12

Non-composite fermion approaches

Analogies play a central role in physics. The CF theory was motivated by the analogy between the fractional and the integral quantum Hall effects. Other analogies were proposed prior to the CF theory. As discussed in Section 5.5, the Laughlin wave function was motivated by the Jastrow wave functions used earlier for ^{4}He superfluids. A "composite boson" approach modeled the physics of the Laughlin wave function after Bose–Einstein condensation. A "hierarchy" approach proposed understanding the FQHE at non-Laughlin fractions by using the Laughlin wave function as the basic building block. Both the hierarchy and composite boson approaches take the Laughlin wave function as their starting point. These ideas, as seen below, are distinct from the CF theory.

This chapter also presents certain other non-CF works. The simple Jastrow form of the Laughlin wave function allows for certain technical simplifications; in particular, a mapping into a classical plasma enables alternative, but $1/m$-specific, derivations of certain properties that were obtained in earlier chapters by other means. In addition, we describe two quantitative approaches for excitations without using composite fermions: Laughlin's wave functions for the quasiparticle and quasihole at $\nu = 1/m$, and Girvin, MacDonald, and Platzman's "single-mode approximation" for neutral excitations.

We also briefly outline the hydrodynamic theory of Conti and Vignale, and Tokatly, which treats the correlated liquid state in the lowest Landau level as a continuous elastic medium, and formulates its collective dynamics in terms of the displacement field. Quantization of this field yields bosonic excitation modes, analogous to the phonons of a crystal. This approach has been applied to the bulk and the edge of incompressible and compressible states.

12.1 Hierarchy scenario

An appealing hierarchical construction was advanced by Haldane [221] and Halperin [230] in an attempt to deal with the non-Laughlin odd-denominator fractions. The basic idea is to build Laughlin-like "daughter" states of the fractionally charged, anyonic quasiparticles of a given "parent" state, and thus construct new states iteratively starting from the Laughlin fractions. All odd-denominator fractions are thus produced. The fractions at any level of the hierarchy are expressed as continued fractions; for example, the fractions at the

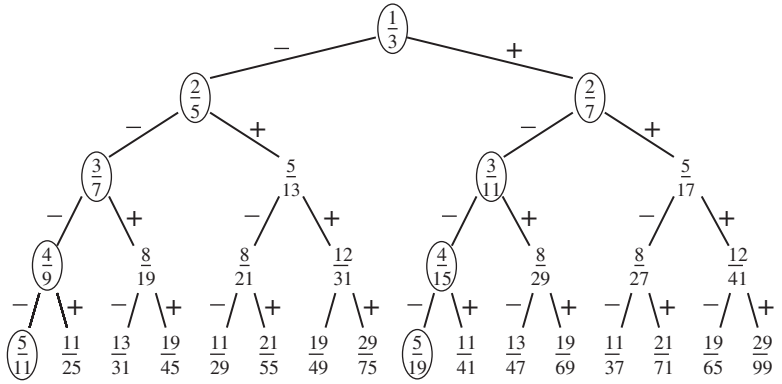

Fig. 12.1. Hierarchical family tree stemming from 1/3. The observed fractions are circumscribed.

fifth level of the hierarchy are given by

$$f = \cfrac{1}{m \pm \cfrac{1}{p_2 \pm \cfrac{1}{p_3 \pm \cfrac{1}{p_4 \pm \cfrac{1}{p_5}}}}}\,, \tag{12.1}$$

where p_j is an even integer. Taking the simplest values $m = 3$ and $p_j = 2$, the hierarchy generates the fractions shown in Fig. 12.1, which have a family tree structure with two children at each generation. The 2^{n-1} fractions at the nth level correspond to different choices of signs in the continued fraction. The convention used in the figure is that the levels of the hierarchy evolve downward, with the positive sign in Eq. (12.1) taking us south-east and the negative sign taking us south-west.

Even though the innovative hierarchy idea appeared natural at first, compatible with the Landau philosophy of working with the deviations from a known ground state, inconsistencies with experiment became evident as more facts were gathered. Essentially, such an approach is expected to be valid *close* to the starting point (the Laughlin state), but an explanation of experimental phenomena in FQHE requires us to go far from it. Let us summarize the problems. (Because it has not been possible to make reliable quantitative predictions in this approach, the comparisons with experiment are of a qualitative nature.)

(a) A fundamental aspect of the hierarchy approach is a parent–daughter relation between fractions. Because each state has two equally plausible daughters (one constructed from quasiparticles and the other from quasiholes), this manifests through a family tree structure for fractions. The appearance of *sequences* of fractions in actual experiments reveals a different underlying structure. (b) The order of stability of fractions predicted by the hierarchy is incompatible with experiment. One would expect all fractions on a given

level to be roughly equally stable. However, while some fractions at very deep levels have been observed, a large number of fractions at earlier levels have not. For example, the tenth generation of the $1/3$ family tree consists of 2^9 fractions (assuming only two daughters at each generation), but only one of them ($10/21$) has been observed. (c) Even if one confines oneself to the observed fractions, the gaps decrease much more slowly than implied by the general considerations of the hierarchy scenario, because a daughter, being a *fractional* quantum Hall state of the quasiparticles, is expected to be much weaker than the parent. (d) For the same reason, the very observation of a large number of fractions is inconsistent with the hierarchy idea.

The CF theory is not a hierarchy. No parent–daughter relationship exists between the fractions of a given sequence $\nu = n/(2pn \pm 1)$, just as none does between the integers of the IQHE. Composite fermions allow us to understand $3/7$ perfectly well without making any reference to $2/5$ (or to any other FQHE state); if one insisted on ascribing a relation to them, the two would be siblings rather than daughter and parent. In the CF theory, all fractions are explained on an equal footing, each fraction stands on its own, and $1/3$ is not the mother of all fractions.[1]

From a theoretical perspective, the hierarchy scenario runs into a conceptual impediment that cannot be overcome within this approach. To illustrate, let us begin with $\nu = 1/3$. As the magnetic field is varied, quasiparticles are produced, one for each additional flux quantum. Due to their nonzero size, however, they are not ideal anyons. They have a well-defined braiding statistics only when the overlap between them is negligible, which, as seen in Chapter 9 (see Figs. 9.5 and 9.6), requires them to be ≥ 10 magnetic lengths apart. At shorter distances, significant corrections to the braiding statistics appear. That presents a difficulty for hierarchy. Simple counting shows that to reach $\nu = 2/5$, which is supposed to be the first "daughter" state, it would be necessary to create half as many quasiparticles as the total number of electrons. At such high densities, the quasiparticles are so strongly overlapping (the interelectron separation is $\sim$3.5 magnetic lengths at $\nu = 1/3$) that the model of quasiparticles with well-defined fractional charge and fractional braiding statistics becomes questionable. An explanation of $10/21$, for which experimental evidence exists (see, for example, Ref. [494]), would require that a macroscopically large number of quasiparticles of $1/3$ make a daughter state at $2/5$; a macroscopically large number of quasiparticles of $2/5$ produce a daughter at $3/7$; and the process be repeated seven more times. If the very first link is broken, the subsequent generations remain unborn.

The point is that the statistics of an emergent entity is not absolute (unlike the Fermi statistics of the electron) but has a limited range of validity. As an illustration, let us consider the more familiar example of a collection of another set of composite particles: ^{4}He atoms. When they are far from one another, they are well described as bosons, and indeed exhibit Bose–Einstein condensation. Let us now imagine applying pressure to bring ^{4}He atoms

[1] Some fractions, such as $4/11$, which are understood as *fractional* QHE of composite fermions, are sometimes thought of as the second level of a CF hierarchy. That hierarchy, however, is distinct from the Haldane–Halperin hierarchy being discussed here. The latter also produces these fractions by appropriate choices of parameters in Eq. (12.1).

closer to one another. Eventually, under extreme pressure, when they begin to overlap strongly, the system goes into a "plasma" phase, wherein electrons are no longer bound to their parent nuclei (as in a white dwarf). A theory in terms of bosonic ^{4}He variables is no longer valid, and indeed no Bose–Einstein condensation would occur. A description of the plasma phase is impossible without knowing the constituents of the ^{4}He boson, and must be formulated in terms of electrons and ^{4}He nuclei.

The anyon language is similarly inapplicable for high densities of the FQHE quasiparticles.[2] The CF theory identifies the more fundamental entities – composite fermions – which provide a valid description at low energies in the entire relevant filling factor range. (For sufficiently high energy states, the composite fermion description also becomes inadequate and one must revert to electrons.)

Haldane's version of the hierarchy [221, 226] assigns, nominally, bosonic braiding statistics to the quasiparticles and quasiholes, but otherwise produces the same structure as the Halperin hierarchy. As underscored by Haldane, the hierarchy construction is valid only under the assumption that the interaction between the quasiparticles is sufficiently strongly repulsive at short distances, i.e., is dominated by the short-range pseudopotentials. That is not borne out by explicit calculation (Section 6.7), which shows that the interaction is only weakly repulsive at short distances, and sometimes even attractive.

The reader may encounter in the literature statements to the effect that the CF and the hierarchy approaches are "equivalent" [538]. That is incorrect, for reasons explained above. The claims of equivalence are based on the concurrence of the two approaches in one narrow aspect, namely in the values of fractional local charge and fractional braiding statistics for quasiparticles. However, not much significance ought to be attached to this fact, given that a determination of these values does not require an understanding of the physical origin of the FQHE; they can be derived from general principles (Su [629]) simply by assuming incompressibility at a fractional filling factor. They are a consequence, not a cause, of incompressibility, and, therefore, it is not surprising that different theories produce the same values for them. The CF and hierarchy approaches attribute different physics to the origin of the FQHE, and have contrasting testable consequences. (Also see the last paragraph of the next section.) In particular, the hierarchy approach does not contain composite fermions and the vast body of physics that follows from them.

12.2 Composite boson approach

Girvin and MacDonald [193] interpreted Laughlin's wave function as a Bose condensate, a notion that was further developed by Zhang, Hansson, and Kivelson [743] and Read [537], and later came to be known as the "composite boson" approach. The idea, roughly, is as follows: For the Laughlin wave function, it appears natural to take the bound state of an electron and an *odd* number (m) of vortices as the fundamental object, which is called a

[2] Its validity is not entirely obvious even in the dilute limit. For ^{4}He atoms, the hard-core repulsion protects their bosonic statistics at low densities. The interaction between the FQHE quasiparticles, on the other hand, is rather weak, and sometimes even attractive (Fig. (6.6)). It is thus ineffective in suppressing overlapping configurations even for dilute densities.

composite boson. Crudely, this amounts to writing the Laughlin wave function as

$$\Psi_{1/m} = \prod_{j<k} (z_j - z_k)^m \Psi_{\rm B}(\{z_i\}) \ , \tag{12.2}$$

where $\Psi_{\rm B}(\{z_i\}) = \exp[-\sum_j |z_j|^2/4]$. Since $\Psi_{\rm B}(\{z_i\})$ is symmetric under exchange and everywhere positive, it is tempting to view $\Psi_{1/m}$ as a Bose condensate of composite bosons.

