid is the fundamental groupoid $\Pi_1(X)$ of a topological space (def. 1.27 below, for introduction see here): For $X$ a topological space, this is the groupoid whose

* objects are the points $x \in X$;

* morphisms $x \overset{[\gamma]}{\longrightarrow} y$ are the homotopy relative boundary-equivalence classes $[\gamma]$ of paths $\gamma : [0,1] \to X$ in $X$, with $\gamma(0) = x$ and $\gamma(1) = y$;

and composition is given, on representatives, by concatenation of paths. Here the class of the reverse path $\bar{\gamma} \,:\, t \mapsto \gamma(1-t)$ constitutes the inverse morphism, making this a groupoid.

If one *chooses* a point $x \in X$, then the corresponding group at that point is the *fundamental group* $\pi_1(X,x) \coloneqq \mathrm{Aut}_{\Pi_1(X)}(x)$ of $X$ at that point.

This highlights one of the reasons for being interested in groupoids over groups: Sometimes this allows to avoid unnatural ad-hoc choices and it serves to streamline and simplify the theory.

A homomorphism between groupoids is the obvious: a function between their underlying objects together with a function between their morphisms which respects source and target objects as well as composition and identity morphisms. If one thinks of the groupoid as a special case of a category, then this is a *functor*. Between groupoids with only a single object this is the same as a group homomorphism.

For example if $f \,:\, X \to Y$ is a continuous function between topological spaces, then postcomposition of paths with this function induces a groupoid homomorphism $f_\ast \,:\, \Pi_1(X) \longrightarrow \Pi_1(Y)$ between the fundamental groupoids from above.

Groupoids with groupoid homomorphisms (functors) between them form a category Grp (def. 1.33 below) which includes the categeory Grp of groups as the full subcategory of the groupoids with a single object. This makes precise how groupoid theory is a genralization of group theory.

However, for groupoids more than for groups one is typically interested in "conjugation actions" on homomorphisms. These are richer for groupoids than for groups, because one may conjugate with a different morphism at each object. If we think of groupoids as special cases of categories, then these "conjugation actions on homomorphisms" are *natural transformations* between functors.

For examples if $f, g : X \longrightarrow Y$ are two continuous functions between topological spaces, and if $\eta : f \Rightarrow g$ is a homotopy from $f$ to $g$, then the homotopy relative boundary classes of the paths $\eta(x, -) : [0,1] \to Y$ constitute a natural transformation between $f_\ast, g_\ast : \Pi_1(X) \to \Pi_y(Y)$ in that for all paths $x_1 \overset{[\gamma]}{\longrightarrow} x_2$ in $X$ we have the "conjugation relation"

$$[\eta(x_1, -)] \cdot [f \circ \gamma] = [g \circ \gamma] \cdot [\eta(x_2, -)] \qquad \text{i.e.} \qquad \begin{array}{ccc} f(x_1) & \overset{[\eta(x_1,-)]}{\longrightarrow} & g(x_1) \\ {}^{[f \circ \gamma]}\downarrow && \downarrow^{[f \circ \gamma]} \\ f(x_1) & \underset{[\eta(x_2,-)]}{\longrightarrow} & g(x_2) \end{array}$$

**Definition 1.26**. **(groupoid)**

A *small groupoid* $\mathcal{G}$ is

1. a set $X$, to be called the *set of objects*;

2. for all pairs of objects $(x, y) \in X \times X$ a set $\mathrm{Hom}(x, y)$, to be called the *set of morphisms with domain or source $x$ and codomain or target $y$*;

3. for all triples of objects $(x, y, z) \in X \times X \times X$ a function

$$\circ_{x,y,z} : \mathrm{Hom}(y, z) \times \mathrm{Hom}(x, y) \longrightarrow \mathrm{Hom}(x, z)$$

   to be called *composition*

4. for all objects $x \in X$ an element

$$\mathrm{id}_x \in \mathrm{Hom}(x, x)$$

   to be called the *identity morphism* on $x$;

5. for all pairs $x, y \in \mathrm{Hom}(x, y)$ of obects a function

$$(-)^{-1} : \mathrm{Hom}(x, y) \longrightarrow \mathrm{Hom}(y, x)$$

   to be called the *inverse-assigning function*

such that

1. (associativity) for all quadruples of objects $x_1, x_2, x_3, x_4 \in X$ and all triples of

morphisms $f \in \mathrm{Hom}(x_1, x_2)$, $g \in \mathrm{Hom}(x_2, x_3)$ and $h \in \mathrm{Hom}(x_3, x_4)$ an equality

$$h \circ (g \circ f) \;=\; (h \circ g) \circ f$$

2. (unitality) for all pairs of objects $x, y \in X$ and all moprhisms $f \in \mathrm{Hom}(x, y)$ equalities

$$\mathrm{id}_y \circ f = f \qquad f \circ \mathrm{id}_x = f$$

3. (invertibility) for all pairs of objects $x, y \in X$ and every morphism $f \in \mathrm{Hom}(x, y)$ equalities

$$f^{-1} \circ f = \mathrm{id}_x \qquad f \circ f^{-1} = \mathrm{id}_y \,.$$

If $\mathcal{G}_1, \mathcal{G}_2$ are two groupoids, then a *homomorphism* or *functor* between them, denoted

$$F : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$$

is

1. a function $F_0 : X_1 \longrightarrow X_2$ between the respective sets of objects;

2. for each pair $x, y \in X_1$ of objects a function

$$F_{x,y} : \mathrm{Hom}_{\mathcal{G}_1}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{G}_2}(F_0(x), F_0(y))$$

   between sets of morphisms

such that

1. (respect for composition) for all triples $x, y, z \in X_1$ and all $f \in \mathrm{Hom}(x, y)$ and $g \in \mathrm{Hom}(y, z)$ an equality

$$F_{y,z}(g) \circ_2 F_{x,y}(f) \;=\; F_{x,z}(g \circ_1 f)$$

2. (respect for identities) for all $x \in X$ an equality

$$F_{x,x}(\mathrm{id}_x) = \mathrm{id}_{F_0(x)} \,.$$

For $\mathcal{G}_1, \mathcal{G}_2$ two groupoids, and for $F, G : \mathcal{G}_1 \to \mathcal{G}_2$ two groupoid homomorphisms/functors, then a *conjugation* or *homotopy* or *natural transformation* (necessarily a natural isomorphism)

$$\eta : F \Rightarrow G$$

is

- for each object $x \in X_1$ of $\mathcal{G}_1$ a morphism $\eta_x \in \mathrm{Hom}_{\mathcal{G}_2}(F(x), G(y))$

such that

- for all $x, y \in X_1$ and $f \in \mathrm{Hom}_{\mathcal{G}_1}(x,y)$ an equality

$$\eta_y \circ_2 F(f) = G(f) \circ \eta_x \qquad \begin{array}{ccc} F(x) & \overset{\eta_x}{\longrightarrow} & G(x) \\ {}^{\mathllap{F(f)}}\downarrow && \downarrow^{\mathrlap{G(f)}} \\ F(y) & \underset{\eta_y}{\longrightarrow} & G(y) \end{array}$$

For $\mathcal{G}_1, \mathcal{G}_2$ two groupoids and $F, G, H : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ three functors between them and $\eta_1 \;:\; F \mathrm{Rihtarrow} G$ and $\eta_2 \;:\; G \Rightarrow H$ conjugation actions/natural isomorphisms between these, there is the composite

$$\eta_2 {:} \eta_1 \;:\; F \Rightarrow H$$

with components the composite of the components

$$(\eta_2 \circ \eta_1)(x) \coloneqq \eta_2(x) \circ \eta_1(x) \;.$$

This yields for any two groupoid a *hom-groupoid*

$$\mathrm{Hom}_{\mathrm{Grpd}}(\mathcal{G}_1, \mathcal{G}_2)$$

whose objects are the groupoid homomorphisms / functors, and whose morphisms are the conjugation actions / natural transformations.

The archetypical example of a groupoid we already encountered above:

**Example 1.27**. **(fundamental groupoid)**

For $X$ a topological space, then its *fundamental groupoid* (as in def. 1.16) has as set of objects the underlying set of $X$, and for $x, y \in X$ two points, the set of homomorphisms is the set of paths from $x$ to \$y4 modulo homotopy relative boundary:

$$\mathrm{Hom}_{\Pi_1(X)}(x,y)(P_{x,y}) / \sim_h$$

and composition is given by concatenation of paths.

**Remark 1.28**. **(groupoids are special cases of categories)**

A small groupoid (def. 1.26) is equivalently a small category in which all morphisms are isomorphisms.

While therefore groupoid theory may be regarded as a special case of category theory, it is noteworthy that the two theories are quite different in character. For example higher groupoid theory is *homotopy theory* which is rich but quite tractable, for instance via tools such as simplicial homotopy theory or homotopy type theory, while higher category theory is intricate and becomes tractable mostly by making recourse to higher groupoid theory in the guise of (infinity,1)-category theory and (infinity,n)-categories.

**Example 1.29**. **(groupoid core of a category)**

For $\mathcal{C}$ any (small) category, then there is a maximal groupoid inside

$$\mathrm{Core}(\mathcal{C}) \hookrightarrow \mathcal{C}$$

sometimes called the *core* of $\mathcal{C}$. This is obtained from $\mathcal{C}$ simply by discarding all those morphisms that are not isomorphisms.

For instance

- For $\mathcal{C} =$ Set then $\mathrm{Core}(\mathrm{Set})$ is the goupoid of sets and bijections between them.

  For $\mathcal{C}$ FinSet then the skeleton of this groupoid (prop. 1.47) is the disjoint union of deloopings (example 1.41) of all the symmetric groups:

  $$\mathrm{Core}(\mathrm{FinSet}) \simeq \underset{n \in \mathbb{N}}{\sqcup} \Sigma(n)$$

- For $\mathcal{C} =$ Vect then $\mathrm{Core}(\mathrm{Vect})$ is the groupoid of vector spaces and linear bijections between them.

  For $\mathcal{C} =$ FinVect then the skeleton of this groupoid is the disjoint union of delooping of all the general linear groups

  $$\mathrm{Core}(\mathrm{FinVect}) \simeq \underset{n \in \mathbb{N}}{\sqcup} \mathrm{GL}(n) \,.$$

**Example 1.30**. (**discrete groupoid**)

For $X$ any set, there is the *discrete groupoid* $\mathrm{Disc}(X)$, whose set of objects is $X$ and whose only morphisms are identity morphisms.

This is also the fundamental groupoid (example 1.27) of the discrete topological space on the set $X$.

**Example 1.31**. (**disjoint union**/**coproduct** of **groupoids**)

Let $\{\mathcal{G}_i\}_{i \in I}$ be a set of groupoids. Then their disjoint union (coproduct) is the groupoid

$$\underset{i \in I}{\sqcup} \mathcal{G}_i$$

whose set of objects is the disjoint union of the sets of objects of the summand groupoids, and whose sets of morphisms between two objects is that of $\mathcal{G}_i$ if both objects are form this groupoid, and is empty otherwise.

**Definitio 1.32**. (**product** of **groupoids**)

Let $\{\mathcal{G}_i\}_{i \in I}$ be a set of groupoids. Their product groupoid is the [groupoid]]

$$\prod_{i \in I} \mathcal{G}_i$$

whose set of objects is the Cartesian product of the sets of objects of the factor groupoids

$$\left(\prod_{i \in I} \mathcal{G}_i\right)_0 \coloneqq \prod_{i \in I} (\mathcal{G}_i)_0$$

and whose set of morphisms between tuples $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ is the corresponding Cartesian product of morphisms, with elements denoted

$$(x_i)_{i \in I} \overset{(f_i)_{i \in I}}{\longrightarrow} (y_i)_{i \in I} \,.$$

For instance if each of the groupoids is the delooping $\mathcal{G}_i = B G_i$ of a group $G_i$ (example 1.40) then the product groupoid is the delooping groupoid of the direct product group:

$$\prod_{i \in I} B G_i \simeq B \prod_{i \in I} G_i \,.$$

As another example, if $\underset{i \in I}{\sqcup} \mathcal{G}_i$ is the coproduct groupoid from example 1.31, and if $\mathcal{G}$ is any groupoid, then a groupoid homomorphism of the form

$$\underset{i \in I}{\sqcup} \mathcal{G}_i \longrightarrow \mathcal{G}$$

is equivalently a tuple $(f_i)_{i \in I}$ of groupoid homomorphisms

$$\mathcal{G}_1 \overset{f_i}{\longrightarrow} \mathcal{G} \,.$$

The analogous statement holds for homotopies between groupoid homomorphisms, and so one find that the hom-groupoid out of a coproduct of groupoids is the product groupoid of the separate hom-groupoids:

$$\mathrm{Hom}_{\mathrm{Grpd}}\left( \underset{i \in I}{\sqcup} \mathcal{G}_i \,,\, \mathcal{G} \right) \;\simeq\; \prod_{i \in I} \mathrm{Hom}_{\mathrm{Grpd}}(\mathcal{G}_1, \mathcal{G}) \,.$$

**Remark 1.33**. ***(1-category of groupoids)***

*From def. 1.26 we see that there is a category whose*

- *objects are the small groupoids;*
- *morphisms are the groupoid homomorphisms (functors).*

But since this 1-category does not reflect the existence of homotopies/natural isomorphisms between homomorphsims/functors of groupoids (def. 1.26) this 1-category is not what one is interested in when considering homotopy theory/higher category theory.