This is to be contrasted with the CF interpretation as one filled Λ level of composite fermions, as expressed in Eq. (5.37), and reproduced here for convenience:

$$\Psi_{1/m} = \prod_{j<k} (z_j - z_k)^{2p} \Phi_1 \ , \tag{12.3}$$

with $m = 2p + 1$. From the CF perspective, it is no more than a coincidence that the Pauli correlations in Φ_1 appear through the binding of precisely one vortex to each electron (which is not the case if Φ_1 is replaced by Φ_n), which combines with the $2p$ vortices of composite fermions to produce a total of $2p + 1$ vortices bound to each electron.

The analogies of Bose–Einstein condensation and the integral quantum Hall effect are equally plausible for the Laughlin ground state wave function at $\nu = 1/m$. A consideration of the physics beyond the $\nu = 1/m$ ground state clarifies that the two views are distinct. The bosonic approach does not extend to fractions other than $\nu = 1/m$.

This offers a curious example of how different approaches may appear "natural" depending on one's perspective. The composite boson interpretation seems most obvious if one takes the Laughlin wave function as the point of departure. Even someone with an overactive imagination would have no reason to write it as in Eq. (5.37) and interpret it as one filled Λ level of composite fermions. On the other hand, if one takes the analogy between the FQHE and the IQHE as the guiding principle, then the composite fermion physics and Eq. (5.37) appear natural.

The question often arises why electrons capture an even, rather than an odd, number of vortices. What bound states are formed is often a complicated issue even in few-body systems, the resolution of which requires detailed microscopic calculations and a careful comparison with experiments. Some insight into why composite fermions are preferred can be gained from the observation that the Pauli repulsion between fermions helps produce desirable correlations.[3]

Some may find it disappointing that the FQHE does not lend itself to an explanation in the familiar language of Bose–Einstein condensation. On the other hand, it can be argued that the lack of BEC and order parameter makes the physics of this quantum liquid more unusual and interesting.

[3] Bosons with strong hard-core repulsive interaction in one dimension emulate fermions for the same reason. The ground state wave function is given by $\Psi^{\rm B} = |\Psi^{\rm F}|$, where $\Psi^{\rm F}$ is the wave function for the Fermi sea (Girardeau [187]). Another example is that of interacting *bosons* in the lowest Landau level, which can be produced by rotating a bosonic trap; they are found, theoretically, to capture a single vortex and behave like composite fermions [69, 96, 98, 427, 541–543, 656, 679].

To summarize: It has not been possible to translate the hierarchy or the composite boson ideas into reliable microscopic theories that are amenable to quantitative tests. Additionally, a large and growing body of experimental facts is incompatible with these ideas. Prominent among these are: the FQHE at non-Laughlin fractions; similarity between the FQHE and the IQHE; appearance of sequences of fractions; filling factor dependence of gaps; existence of composite fermions; the compressible CF Fermi sea ($\nu = 1/2$); the effective magnetic field; the paired CF state ($\nu = 5/2$); non-fully spin-polarized FQHE; and a plethora of excitations and other phenomena.

12.3 Response to Laughlin's critique

Laughlin expressed reservations about the CF theory in his 1998 Nobel Lecture (Laughlin [372]), which have caused some confusion and, therefore, deserve clarification.[4] To quote:

> Fractional quantum Hall quasiparticles are the elementary excitations of a distinct state of matter that cannot be deformed into noninteracting electrons without crossing a phase boundary. That means that they are different from electrons in the only sensible way we have of defining different, and in particular are not adiabatic images of electrons the way quasiparticle excitations of metals and band insulators are. Some composite fermion enthusiasts claim otherwise – that these particles are nothing more than screened electrons (Jain, 1989) – but this is incorrect. The alleged screening process always runs afoul of a phase boundary at some point, in the process doing some great violence to the ground state and low-lying excitations. I emphasize these things because there is a regrettable tendency in solid-state physics to equate an understanding of nature with an ability to model, an attitude that sometimes leads to overlooking or misinterpreting the higher organizing principle actually responsible for an effect. In the case of the integral or fractional quantum Hall effects, the essential thing is the accuracy of quantization. No amount of modeling done on any computer, existing or contemplated, will ever explain this accuracy by itself. Only a thermodynamic principle can do this. The idea that the quasiparticle is only a screened electron is unfortunately incompatible with the key principle at work in these experiments. If carefully analyzed it leads to the false conclusion that the Hall conductance is integrally quantized.

We take the two principal objections raised in this paragraph one by one. Composite fermions have indeed been described as "screened electrons." Screening is perturbative in many familiar applications, but it has been stressed (see Section 9.4 and Ref. [201]) that the screening of an electron into a composite fermion, which occurs through the binding of exactly an even number of vortices to electrons, is quantized and nonperturbative. A vortex cannot be attached adiabatically – either all or none of it is bound to an electron. Composite fermions are thus topologically distinct from electrons and cannot be obtained from electrons in any perturbative treatment. This is also the main point of the topological Chern–Simons approaches. Therefore, the objection following from an assumed adiabatic connectivity between composite fermions and electrons is not valid. (The last statement

[4] This section owes a great deal to many discussions with Fred Goldhaber.

in Laughlin's critique cannot be addressed in the absence of explicit details of the stated analysis.)

Let us next turn to the question of "modeling" versus "understanding." Laughlin rightly notes that modeling cannot replace understanding. That would be a serious criticism, for example, against presenting the exact numerical solutions of the Schrödinger equation as an "explanation" of the FQHE. That is not the case with the composite fermion theory. In fact, the question, "What thermodynamic principle explains the vast body of experimental facts pertaining to the FQHE and other related phenomena?", has only one answer: "The formation of composite fermions." Composite fermions provide an understanding of the origin of incompressibility at fractional fillings and a multitude of other facts without resorting to any computer modeling. The usefulness of computer modeling lies in that it yields a detailed and undeniable confirmation of the CF principle. Being exact, computer studies are especially powerful in the FQHE, which is why Laughlin tested and confirmed his own wave function in numerical calculations on small systems [369]. The statement about computer calculations being incapable of explaining the accuracy of the Hall quantization is valid, but again not an objection to the CF theory. The CF principle produces incompressibility at certain special fractional fillings; such incompressibility, as first pointed out by Laughlin, leads to a precisely quantized Hall resistance. The exactness of Hall quantization is not related to the quantitative accuracy of the wave functions.

Laughlin does acknowledge in the same article the explanation of the half-filled Landau level state as a Fermi sea of composite fermions. To quote: "And of course there is the discovery of the strange Fermi surface at half-filling and its explanation in terms of composite fermions by Bert Halperin, Patrick Lee, and Nick Read [231] that is now defining the intellectual frontier in this field."

Laughlin also stresses elsewhere in the article the appearance of an induced gauge interaction between the "quasiparticle defects" of an incompressible FQHE state, revealed by their fractional braiding statistics. He suggests the possibility that the gauge interaction of the standard model of particle physics might similarly be an emergent phenomenon in a more fundamental theory that does not postulate this gauge interaction at the outset. This is an interesting idea. But the braiding statistics of the quasiparticles defects, which is feeble and as yet unobserved,[5] is itself a consequence of a much more robust emergent gauge interaction between composite fermions (Section 9.8.2). The latter causes a substantial effective reduction of the external magnetic field, which has been verified through its numerous experimental consequences. The "quarks" of the FQHE are not the quasiparticle defects of an incompressible state of composite fermions but composite fermions themselves.[6]

[5] As discussed in Section 9.8.3, the effect of the statistical gauge interaction between distant quasiparticles is negligible compared with the influence of the strong magnetic field perpendicular to the Hall plane. We might envision making the statistical interaction strong by creating a high density of quasiparticles, but then the whole idea of fractional braiding statistics falls apart.

[6] Yang and Mills originally applied the concept of non-Abelian gauge interaction to the "wrong" particles, the nucleons, rather than quarks (which were discovered only later).

12.4 Two-dimensional one-component plasma (2DOCP)

Due to its simple Jastrow form, the Laughlin wave function is amenable to a mapping into the statistical mechanics of a two-dimensional one-component classical plasma. The probability of finding particles in a configuration $\{z_j\}$ is proportional to

$$|\Psi(\{z_j\})|^2 = \exp\left[2m \sum_{k<j} \ln|z_k - z_j| - \frac{1}{2\ell^2} \sum_k |z_k|^2 \right] . \tag{12.4}$$

This quantity can be interpreted as the Boltzmann probability factor for a classical gas of charged "particles" in a uniform neutralizing background in two dimensions, namely a two-dimensional one-component plasma (2DOCP). These "particles" live in a strictly two-dimensional space, and are distinct from electrons, which are three-dimensional particles confined to move in two dimensions. To distinguish the two, we refer to the former as "particles" (with quotation marks). The number of "particles" is the same as the number of electrons, along with an exact correspondence between their positions. But the "particles" do not have the same charge and interaction as the electrons.

To see the mapping between the two problems, we recall some facts from 2D classical electrodynamics. Gauss's law in two dimensions is

$$\nabla^2 V(\boldsymbol{r}) = -4\pi\rho(\boldsymbol{r}) . \tag{12.5}$$

The potential of a point charge of unit strength at the origin satisfies

$$\nabla^2 V(\boldsymbol{r}) = -4\pi\delta^{(2)}(\boldsymbol{r}) . \tag{12.6}$$

The identity

$$\nabla^2 \ln(r) = 2\pi\delta^{(2)}(\boldsymbol{r}) , \tag{12.7}$$

is analogous to $\nabla^2(1/r) = -4\pi\delta^{(3)}(\boldsymbol{r})$ of three dimensions, and gives the form of the Coulomb potential in two dimensions as

$$V(\boldsymbol{r}) = -2\ln r . \tag{12.8}$$

Proof We prove Eq. (12.7). We have

$$\nabla^2 \ln r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\ln r \right) = 0 \tag{12.9}$$

for $r \neq 0$. (For $r = 0$, the quantity inside the brackets is ill defined.) Now consider a closed loop C around the origin enclosing an area σ. Then

$$\int_\sigma \mathrm{d}\boldsymbol{a}\, \nabla^2 \ln r = \int_\sigma \mathrm{d}\boldsymbol{a}\, \nabla \cdot (\nabla \ln r) \tag{12.10}$$

$$= \oint_C \nabla \ln r \times \mathrm{d}\boldsymbol{l} \tag{12.11}$$

$$= \oint_C \frac{\hat{\boldsymbol{r}}}{r} \times (\mathrm{d}r\, \hat{\boldsymbol{r}} + r\, \mathrm{d}\theta\, \hat{\boldsymbol{\theta}}) \tag{12.12}$$

$$= 2\pi \hat{\boldsymbol{z}} \, . \tag{12.13}$$

Here, as usual, the area element $\mathrm{d}\boldsymbol{a}$ points perpendicular to the plane (in the z direction), and we have used Stokes's theorem:[7]

$$\int_\sigma \mathrm{d}\boldsymbol{a}\, \nabla \cdot \boldsymbol{A} = \oint_C \boldsymbol{A} \times \mathrm{d}\boldsymbol{l} \, . \tag{12.15}$$

□

For a uniform charge density, we have

$$\nabla^2 V(\boldsymbol{r}) = -4\pi \bar{\rho} \, , \tag{12.16}$$

the solution for which is

$$V(\boldsymbol{r}) = -\pi \bar{\rho} r^2 \, , \tag{12.17}$$

because $\nabla^2 r^2 = (\partial_x^2 + \partial_y^2)(x^2 + y^2) = 4$.

The electrostatic energy of particles of charge e_j interacting with a uniform background charge density $\bar{\rho}$ is given by

$$E(\{\boldsymbol{r}_j\}) = -2 \sum_{j<k} e_j e_k \ln |\boldsymbol{r}_j - \boldsymbol{r}_k| - \pi \bar{\rho} \sum_i e_i |\boldsymbol{r}_i|^2 \, . \tag{12.18}$$

For "particles" of charge

$$e_j = -1 \tag{12.19}$$

in a uniform background charge density

$$\bar{\rho} = \frac{1}{2\pi m \ell^2} \, , \tag{12.20}$$

[7] This is the two-dimensional version of the divergence theorem. It can be derived from the usual form of Stokes's theorem

$$\int_\sigma (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{a} = \oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} \tag{12.14}$$

by substituting $\boldsymbol{A} \to \boldsymbol{A} \times \boldsymbol{B}$, where $\boldsymbol{B}$ is a constant vector in the 2D plane.

we have

$$E(\{\boldsymbol{r}_j\}) = -2\sum_{j<k}\ln|\boldsymbol{r}_j-\boldsymbol{r}_k| + \frac{1}{2m\ell^2}\sum_i |\boldsymbol{r}_i|^2 \,. \tag{12.21}$$

We can therefore write

$$|\Psi_{1/m}(\{z_j\})|^2 = \exp[-\beta E(\{z_j\})] \,, \tag{12.22}$$

with

$$\beta = m \,. \tag{12.23}$$

Various diagonal correlation functions for $\Psi_{1/m}$ are the same as the corresponding correlation functions of the 2DOCP. For example, the pair correlation function is given by

$$\begin{aligned} g(\boldsymbol{r}_1,\boldsymbol{r}_2) &= \frac{N(N-1)}{\rho^2}\int \mathrm{d}^2\boldsymbol{r}_3\ldots\mathrm{d}^2\boldsymbol{r}_N\,|\Psi_{1/m}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)|^2 \\ &= \frac{N(N-1)}{\rho^2}\int \mathrm{d}^2\boldsymbol{r}_3\ldots\mathrm{d}^2\boldsymbol{r}_N\,\exp[-\beta E(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)] \,. \end{aligned} \tag{12.24}$$

What does the analogy to the 2DOCP tell us?

- It is intuitively obvious that the classical plasma is overall charge neutral. The wave function Ψ, therefore, describes a uniform density system.
- The charge neutrality implies that the number density of the "particles" is equal to $\bar{\rho}$. Because the "particles" and electrons have the same number density, the density of electrons is also $\bar{\rho} = 1/(2\pi m\ell^2)$. The filling factor of the state described by $\Psi_{1/m}$ is

$$\nu = \frac{\rho\phi_0}{B} = 2\pi\ell^2\rho = \frac{1}{m} \,. \tag{12.25}$$

- The overall charge neutrality, by itself, does not imply that the state is a liquid; that would require a calculation of the pair correlation function. One can, however, make certain statements from the intuitive insight gained by mapping into the 2DOCP. Because the exponent m is inversely proportional to the temperature, we expect the "particles" to form a crystal for large m, which would imply a crystal state also for electrons. As m is reduced (i.e., the temperature is increased), the crystal eventually melts. From Monte Carlo simulations of the 2DOCP [48] this transition is known to occur at $m \approx 72$. The Laughlin wave function with $m > 72$ thus describes a crystal. This crystal has no relevance to the actual crystal state at low fillings (more on that in Chapter 15), because the Laughlin wave function loses its validity well before the filling factor reaches such small values.

12.5 Charged excitations at $\nu = 1/m$

We introduced in Section 5.5 Laughlin's wave function for the $\nu = 1/m$ ground state. He also wrote trial wave functions for the quasihole and quasiparticle excitations of this state.

12.5.1 Laughlin's trial wave functions

Laughlin showed that an excited state can be obtained by inserting a point flux tube into an incompressible state and adiabatically increasing the flux through it from zero to one flux quantum (Section 9.3.4). This thought experiment produces, in principle, the exact excitation, and suggests a construction for approximate trial wave functions for it. A single particle state evolves according to

$$\eta_l \to \eta_{l\pm 1} \,, \tag{12.26}$$

where

$$\eta_l = (2\pi 2^l l!)^{-1/2} z^l \mathrm{e}^{-\frac{1}{4}|z|^2} \,, \tag{12.27}$$

and the sign depends on the direction of the inserted flux, which has been taken to be at the origin. An approximate trial wave function can be constructed by first expressing the ground state wave function in terms of basis functions:

$$\Psi_{1/m} = \sum_{\{l_j\}} C_{\{l_j\}} \left(\prod_j \eta_{l_j}(z_j) \right) \tag{12.28}$$

and then making the above replacement for each particle. The resulting wave function is still too complicated. Laughlin further simplified the problem by not worrying about the prefactors but only the exponents, to write

$$\Psi_{1/m}^{\mathrm{L-qh}} = \mathrm{e}^{-\frac{1}{4}\sum_l |z_l|^2} \left(\prod_i z_i \right) \prod_{j<k} (z_j - z_k)^m \tag{12.29}$$

for the quasihole, and

$$\Psi_{1/m}^{\mathrm{L-qp}} = \mathrm{e}^{-\frac{1}{4}\sum_l |z_l|^2} \left(\prod_i \frac{\partial}{\partial z_i} \right) \prod_{j<k} (z_j - z_k)^m \tag{12.30}$$

for the quasiparticle.[8] These describe a quasihole and a quasiparticle at the origin.

As discussed in Section 9.3.5, the local charge of the quasiparticle or the quasihole can be calculated from its wave functions, which, in general, requires evaluation of multi-dimensional integrals. The plasma analogy provides a simple method for obtaining the local charge of the quasihole at $\nu = 1/m$. For a quasihole at $\boldsymbol{r}_0$, with $\Psi_{1/m}^{\mathrm{L-qh}} = \prod_j \left(z_j - z_0\right) \Psi_{1/m}$, we have

$$|\Psi_{1/m}^{\mathrm{L-qh}}|^2 = \mathrm{e}^{-\beta E} \,, \tag{12.31}$$

where $\beta = m$, and

$$E(\{\boldsymbol{r}_j\}) = -2 \sum_{j<k} \ln |\boldsymbol{r}_j - \boldsymbol{r}_k| - \frac{2}{m} \sum_j \ln |\boldsymbol{r}_j - \boldsymbol{r}_0| + \frac{1}{2m\ell^2} \sum_i |\boldsymbol{r}_i|^2 \,. \tag{12.32}$$

[8] This excitation was originally called a "quasielectron" [369]. Because it is fundamentally distinct from an electron, we prefer the name "quasiparticle."

This is the energy of "particles" of charge -1 in a uniform background, as before, but with an additional external charge $-1/m$ at $\boldsymbol{r}_0$. The "particles" are repelled by this external charge, and because the plasma screens completely, precisely $1/m$ of a "particle" is missing from the vicinity of $\boldsymbol{r}_0$. In the electron system, precisely $1/m$ of an electron is missing from the neighborhood of $\boldsymbol{r}_0$, giving the fractional charge $e^* = e/m$. The local charge of Laughlin's *quasiparticle* ($\Psi_{1/m}^{\mathrm{L-qp}}$) at $\nu = 1/m$ cannot be obtained by this method, but Monte Carlo studies [345] have demonstrated that it has the correct charge $e^* = -e/3$.

The plasma analogy does not apply to the general FQHE states, or even to improved wave functions at $\nu = 1/m$. (Recall that the Laughlin wave function, although very good, is not exact.) Wave function independent derivations at the fractions $\nu = n/(2pn \pm 1)$ are given in Section 5.10 for uniform density, and in Section 9.3 for the local charges of the quasiparticle and quasihole.

12.5.2 Comparison with the CF theory

The CF-quasiparticles and CF-quasiholes for the general FQHE state are described in Section 5.9.4 (Eqs. 5.60 and 5.61). We consider here the special cases of CF-quasiparticle and CF-quasihole at $\nu = 1/m$, shown pictorially in Fig. 12.2, and ask how their wave functions compare with Laughlin's.

A single quasihole The wave function for a CF-quasihole is related to the wave function of the state at $\nu = 1$ with one electron removed. If we remove the electron at the origin (i.e., from the zero angular momentum orbital), the wave function is given by

$$\Phi_1^{\mathrm{hole}} = \begin{vmatrix} z_1 & z_2 & z_3 & \cdot & \cdot \\ z_1^2 & z_2^2 & z_3^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \exp\left[-\frac{1}{4}\sum_i |z_i|^2\right] \tag{12.33}$$

$$= \left(\prod_j z_j\right) \prod_{j<k} (z_j - z_k) \exp\left[-\frac{1}{4}\sum_i |z_i|^2\right]. \tag{12.34}$$

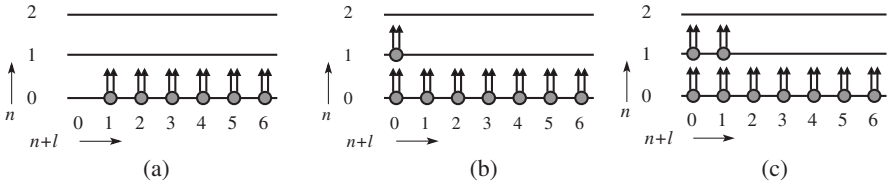

Fig. 12.2. (a) A single CF-quasihole at the origin. (b) A single CF-quasiparticle at the origin. (c) Two CF-quasiparticles at the origin. The disk geometry is assumed, and the filling factor is $\nu = 1/3$. The angular momentum is denoted by l, and the Λ level index by n.

The wave function for the CF-quasihole at the origin is, then,

$$\Psi^{\mathrm{CF-qh}}_{\frac{1}{2p+1}} = \left(\prod_j z_j\right) \Psi_{\frac{1}{2p+1}} \,, \tag{12.35}$$

which matches Laughlin's ansatz in Eq. (12.29), but has a physical interpretation as a missing composite fermion. The wave function is obtained more elegantly than the steps leading to Eq. (12.29).