In order to obtain the right notion of category of groupoids that does reflect homotopies, we first consider now the *horizontal* composition of homotopies/natural transformations.

**Lemma 1.34**. ***(horizontal composition of homotopies with morphisms)***

*Let $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$ be groupoid and let*

$$\mathcal{G}_1 \overset{F_1}{\longrightarrow} \mathcal{G}_2 \quad \underset{\overset{\longrightarrow}{F_2}}{\overset{\overset{F'_2}{\longrightarrow}}{\Downarrow}} \eta \;\; \mathcal{G}_3 \overset{F_3}{\longrightarrow} \mathcal{G}_3$$

*be morphisms and a homotopy $\eta$. Then there is a homotopy*

$$\mathcal{G}_1 \qquad \underset{\overset{\longrightarrow}{F_3 \circ F'_2 \circ F_1}}{\overset{\overset{F_3 \circ F'_2 \circ F_1}{\longrightarrow}}{\Downarrow}} F_2 \cdot \eta_\cdot F_1 \;\; \mathcal{G}_2$$

*between the respective composites, with components given by*

$$(F_2 \cdot \eta \cdot F_1)(x) \; \coloneqq \; F_2(\eta(F_1(x))) \; .$$

*This operation constitutes a groupoid homomorphism/functor*

$$F_3 \cdot (-) \cdot F_1 \; : \; \mathrm{Hom}_{\mathrm{Grpd}}(\mathcal{G}_2, \mathcal{G}_3) \longrightarrow \mathrm{Hom}_{\mathrm{Grp}}(\mathcal{G}_1, \mathcal{G}_4) \; .$$

***Proof***. The respect for identities is clear. To see the respect for composition, let

$$\begin{array}{c} \overset{F}{\longrightarrow} \\ \Downarrow \eta_1 \\ \mathcal{G}_2 \;\; \overset{G}{\longrightarrow} \;\; \mathcal{G}_3 \\ \Downarrow \eta_2 \\ \underset{H}{\longrightarrow} \end{array}$$

be two composable homotopies. We need to show that

$$F_3 \cdot (\eta_2 \circ \eta_1 \cdot F_1 = (F_3 \cdot \eta_2 \cdot F_1) \circ (F_3 \cdot \eta_1 \cdot F_1) \; .$$

Now for $x$ any object of $\mathcal{G}_1$ we find

$$\begin{aligned} (F_3 \cdot (\eta_2 \circ \eta_1 \cdot F_1)(x) & \coloneqq F_2((\eta_2 \circ \eta_1)(F_1(X))) \\ & \coloneqq F_3(\eta_2(F_1(x)) \circ \eta_1(F_1(x))) \\ & = F_2(\eta_2(F_1(x))) \circ F_2(\eta_1(F_1(X))) \\ & = ((F_3 \cdot \eta_2 \cdot F_1) \circ (F_3 \cdot \eta_1 \cdot F_1))(x) \end{aligned} \; .$$

Here all steps are unwinding of the definition of horizontal and of ordinary (vbertical) composition of homotopies, except the third equality, which is the functoriality of $F_2$. ■

**Lemma 1.35**. ***(horizontal composition of homotopies)***

*Consider a diagram of groupoids, groupoid homomorphsims (functors) and homotopies (natural transformations) as follows:*

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, shift left=2, "F_1"] \arrow[r, shift right=2, "F'_1"'] \arrow[r, phantom, "\Downarrow \eta_1"] & \mathcal{G}_2 \arrow[r, shift left=2, "F_2"] \arrow[r, shift right=2, "F'_2"'] \arrow[r, phantom, "\Downarrow \eta_2"] & \mathcal{G}_3 \end{tikzcd}$$

*The horizontal composition of the homotopies to a single homotopy of the form*

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, shift left=2, "F_2 \circ F_1"] \arrow[r, shift right=2, "F'_2 \circ F'_1"'] \arrow[r, phantom, "\Downarrow \eta_2 \cdot \eta_1"] & \mathcal{G}_3 \end{tikzcd}$$

*may be defined in temrs of the horizontal composition of homotopies with morphisms (lemma 1.34) and the ("vertical") composition of homotopies with themselves, in two different ways, namely by decomposing the above diagram as*

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, shift left=2, "F_1"] \arrow[r, shift right=2, "F'_1"'] \arrow[r, phantom, "\Downarrow \eta_1"] & \mathcal{G}_2 \arrow[r, "F_2"] & \mathcal{G}_3 \\ \mathcal{G}_1 \arrow[r, "F'_1"'] & \mathcal{G}_2 \arrow[r, shift left=2, "F_2"] \arrow[r, shift right=2, "F'_2"'] \arrow[r, phantom, "\Downarrow \eta_2"] & \mathcal{G}_3 \end{tikzcd}$$

*or as*

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, "F_1"] & \mathcal{G}_2 \arrow[r, shift left=2, "F_2"] \arrow[r, shift right=2, "F'_2"'] \arrow[r, phantom, "\Downarrow \eta_2"] & \mathcal{G}_3 \\ \mathcal{G}_1 \arrow[r, shift left=2, "F_1"] \arrow[r, shift right=2, "F'_1"'] \arrow[r, phantom, "\Downarrow \eta_1"] & \mathcal{G}_2 \arrow[r, "F'_2"'] & \mathcal{G}_3 \end{tikzcd}$$

*In the first case we get*

$$\eta_2 \cdot \eta_1 \;\coloneqq\; (\eta_2 \cdot F'_1) \circ (F_2 \cdot \eta_1)$$

*while in the second case we get*

$$\eta_2 \cdot \eta_1 \;\coloneqq\; (F'_2 \cdot \eta_1) \circ (\eta_2 \cdot F_1) \,.$$

*These two definitions coincide.*

***Proof***. For $x$ an object of $\mathcal{G}_1$, then we need that the following square diagram commutes in $\mathcal{G}_3$

$$
\begin{array}{ccccccc}
F_2(F_1(x)) & \xrightarrow{(F_2\cdot\eta_1)(x)} & F_2(F'_1(x)) & & F_2(F_1(x)) & \xrightarrow{F_2(\eta_1(x))} & F_2(F'_1(x)) \\
{}^{(\eta_2\cdot F_1)(x)}\downarrow & & \downarrow^{(\eta_2\cdot F'_1)(x)} & = & {}^{\eta_2(F_1(x))}\downarrow & & \downarrow^{\eta_2(F'_1(x))} \\
F'_2(F_1(x)) & \underset{(F'_2\cdot\eta_1)(x)}{\longrightarrow} & F'_2(F'_1(y)) & & F'_2(F_1(x)) & \underset{F'_2(\eta_1(x))}{\longrightarrow} & F'_2(F'_1(y))
\end{array}
$$

But the ommutativity of the square on the right is the defining compatibility condition on the components of $\eta_2$ applied to the morphism $\eta_1(x)$ in $\mathcal{G}_2$. ∎

**Proposition 1.36**. ***(horizontal composition with homotopy is natural transformation)***

*Consider groupoids, homomorphisms and homotopies of the form*

$$
\mathcal{G}_1 \underoverset{\underset{F'_1}{\longrightarrow}}{\overset{F_1}{\longrightarrow}}{\Downarrow \eta_1} \mathcal{G}_2 \qquad \mathcal{G}_3 \underoverset{\underset{F'_3}{\longrightarrow}}{\overset{F_3}{\longrightarrow}}{\Downarrow \eta_3} \mathcal{G}_4 \,.
$$

*Then horizontal composition with the homotopies (lemma 1.35) constitutes a natural transformation between the functors of horizontal composition with morphisms (lemma 1.34)*

$$
(\eta_3 \cdot (-) \cdot \eta_1) \;:\; (F_3 \cdot (-) \cdot F_1) \;\Rightarrow\; (F'_3(-) \cdot F'_1) \;:\; \mathrm{Hom}_{\mathrm{Grpd}}(\mathcal{G}_2, \mathcal{G}_2) \longrightarrow \mathrm{Hom}_{\mathrm{Grpd}}(\mathcal{G}_1, \mathcal{G}_4) \,.
$$

***Proof***. By lemma 1.35. ∎

It first of all follows that the following makes sense

**Definition 1.37**. **(homotopy category of groupoids)**

There is also the homotopy category $\mathrm{Ho}(\mathrm{Grpd})$ whose

- objects are small groupoids;
- morphisms are equivalence classes of groupoid homomorphisms modulo homotopy (i.e. functors modulo natural transformations).

This is usually denoted $\mathrm{Ho}(\mathrm{Grpd})$.

Of course what the above really means is that, without quotienting out homotopies, groupoids form a 2-category, in fact a (2,1)-category, in fact an enriched category which is enriched over the naive 1-category of groupoids from remark 1.33, hece a strict 2-category with hom-groupoids.

**Definition 1.38**. **(equivalence of groupoids)**

Given two groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$, then a homomorphism

$$
F : \mathcal{G}_1 \longrightarrow \mathcal{G}_2
$$

is an *equivalence* it it is an isomorphism in the homotopy category $\mathrm{Ho}(\mathrm{Grpd})$ (def. 1.37), hence if there exists a homomorphism the other way around

$$G : \mathcal{G}_2 \longrightarrow \mathcal{G}_1$$

and a homotopy/natural transformations of the form

$$G \circ F \simeq \mathrm{id}_{\mathcal{G}_1} \qquad F \circ G \simeq \mathrm{id}_{\mathcal{G}_2} \;.$$

**Example 1.39**. (**(2,1)-functoriality of fundamental groupoid**)

If $X$ and $Y$ are topological spaces and $f : X \longrightarrow Y$ is a continuous function between them, then this induces a groupoid homomorphism (functor) between the respective fundamental groupoids (def. 1.27)

$$F_f : \Pi_1(X) \longrightarrow \Pi_1(Y)$$

given on objects by the underlying function of $f$

$$(F_f)_0 \coloneqq f$$

and given on the class of a path by the evident postcomposition with $f$

$$(F_f)_{x,y} : (x \overset{[\gamma]}{\longrightarrow} y) \;\mapsto\; (f(x) \overset{[f \circ \gamma]}{\longrightarrow} f(y)) \;.$$

This construction clearly respects identity morphisms and composition and hence is itself a functor of the form

$$\Pi_1 : \mathrm{Top} \longrightarrow \mathrm{Grpd}_1$$

from the category Top of topological space to the 1-category Grpd of groupoids.

But more is true: If $f, g : X \longrightarrow Y$ are two continuous function and

$$\eta : f \Rightarrow g$$

is a left homotopy between them, hence a continuous function

$$\eta : X \times [0,1] \longrightarrow Y$$

such that $\eta(-,0) = f$ and $\eta(-,1) = g$, then this induces a homotopy between the above groupoid homomorphisms (a natural transformation of functors).

This shows that the fundamental groupoid functor in fact descends to homotopy categories

$$\Pi_1 : \mathrm{Ho}(\mathrm{Top}) \longrightarrow \mathrm{Ho}(\mathrm{Grpd}) \;.$$

(In fact this means it even extends to a (2,1)-functor from the (2,1)-category of topological spaces, continuous functions, and higher homotopy-classes of left homotopues, to that of groupoids.)