A single quasiparticle A CF-quasiparticle is a single composite fermion in an otherwise empty Λ level, shown in Fig. 12.2(b). The wave function for the state containing an additional electron in the second Landau level (at the origin) is

$$\begin{aligned}
\Phi_1^{\mathrm{particle}} &= \begin{vmatrix} z_1^* & z_2^* & z_3^* & . & . \\ 1 & 1 & 1 & . & . \\ z_1 & z_2 & z_3 & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{vmatrix} \exp\left[-\frac{1}{4}\sum_i |z_i|^2\right] \\
&= \sum_{i=1}^{N} (-1)^{i+1} z_i^* \prod_{j<k}{}' (z_j - z_k) \exp\left[-\frac{1}{4}\sum_i |z_i|^2\right] \\
&= \left[\sum_{i=1}^{N} \frac{z_i^*}{\prod_j{}'(z_i - z_j)}\right] \Phi_1 \,,
\end{aligned} \tag{12.36}$$

where the prime denotes the condition $j,k \neq i$. To obtain the wave function for the CF-quasiparticle, we composite-fermionize $\Phi_1^{\mathrm{particle}}$ by first multiplying it by $\prod_{j<k}(z_j - z_k)^{2p}$ and then projecting the product into the lowest Landau level using the methods in Section 5.14. This yields

$$\begin{aligned}
\Psi^{\mathrm{CF-qp}}_{\frac{1}{2p+1}} &= \exp\left[-\frac{1}{4}\sum_i |z_i|^2\right] \left[\sum_{i=1}^{N} \frac{2\frac{\partial}{\partial z_i}}{\prod_j{}'(z_j - z_i)}\right] \prod_{l<m} (z_l - z_m)^{2p+1} \\
&= \sum_{i=1}^{N} \frac{2(2p+1)\sum_k{}'(z_i - z_k)^{-1}}{\prod_j{}'(z_j - z_i)} \Psi_{\frac{1}{2p+1}} \,.
\end{aligned} \tag{12.37}$$

(The presence of $(z_j - z_k)$ factors in the denominator is not a problem because they are canceled by similar factors in the numerator.) The CF-quasiparticle wave function is different from Laughlin's. The comparisons in Table 12.1 and in Fig. 12.3 demonstrate this wave function to be better. Similar conclusions are reached by other studies (Girlich and M. Hellmund [188]; Kasner and Apel [328]; Melik-Alaverdian and Bonesteel [436]).

Several other qualitative facts, including the asymmetry between the quasiparticle and the quasihole, find an intuitive explanation in CF theory. These are discussed in Section 5.9.4 in a more general context.

Table 12.1. *Testing two trial wave functions for the $\nu = 1/3$ quasiparticle*

N	D	CF	Laughlin
3	3	1	1
4	11	0.9969	0.9987
5	46	0.9930	0.9967
6	217	0.9941	0.9885
7	1069	0.9828	0.9651
8	5529	0.9671	0.9365

Notes: Exact wave function for quasiparticle excitation obtained from diagonalization of Coulomb Hamiltonian. Its overlap with trial wave functions Eq. (12.30) (Laughlin) and Eq. (12.37) (CF).

Disk geometry with symmetric gauge is used. D is dimension of Fock space in the lowest Landau level.

Source: Dev and Jain [116]

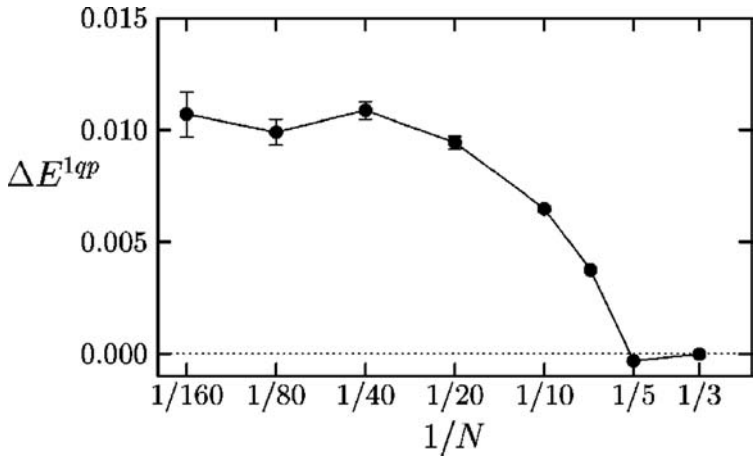

Fig. 12.3. $\Delta E^{1\mathrm{qp}} = E_{\mathrm{L}}^{1\mathrm{qp}} - E_{\mathrm{CF}}^{1\mathrm{qp}}$ is the difference between the energies of two trial wave functions for the quasiparticle at $\nu = 1/3$ given in Eqs. (12.30) and (12.37), for up to $N = 160$ electrons. $E_{\mathrm{L}}^{1\mathrm{qp}}$ is the energy of Laughlin's quasiparticle wave function, and $E_{\mathrm{CF}}^{1\mathrm{qp}}$ is the energy of the CF-quasiparticle. The former is approximately $0.011e^2/\epsilon\ell$ higher in the thermodynamic limit, which is about 15% of the energy of the quasiparticle ($\sim 0.07e^2/\epsilon\ell$). Source: G.-S. Jeon and J. K. Jain, *Phys. Rev. B* **68**, 165346 (2003). (Reprinted with permission.)

Two quasiparticles A generalization of Laughlin's construction to two quasiparticles suggests the wave function (see, for example, Kjønsberg and Myrheim [345])

$$\Psi_{\mathrm{L}}^{\text{2-L-qp}} = \mathrm{e}^{-\sum_j |z_j|^2/4} \prod_l \left(2\frac{\partial}{\partial z_l}\right)\left(2\frac{\partial}{\partial z_l}\right) \prod_{j<k} (z_j - z_k)^3 \ . \tag{12.38}$$

In contrast, two CF-quasiparticles at the origin, shown in Fig. 12.2(c), are described by the wave function

$$\Psi^{\text{2-CF-qp}} = \mathcal{P} \prod_{j<k} (z_j - z_k)^2 \begin{vmatrix} z_1^* & z_2^* & \cdots \\ z_1^* z_1 & z_2^* z_2 & \cdots \\ 1 & 1 & \cdots \\ z_1 & z_2 & \cdots \\ \vdots & \vdots & \cdots \\ z_1^{N-3} & z_2^{N-3} & \cdots \end{vmatrix} \times \exp\left[-\frac{1}{4}\sum_j |z_j|^2\right] . \tag{12.39}$$

The explicit LLL projected form can be written down as before, but is not shown here. The collection of two CF-quasiparticles has lower energy than that of two Lauglin-quasiparticles. The energy difference is estimated [292] to be $\approx 0.16 e^2/\epsilon\ell$, which is roughly equal to twice the energy required to create a single CF-quasiparticle. A qualitative difference appears between the two approaches at the level of two quasiparticles. The two wave functions considered above have different total angular momenta, $L = 3N^2 - 7N + 4$ (CF) and $L = 3N^2 - 7N$ (Laughlin). (The largest occupied single electron orbital has the same angular momentum for the two states, though.) Construction of a two quasiparticles state with an angular momentum $L = 3N^2 - 7N + 4$ is not obvious within Laughlin's approach.

Kjønsberg and Myrheim [345] calculate the braiding statistics of the 1/3 quasiparticles using the Laughlin wave function and note (Fig. 12.4) that it does not produce a well-defined value in the limit when the quasiparticles are far separated, indicating that this wave function does not capture the long-distance behavior of the actual quasiparticle sufficiently accurately for this purpose. CF quasiparticles, on the other hand, possess a well-defined braiding statistics (Kjønsberg and Leinaas [346]; Jeon, Graham, and Jain [293, 294]).

12.6 Neutral excitations: Girvin–MacDonald–Platzman theory

The neutral excitations of FQHE states are understood as CF-excitons (Fig. 5.4), which have been confirmed to be extremely accurate (Fig. 6.1). This section presents another model for the neutral excitations.

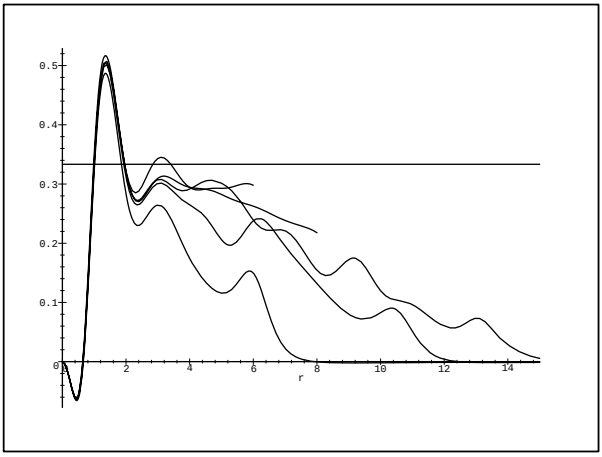

Fig. 12.4. The braiding statistics parameter (shown on the y-axis) for the quasiparticle at $\nu = 1/3$, using Laughlin's trial wave function (Eq. 12.30), determined from a Monte Carlo evaluation of the Berry phase. The x-axis label is r, which is related to the distance d between the quasiparticles as $d/\ell = 2\sqrt{2}\ r$. The lowest curve is for 20 electrons, the next for 50, and the third curve from the bottom is for 75 electrons. The results for 100 and 200 electrons terminate at $r = 8$ and $r = 6$ due to numerical problems, and the height of the horizontal line is 1/3. Source: H. Kjønsberg and J. Myrheim, *Int. J. Mod. Phys. A* **14**, 537 (1999). (Reprinted with permission.)

A trial wave function for a phonon-like excitation of a Bose superfluid can be constructed as (Bijl [31]; Feynman [160])

$$\phi_{\boldsymbol{k}} = \frac{1}{\sqrt{N}} \rho_{\boldsymbol{k}} \phi_{\text{ground}} \ , \tag{12.40}$$

where ϕ_{ground} is the ground state wave function, and

$$\rho(\boldsymbol{r}) = \sum_{j=1}^{N} \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{r}_j} \tag{12.41}$$

is the density operator. Clearly, $\phi_{\boldsymbol{k}}$ describes a collective "density wave" excitation, analogous to the "phonon" of a crystal. Because $\phi_{\boldsymbol{k}}$ is closely related to the ground state wave function, it is likely to have favorable correlations and low energy. Its orthogonalilty to ϕ_{ground},

$$\langle \phi_{\text{ground}} | \phi_{\boldsymbol{k}} \rangle = \frac{1}{\sqrt{N}} \int \mathrm{d}^2\boldsymbol{r}\, \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \langle \phi_{\text{ground}} | \rho(\boldsymbol{r}) | \phi_{\text{ground}} \rangle = 0 \ , \tag{12.42}$$

follows, for $\boldsymbol{k} \neq 0$, because $\langle \phi_{\text{ground}} | \rho(\boldsymbol{r}) | \phi_{\text{ground}} \rangle =$ constant for a spatially uniform ground state. The energy of this wave function can be related to the static structure factor, which, in turn, can be obtained from neutron scattering experiments. A peak in the structure factor produces a minimum in the dispersion, which is called the roton minimum. Such

a minimum is observed experimentally, but the energy at the minimum is a factor of two lower than that predicted by the above wave function. A quantitative understanding of the dispersion of the collective excitation has been one of the triumphs of the theory of superfluidity, which has required inclusion of backflow corrections (Feynman and Cohen [162]) and detailed variational and Green's function Monte Carlo studies (see, for example, Manousakis and Pandharipande [429]).