As a direct consequence it follows that if there is a homotopy equivalence

$$X \simeq_h Y$$

between topological spaces, then there is an induced equivalence of groupoids betwee their fundamental groupoids

$$\Pi_1(X) \simeq \Pi_1(Y) \,.$$

Hence the fundamental groupoid is a *homotopy invariant* of topological spaces. Of course by prop. 1.46 the fundamental groupoid is equivalent, as a groupoid, to the disjoint union of the deloopings of all the fundamental groups of the given topological spaces, one for each connected component, and hence this is equivalently the statement that the set of connected components and the fundamental groups of a topological space are homotopy invariants.

**Example 1.40**. **(delooping of a group)**

Let $G$ be a group. Then there is a groupoid, denoted $B G$, with a single object $p$, with morphisms

$$\mathrm{Hom}_{BG}(p,p) \coloneqq G$$

the elements of $G$, with composition the multiplication in $G$, with identity morphism the neutral element in $G$ and with inverse morphisms the inverse elements in $G$.

This is also called the *delooping* of $G$ (because the loop space object of $B G$ at the unique point is the given group: $\Omega B G \simeq G$).

For $G_1, G_2$ two groups, then there is a natural bijection between group homomorphisms $\phi : G_1 \to G_2$ and groupoid homomorphisms $G G_1 \to B_{G 2}$: the latter are all of the form $B\phi$, with $(B\phi)_0$ uniquely fixed and $(B\phi)_{p,p} = \phi$.

This means that the construction $B(-)$ is a fully faithful functor

$$B(-) \;:\; \mathrm{Grp} \hookrightarrow \mathrm{Grpd}_1$$

into from the category Grp of groups to the 1-category of groupoids.

But beware that this functor is not fully faithful when homotopies of groupoids are taken into acount, because there are in general non-trivial homotopies between morphims of the form

$$B\phi_1, B\phi_2 \;:\; B G \longrightarrow B H$$

By definition, such a homotopy (natural transformation) $\eta : B\phi_1 \Rightarrow B\phi_2$ is a choice of a single elemet $\eta_p \in H$ such that for all $g \in G$ we have

$$\phi_2(g) = h \cdot \phi_1(g) \cdot h^{-1} \qquad\qquad \begin{array}{ccc} p & \overset{h}{\longrightarrow} & p \\ {}^{\mathllap{\phi_1(g)}}\downarrow && \downarrow^{\mathrlap{\phi_2(g)}} \\ p & \underset{h}{\longrightarrow} & p \end{array}$$

hence such that

$$\phi_2 = \mathrm{Ad}_h \circ \phi_1 \,.$$

Therefore notably the induced functor

$$B(-) \,:\, \mathrm{Grp} \longrightarrow \mathrm{Ho}(\mathrm{Grp})$$

to the homotopy category of groupoids is not fully faithful.

But since $B G$ is canonically a pointed object in groupoids, we may also regard delooping as a functor

$$B(-) \,:\, \mathrm{Grp} \longrightarrow \mathrm{Grpd}^{\ast/}$$

to the category of pointed objects of Grpd. Since groupoid homomorphisms $B G_1 \to B G_2$ necessarily preserve the basepoint, this makes no difference at this point. But as we now pass to the homotopy category

$$B(-) \,:\, \mathrm{Grp} \hookrightarrow \mathrm{Ho}(\mathrm{Grpd}^{\ast/})$$

then also the homotopies are required to preserve the absepoint, and for homotopies between homomorphisms between delooped groups this means, since there only is a single point, that these homotopies are all trivial. Hence regarded this way the functor is a fully faithful functor again, hence an equivalence of categories onto its essential image. By prop. 1.47 below this essential image consists precisely of the (pointed) connected groupoids:

*Groups are equivalently pointed connected groupoids*.

**Example 1.41. (disjoint union of delooping groupoids)**

Let $\{G_i\}_{i \in I}$ be a set of groups. Then there is a groupoid $\underset{i \in I}{\sqcup} B G_i$ which is the disjoint union groupoid (example 1.31) of the delooping groupoids $B G_i$ (example 1.40).

Its set of objects is the index set $I$, and

$$\mathrm{Hom}(i,j) = \left\{ \begin{array}{ccc} G_i &|& i = j \\ \emptyset &|& \text{otherwise} \end{array} \right.$$

**Definition 1.42. (connected components of a groupoid)**

Given a groupoid $\mathcal{G}$ with set of objects $X$, then the relation “there exists a morphism from $x$ to $y$”, i.e.

$$(x \sim y) \;\coloneqq\; (\mathrm{Hom}(x,y) \neq \emptyset)$$

is clearly an equivalence relation on $X$. The corresponding set of equivalence classes is denoted

$$\pi_0(\mathcal{G})$$

and called the set of *connected components* of $\mathcal{G}$.

**Definition 1.43**. (**automorphism groups**)

Given a groupoid $\mathcal{G}$ and an object $x$, then under composition the set $\mathrm{Hom}_{\mathcal{G}}(x, x)$ forms a group. This is called the *automorphism group* $\mathrm{Aut}_{\mathcal{G}}(x)$ or *vertex group* or *isotropy group* of $x$ in $\mathcal{G}$.

For each object $x$ in a groupoid $\mathcal{G}$, there is a canonical groupoid homomorphism

$$B\,\mathrm{Aut}_{\mathcal{G}}(x) \hookrightarrow \mathcal{G}$$

from the delooping groupoid (def. 1.40) of the automorphism group. This takes the unique object of $B\,\mathrm{Aut}_{\mathcal{G}}(x)$ to $x$ and takes every automorphism of $x$ "to itself", regarded now again as a morphism in $\mathcal{G}$.

**Definition 1.44**. ***(weak homotopy equivalence of groupoids)***

*Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be groupoids. Then a morphism (functor)*

$$F : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$$

*is called a weak homotopy equivalence if*

1. *it induces a bijection on connected components (def. 1.42):*

   $$\pi_0(F) : \pi_0(\mathcal{G}_1) \xrightarrow{\simeq} \pi_0(\mathcal{G}_2)$$

2. *for each object $x$ of $\mathcal{G}_1$ the morphism*

   $$F_{x,x} : \mathrm{Aut}_{\mathcal{G}_1}(x) \xrightarrow{\simeq} \mathrm{Aut}_{\mathcal{G}_2}(F_0(X))$$

   *is an isomorphism of automorphism groups (def. 1.43)*

**Lemma 1.45**. ***(automorphism group depends on basepoint only up to conjugation)***

*For $\mathcal{G}$ a groupoid, let $x$ and $y$ be two objects in the same connected component (def. 1.42). Then there is a group isomorphism*

$$\mathrm{Aut}_{\mathcal{G}}(x) \simeq \mathrm{Aut}_{\mathcal{G}}(y)$$

*between their automorphism groups (def. 1.43).*

***Proof***. By assumption, there exists some morphism from $x$ to $y$

$$x \overset{f}{\longrightarrow} y\,.$$

The operation of conjugation with this morphism

$$\begin{array}{ccc} \mathrm{Aut}_{\mathcal{G}}(x) & \overset{\mathrm{Ad}_f}{\longrightarrow} & \mathrm{Aut}_{\mathcal{G}}(y) \\ g & \mapsto & f^{-1} \circ g \circ f \end{array}$$

is clearly a group isomorphism as required. ▮

**Lemma 1.46**. ***(equivalences between disjoint unions of delooping groupoids)***

*Let $\{G_i\}_{i \in I}$ and $\{H_j\}_{j \in J}$ be sets of groups and consider a homomorphism (functor)*

$$F \;:\; \mathrm{sqcup}_{i \in I} G_i \longrightarrow \underset{j \in J}{\sqcup} H_j$$

*between the corresponding disjoint unions of delooping groupoids (example 1.40).*

*Then the following are equivalent:*

1. *$F$ is an equivalence of groupoids (def. 1.38);*
2. *$F$ is a weak homotopy equivalence (def. 1.44).*

***Proof***. The implication 2) $\Rightarrow 1)$ is immediate.

In the other direction, assume that $F$ is an equivalence of groupoids, and let $G$ be an inverse up to natural isomorphism. It is clear that both induces bijections on connected components. To see that both are isomorphisms of automorphisms groups, observe that the conditions for the natural isomorphisms

$$\alpha \;:\; G \circ F \Rightarrow \mathrm{id} \qquad \beta \;:\; F \circ G \Rightarrow \mathrm{id}$$

are in each separate delooping groupoid $B H_j$ of the form

$$\begin{array}{ccc} * & \overset{\alpha}{\longrightarrow} & * \\ {}^{\mathllap{G_{F_0(i),F_0(i)}(F_{i,i}(f))}}\downarrow && \downarrow^{\mathrlap{\mathrm{id}}} \\ * & \underset{\alpha}{\longrightarrow} & * \end{array} \qquad \begin{array}{ccc} * & \overset{\beta}{\longrightarrow} & * \\ {}^{\mathllap{F_{G_0(j),G_0(j)}(G_{j,j}(f))}}\downarrow && \downarrow^{\mathrlap{\mathrm{id}}} \\ * & \underset{\beta}{\longrightarrow} & * \end{array}$$

since there is only a single object. But this means $F_{i,i}$ and $F_{j,j}$ are group isomorphisms. ▮

**Proposition 1.47**. ***(every groupoid is equivalent to a disjoint union of group deloopings)***

*Assuming the axiom of choice, then:*

*For $\mathcal{G}$ any groupoid, then there exists a set $\{G_i\}_{i \in I}$ of groups and an equivalence of groupoids (def. 1.38)*

$$\mathcal{G} \simeq \underset{i \in I}{\sqcup} B G_i$$

*between $\mathcal{G}$ and a disjoint union of delooping groupoids (example 1.41). This is called a skeleton of $\mathcal{G}$.*

*Concretely, this exists for $I = \pi_0(\mathcal{G})$ the set of connected components of $\mathcal{G}$ (def. 1.42) and for $G_i \coloneqq \mathrm{Aut}_{\mathcal{G}}(x)$ the automorphism group (def. 1.43) of any object $x$ in the given connected component.*

***Proof***. Using the axiom of choice we may find a set $\{x_i\}_{i \in \pi_0(\mathcal{G})}$ of objects of $\mathcal{G}$, with $x_i$ being in the connected component $i \in \pi_0(\mathcal{G})$.

This choice induces a functor

$$\mathrm{inc} : \underset{i \in \pi_0(\mathcal{G})}{\sqcup} \mathrm{Aut}_{\mathcal{G}}(x_i) \longrightarrow \mathcal{G}$$

which takes each object and morphism "to itself".

Now using the axiom of choice once more, we choose in each connected component $i \in \pi_0(\mathcal{G})$ and for each object $y$ in that connected component a morphism

$$x_i \overset{f_{x_i,y}}{\longrightarrow} y \,.$$

Using this we obtain a functor the other way around

$$p : \mathcal{G} \longrightarrow \underset{i \in \pi_0(\mathcal{G})}{\sqcup} \mathrm{Aut}_{\mathcal{G}}(x_i)$$

which sends each object to its connected component, and which for pairs of objects $y$, $z$ of $\mathcal{G}$ is given by conjugation with the morphisms choosen above:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{G}}(y,z) & \overset{p_{y,z}}{\longrightarrow} & & \mathrm{Aut}_{\mathcal{G}}(x_i) & \\
y & & y & \overset{f_{x_i,y}}{\longleftarrow} & x_i \\
{}^{f}\downarrow & \mapsto & {}^{f}\downarrow & & \\
z & & z & \underset{f^{-1}_{x_i,z}}{\longrightarrow} & x_i
\end{array} \,.$$

It is now sufficient to show that there are conjugations/natural isomorphisms

$$p \circ \mathrm{inc} \simeq \mathrm{id} \qquad \mathrm{inc} \circ p \simeq \mathrm{id} \,.$$

For the first this is immediate, since we even have equality

$$p \circ \mathrm{inc} \; = \; \mathrm{id} \,.$$

For the second we observe that choosing

$$\eta(y) \coloneqq f_{x_i,y}$$

yields a naturality square by the above construction:

$$
\begin{array}{ccc}
x_i & \overset{f_{x_i,y}}{\longrightarrow} & y \\
{}^{\mathllap{f_{x_i,z}\circ f \circ f_{x_i,y}^{-1}}}\downarrow && \downarrow^{\mathrlap{f}} \\
x_i & \underset{f_{x_i,z}}{\longrightarrow} & z
\end{array}
.
$$

■

**Proposition 1.48**. ***(weak homotopy equivalence is equivalence of groupoids)***

*Let $F : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ be a homomorphism of groupoids.*

*Assuming the axiom of choice then the following are equivalent:*

1. *$F$ is an equivalence of groupoids (def. 1.38);*
2. *$F$ is a weak homotopy equivalence in that*
    1. *it induces an bijection of sets of connected components (def. 1.42);*

    $$\pi_0(F) \, : \, \pi_0(\mathcal{G}_1) \overset{\simeq}{\longrightarrow} \pi_0(\mathcal{G}_0) \,,$$

3. *for each object $x \in \mathcal{G}_1$ it induces an isomorphis of automorphism groups (def. 1.43):*

    $$F_{x,x} \, : \, \mathrm{Aut}_{\mathcal{G}_1}(x) \overset{\simeq}{\longrightarrow} \mathrm{Aut}_{\mathcal{G}_2}(F_0(x)) \,.$$

***Proof***. In one direction, if $F$ has an inverse up to natural isomorphism, then this induces by definition a bijection on connected components, and it induces isomorphism on homotopy groups by lemma 1.45.

In the other direction, choose equivalences to skeleta as in prop. 1.47 to get a commuting diagram in the 1-category of groupoids as follows:

$$
\begin{array}{ccc}
\mathcal{G}_1 & \underoverset{\simeq}{\mathrm{inc}_1}{\longleftarrow} & \underset{i \in \pi_0(\mathcal{G}_1)}{\sqcup} \mathrm{Aut}_{\mathcal{G}_1}(x_i) \\
{}^{\mathllap{F}}\downarrow && \downarrow^{\mathrlap{\tilde F}} \\
\mathcal{G}_2 & \underoverset{\mathrm{inc}_2}{\simeq}{\longleftarrow} & \underset{i \in \pi_0(\mathcal{G}_1)}{\sqcup} \mathrm{Aut}_{\mathcal{G}_2}(F_0(x_i))
\end{array}
.
$$

Here $\mathrm{inc}_1$ and $\mathrm{inc}_2$ are equivalences of groupoids by prop. 1.47. Moreover, by assumption that $F$ is a weak homotopy equivalence $\tilde F$ is the union of of deloopings of isomorphisms of groups, and hence has a strict inverse, in particular a homotopy inverse, hence is in particular an euivalence of groupoids.

In conclusion, when regarded as a diagram in the homotopy category $\mathrm{Ho}(\mathrm{Grpd})$ (def. 1.37), the top, bottom and right moprhism of the above diagram are isomorphisms. It follows that also $f$ is an isomorphism in $\mathrm{Ho}(\mathrm{Grpd})$. But this means exactly that it is a homotopy equivalence of groupoids, by def. 1.38. ■

# 2. Covering spaces

A *covering space* (def. 2.1 below) is a "continuous fiber bundle of sets" over a topological space, in just the same way as a topological vector bundle is a "continuous fiber bundle of vector spaces".

**Definition 2.1. (covering space)**

Let $X$ be a topological space. A *covering space* of $X$ is a continuous function

$$p : E \longrightarrow X$$

such that there exists an open cover $\underset{i}{\sqcup} U_i \to X$, such that restricted to each $U_i$ then $E \to X$ is homeomorphic over $U_i$ to the product topological space (this example) of $U_i$ with the discrete topological space (this example) on a set $F_i$,

In summary this says that $p : E \to X$ is a covering space if there exists a pullback diagram in Top of the form

$$\begin{array}{ccc} \underset{i}{\sqcup} U_i \times \mathrm{Disc}(F_i) & \longrightarrow & E \\ \downarrow & (\mathrm{pb}) & \downarrow^{p} \\ \underset{i \in I}{\sqcup} U_i & \longrightarrow & X \end{array} .$$

For $x \in U_i \subset X$ a point, then the elements in $F_x = F_i$ are called the *leaves* of the covering at $x$.

Given two covering spaces $p_i : E_i \to X$ , then a homomorphism between them is a continuous function $f : E_1 \to E_2$ between the total covering spaces, which respects the fibers in that the following diagram commutes

$$\begin{array}{ccc} E_1 & & \overset{f}{\longrightarrow} & & E_2 \\ & {}_{p_1}\searrow & & \swarrow_{p_2} \\ & & X \end{array} .$$

This defines a category $\mathrm{Cov}(X)$, the *category of covering spaces* over $X$, whose

- objects are the covering spaces over $X$;
- morphisms are the homomorphisms between these.

**Example 2.2. (trivial covering space)**

For $X$ a topological space and $S$ a set with $\mathrm{Disc}(S)$ the discrete topological space on that set, then the projection out of the *poduct topological space?*

$$\mathrm{pr}_1 \; : \; X \times \mathrm{Disc}(S) \longrightarrow X$$

is a covering space, called the *trivial covering space* over $X$ with fiber $\mathrm{Disc}(S)$.

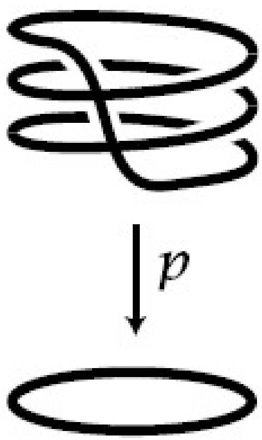
If $E \overset{p}{\longrightarrow} X$ is any covering space, then an isomorphism of covering spaces of the form

$$\begin{tikzcd} E \arrow[rr, "\simeq"] \arrow[dr, "p"'] & & X \times \mathrm{Disc}(S) \arrow[dl, "\mathrm{pr}_2"] \\ & X & \end{tikzcd}$$

is called a *trivialization* of $E \overset{p}{\to} X$.

It is in this sense that evry coverin space $E$ is, by definition, locally trvializable.

**Example 2.3. (covering of circle by circle)**



Regard the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ as the topological subspace of elements of unit absolute value in the complex plane. For $k \in \mathbb{N}$, consider the continuous function

$$p \coloneqq (-)^k : S^1 \longrightarrow S^1$$

given by taking a complex number to its $k$th power. This may be thought of as the result of "winding the circle $k$ times around itself". Precisely, for $k \geq 1$ this is a covering space (def. 2.1) with $k$ leaves at each point.

*graphics grabbed from Hatcher*

**Example 2.4. (covering of circle by real line)**

Consider the continuous function

$$\exp(2\pi i(-)) : \mathbb{R}^1 \longrightarrow S^1$$

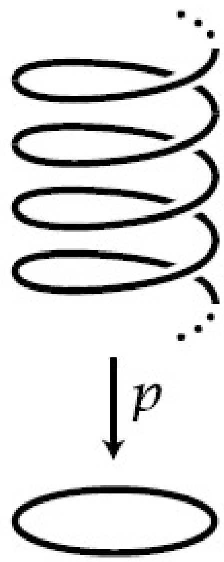
from the real line to the circle, which,

1. with the circle regarded as the unit circle in the complex plane $\mathbb{C}$, is given by

$$t \mapsto \exp(2\pi i t)$$

2. with the circle regarded as the unit circle in $\mathbb{R}^2$, is given by

$$t \mapsto (\cos(2\pi t), \sin(2\pi t)) \,.$$



We may think of this as the result of "winding the line around the circle ad infinitum". Precisely, this is a covering space (def. 2.1) with the leaves at each point forming the set $\mathbb{Z}$ of natural numbers.

Here are some basic properties of covering spaces:

**Proposition 2.5**. ***(covering projections are open maps)***

*If $p : E \to X$ is a covering space projection, then $p$ is an open map.*

***Proof***. By definition of covering space there exists an open cover $\{U_i \subset X\}_{i \in I}$ and homeomorphisms $p^{-1}(U_i) \simeq U_i \times \mathrm{Disc}(F_i)$ for all $i \in I$. Since the projections out of a product topological space are open maps (this prop.), it follows that $p$ is an open map when restricted to any of the $p^{-1}(U_i)$. But a general open subset $W \subset E$ is the union of its restrictions to these subspaces:

$$W = \underset{i \in I}{\cup} (W \cap p^{-1}(U_i)) \,.$$

Since images preserve unions (this prop.) it follows that

$$p(W) = \underset{i \in I}{\cup} p(W \cap p^{-1}(U_i))$$

is a union of open sets, and hence itself open. ▮

**Lemma 2.6**. ***(fiber-wise diagonal of covering space is open and closed)***

*Let $E \overset{p}{\to} X$ be a covering space. Consider the fiber product*

$$E \times_X E \coloneqq \{(e_1, e_2) \in E \times E \;|\; p(e_1) = p(e_2)\}$$

*hence (by the discussion at Top - Univeral constructions) the topological subspace of the product space $E \times E$, as shown on the right. By the universal property of the fiber product, there is the diagonal continuous function*

$$\begin{array}{ccc} E & \longrightarrow & E \times_X E \\ e & \mapsto & (e,e) \end{array} \,.$$

*Then the image of $E$ under this function is an open subset and a closed subset:*

$$\Delta(E) \subset E \times_X E \quad \text{is open and closed} \,.$$

***Proof***. First to see that it is an open subset. It is sufficient to show that for any $e \in E$ there exists an open neighbourhood of $(e,e) \in E \times_X E$.

Now by definition of covering spaces, there exists an open neighbourhood $U_{p(e)} \subset X$ of $p(e) \in X$ such that

$$
\begin{array}{ccccc}
U_{p(e)} \times \mathrm{Disc}(p^{-1}(p(e))) && \xrightarrow{\simeq} && E|_{U_{p(e)}} \\
& {}_{\mathrm{pr}_1}\searrow && \swarrow_{p} \\
&& U_{p(e)}
\end{array}
\,.
$$

It follows that $U_{p(e)} \times \{e\} \subset E$ is an open neighbourhood. Hence by the nature of the product topology, $U_{p(e)} \times U_{p(e)} \subset E \times E$ is an open neighbourhood of $(e,e)$ in $E \times E$ and hence by the nature of the subspace topology the restriction

$$
(E \times_X E) \cap (U_{p(e)} \times U_{p(e)}) \subset E \times_X E
$$

is an open neighbourhood of $(e,e)$ in $E \times_X E$.

Now to see that the diagonal is closed, hence that the complement $(E \times_X E) \setminus \Delta(E)$ is an open subset, it is sufficient to show that every point $(e_1, e_2)$ with $e_1 \neq e_2$ but $p(e_1) = p(e_2)$ has an open neighbourhood in this complement.

As before, there is an open neighbourhood $U \subset X$ of $p(e_1) = p(e_2)$ over which the cover trivializes, and hence $U \times \{e_1\}, U \times \{e_2\} \subset E$ are open neighbourhoods of $e_1$ and $e_2$, respectively. These are disjoint by the assumption that $e_1 \neq e_2$. As above, this means that the intersection

$$
(E \times_X E) \cap ((U \times \{e_1\}) \times (U \times \{e_2\})) \subset (E \times_X E) \setminus \Delta(E)
$$

is an open subset of the complement of the diagonal in the fiber product. ▮

## Lifting properties

If $E \overset{p}{\longrightarrow} X$ is any continuous function (possibly a covering space or a topological vector bundle) then a *section* is a continuous function $\sigma : X \to E$ which sends each point in the base to a point in the fiber above it, hence which makes this diagram commute:

$$
\begin{array}{ccc}
&& E \\
& {}^{\sigma}\nearrow & \downarrow^{p} \\
X & = & X
\end{array}
.
$$

We may think of this as “lifting” each point in the base to point in the fibers “through” the projection map $p$. More generally if $Y \hookrightarrow X$ is a subspace, we may consider such lifts only over $Y$

$$
\begin{array}{ccc}
&& E \\
& {}^{\sigma}\nearrow & \downarrow^{p} \\
Y & \hookrightarrow & X
\end{array}
$$

sometimes called a "local section". But this suggests that for $Y \xrightarrow{f} X$ any continuous function, we consider "lifting its image through $p$"

$$\begin{array}{ccc} & & E \\ & {}^{\sigma}\nearrow & \downarrow^{p} \\ Y & \underset{f}{\longrightarrow} & X \end{array}.$$

For example if $Y = [0,1]$ is the topological interval, then $f : [0,1] \to X$ is a path in the base space $X$, and a lift through $p$ of this is a path in the total space which "runs above" the given path. Such lifts of paths through covering projections is the topic of *monodromy* below.

Here it is usually of interest to consider the lifting problem subject to some constraint. For instance we will want to consider lifts of paths $\gamma : [0,1] \to X$ through a covering projection, subject to the condition that the starting point $\gamma(0)$ is lifted to a prescribed point $p \in E$.