The above prescription allows construction of a trial wave function for an excited state from the knowledge of the ground state wave function. It may seem that we are getting excited states for free, but that is not the case. We are making the assumption that the low-energy neutral excitation is a density wave, which may seem natural, but must be verified.

Girvin, MacDonald, and Platzman (GMP) [191, 192] extend this idea to the neutral excitation of the FQHE, and we describe below their theory. We begin by writing the wave function

$$\phi_{\boldsymbol{k}} = \frac{1}{\sqrt{N}} \rho_{\boldsymbol{k}} \Psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \tag{12.43}$$

where Ψ is an ansatz for the FQHE ground state wave function. Because the number of electrons remains unchanged, this describes a neutral excitation. A shortcoming of this wave function is that it is not restricted to the lowest Landau level, which makes it inappropriate in very high magnetic fields. Since we are constructing a *trial* wave function, we project it into the lowest Landau level to obtain the GMP wave function:

$$\begin{aligned} \bar{\phi}_{\boldsymbol{k}} &= \mathcal{P}_{\mathrm{LLL}} \phi_{\boldsymbol{k}} \\ &= \frac{1}{\sqrt{N}} \bar{\rho}_{\boldsymbol{k}} \Psi(\boldsymbol{r}_1, ..., \boldsymbol{r}_N) \ , \end{aligned} \tag{12.44}$$

with

$$\begin{aligned} \bar{\rho}(\boldsymbol{r}) &= \mathcal{P}_{\mathrm{LLL}} \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_j} \mathcal{P}_{\mathrm{LLL}} \\ &= \mathcal{P}_{\mathrm{LLL}} \sum_{j=1}^{N} \exp\left[-\frac{\mathrm{i}}{2} k \bar{z}_j\right] \exp\left[-\frac{\mathrm{i}}{2} \bar{k} z_j\right] \mathcal{P}_{\mathrm{LLL}} \\ &= \sum_{j=1}^{N} \exp\left[-\mathrm{i}k \frac{\partial}{\partial z_j}\right] \exp\left[-\frac{\mathrm{i}}{2} \bar{k} z_j\right] \ , \end{aligned} \tag{12.45}$$

where $z = x - \mathrm{i}y$, $k = k_x - \mathrm{i}k_y$, and it is understood that the derivatives do not act on the Gaussian factor. This is known as the "single-mode approximation" (SMA), and is equivalent to assuming that the oscillator strength is exhausted by a single excitation.

All we need to do now is evaluate the energy of the GMP wave function. That can be done exactly in numerical studies on finite systems. Another method is based on the result,

proved below, that, as in in the case of helium superfluid, the energy of the GMP wave function can be expressed entirely as a function of the static structure factor of the ground state. The latter, in turn, can be computed with the help of either a candidate wave function or some other approximate technique.

It is convenient to work with $\bar{V}$, the projected Coulomb energy. The projection is most easily carried out in the Fourier space, where

$$\begin{aligned} V &= \frac{1}{2}\int \frac{\mathrm{d}^2\boldsymbol{q}}{(2\pi)^2} V(q) \sum_{\mathrm{i}\neq j} \mathrm{e}^{i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)} \\ &= \frac{1}{2}\int \frac{\mathrm{d}^2\boldsymbol{q}}{(2\pi)^2} V(q) \left(\rho^\dagger_{\boldsymbol{q}}\rho_{\boldsymbol{q}} - N\right) . \end{aligned} \tag{12.46}$$

A result from Exercise 12.7 (Eq. E12.18) gives the projected interaction:

$$\bar{V} = \frac{1}{2}\int \frac{\mathrm{d}^2\boldsymbol{q}}{(2\pi)^2} V(q) \left(\bar{\rho}^\dagger_{\boldsymbol{q}}\bar{\rho}_{\boldsymbol{q}} - N\mathrm{e}^{-q\bar{q}/2}\right) . \tag{12.47}$$

The energy expectation value of the GMP wave function, measured from the ground state energy, is given by

$$\begin{aligned} \Delta_{\boldsymbol{k}} &= \frac{\langle\bar{\phi}_{\boldsymbol{k}}|\bar{V} - E_0|\bar{\phi}_{\boldsymbol{k}}\rangle}{\langle\bar{\phi}_{\boldsymbol{k}}|\bar{\phi}_{\boldsymbol{k}}\rangle} \\ &= \frac{\bar{f}(k)}{\bar{S}(k)} , \end{aligned} \tag{12.48}$$

where

$$E_0 = \frac{\langle\Psi|\bar{V}|\Psi\rangle}{\langle\Psi|\Psi\rangle}, \tag{12.49}$$

$$\bar{S}(k) = \frac{1}{N}\langle\Psi|\bar{\rho}^\dagger_{\boldsymbol{k}}\bar{\rho}_{\boldsymbol{k}}|\Psi\rangle, \tag{12.50}$$

and

$$\bar{f}(k) = \frac{1}{N}\langle\Psi|\bar{\rho}^\dagger_{\boldsymbol{k}}[\bar{V},\bar{\rho}_{\boldsymbol{k}}]|\Psi\rangle . \tag{12.51}$$

$\bar{S}(k)$ is the projected structure factor and $\bar{f}(k)$ is the projected oscillator strength.

We first consider $\bar{S}(k)$. The quantity that is most readily calculated from a given wave function is the structure factor

$$S(k) = \frac{1}{N}\langle\Psi|\rho^\dagger_{\boldsymbol{k}}\rho_{\boldsymbol{k}}|\Psi\rangle = \frac{1}{N}\langle\Psi|\mathcal{P}_{\mathrm{LLL}}(\rho^\dagger_{\boldsymbol{k}}\rho_{\boldsymbol{k}})|\Psi\rangle , \tag{12.52}$$

where the last equation follows because Ψ is assumed to be strictly in the lowest Landau level. $\bar{S}(k)$ can be obtained from $S(k)$ using the relation

$$\bar{S}(k) = S(k) - (1 - \mathrm{e}^{-k\bar{k}/2}) , \tag{12.53}$$

which follows from Eq. (E12.18) derived in Exercise 12.7.

Using $\bar{\rho}_{\boldsymbol{q}}^{\dagger} = \bar{\rho}_{-\boldsymbol{q}}$, the expression for the projected oscillator strength can be rewritten as

$$\bar{f}(k) = \frac{1}{2N}\langle\Psi|[\bar{\rho}_{\boldsymbol{k}}^{\dagger}, [\bar{V}, \bar{\rho}_{\boldsymbol{k}}]]|\Psi\rangle \ . \tag{12.54}$$

The double commutator can be evaluated with the help of Eqs. (12.47) and (E12.17) to give

$$\begin{aligned}\bar{f}(k) = \frac{1}{2}\sum_{\boldsymbol{q}} V(q)\left(\mathrm{e}^{\frac{q^* k}{2}} - \mathrm{e}^{\frac{qk^*}{2}}\right) \\ \times \left[\bar{S}(q)\mathrm{e}^{-\frac{k^2}{2}}\left(\mathrm{e}^{-\frac{k^* q}{2}} - \mathrm{e}^{-\frac{kq^*}{2}}\right) + \bar{S}(k+q)\left(\mathrm{e}^{\frac{k^* q}{2}} - \mathrm{e}^{\frac{kq^*}{2}}\right)\right] \ .\end{aligned} \tag{12.55}$$

Thus, the energy of the GMP excitation, Eq. (12.48), can be obtained from the knowledge of the static structure factor.

Let us ask what the SMA gives for $\nu = 1$. This state has no excitations within the lowest Landau level, as is indicated by the vanishing of $\bar{S}_{\boldsymbol{k}}$. We therefore switch back to the "unprojected" SMA, for which

$$\Delta_{\boldsymbol{k}} = \frac{f(k)}{S(k)} \ , \tag{12.56}$$

with

$$S(k) = \frac{1}{N}\langle\Psi|\rho_{\boldsymbol{k}}^{\dagger}\rho_{\boldsymbol{k}}|\Psi\rangle \ , \tag{12.57}$$

and

$$f(k) = \frac{1}{N}\langle\Psi|\rho_{\boldsymbol{k}}^{\dagger}[H, \rho_{\boldsymbol{k}}]|\Psi\rangle \ , \tag{12.58}$$

where $H = K + V$ is the sum of kinetic and interaction energies. Since the density commutes with V (without projection), the commutator reduces to $[K, \rho_{\boldsymbol{k}}]$, which can be evaluated (Exercise 12.9) to give

$$f(k) = \frac{\hbar^2 k^2}{2m_{\mathrm{b}}} \ . \tag{12.59}$$

The gap is then given by

$$\Delta_{\boldsymbol{k}} = \frac{\hbar^2 k^2}{2m_{\mathrm{b}} S(k)} = \frac{\hbar^2 k^2}{2m_{\mathrm{b}}[1 - \mathrm{e}^{k^2\ell^2/2}]} \ . \tag{12.60}$$

The $k\ell \to 0$ limit,

$$\Delta_{\boldsymbol{k}} = \frac{\hbar^2}{m_{\mathrm{b}}\ell^2} = \hbar\omega_{\mathrm{c}} \ , \tag{12.61}$$

is consistent with Kohn's theorem, which states that, in the presence of a magnetic field, given the exact ground state of the interacting system (Ψ), the state $\rho_{\boldsymbol{k}}\Psi$ is an exact eigenstate, in the limit $\boldsymbol{k} \to 0$, with eigenenergy $\hbar\omega_{\mathrm{c}}$.