Since such a point is equivalently a continuous function $\mathrm{const}_p : * \to X$, this is the same as asking for a continuous function $\sigma$ that makes both triangles in the following diagram commute:

$$\begin{array}{ccc} * & \overset{\mathrm{const}_p}{\longrightarrow} & E \\ {}^{\mathrm{const}_0}\downarrow & {}^{\sigma}\nearrow & \downarrow^{p} \\ [0,1] & \underset{\gamma}{\longrightarrow} & X \end{array}.$$

This is an example of a general situation which plays a central role in homotopy theory: We say that a square commuting diagram

$$\begin{array}{ccc} A & \longrightarrow & E \\ {}^{i}\downarrow & & \downarrow^{p} \\ B & \longrightarrow & X \end{array}$$

is a *lifting problem* and that a diagonal morphism

$$\begin{array}{ccc} A & \longrightarrow & E \\ {}^{i}\downarrow & \nearrow & \downarrow^{p} \\ B & \longrightarrow & X \end{array}$$

such that both resulting triangles commute is a *lift*. If such a lift exists for $i$ taken from some class of morphisms, then one says that $p$ has the *right lifting property* against this class.

We now discuss some right lifting properties satisfied by covering spaces:

1. homotopy-lifting propery,

2. the lifting theorem.

These lifting properties will be used in below for the computation of fundamental groups of some topological spaces.

**Lemma 2.7**. ***(lifts out of connected space into covering spaces are unique relative to any point)***

*Let*

1. $E \overset{p}{\to} X$ *be a covering space,*
2. $Y$ *a connected topological space*
3. $f : Y \longrightarrow X$ *a continuous function.*
4. $\hat{f}_1, \hat{f}_2 : Y \longrightarrow E$ *two lifts of* $f$*, in that the following diagram commutes:*

$$\begin{array}{ccc} & & E \\ & {}^{\hat{f}_i}\nearrow & \downarrow^{p} \\ Y & \underset{f}{\longrightarrow} & X \end{array}$$

*for* $i \in \{1, 2\}$*.*

*If there exists* $y \in Y$ *such that* $\hat{f}_1(y) = \hat{f}_2(y)$ *then the two lifts already agree everywhere:* $\hat{f}_1 = \hat{f}_2$*.*

***Proof***. By the universal property of the fiber product

$$E \times_X E \coloneqq \{(e_1, e_2) \in E \times E \mid p(e_1) = p(e_2)\} \subset E \times E$$

the two lifts determine a single continuous function of the form

$$(\hat{f}_1, \hat{f}_2) : Y \longrightarrow E \times_X E \,.$$

Write

$$\Delta(E) \coloneqq \{(e, e) \in E \times_X E \mid e \in E\}$$

for the diagonal on $E$ in the fiber product. By lemma 2.6 this is an open subset and a closed subset of the fiber product space. Hence by continuity of $(\hat{f}_1, \hat{f}_2)$ also its pre-image

$$(\hat{f}_1, \hat{f}_2)^{-1}(\Delta(E)) \subset Y$$

is both closed and open, hence also its complement is open in $Y$.

Moreover, the assumption that the functions $\hat{f}_1$ and $\hat{f}_2$ agree in at least one point means that the above pre-image is non-empty. Therefore the assumption that $Y$ is connected implies that this pre-image coincides with all of $Y$. This is the statement to be proven. ▮

**Lemma 2.8**. ***(path lifting property)***

*Let $p : E \to X$ be any covering space. Given*

1. *$\gamma : [0,1] \to X$ a path in $X$,*
2. *$\hat{x}_0 \in E$ be a lift of its starting point, hence such that $p(\hat{x}_0) = \gamma(0)$*

*then there exists a unique path $\hat{\gamma} : [0,1] \to E$ such that*

1. *it is a lift of the original path: $p \circ \hat{\gamma} = \gamma$;*
2. *it starts at the given lifted point: $\hat{\gamma}(0) = \hat{x}_0$.*

*In other words, every commuting diagram in Top of the form*

$$\begin{array}{ccc} \{0\} & \overset{\hat{x}_0}{\longrightarrow} & E \\ \downarrow & & \downarrow^{p} \\ [0,1] & \underset{\gamma}{\longrightarrow} & X \end{array}$$

*has a unique lift:*

$$\begin{array}{ccc} \{0\} & \overset{\hat{x}_0}{\longrightarrow} & E \\ \downarrow & {}^{\hat{\gamma}}\nearrow & \downarrow^{p} \\ [0,1] & \underset{\gamma}{\longrightarrow} & X \end{array}.$$

***Proof***. First consider the case that the covering space is trival, hence of the Cartesian product form

$$\mathrm{pr}_1 \;:\; X \times \mathrm{Disc}(S) \longrightarrow X \,.$$

By the universal property of the product topological spaces in this case a lift $\hat{\gamma} : [0,1] \to X \times \mathrm{Disc}(S)$ is equivalently a pair of continuous functions

$$\mathrm{pr}_1(\hat{\gamma}) : [0,1] \to X \qquad \mathrm{pr}_2(\hat{\gamma}) : [0,1] \to \mathrm{Disc}(S) \,,$$

Now the lifting condition explicitly fixes $\mathrm{pr}_1(\hat{\gamma}) = \gamma$. Moreover, a continuous function into a discrete topological space $\mathrm{Disc}(S)$ is locally constant, and since $[0,1]$ is a connected topological space this means that $\mathrm{pr}_2(\hat{\gamma})$ is in fact a constant function (this example), hence uniquely fixed to be $\mathrm{pr}_2(\hat{\gamma}) = \hat{x}_0$.

This shows the statement for the case of trivial covering spaces.

Now consider any covering space $p : E \to X$. By definition of covering spaces, there exists for every point $x \in X$ a open neighbourhood $U_x \subset X$ such that the restriction of $E$ to $U_x$ becomes a trivial covering space:

$$p^{-1}(U_x) \simeq U_x \times \mathrm{Disc}(p^{-1}(x)) \;.$$

Consider such a choice

$$\{U_x \subset X\}_{x \in X} \;.$$

This is an open cover of $X$. Accordingly, the pre-images

$$\{\gamma^{-1}(U_x) \subset [0,1]\}_{x \in X}$$

constitute an open cover of the topological interval $[0,1]$.

Now the closed interval is a compact topological space, so that this cover has a finite open subcover. By the Euclidean metric topology, each element in this finite subcover is a disjoint union of open intervals. The collection of all these open intervals is an open refinement of the original cover, and by compactness it once more has a finite subcover, now such that each element of the subcover is guaranteed to be a single open interval.

This means that we find a finite number of points

$$t_0 < t_1 < \cdots <_{n+1} \;\in [0,1]$$

with $t_0 = 0$ and $t_{n+1} = 1$ such that for all $0 < j \leq n$ there is $x_j \in X$ such that the corresponding path segment

$$\gamma([t_j, t_{j+1}]) \subset X$$

is contained in $U_{x_j}$ from above.

Now assume that $\hat{\gamma}|_{[0,t_j]}$ has been found. Then by the triviality of the covering space over $U_{x_j}$ and the first argument above, there is a unique lift of $\gamma|_{[t_j, t_{j+1}]}$ to a continuous function $\hat{\gamma}|_{[t_j,t_{j+1}]}$ with starting point $\hat{\gamma}(t_j)$. Since $[0, t_{j+1}]$ is the pushout $[0,t_j] \underset{\{t_j\}}{\sqcup} [t_j, t_{j+1}]$ (this example), it follows that $\hat{\gamma}|_{[0,t_j]}$ and $\hat{\gamma}|_{[t_j,t_{j+1}]}$ uniquely glue to a continuous function $\hat{\gamma}|_{[0,t_{j+1}]}$ which lifts $\gamma|_{[0,t_{j+1}]}$.

By induction over $j$, this yields the required lift $\hat{\gamma}$.

Conversely, given any lift, $\hat{\gamma}$, then its restrictions $\hat{\gamma}|_{[t_j,t_{j+1}]}$ are uniquely fixed by the above inductive argument. Therefore also the total lift is unique. Altrnatively, uniqueness of the lifts is a special case of lemma 2.7. ∎

**Proposition 2.9**. ***(homotopy lifting property of covering spaces)***

*Let*

1. $E \overset{p}{\to} X$ *be a covering space;*

2. $Y$ *a locally path-connected topological space.*

*Then every lifting problem of the form*

$$\begin{array}{ccc} Y & \overset{\hat f}{\longrightarrow} & E \\ {}^{\mathllap{(\mathrm{id}_Y, \mathrm{const}_0)}}\downarrow && \downarrow^{\mathrlap{p}} \\ Y \times [0,1] & \underset{\eta}{\longrightarrow} & X \end{array}$$

*has a unique lift*

$$\begin{array}{ccc} Y & \overset{\hat f}{\longrightarrow} & E \\ {}^{\mathllap{(\mathrm{id}_Y, \mathrm{const}_0)}}\downarrow & {}^{\hat \eta}\nearrow & \downarrow^{\mathrlap{p}} \\ Y \times [0,1] & \underset{\eta}{\longrightarrow} & X \end{array}.$$

***Proof***. For every point $y \in Y$ the situation restricts to that of path lifting

$$\begin{array}{ccccc} \ast & \overset{\mathrm{const}_y}{\longrightarrow} & Y & \overset{f}{\longrightarrow} & E \\ {}^{\mathllap{\mathrm{const}_0}}\downarrow && {}^{\mathllap{(\mathrm{id}_Y, \mathrm{const}_0)}}\downarrow & {}^{\hat \eta}\nearrow & \downarrow^{\mathrlap{p}} \\ [0,1] & \underset{(\mathrm{const}_y, \mathrm{id})}{\longrightarrow} & Y \times [0,1] & \underset{\eta}{\longrightarrow} & X \end{array}.$$

This has a unique lift $\hat \eta_y$ by lemma 2.8. Hence if a continuous lift of $\eta$ does exist, it must be given by

$$\hat \eta(y,t) = \hat \eta_y(t)$$

and so it only remains to see that this function is continuous.

To that end, let $\{U_i \subset X\}_{i \in I}$ be an open cover over which the covering space trivializes. Then $\{\eta^{-1}(U_i) \subset Y \times [0,1]\}$ is an open cover. Since $Y$ is assumed to be locally connected, so is the product space $Y \times [0,1]$, and hence this cover is refined by a cover of connected open subsets $\{V_i \subset Y \times [0,1]\}_{j \in J}$.

By lemma 2.7 over these $\hat f$ is constant on one leaf, and hence so is $\hat \eta$. This constant lift is continuous.

This shows that $\hat \eta$ restricts to a continuous function over an open cover of $Y \times [0,1]$ and thus is itself continuous. ▮

**Example 2.10**. **(lift homotopy of paths for given lifts of paths)**

Let $p : E \to X$ be a covering space. Then given a homotopy relative the starting point between two paths in $X$,

$$\eta : \gamma_1 \Rightarrow \gamma_2$$

there is for every lift $\hat{\gamma}_1, \hat{\gamma}_2$ of these two paths to paths in $E$ with the same starting point a unique homotopy

$$\hat{\eta} : \hat{\gamma}_1 \Rightarrow \hat{\gamma}_2$$

between the lifted paths that lifts the given homotopy:

For commuting squares of the form

$$\begin{array}{ccc} ([0,1] \times \{0\}) \cup (\{0,1\} \times [0,1]) & \overset{(\gamma_1, \gamma_2)}{\longrightarrow} & E \\ \downarrow & {}^{\hat{\eta}}\nearrow & \downarrow^{p} \\ [0,1] \times [0,1] & \underset{\eta}{\longrightarrow} & X \end{array}$$

there is a unique diagonal lift in the lower diagram, as shown.

Moreover if the homotopy $\eta$ also fixes the endpoint, then so does the lifted homotopy $\hat{\eta}$.

***Proof***. There are horizontal homeomorphisms such that the following diagram commutes

$$\begin{array}{ccc} [0,1] & \overset{\simeq}{\longrightarrow} & ([0,1] \times \{0\}) \cap (\{0,1\} \times [0,1]) \\ \downarrow & & \downarrow \\ [0,1] \times [0,1] & \underset{\simeq}{\longrightarrow} & [0,1] \times [0,1] \end{array} \,.$$

With this the statement follows from 2.9. ■

**Example 2.11**. Let $(E, e) \overset{p}{\longrightarrow} (X, x)$ be a pointed covering space and let $f : (Y, y) \longrightarrow (X, x)$ be a point-preserving continuous function such that the image of the fundamental group of $(Y, y)$ is contained within the image of the fundamental group of $(E, e)$ in that of $(X, x)$:

$$f_*(\pi_1(Y, y)) \subset p_*(\pi_1(E, e)) \quad \subset \pi_1(X, x) \,.$$

Then for $\ell_Y$ a path in $(Y, y)$ that happens to be a loop, every lift of its image path $f \circ \ell$ in $(X, x)$ to a path $\widehat{f \circ \ell_Y}$ in $(E, e)$ is also a loop there.

***Proof***. By assumption, there is a loop $\ell_E$ in $(E, e)$ and a homotopy fixing the endpoints of the form

$$\eta_X : p \circ \ell_E \Rightarrow f \circ \ell_Y \,.$$

Then by the homotopy lifting property as in example 2.10, there is a homotopy in $(E, e)$ relative to the basepoint

$$\eta_E : \ell_E \Rightarrow \widehat{f \circ \ell_Y}$$

and lifting the homotopy $\eta_X$. Therefore $\eta_E$ is in fact a homotopy between loops, and so $\widehat{f \circ \ell_Y}$ is indeed a loop. ▮

**Proposition 2.12**. ***(lifting theorem)***

*Let*

1. $p : E \to X$ *be a* *covering space;*
2. $e \in E$ *a point, with* $x \coloneqq p(e)$ *denoting its image,*
3. $Y$ *be a* *connected* *and* *locally path-connected* *topological space;*
4. $y \in Y$ *a point*
5. $f : (Y, y) \longrightarrow (X, x)$ *a* *continuous function* *such that* $f(y) = x$.

*Then the following are equivalent:*

1. *There exists a unique lift* $\hat f$ *in the diagram*

$$\begin{array}{ccc} && (E,e) \\ & {}^{\hat f}\nearrow & \downarrow^{p} \\ (Y,y) & \underset{f}{\longrightarrow} & (X,x) \end{array}$$

   *of* *pointed topological spaces.*

2. *The* *image* *of the* *fundamental group* *of* $Y$ *under* $f$ *in that of* $X$ *is contained in the image of the fundamental group of* $E$ *under* $p$:

$$f_*(\pi_1(Y, y)) \subset p_*(\pi_1(E, e))$$

*Moreover, if* $Y$ *is path-connected, then the lift in the first item is unique.*

***Proof***. The implication $1) \Rightarrow 2)$ is immediate. We need to show that the second statement already implies the first.

Since $Y$ is connected and locally path-connected, it is also a path-connected topological space (this prop.). Hence for every point $y' \in Y$ there exists a path $\gamma$ connecting $y$ with $y'$ and hence a path $f \circ \gamma$ connecting $x$ with $f(y')$. By the path-lifting property (lemma 2.8) this has a unique lift

$$\begin{array}{ccc} \{0\} & \overset{e}{\longrightarrow} & E \\ \downarrow & {}^{\widehat{f \circ \gamma}}\nearrow & \downarrow^{p}. \\ [0,1] & \underset{f \circ \gamma}{\longrightarrow} & X \end{array}$$

Therefore

$$\hat{f}(y') \coloneqq \widehat{f \circ \gamma}(1)$$

is a lift of $f(y')$.

We claim now that this pointwise construction is independent of the choice $\gamma$, and that as a function of $y'$ it is indeed continuous. This will prove the claim.

Now by the path lifting lemma 2.8 the lift $\widehat{\mathrm{f} \circ \gamma}$ is unique given $f \circ \gamma$, and hence $\hat{f}(y')$ depends at most on the choice of $\gamma$.

Hence let $\gamma' : [0,1] \to Y$ be another path in $Y$ that connects $y$ with $y'$. We need to show that then $\widehat{f \circ \gamma'} = \widehat{f \circ \gamma}$.

First observe that if $\gamma'$ is related to $\gamma$ by a homotopy, so that then also $f \circ \gamma'$ is related to $f \circ \gamma$ by a homotopy, then this is the statement of the homotopy lifting property as in example 2.10.

Next write $\bar{\gamma}' \cdot \gamma$ for the path concatenation of the path $\gamma$ with the reverse path of the path $\gamma'$, hence a loop in $Y$, so that $f \circ (\bar{\gamma}' \cdot \gamma)$ is a loop in $X$. The assumption that $f_*(\pi_1(Y,y)) \subset p_*(\pi_1(E,e))$ implies (example 2.11) that the path $\widehat{f \circ (\bar{\gamma}' \cdot \gamma)}$ which lifts this loop to $E$ is itself a loop in $E$.

By uniqueness of path lifting, this means that the lift of $f \circ (\gamma' \cdot (\bar{\gamma}' \cdot \gamma))$ coincides with that of $f \circ \gamma'$. But $\bar{\gamma}' \cdot (\gamma' \cdot \gamma)$ is homotopic (via reparameterization) to just $\gamma$. Hence it follows now with the first statement that the lift of $f \circ \gamma'$ indeed coincides with that of $f \circ \gamma$.

This shows that the above prescription for $\hat{f}$ is well defined.

It only remains to show that the function $\hat{f}$ obtained this way is continuous.

Let $y' \in Y$ be a point and $W_{\hat{f}(y')} \subset E$ an open neighbourhood of its image in $E$. It is sufficient to see that there is an open neighbourhood $V_{y'} \subset Y$ such that $\hat{f}(V_y) \subset W_{\hat{f}(y')}$.

Let $U_{f(y')} \subset X$ be an open neighbourhood over which $p$ trivializes. Then the restriction

$$p^{-1}(U_{f(y')}) \cap W_{\hat{f}(y')} \;\subset\; U_{f(y')} \times \mathrm{Disc}(p^{-1}(f(y')))$$

is an open subset of the product space. Consider its further restriction

$$\left(U_{f(y')} \times \{\hat{f}(y')\}\right) \cap \left(p^{-1}(U_{f(y')}) \cap W_{\hat{f}(y')}\right)$$

to the leaf

$$U_{f(y')} \times \{\hat{f}(y')\} \subset U_{f(y')} \times p^{-1}(f(y'))$$

which is itself an open subset. Since $p$ is an open map (this prop.), the subset

$$p\Big(\big(U_{f(y')} \times \{\hat{f}(y')\}\big) \cap \big(p^{-1}(U_{f(y')}) \cap W_{\hat{f}(y')}\big)\Big) \subset X$$

is open, hence so is its pre-image

$$f^{-1}\Big(p\Big(\big(U_{f(y')} \times \{\hat{f}(y')\}\big) \cap \big(p^{-1}(U_{f(y')}) \cap W_{\hat{f}(y')}\big)\Big)\Big) \subset Y .$$

Since $Y$ is assumed to be locally path-connected, there exists a path-connected open neighbourhood

$$V_{y'} \subset f^{-1}\Big(p\Big(\big(U_{f(y')} \times \{\hat{f}(y')\}\big) \cap \big(p^{-1}(U_{f(y')}) \cap W_{\hat{f}(y')}\big)\Big)\Big) .$$

By the uniqueness of pah lifting, the image of that under $\hat{f}$ is

$$\begin{aligned} \hat{f}(V_{y'}) &= f(V_{y'}) \times \{\hat{f}(y')\} \\ &\subset p\Big(\big(U_{f(y')} \times \{\hat{f}(y')\}\big) \cap \big(p^{-1}(U_{f(y')}) \cap W_{\hat{f}(y')}\big)\Big) \times \{\hat{f}(y')\} \\ &\simeq \big(U_{f(y')} \times \{\hat{f}(y')\}\big) \cap \big(p^{-1}(U_{f(y')}) \cap W_{\hat{f}(y')}\big) \\ &\subset W_{\hat{f}(y')} \end{aligned} .$$

This shows that the lifted function is continuous. Finally that this continuous lift is unique is the statement of lemma 2.7. ■

# Monodromy

Since the lift of a path through a covering space projection is unique once the lift of the starting point is chosen (lemma 2.8) every path in the base space determines a function between the fiber sets over its endpoints. By the homotopy lifting property of covering spaces as in example 2.10 this function only depends on the equivalence class of the path under homotopy relative boundary. Therefore this fiber-assignment is in fact an *action* of the fundamental groupoid of the base space on sets, called a *groupoid representation* (def. 2.13 below). In particular, associated with any homotopy-class of a loop, hence of an element in the fundamental group, there is associated a bijection of the fiber over the loop's basepoint with itself, hence a permutation representation of the fundamental group. This is called the *monodromy* of the covering space. It is a measure for how the coverign space fails to be globally trivial.

In fact the fundamental theorem of covering spaces (prop. 2.22) below says that the monodromy representation characterizes the covering spaces completely and faithfully. This means that covering spaces may be dealt with completely with tools from group theory and representation theory, a fact that we make use of in the computation of examples below.

**Definition 2.13. (groupoid representation)**

Let $\mathcal{G}$ be a groupoid. Then:

A linear representation of $\mathcal{G}$ is a groupoid homomorphism (functor)

$$\rho : \mathcal{G} \longrightarrow \mathrm{Core}(\mathrm{Vect})$$

to the groupoid core of the category Vect of vector spaces (example 1.29). Hence this is

1. For each object $x$ of $\mathcal{G}$ a vector space $V_x$;

2. for each morphism $x \overset{f}{\longrightarrow} y$ of $\mathcal{G}$ a linear map $\rho(f) : V_x \to V_y$

such that

1. (respect for composition) for all composable morphisms $x \overset{f}{\to} y \overset{g}{\to} z$ in the groupoid we have an equality

$$\rho(g) \circ \rho(f) = \rho(g \circ f)$$

2. (respect for identities) for each object $x$ of the groupoid we have an equality

$$\rho(\mathrm{id}_x) = \mathrm{id}_{V_x} \;.$$

Similarly a *permutation representation* of $\mathcal{G}$ is a groupoid homomorphism (functor)

$$\rho : \mathcal{G} \longrightarrow \mathrm{Core}(\mathrm{Set})$$

to the groupoid core of Set. Hence this is

1. For each object $x$ of $\mathcal{G}$ a set $S_x$;

2. for each morphism $x \overset{f}{\longrightarrow} y$ of $\mathcal{G}$ a function $\rho(f) : S_x \to S_y$

such that composition and identities are respected, as above.

For $\rho_1$ and $\rho_2$ two such representations, then a homomorphism of representations

$$\phi : \rho_1 \longrightarrow \rho_2$$

is a natural transformation between these functors, hence is

- for each object $x$ of the groupoid a (linear) function

$$(V_1)_x \overset{\phi(x)}{\longrightarrow} (V_2)_x$$

- such that for all morphisms $x \overset{f}{\longrightarrow} y$ we have

$$\phi(y) \circ \rho_1(f) = \rho_2(x) \circ \phi(x) \qquad \begin{array}{ccc} (V_1)_x & \overset{\phi(x)}{\longrightarrow} & (V_2)_x \\ {}^{\mathllap{\rho_1(f)}}\downarrow && \downarrow^{\mathrlap{\phi_2(f)}} \\ (V_1)_y & \underset{\phi(y)}{\longrightarrow} & (V_2)_y \end{array}$$

By def. 1.26 the representations of $\mathcal{G}$ in $\mathrm{Core}(\mathcal{C})$ and homomorphisms between them constitute a groupoid called the *representation groupoid*

$$\mathrm{Rep}(\mathcal{G}) \;\coloneqq\; \mathrm{Hom}_{\mathrm{Grpd}}(\mathcal{G}, \mathrm{Core}(\mathcal{C})) \;.$$

**Example/Definition 2.14. (group representations are groupoid representations of delooping groupoids)**

If here `\mathcal(G) = B G` is the delooping groupoid of a group $G$ (example 1.40), then a groupoid representation of $BG$ is a *group representation* of $G$ (def. 2.13), and one writes

$$\mathrm{Rep}(G) \coloneqq \mathrm{Rep}(BG)$$

for the representation groupoid.

For each object $x \in X$ the canonical inclusion of the delooping groupoid of the automorphism group (from def. 1.43)

$$\mathrm{inc}_x \;:\; B\,\mathrm{Aut}_{\mathrm{math}} \hookrightarrow \mathcal{G}$$

induces by precomposition a homomorphism of representation groupoids:

$$\mathrm{Hom}(\mathrm{inc}_x, \mathrm{Core}(\mathcal{C})) \;:\; \mathrm{Rep}(\mathcal{G}, \mathcal{C}) \longrightarrow \mathrm{Rep}(\mathrm{Aut}_{\mathcal{G}}(x), \mathcal{C}) \;.$$

We say that a groupoid representation is *faithful* or *free* if for all objects $x$ its restriction to a group representation of $\mathrm{Azt}_{\mathcal{G}}(x)$ this way is transitive or free, respectively.