12.6.1 SMA on a sphere

To construct the GMP wave function in the spherical geometry, we need an appropriate form for the density operator. In the planar geometry, we defined

$$\rho(\boldsymbol{r}) = \sum_{j=1}^{N} \delta^{(2)}(\boldsymbol{r} - \boldsymbol{r}_j) = \sum_{j=1}^{N} \int \frac{\mathrm{d}^2\boldsymbol{q}}{(2\pi)^2} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r}_j)} \equiv \int \frac{\mathrm{d}^2\boldsymbol{q}}{(2\pi)^2} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \rho_{\boldsymbol{q}} \ . \tag{12.62}$$

In the spherical geometry, we write

$$\begin{aligned} \rho(\boldsymbol{\Omega}) &= \sum_{j=1}^{N} \delta^{(2)}(\boldsymbol{\Omega} - \boldsymbol{\Omega}_j) \\ &= \sum_{j=1}^{N} \sum_{L=0}^{\infty} \sum_{M=-L}^{L} Y^*_{L,M}(\boldsymbol{\Omega}) Y_{L,M}(\boldsymbol{\Omega}_j) \\ &\equiv \sum_{L=0}^{\infty} \sum_{M=-L}^{L} Y^*_{L,M}(\boldsymbol{\Omega}) \rho_{LM} \ , \end{aligned} \tag{12.63}$$

where

$$\rho_{LM} = \sum_{j=1}^{N} Y_{L,M}(\boldsymbol{\Omega}_j) \ . \tag{12.64}$$

The GMP wave function for the excitation is then given by

$$\phi_{LM} = \frac{1}{\sqrt{N}} \mathcal{P}_{\mathrm{LLL}} \rho_{LM} \Psi(\boldsymbol{\Omega}_1, \ldots, \boldsymbol{\Omega}_N) \ , \tag{12.65}$$

where $\mathcal{P}_{\mathrm{LLL}}$ projects the state into the lowest Landau level.

Recall that the angular momentum operator for $Q = 0$ is

$$\boldsymbol{L} = \boldsymbol{\Lambda} + Q\boldsymbol{\Omega} = \boldsymbol{\Lambda} \ . \tag{12.66}$$

Because the spherical harmonics transform as vectors under rotations generated by $\boldsymbol{\Lambda}$, and commute with $\boldsymbol{\Omega}$, we have

$$[L_z, \rho_{LM}] = M \rho_{LM} \tag{12.67}$$

and

$$[L_\pm, \rho_{LM}] = \sqrt{L(L+1) - M(M \pm 1)}\ \rho_{LM\pm1} \ . \tag{12.68}$$

It further follows that

$$\boldsymbol{L}^2 \rho_{LM} \Psi = L(L+1) \rho_{LM} \Psi \tag{12.69}$$

and

$$L_z \rho_{LM} \Psi = M \rho_{LM} \Psi \ , \tag{12.70}$$

Table 12.2. *Roton gaps from the CF and the Girvin–MacDonald–Platzman theories*

ν	CF	GMP
1/3	0.063(3)	0.078
1/5	0.0095(6)	0.017
1/7	0.0009(5)	0.0063

Notes: Energy of the primary roton in units of $e^2/\epsilon\ell$. CF energies obtained by extrapolation of finite system results.

Same ground state wave functions used in both studies.

Source: Jain and Kamilla [281]; Girvin, MacDonald, and Platzman [191, 192].

where we have used that Ψ has $L = 0$. The GMP wave function thus has a definite symmetry under rotation.

To compare the energy Δ_L for systems with different sizes, or with Δ_q of the planar geometry, one converts L into $\boldsymbol{q}$ with the help of the relation [226] $L = qR$, where R is the radius of the sphere. It can be verified that, with this identification, the energy of the (unprojected) GMP excitation on the sphere at $\nu = 1$ becomes identical to Eq. (12.60).

12.6.2 Testing the SMA

The SMA needs a ground state wave function. In exact diagonalization studies, we can use the exact ground state wave function to construct the GMP wave function, and evaluate its energy exactly. Figure 12.5 shows the dispersion of the GMP excitation (solid dots) for $\nu = 1/3$, $2/5$ and $3/7$, evaluated by and He, Simon and Halperin [247] and Platzman He [521]. The spherical geometry is used in these studies.

Calculation of the GMP dispersion for larger systems requires the static structure factor (or the pair correlation function), which is obtained from a trial wave function for the ground state. Initially, Girvin, MacDonald, and Platzman applied SMA to $\nu = 1/(2p+1)$ because of the availability of the Laughlin wave function. Dispersions of the GMP mode at other fractions of the type $\nu = n/(2n+1)$ have also been calculated (Park and Jain [505]; Scarola, Park and Jain [567]), using the ground state wave functions of Eq. (5.32). Table 12.2 gives the minimum energy required to create a neutral excitation, evaluated both from the GMP and the CF-exciton models. These studies show that the GMP theory works well at low and intermediate wave vectors, especially at the Laughlin fractions, where it successfully predicts a roton minimum.

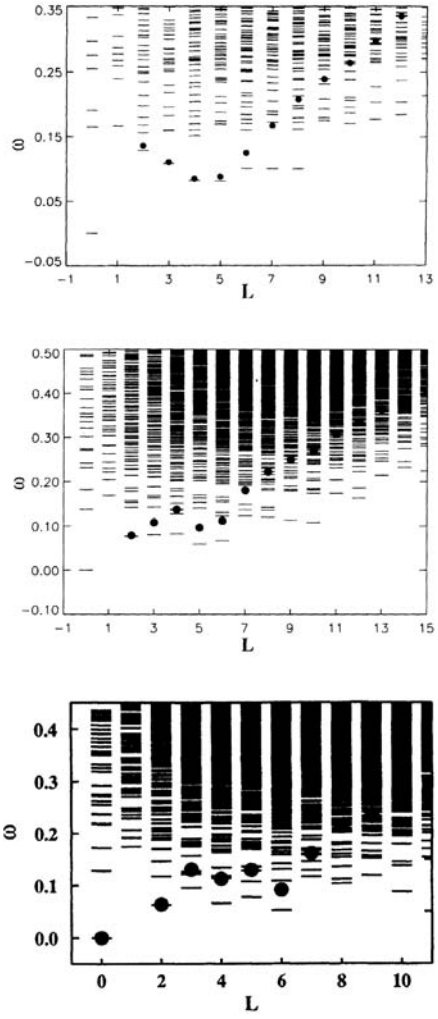

Fig. 12.5. Energy dispersion of the GMP excitation (dots) at $\nu = 1/3$ for 9 particles, at $\nu = 2/5$ for 10 particles, and at $\nu = 3/7$ for 12 particles. The dashes show the exact Coulomb eigenenergies. Sources: P. M. Platzman and S. He, *Phys. Rev. B* **49**, 13674 (1994); S. He, S. H. Simon, and B. I. Halperin, *Phys. Rev. B* **50**, 1823 (1994). (Reprinted with permission.)

12.7 Conti–Vignale–Tokatly continuum-elasticity theory

Classical liquids and solids also exhibit a collective response to external perturbations, which can be described, in the long-wavelength limit, by treating them as a continuous elastic medium. In the well-developed classical elasticity theory, the dynamics of the system is described in terms of the displacement field, $\boldsymbol{u}(\boldsymbol{r}, t)$, the deviation of a volume element

from its equilibrium position. Its equations of motion involve the bulk modulus and the shear modulus, which are to be fixed phenomenologically by comparing to either experiments or microscopic calculations.

Conti and Vignale [95, 196, 657] take the view that the LLL electron liquid can also be treated analogously, but with frequency and wave vector dependent visco-elastic constants. The current density and the density can be expressed in terms of $\boldsymbol{u}$ (for small perturbations) as

$$\boldsymbol{j}(\boldsymbol{r},t) = \rho_0 \partial_t \boldsymbol{u}(\boldsymbol{r},t) \tag{12.71}$$

$$\delta\rho = -\rho_0 \boldsymbol{\nabla}\cdot\boldsymbol{u}\ , \tag{12.72}$$

where $\boldsymbol{v} = \partial_t \boldsymbol{u}$ is the velocity field and ρ_0 is the unperturbed density, which is uniform in space and constant in time. Equation (12.72) follows from the linearized continuity equation

$$\partial_t \rho + \boldsymbol{\nabla}\cdot\boldsymbol{j} \simeq \partial_t \delta\rho + \rho_0 \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0\ . \tag{12.73}$$

From an analysis of the equations of motion for $\boldsymbol{u}(\boldsymbol{r},t)$ from classical elasticity theory in the limit of high magnetic field, Conti and Vignale conclude that the bulk modulus B must diverge as

$$B \propto \frac{1}{q^4} \tag{12.74}$$

at small wave vectors to produce a nonzero gap for the exciton in the long-wavelength limit. Furthermore, from a calculation of the static structure factor, they relate the shear modulus S (which they assume to be independent of frequency) to Δ, the $q = 0$ gap of the exciton:

$$S = \frac{1-\nu}{4\nu}\rho_0 \Delta\ . \tag{12.75}$$

By taking S to be the shear modulus of the Wigner crystal, $S = 0.097\,75\rho_0\sqrt{\nu}e^2/\epsilon\ell$ [35], this predicts $\Delta = 0.11$, 0.044, and $0.025e^2/\epsilon\ell$ at $\nu = 1/3$, 1/5, and 1/7, which are in decent agreement with the exact diagonalization results.

Conti and Vignale [95] write a Lagrangian for the displacement field (the extremization of which produces the equations of motion) and quantize the displacement field in the standard manner to obtain a Hamiltonian of independent bosons with wave vector dependent energy:

$$H = \sum_{\boldsymbol{q}} \hbar\omega_q \left(b_{\boldsymbol{q}}^\dagger b_{\boldsymbol{q}} + \frac{1}{2} \right)\ . \tag{12.76}$$

The boson is identified with the intra-LL collective mode (or the inter-Λ level CF exciton). The FQHE liquid is thus dynamically equivalent to a set of noninteracting bosons, which are analogous to the phonons of a Wigner crystal.

Tokatly [638, 639] has assigned different q, ω dependences to the elastic moduli. He notes that Kohn's theorem, which requires stress to vanish for rigid displacements, forbids divergences in the $q \to 0$ limit. The f-sum rule, on the other hand, requires finiteness of

the bulk and shear moduli in the $\omega \to \infty$ limit. He has shown that a consistent picture can be obtained by taking a constant bulk modulus K, and a shear modulus of the form

$$S(\omega, q \to 0) \propto \frac{\omega^2}{\omega^2 - \Delta^2} \tag{12.77}$$

with a divergence at $\omega \to \Delta$. It vanishes in the $\omega \to 0$ limit, as expected for a liquid. With these choices the f-sum rule and Kohn's theorem can be satisfied, and the collective mode dispersion has a roton minimum.

The Conti–Vignale and the Tokatly theories can be shown [640] to be related, in the long-wavelength limit and for frequencies close to Δ, by a canonical transformation that preserves the form of the equations of motion but transfers the divergence from the shear modulus to the bulk modulus or vice versa. They predict identical relation (Eq. 12.75) between the high-frequency shear modulus and the gap. Tokatly has also derived the continuum elasticity theory starting from the CFCS theory, by linearizing the equation of motion for the Wigner function (which is the same as the semiclassical Boltzmann equation).