Here the representation $\rho$ of a group $G$ on some set $S$

1. *transitive* if for all pairs of elements $s_1, s_2 \in S$ there is a $g \in G$ such that $\rho(g)(s_1) = s_2$;

2. *free* if whenever $g(s) = s$ holds for all $s \in S$ then $g$ is the neutral elements.

**Proposition 2.15**. ***(groupoid representations are products of group representations)***

*Assuming the axiom of choice then the following holds:*

*Let $\mathcal{G}$ be a groupoid. Then its groupoid of groupoid representations $\mathrm{Rep}(\mathcal{G})$ (def. 2.13) is equivalent (def. 1.38) to the product groupoid (example 1.32) indexed by the set of connected components $\pi_0(\mathcal{G})$ (def. 1.42) of group representations* *(example 2.14) of the automorphism group $G_i \coloneqq \mathrm{Aut}_{\mathcal{G}}(x_i)$ (def. 1.43) for $x_i$ any object in the $i$th connected component:*

$$\mathrm{Rep}(\mathcal{G}) \;\simeq\; \prod_{i \in \pi_0(\mathcal{G})} \mathrm{Rep}(G_i) \;.$$

***Proof***. Let $\mathcal{C}$ be the category that the representation is on (e.g. $\mathcal{C} =$ Set for permutation representations). Then by definition

$$\mathrm{Rep}(\mathcal{G}) = \mathrm{Hom}_{\mathrm{Grpd}}(\mathcal{G}, \mathrm{Core}(\mathcal{C})) \;.$$

Consider the injection functor of the skeleton from lemma 1.46

$$\mathrm{inc} \;:\; \underset{i \in \pi_0(\mathcal{G})}{\sqcup} B G_i \longrightarrow \mathcal{G} \;.$$

By lemma 1.34 the pre-composition with this constitutes a functor

$$\mathrm{inc}^\ast \;:\; \mathrm{Hom}(\mathcal{G}, \mathcal{C}) \longrightarrow \mathrm{Hom}( \underset{i \in \pi_0(\mathcal{G})}{\sqcup} B G_i, \mathcal{C})$$

and by combining lemma 1.46 with lemma 1.36 this is an equivalence of groupoids. Finally, by example 1.32 the groupoid on the right is the product groupoid as claimed. ▮

**Definition 2.16**. **(monodromy of a covering space)**

Let $X$ be a topological space and $E \overset{p}{\to} X$ a covering space (def. 2.1). Write $\Pi_1(X)$ for the fundamental groupoid of $X$ (example 1.27).

Define a groupoid homomorphism

$$\mathrm{Fib}_E \;:\; \Pi_1(X) \longrightarrow \mathrm{Core}(\mathrm{Set})$$

to the groupoid core of the category Set of sets (example 2.13), hence a permutation groupoid representation (example 2.13), as follows:

1. to a point $x \in X$ assign the fiber $p^{-1}(\{x\}) \in \mathrm{Set}$;

2. to the homotopy class of a path $\gamma$ connecting $x \coloneqq \gamma(0)$ with $y \coloneqq \gamma(1)$ in $X$ assign the function $p^{-1}(\{x\}) \longrightarrow p^{-1}(\{y\})$ which takes $\hat{x} \in p^{-1}(\{x\})$ to the endpoint of a path $\hat{\gamma}$ in $E$ which lifts $\gamma$ through $p$ with starting point $\hat{\gamma}(0) = \hat{x}$

$$\begin{aligned} p^{-1}(x) &\longrightarrow p^{-1}(y) \\ (\hat{x} = \hat{\gamma}(0)) &\mapsto \hat{\gamma}(1) \end{aligned} \;.$$

This construction is well defined for a given representative $\gamma$ due to the unique path-lifting property of covering spaces (lemma 2.8) and it is independent of the choice of $\gamma$ in the given homotopy class of paths due to the homotopy lifting property (example 2.10).
Similarly, these two lifting properties give that this construction respects

composition in $\Pi_1(X)$ and hence is indeed a homomorphism of groupoids (a functor).

**Proposition 2.17**. *Given a homomorphism between two covering spaces $E_i \overset{p_i}{\to} X$, hence a continuous function $f : E_1 \to E_2$ which respects fibers in that the diagram*

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

*commutes, then the component functions*

$$f\vert_{\{x\}} \;:\; p_1^{-1}(\{x\}) \longrightarrow p_2^{-1}(\{x\})$$

*are compatible with the monodromy $\mathrm{Fib}_E$ (def. 2.16) along any path $\gamma$ between points $x$ and $y$ from def. 2.16 in that the following diagrams of sets commute*

$$\begin{tikzcd} p_1^{-1}(x) \arrow[r, "f\vert_{\{x\}}"] \arrow[d, "\mathrm{Fib}_{E_1}([\gamma])"'] & p_2^{-1}(x) \arrow[d, "\mathrm{Fib}_{E_2}([\gamma])"] \\ p_1^{-1}(y) \arrow[r, "f\vert_{\{y\}}"'] & p_2^{-1}(\{y\}) \end{tikzcd}$$

*This means that $f$ induces a homotopy (natural transformation) between the monodromy homomorphisms (functors)*

$$\begin{tikzcd} \Pi_1(X) \arrow[r, bend left, "\mathrm{Fib}_{E_1}"] \arrow[r, bend right, "\mathrm{Fib}_{E_2}"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{Core}(\mathrm{Set}) \end{tikzcd}$$

*of $E_1$ and $E_2$, respectively, and hence that constructing monodromy is itself a functor from the category of covering spaces of $X$ to that of permutation representations of the fundamental groupoid of $X$:*

$$\mathrm{Fib} \;:\; \mathrm{Cov}(X) \longrightarrow \mathrm{Rep}(\Pi_1(X), \mathrm{Set}) \;.$$

**Example 2.18**. **(three-sheeted covers of the circle)**

There are, up to isomorphism, three different 3-sheeted covering spaces of the circle $S^1$.

The one from example 2.3 for $k = 3$. Another one. And the trivial one. Their corresponding permutation actions according to def. 2.16 may be seen from the pictures on the right.

*graphics grabbed from Hatcher*

**Example 2.19**. **(fundamental groupoid of covering space)**

Let $E \overset{p}{\longrightarrow} X$ be a covering space.

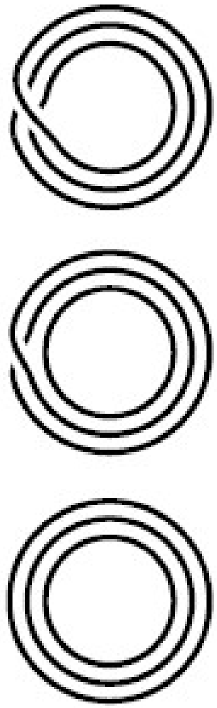
Then the fundamental groupoid $\Pi_1(E)$ of the total space $E$ is the groupoid

whose

* objects are pairs $(x, \hat x)$ consisting of a point $x \in X$ and en element $\hat x \in \mathrm{Fib}_E(x)$;
* morphisms $[\hat \gamma] : (x, \hat x) \to (x', \hat x')$ are morphisms $[\gamma] : x \to x'$ in $\Pi_1(X)$ such that $\mathrm{Fib}_E([\gamma])(\hat x) = \hat x'$.

This is also called the *Grothendieck construction* of the monodromy functor $\mathrm{Fib}_E \,:\, \Pi_1(X) \to \mathrm{Core}(\mathrm{Set})$, and denoted

$$\Pi_1(E) \;\simeq\; \int_{\Pi_1(X)} \mathrm{Fib}_E \,.$$



***Proof***. By the uniqueness of the path-lifting, lemma 2.8 and the very definition of the monodromy functor. ▮

**Definition 2.20**. **(reconstruction of covering spaces from monodromy)**

Let

1. $(X, \tau)$ be a locally path-connected semi-locally simply connected topological space,
2. $\rho \in \mathrm{Rep}(\Pi_1(X), \mathrm{Set})$ a permutation representation of its fundamental groupoid.

Consider the disjoint union set of all the sets appearing in this representation

$$E(\rho) \;\coloneqq\; \underset{x \in X}{\sqcup} \rho(x)$$

For

1. $U \subset X$ an open subset
   1. which is path-connected
   2. for which every element of the fundamental group $\pi_1(U, x)$ becomes trivial under $\pi_1(U, x) \to \pi_1(X, x)$,
2. for $\hat x \in \rho(x)$ with $x \in U$

consider the subset

$$V_{U,\hat{x}} \coloneqq \left\{ \rho(\gamma)(\hat{x}) \mid x' \in U\,,\;\; \gamma \, \text{path from}\, x \,\text{to}\, x' \right\} \;\subset\; E(\rho)\,.$$

The collection of these defines a base for a topology (prop. 2.21 below). Write $\tau_\rho$ for the corresponding topology. Then

$$(E(\rho), \tau_\rho)$$

is a topological space. It canonically comes with the function

$$\begin{array}{ccc} E(\rho) & \overset{p}{\longrightarrow} & X \\ \hat{x} \in \rho(x) & \mapsto & x \end{array}\,.$$

Finally, for

$$f \colon \rho_1 \longrightarrow \rho_2$$

a homomorphism of permutation representations, there is the evident induced function

$$\begin{array}{ccc} E(\rho_1) & \overset{\mathrm{Rec}(f)}{\longrightarrow} & E(\rho_2) \\ (\hat{x} \in \rho_1(x)) & \mapsto & (f_x(\hat{x}) \in \rho_2(x)) \end{array}\,.$$

**Proposition 2.21**. *The construction $\rho \mapsto E(\rho)$ in def. 2.20 is well defined and yields a covering space of $X$.*

*Moreover, the construction $f \mapsto \mathrm{Rec}(f)$ yields a homomorphism of covering spaces.*

***Proof***. First to see that we indeed have a topology, we need to check (by this prop.) that every point is contained in some base element, and that every point in the intersection of two base elements has a base neighbourhood that is still contained in that intersection.

So let $x \in X$ be a point. By the assumption that $X$ is semi-locally simply connected there exists an open neighbourhood $U_x \subset X$ such that every loop in $U_x$ on $x$ is contractible in $X$. By the assumption that $X$ is a locally path-connected topological space, this contains an open neighbourhood $U'_x \subset U_x$ which is path connected and, as every subset of $U_x$, it still has the property that every loop in $U'_x$ based on $x$ is contractible as a loop in $X$. Now let $\hat{x} \in E$ be any point over $x$, then it is contained in the base open $V_{U'_x, x}$.

The argument for the base open neighbourhoods contained in intersections is similar.

Then we need to see that $p \colon E(\rho) \to X$ is a continuous function. Since taking pre-images preserves unions (this prop.), and since by semi-local simply connectedness and local path connectedness every neighbourhood contains an open neighbourhood $U \subset X$ that labels a base open, it is sufficient to see that $p^{-1}(U)$ is a base open. But by the very assumption on $U$, there is a unique morphism in $\Pi_1(X)$ from any point $x \in U$ to any other point in $U$, so that $\rho$ applied to these paths establishes a bijection of sets

$$p^{-1}(U) \;\simeq\; \underset{\hat{x} \in \rho(x)}{\sqcup} V_{U,\hat{x}} \;\simeq\; U \times \rho(x)\,,$$

thus exhibiting $p^{-1}(U)$ as a union of base opens.

Finally we need to see that this continuous function $p$ is a covering projection, hence that every point $x \in X$ has a neighbourhood $U$ such that $p^{-1}(U) \simeq U \times \rho(x)$. But this is again the case for those $U$ all whose loops are contractible in $X$, by the above identification via $\rho$, and these exist around every point by semi-local simply-connetedness of $X$.

This shows that $p : E(\rho) \to X$ is a covering space. It remains to see that $\mathrm{Rec}(f) : E(\rho_1) \to E(\rho_2)$ is a homomorphism of covering spaces. Now by construction it is immediate that this is a function over $X$, in that this diagram commutes:

$$\begin{array}{ccccc} E(\rho_1) && \overset{\mathrm{Rec}(f)}{\longrightarrow} && E(\rho_2) \\ & \searrow && \swarrow \\ && X \end{array}\,.$$

So it only remains to see that $\mathrm{Rec}(f)$ is a continuous function. So consider $V_{U, y_2 \in \rho_2(x)}$ a base open of $E(\rho_2)$. By naturality of $f$

$$\begin{array}{ccc} \rho_1(x') & \overset{f_{x'}}{\longrightarrow} & \rho_2(x') \\ {}^{\mathllap{\rho_1(\gamma)}}\uparrow^{\mathrlap{\simeq}} && {}^{\mathllap{\simeq}}\uparrow^{\mathrlap{\rho_2(\gamma)}} \\ \rho_1(x) & \underset{f_x}{\longrightarrow} & \rho_2(x) \end{array}$$

its pre-image under $\mathrm{Rec}(f)$ is

$$\mathrm{Rec}(f)^{-1}(V_{U, y_2 \in \rho_2(x)}) = \underset{y_1 \in f^{-1}(y_2)}{\sqcup} V_{U, y_1 \in \rho_1(x)}$$

and hence a union of base opens. ▮

**Proposition 2.22**. ***(fundamental theorem of covering spaces)***

*Let $X$ be a locally path-connected and semi-locally simply-connected topological space. Then the operations on*

1. *extracting the monodromy $\mathrm{Fib}_E$ of a covering space $E$ over $X$*

2. *reconstructing a covering space from monodromy $\mathrm{Rec}(\rho)$*

*constitute an equivalence of categories*

$$\mathrm{Cov}(X) \underset{\mathrm{Fib}}{\overset{\mathrm{Rec}}{\underset{\longrightarrow}{\longleftarrow}}} \mathrm{Set}^{\Pi_1(X)} \;.$$

***Proof***. Given $\rho \in \mathrm{Set}^{\Pi_1(X)}$ a permutation representation, we need to exhibit a natural isomorphism of permutation representations.

$$\eta_\rho \;:\; \rho \longrightarrow \mathrm{Fib}(\mathrm{Rec}(\rho))$$

First consider what the right hand side is like: By this def. of $\mathrm{Rec}$ and this def. of $\mathrm{Fib}$ we have for every $x \in X$ an actual equality

$$\mathrm{Fib}(\mathrm{Rec}(\rho))(x) = \rho(x) \;.$$

To similarly understand the value of $\mathrm{Fib}(\mathrm{Rec}(\rho))$ on morphisms $[\gamma] \in \Pi_1(X)$, let $\gamma : [0,1] \to X$ be a representing path in $X$. We find, by the Lebesgue number lemma as in the proof of this lemmapace#CoveringSpacePathLifting), a finite number of paths $\{\gamma_i\}_{i \in \{1,n\}}$ such that

1. regarded as morphisms $[\gamma_i]$ in $\Pi_1(X)$ they compose to $[\gamma]$:

$$[\gamma] = [\gamma_n] \circ \cdots \circ [\gamma_2] \circ [\gamma_1]$$

2. each $\gamma_i$ factors through an open subset $U_i \subset X$ over which $\mathrm{Rec}(\rho)$ trivializes.

Hence by functoriality of $\mathrm{Fib}(\mathrm{Rec}(\rho))$ it is sufficient to understand its value on these paths $\gamma_i$. But on these we have again by direct unwinding of the definitions that

$$\mathrm{Fib}(\mathrm{Rec}(\rho))([\gamma_i]) = \rho([\gamma_i]) \;.$$

This means that if we take

$$\eta_\rho(x) : \rho(x) \overset{=}{\longrightarrow} \mathrm{Fib}(\mathrm{Rec}(\rho))$$

to be the above identification, then this is a natural transformation and hence in a particular a natural isomorphism, as required.

Conversely, given $E \in \mathrm{Cov}(X)$ a covering space, we need to exhibit a natural isomorphism of covering spaces of the form

$$\epsilon_E \;:\; \mathrm{Rec}(\mathrm{Fib}(E)) \longrightarrow E \;.$$

Again by this def. of $\mathrm{Rec}$ and this def. of $\mathrm{Fib}$ the underlying set of $\mathrm{Rec}(\mathrm{Fib}(E))$ is actually equal to that of $E$, hence it is sufficient to check that this identity function on underlying sets is a homeomorphism of topological spaces.

By the assumption that $X$ is locally path-connected and semi-locally simply connected, it is sufficient to check for $U \subset X$ an open path-connected subset and $x \in X$ a point with the property that $\pi_1(U,x) \to \pi_1(X,x)$ lands is constant on the trivial element, that the open subsets of $E$ of the form $U \times \{\hat{x}\} \subset p^{-1}(U)$ form a basis for the topology of $\mathrm{Rec}(\mathrm{Fib}(E))$. But this is the case by definition of $\mathrm{Rec}$.

This proves the equivalence.

(Notice that the assumption of local path-connectedness and semi-local simply-connectedness of $X$ is used only to guarantee that the functor $\mathrm{Rec}$ exists in the first place.) ▮

# 3. Examples

## Fundamental groups

**Proposition 3.1**. ***(fundamental group of the circle is the integers)***

*The fundamental group $\pi_1$ of the circle $S^1$ is the additive group of integers:*

$$\pi_1(S^1) \xrightarrow{\simeq} \mathbb{Z}$$

*and the isomorphism is given by assigning winding number.*

Here in the context of topological homotopy theory the circle $S^1$ is the topological subspace $S^1 = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$ of the Euclidean plane with its metric topology, or any topological space of the same homotopy type. More generally, the circle in question is, as a homotopy type, the homotopy pushout

$$S^1 \simeq * \coprod_{* \sqcup *} * \,,$$

hence the homotopy type with the universal property that it makes a homotopy commuting diagram of the form

$$\begin{array}{ccc} * \sqcup * & \longrightarrow & * \\ \downarrow & \swArrow & \downarrow \\ * & \longrightarrow & S^1 \end{array} .$$

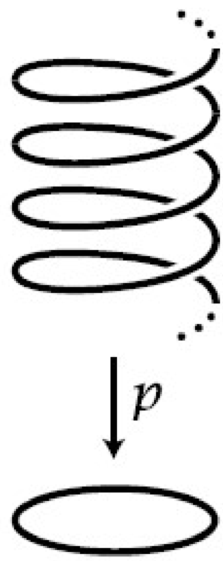
***Proof***. The universal covering space $\widehat{S^1}$ of $S^1$ is the real line (by this example):

$$p \coloneqq (\cos(2\pi(-)), \sin(2\pi(-))) \;:\; \mathbb{R}^1 \longrightarrow S^1 \,.$$

Since the circle is locally path-connected (this example) and semi-locally simply connected (this example) the fundamental theorem of covering spaces applies and gives that the automorphism group of $\mathbb{R}^1$ over $S^1$ equals the automorphism group of its monodromy permutation representation:

$$\mathrm{Aut}_{\mathrm{Cov}(S^1)}(\mathbb{R}^1) \;\simeq\; \mathrm{Aut}_{\pi_1(S^1)\mathrm{Set}}(\mathrm{Fib}_{S^1}) \,.$$

Moreover, as a corollary of the fundamental theorem of covering spaces we have that the monodromy representation of a universal covering space is given by the




action of the fundamental group $\pi_1(S)$ on itself (this prop.).

But the automorphism group of any group regarded as an action on itself by left multiplication is canonically isomorphic to that group itself (by this example), hence we have

$$\mathrm{Aut}_{\pi_1(S^1)\mathrm{Set}}(\mathrm{Fib}_{S^1}) \;\simeq\; \mathrm{Aut}_{\pi_1(S^1)\mathrm{Set}}(\pi_1(S^1)) \;\simeq\; \pi_1(S^1)\;.$$

Therefore to conclude the proof it is now sufficient to show that

$$\mathrm{Aut}_{\mathrm{Cov}(S^1)}(\mathbb{R}^1) \simeq \mathbb{Z}\;.$$

To that end, consider a homeomorphism of the form

$$\begin{tikzcd} \mathbb{R}^1 \arrow[rr, "f", "\simeq"'] \arrow[dr, "p"'] & & \mathbb{R}^1 \arrow[dl, "p"] \\ & S^1 & \end{tikzcd}\;.$$

Let $s \in S^1$ be any point, and consider the restriction of $f$ to the fibers over the complement:

$$\begin{tikzcd} p^{-1}(S^1 \setminus \{s\}) \arrow[rr, "f", "\simeq"'] \arrow[dr, "p"'] & & p^{-1}(S^1 \setminus \{s\}) \arrow[dl, "p"] \\ & S^1 \setminus \{s\} & \end{tikzcd}\;.$$

By the covering space property we have (via this example) a homeomorphism

$$p^{-1}(S^1 \setminus \{s\}) \simeq (0,1) \times \mathrm{Disc}(\mathbb{Z})\;.$$

Therefore, up to homeomorphism, the restricted function is of the form

$$\begin{tikzcd} (0,1) \times \mathrm{Disc}(\mathbb{Z}) \arrow[rr, "f", "\simeq"'] \arrow[dr, "\mathrm{pr}_1"'] & & (0,1) \times \mathrm{Disc}(\mathbb{Z}) \arrow[dl, "\mathrm{pr}_1"] \\ & (0,1) & \end{tikzcd}\;.$$

By the universal property of the product topological space this means that $f$ is equivalently given by its two components

$$(0,1) \times \mathrm{Disc}(\mathbb{Z}) \xrightarrow{\mathrm{pr}_1 \circ f} (0,1) \qquad (0,1) \times \mathrm{Disc}(\mathbb{Z}) \xrightarrow{\mathrm{pr}_2 \circ f} \mathrm{Disc}(\mathbb{Z})\;.$$

By the commutativity of the above diagram, the first component is fixed to be $\mathrm{pr}_1$. Moreover, by the fact that $\mathrm{Disc}(\mathbb{Z})$ is a discrete space it follows that the second component is a locally constant function (by this example). Therefore, since the product space with a discrete space is a disjoint union space (via this example)

$$(0,1) \times \mathrm{Disc}(\mathbb{Z}) \simeq \underset{n \in \mathbb{Z}}{\sqcup} (0,1)$$

and since the disjoint summands $(0,1)$ are connected topological spaces (this example), it follows that the second component is a constant function on each of these summands (by this example).

Finally, since every function out of a discrete topological space is continuous, it follows in conclusion that the restriction of $f$ to the fibers over $S^1 \setminus \{s\}$ is entirely encoded in an endofunction of the set of integers

$$\phi : \mathbb{Z} \to \mathbb{Z}$$

by

$$\begin{array}{ccc} S^1 \setminus \{s\} \times \mathrm{Disc}(\mathbb{Z}) & \overset{f}{\longrightarrow} & S^1 \setminus \{s\} \times \mathrm{Disc}(\mathbb{Z}) \\ (t,k) & \mapsto & (t, \phi(k)) \end{array} .$$

Now let $s' \in S^1$ be another point, distinct from $s$. The same analysis as above applies now to the restriction of $f$ to $S^1 \setminus \{s'\}$ and yields a function

$$\phi' : \mathbb{Z} \longrightarrow \mathbb{Z} .$$

Since

$$\left\{ p^{-1}(S^1 \setminus \{s\}) \subset \mathbb{R}^1 , p^{-1}(S^1 \setminus \{s'\}) \subset \mathbb{R}^1 \right\}$$

is an open cover of $\mathbb{R}^1$, it follows that $f$ is unqiuely fixed by its restrictions to these two subsets.

Now unwinding the definition of $p$ shows that the condition that the two restrictions coincide on the intersection $S^1 \setminus \{s, s'\}$ implies that there is $n \in \mathbb{Z}$ such that $\phi(k) = k + n$ and $\phi'(k) = k + n$.

This shows that $\mathrm{Aut}_{\mathrm{Cov}(S^1)}(\mathbb{R}^1) \simeq \mathbb{Z}$. ▮

# Higher homotopy groups

(…)

---

This concludes the introduction to basic homotopy theory.

For introduction to more general and abstract homotopy theory see at *Introduction to Homotopy Theory*.

An incarnation of homotopy theory in linear algebra is *homological algebra*. For introduction to that see at *Introduction to Homological Algebra*.

---

# 4. References

A textbook account is in


- Tammo tom Dieck, sections 2 an 3 of *Algebraic Topology*, EMS 2006 (pdf)


Lecture notes include


- Jesper Møller, *The fundamental group and covering spaces* (2011) (pdf)


*Revised on July 17, 2017 18:18:28 by Urs Schreiber*