12.8 Search for a model interaction

Model interactions have been constructed for which some simple FQHE wave functions are the exact ground states. We discuss four cases.

12.8.1 $\nu = 1/3$

Haldane [221] constructs a model interaction for which the Laughlin wave function for the $\nu = 1/m$ state is the exact nondegenerate ground state. The model is based on the observation that, at $\nu = 1/m$, Laughlin's wave function is the unique wave function that contains no pairs with relative angular momentum less than m. To see this, we only need to note that, apart from the Gaussian factor, the wave function of a pair with relative angular momentum m is given by $(z_i - z_j)^m$. How does this help? Let us consider the model

$$V_l = 0, \qquad l \geq m\,, \tag{12.78}$$

where V_l are the Haldane pseudopotentials. The actual values of the nonzero V_l's are not relevant, except that they are taken to be positive, as would be the case for a repulsive interaction. For this model, the energy of pairs with relative angular momenta $l \geq m$ is identically zero. In particular, the energy of the Laughlin wave function vanishes. The Laughlin wave function is thus an eigenstate for this model. It is also a ground state, since negative energies are not possible.

We still need to prove uniqueness. The general wave function excluding pairs of angular momentum $l < m$ has the form

$$F_S[\{z_j\}] \prod_{i<k} (z_i - z_k)^m \exp\left[-\sum_l \frac{|z_l|^2}{4\ell^2} \right], \tag{12.79}$$

where $F_S[\{z_j\}]$ is an arbitrary symmetric polynomial. At $\nu = 1/m$ we must have $F = 1$; otherwise, F supplies additional powers of z_j, thereby decreasing the filling factor. This proves that, at $\nu = 1/m$, Laughlin's wave function is the unique ground state for the model Hamiltonian.

Trugman and Kivelson [646] construct a real-space interaction for which Laughlin's $\nu = 1/3$ wave function is the ground state

$$V_{\rm TK}(\boldsymbol{r}) = \alpha \nabla^2 \delta^{(2)}(\boldsymbol{r}) \ . \tag{12.80}$$

The seemingly strange Laplacian of a delta function can be a defined through a limiting procedure, but there is no need to do that. The interaction is perfectly well defined in terms of matrix elements. The Haldane pseudopotentials for this interaction are given by

$$\begin{aligned} V_l &= \langle l | V_{\rm TK}(\boldsymbol{r}) | l \rangle \\ &= \int \mathrm{d}^2 \boldsymbol{r} |\psi_l(\boldsymbol{r})|^2 \alpha \nabla^2 \delta^{(2)}(\boldsymbol{r}) \\ &= \int \mathrm{d}^2 \boldsymbol{r}\, \alpha \delta^{(2)}(\boldsymbol{r}) \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \frac{|\psi_l(\boldsymbol{r})|^2}{r} \ , \end{aligned} \tag{12.81}$$

where $\boldsymbol{r}$ is the relative coordinate, and $\psi_l(\boldsymbol{r})$ is the wave function for two electrons in relative angular momentum l state. The matrix element vanishes provided $|\psi_l(r)|^2 \propto r^{2+\epsilon}$ as $r \to 0$, with $\epsilon > 0$. Because $\psi_l(\boldsymbol{r}) \sim r^l$, we have $V_l = 0$ for $l \geq 2$. The rest of the proof follows as before. (We recall that the even pseudopotentials are not relevant for fully spin-polarized electrons.)

The above discussion also implies that for $\nu > 1/3$, *all* pairs cannot possibly have a relative angular momentum of three or greater. Therefore, the argument cannot be generalized to filling factors $\nu = n/(2n+1)$ with $n > 1$, even though, as demonstrated by numerical calculations (Gros and MacDonald [217]), the hard-core model exhibits FQHE at these filling factors.

12.8.2 $\nu = 1/2$

Greiter, Wen, and Wilczek [213] write a Hamiltonian involving a three-body interaction for which the Moore–Read Pfaffian wave function (Eq. 7.11) is the exact ground state. To see this, let us consider (charged) *bosons* in the lowest Landau level. Any bosonic wave function that vanishes when any two bosons coincide is a zero-energy eigenstate of the interaction $H = V \sum_{i<j} \delta^{(2)}\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)$. The lowest degree polynomial with that property (in addition to being symmetric under exchange) is the $\nu = 1/2$ wave function $\prod_{j<k}(z_j - z_k)^2$ (suppressing the Gaussian factor for simplicity), because the lowest degree polynomial that vanishes upon particle coincidence is $\prod_{j<k}(z_j - z_k)$, and the minimum one must do to symmetrize it is to supply an additional factor of $\prod_{j<k}(z_j - z_k)$, the lowest degree polynomial that is completely antisymmetric. Many zero energy eigenstates of H of the form

$F[\{z_i\}]\prod_{j<k}(z_j - z_k)^2$, $F[\{z_i\}]$ symmetric under exchange, exist for $\nu < 1/2$, and none for $\nu > 1/2$. Greiter, Wen, and Wilczek consider a less restrictive interaction:

$$H_{\rm Pf} = V \sum_{i<j<k} \delta^{(2)}\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right) \delta^{(2)}\left(\boldsymbol{r}_i - \boldsymbol{r}_k\right) \ , \tag{12.82}$$

which imposes a penalty only when *three* bosons coincide. The $\nu = 1$ bosonic Pfaffian wave function

$$\Psi_1^{\rm Pf,boson} = {\rm Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}(z_i - z_j) \exp\left[-\frac{1}{4}\sum_k |z_k|^2\right] \tag{12.83}$$

has the property that it vanishes when three bosons coincide (although not when two do), because the Pfaffian factor only removes pairwise zeroes from $\prod_{j<k}(z_j - z_k)$. The wave function is, thus, a zero-energy eigenstate of $H_{\rm Pf}$. (This property remains valid for the multi-quasihole wave functions discussed in Section 9.9.) Further thought will convince the reader that this is the lowest degree polynomial that vanishes upon three-boson coincidences, and thus the unique bosonic ground state of $H_{\rm Pf}$ at $\nu = 1$. $\Psi_1^{\rm Pf,boson}$ can thus be obtained by exact diagonalization. The fermionic wave function $\Psi_{1/2}^{\rm Pf}$ is the ground state for a three-body interaction that involves appropriate derivatives of the delta function interaction.

The three-body interaction of Eq. (12.82) has a natural generalization to the spherical geometry (Read and Rezayi [539]). The $\nu = 1$ bosonic and the the $\nu = 1/2$ fermionic Pfaffian states correspond to $2Q = N - 2$ and $2Q = 2N - 3$, respectively. For bosons, the closest approach of three particles corresponds to the maximum total angular momentum for the triplet, which is $L_{\rm max} = 3Q$ (Q being the orbital angular momentum for each boson). An elimination of such configurations is equivalent to an avoidance of triplet coincidences. The Hamiltonian in the spherical geometry can thus be equivalently written as

$$H = V \sum_{i<j<k} P_{ijk}(L_{\rm max}) \ , \tag{12.84}$$

where $P_{ijk}(L_{\rm max})$ is the projection operator onto a triplet of orbital angular momentum $L_{\rm max}$. This Hamiltonian imposes a penalty for triplet angular momentum of $L_{\rm max} = 3Q$, and obtains the Pfaffian as the exact zero-energy ground state. For fermions, the closest approach of three particles corresponds to the maximum total angular momentum $L_{\rm max} = 3Q - 3$; an elimination of such configurations produces the spherical version of the Moore–Read wave function at $\nu = 1/2$.

12.8.3 $\nu = 2/5$

A model can be constructed for which the unprojected wave function $\Phi_2\Phi_1^2$ for the $\nu = 2/5$ ground state is exact [275]. The model truncates the Hilbert space to the lowest two Landau levels ($n = 0$ and $n = 1$), takes them to be degenerate, and considers the Trugman–Kivelson interaction, the zero-energy eigenstates of which vanish with a third-order zero when two particles approach one another. The most general antisymmetric wave function of this type

is $\Phi_1^2\Phi_{\nu^*}$, where Φ_{ν^*} is an antisymmetric function with at most one power of $\bar{z}_j$ (aside from the Gaussian factor), i.e., Φ_{ν^*} is restricted to the lowest two Landau levels. For $\nu^* < 2$ there are many choices for such a wave function; for $\nu^* = 2$ there is a unique choice (Φ_2); and for $\nu^* > 2$ no such wave function exists. $\Phi_1^2\Phi_2$ is thus the unique ground state for this model at $\nu = 2/5$. Rezayi and MacDonald [548] find, for a six-particle system, that the ground state evolves smoothly as the Landau level spacing is varied from zero to infinity, indicating the absence of a level-crossing transition for this system during the process of adiabatic projection into the lowest Landau level.

12.8.4 Spin-singlet $\nu = 1/2$

The Haldane–Rezayi wave function [224, 225] of Eq. (E7.6) avoids relative angular momentum $m = 1$ for all pairs – for like-spin electrons due to the Fermi statistics, and for unlike-spin electrons due to the Jastrow factor. It does allow, however, pairs with relative angular momentum $m = 0$. It is an exact zero-energy eigenstate for the interaction $V_j = V_1\delta_{j1}$, which is called the "hollow-core" interaction model, because it does not penalize two electrons at their closest approach (because $V_0 = 0$).

Summary A model that can be solved exactly to produce the qualitative phenomenology of the FQHE does not exist. We have sometimes succeeded, as in the popular television quiz show "Jeopardy," in formulating a question to which an already-known simple wave function is the answer. This approach has limited applicability, however. To summarize what we have so far: (a) One model gives only one fraction. (b) Only certain simple wave functions are obtained.[9] (c) No model has been solved for all eigenstates and eigenenergies even at a single filling factor; even the first excited state eludes exact solution. The most important use of such models has been in numerically generating certain trial wave functions by exact diagonalization. Fortunately, a wave function derives its legitimacy not from being the exact solution for a model interaction, but from being an accurate representation of the actual (Coulomb) solution.[10]

Exercises

12.1 A superficial "derivation" of the Laughlin wave function is as follows. Perform a Chern–Simons transformation in which m flux quanta are attached to each electron, and show that the electron wave function is related to the wave function in the transformed problem, $\Psi_{\rm CS}$, by

$$\Psi = \prod_{j<k}\left(\frac{z_j - z_k}{|z_j - z_k|}\right)^m \Psi_{\rm CS}\ . \tag{E12.1}$$

[9] In principle, we can always concoct a model for which a given wave function is the ground state. For example, any wave function $|\Psi\rangle$ is the exact, nondegenerate ground state of the Hamiltonian $H = -|\Psi\rangle\langle\Psi|$. This information, however, is of no use.

[10] "I would rather be approximately right than exactly wrong." A Wall Street broker.

Demonstrate that the assumptions: (i) $\Psi_{\rm CS}$ has no vortices (i.e., is everywhere non-negative), and (ii) Ψ is in the lowest Landau level, uniquely fix the form of $\Psi_{\rm CS}$ and yield the Laughlin wave function for Ψ. (Note that the form of the interaction did not play any role. A "real" derivation would tell us, for the Coulomb interaction, why Ψ is not the ground state at large m, and what are the corrections to it for small m.)

12.2 Consider a Laughlin-like wave function for bosons

$$\Psi_{\rm B} = \prod_{j<k} |z_j - z_k|^m \exp\left[-\frac{1}{4}\sum_l |z_l|^2\right] . \tag{E12.2}$$

This wave function has the same charge density as $\Psi_{1/m}$, therefore describes a state of charged bosons at $\nu = 1/m$. Consider the off-diagonal element of the one-particle reduced density matrix (Appendix H)

$$\rho_1(z,z') = N\frac{\int {\rm d}^2 z_2, \ldots, {\rm d}^2 z_N \Psi_B^*(z, z_2, \ldots, z_N)\Psi_B(z', z_2, \ldots, z_N)}{\int {\rm d}^2 z_1, \ldots, {\rm d}^2 z_N \Psi_B^*(z_1, z_2, \cdots, z_N)\Psi_B(z_1, z_2, \ldots, z_N)} . \tag{E12.3}$$

The plasma analogy enables a determination of its long-distance behavior. Show that ${\rm e}^{(m/2)\ln|z-z'|}\rho(z,z')$ is proportional to the partition function for the classical problem which has charge -1 particles at z_j and charge $-1/2$ impurities at fixed positions z and z'. Because of the complete screening of the impurities by the plasma, this function ought to be independent of z and z' in the limit $|z - z'| \to \infty$, implying that in this limit we have $\rho(z,z') \sim |z-z'|^{-m/2}$, i.e., an algebraic off-diagonal long-range order (ODLRO). For the fermionic wave function $\Psi_{1/m}$, the additional phases destroy the algebraic ODLRO, producing a Gaussian fall off for $\rho_1(z,z')$. (Source: Girvin and MacDonald [193].)

12.3 This exercise concerns the result that Jastrow wave functions can exhibit Bose–Einstein condensation, which is crucial if they are to be meaningful for a study of BEC. Consider the following wave function for bosons (Bijl [31]; Jastrow [289]):

$$\Psi_N = {\rm e}^{-\frac{\beta}{2}\sum_{i<j} u(r_{ij})} , \tag{E12.4}$$

where $u(r) = \infty$ for $r < a$ (which implies vanishing probability of two bosons approaching closer than distance a, as appropriate for an infinite repulsive interaction at short distances), and has a power law behavior for $r > a$. The parameter β depends on the strength of the interaction. Define the normalization factor

$$Q_N = \int {\rm d}\boldsymbol{r}_1 \cdots {\rm d}\boldsymbol{r}_N \, {\rm e}^{-\beta\sum_{j<k} u(r_{jk})} , \tag{E12.5}$$

which can be interpreted, apart from a factor of $N!$, as the classical partition function of a system of particles with an interaction $2u(r)$, with β interpreted as the inverse temperature. The number of bosons in the zero momentum state is given

by (Appendix H)

$$\begin{aligned} n_0 &= V \lim_{|\boldsymbol{r}-\boldsymbol{r}'|\to\infty} \langle \hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}')\rangle \\ &= V\frac{1}{V^2}\int' \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\langle \hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}')\rangle \\ &= \frac{N}{V}\frac{\mathcal{Q}_{N+1}}{\mathcal{Q}_N}\,, \end{aligned} \tag{E12.6}$$

where the prime on the integral sign denotes the condition $|\boldsymbol{r}-\boldsymbol{r}'| > a$ and

$$\mathcal{Q}_{N+1} \equiv \int' \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\,\mathrm{d}\boldsymbol{r}_2\ldots\mathrm{d}\boldsymbol{r}_N\,\Psi(\boldsymbol{r},\boldsymbol{r}_2,\ldots\boldsymbol{r}_N)\Psi(\boldsymbol{r}',\boldsymbol{r}_2,\ldots\boldsymbol{r}_N)\,. \tag{E12.7}$$

(i) Show that

$$\mathcal{Q}_{N+1} \geq Q_{N+1}\,\mathrm{e}^{-\phi-\Delta}\,, \tag{E12.8}$$

where ϕ and Δ are positive, finite numbers defined by

$$\sum_{j=2}^{N} u(|\boldsymbol{r}-\boldsymbol{r}_j|) \geq -\phi \tag{E12.9}$$

$$\min[u(r)] = -\Delta\,. \tag{E12.10}$$

These conditions are not satisfied by all $u(r)$. Equation (E12.10) requires that $u(r)$ be bounded from below, and Eq. (E12.9) imposes a constraint on the long-range form of $u(r)$, derived below.

(ii) Next, show that

$$\frac{(N+1)Q_N}{Q_{N+1}} = \mathrm{e}^{\beta\mu} = z\,, \tag{E12.11}$$

where μ is the chemical potential of the equivalent classical problem, and z is the activity.

(iii) Derive the inequality

$$\frac{n_0}{N} \geq \frac{N}{V}z^{-1}\,\mathrm{e}^{-2\phi-\Delta}\,. \tag{E12.12}$$

If the chemical potential does not diverge, which is the case for many reasonable interactions $u(r)$, and ϕ and Δ are finite numbers, then the system exhibits BEC.

(iv) Obtain the condition that ϕ is finite provided $|u(r)| \leq A/r^{d+\epsilon}$ in d dimensions.[11]

(Source: Reatto [540].)

12.4 Show that the two methods of LLL projection (Section 5.14) produce the same wave function (apart from an overall constant) for a single CF-quasiparticle at $\nu = 1/(2p+1)$.

[11] An unexpected corollary is what is known as "supersolidity" (Chester [74]). For many choices of $u(r)$ which satisfy the conditions for ODLRO, the equivalent classical system is known to describe a crystal at sufficiently large β, or low temperatures (which melts at a certain critical value of β as β is decreased). That demonstrates, in principle, the possibility of BEC in a crystal, i.e., of the coexistence of diagonal and off-diagonal long-range orders.

12.5 Consider the spin-singlet state

$$\Psi_{2/5} = \Phi_{1,1}\Phi_1^2 \,, \tag{E12.13}$$

which is a generalization of the Laughlin wave function to a two-component system (Halperin [229]). Using the plasma analogy:
(i) Show that $\Psi_{2/5}$ has uniform density.
(ii) Obtain the charge of the vortex

$$\prod_{j=1}^{N/2}(z_j - \eta)\Psi_{2/5} \,. \tag{E12.14}$$

(iii) Show that $\Psi_{2/5}$ is the unique zero-energy ground state in the lowest Landau level (assuming zero Zeeman energy) for the hard-core interaction

$$V_0 = V_1 > 0, \qquad V_2 = V_3 = \cdots = 0 \,. \tag{E12.15}$$

12.6 The "hole" wave function at $\nu = 1$, in which one electron is missing from the $m = 0$ orbital (Fig. 12.2), is given by Eq. (12.35) with $p = 0$.
(i) Show that, for a finite N, this wave function is not unique at the corresponding total angular momentum.
(ii) Now model this state in terms of composite fermions carrying two vortices, and show that a minimization of the CF kinetic energy produces a unique wave function. Write this wave function (without explicit projection).

This wave function has been shown to be a much better representation of the exact state than the "hole" of Fig. 12.2 (Jeon *et al.* [298]). The wave function of Oaknin *et al.* [483], reproduced in Eq. (E3.33), is also excellent.

12.7 Derive the following properties of the projected density operator:

$$\bar{\rho}_{\boldsymbol{q}}^{\dagger} = \bar{\rho}_{-\boldsymbol{q}}, \tag{E12.16}$$

$$[\bar{\rho}_{\boldsymbol{k}}, \bar{\rho}_{\boldsymbol{q}}] = (\mathrm{e}^{\bar{k}q/2} - \mathrm{e}^{k\bar{q}/2})\bar{\rho}_{\boldsymbol{k}+\boldsymbol{q}}, \tag{E12.17}$$

$$\mathcal{P}_{\mathrm{LLL}}\rho_{\boldsymbol{q}}^{\dagger}\rho_{\boldsymbol{q}} = \bar{\rho}_{\boldsymbol{q}}^{\dagger}\bar{\rho}_{\boldsymbol{q}} + (1 - \mathrm{e}^{-q\bar{q}/2}). \tag{E12.18}$$

Hint: Use the identity $\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^B\mathrm{e}^A\mathrm{e}^{[A,B]}$ (Appendix B) to reorder terms.

12.8 Using the symmetry under $\boldsymbol{k} \to -\boldsymbol{k}$, show that the oscillator strength can be written as a double commutator:

$$f(k) = \frac{1}{2N}\langle\Psi|[\rho_{\boldsymbol{k}}^{\dagger}, [H, \rho_{\boldsymbol{k}}]]|\Psi\rangle \,. \tag{E12.19}$$

Since the density commutes with V (without projection), the commutator reduces to $[K, \rho_{\boldsymbol{k}}]$, where $K = \sum_j \boldsymbol{\pi}_j^2/2m_{\mathrm{b}}$, and $\boldsymbol{\pi} = \boldsymbol{p} + e\boldsymbol{A}/c$. Evaluate the commutators to derive Eq. (12.59).

12.9 Kohn's theorem is derived in this exercise. Consider

$$H = \frac{1}{m_\mathrm{b}} \sum_j \boldsymbol{\pi}_j^2 + V, \tag{E12.20}$$

where $\boldsymbol{\pi} = \boldsymbol{p} + e\boldsymbol{A}/c$, and define $\boldsymbol{\Pi} = \sum_j \boldsymbol{\pi}_j$, the kinetic momentum of the whole system. Now calculate the commutator

$$[H, \Pi_\pm] = \pm\hbar\omega_\mathrm{c}\Pi_\pm \,, \tag{E12.21}$$

where

$$\Pi_\pm = \Pi_x \pm \mathrm{i}\Pi_y \,. \tag{E12.22}$$

This, incidentally, is the operator form for the Lorentz equation for the whole system:

$$\frac{\mathrm{d}\boldsymbol{\Pi}}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[H, \boldsymbol{\Pi}] = -\frac{e}{m_\mathrm{b}c}\boldsymbol{\Pi} \times \boldsymbol{B} \,. \tag{E12.23}$$

Show that $\Pi_+\Psi$ is an exact eigenstate with energy $\hbar\omega_\mathrm{c}$ above the ground state. Further, using the symmetric gauge, show that Π_+ is proportional to the center of mass ladder operator $A^\dagger$ of Eq. (3.247). (Source: Kohn [351].)

12.10 Use the explicit form for the spherical harmonics, L_z and $L_\pm$ (Eqs. 3.131 and 3.130) to confirm the commutation relations in Eqs. (12.67) and (12.68) for a single particle with $L = 1$.