5

Foundations of the composite fermion theory

This chapter introduces the basic principles of the composite fermion theory. It should really be called the "composite fermion model" or the "composite fermion hypothesis" in this chapter. The extensive scrutiny and testing that elevate it to the status of a "theory" are topics of subsequent chapters.

5.1 The great FQHE mystery

Not often does nature present us with a mystery as well defined as the phenomenon of the fractional quantum Hall effect. The questions that theory is challenged to answer could not be more sharply posed.

- What is the physics of this quantum fluid? The appearance of precise quantum numbers and dissipationless transport in a dirty solid state system containing many electrons is a signature of cooperative behavior. What are the correlations in the FQHE state and why do they manifest themselves in such a rich, yet stunningly simple fashion? Why do gaps open at certain fractional fillings of the lowest Landau level? Experiments point to something unique and special about ground states at certain special filling factors. What order brings about this uniqueness?
- A tremendous amount of factual information is contained in the fractions that are observed and the order in which they appear. That imposes rigorous constraints on theory. With the proliferation of fractions, certain striking patterns have emerged. Many fractions are conspicuous by their absence. For example, the simplest fraction $f = 1/2$ has not been observed. In fact, no sub-unity fractions with even denominators have been observed. Another remarkable aspect is that fractions are not isolated but belong to certain sequences. For example, in Fig. 2.7, the fractions $1/3$, $2/5$, $3/7$, $4/9$, $5/11$, ... follow the sequence $f = n/(2n+1)$. These observations lead to the following questions: Why are some fractions seen but not others? Why do they appear in sequences? What determines the order of their stability? Why are even denominator fractions absent (with one exception)?
- What is the microscopic description of this state? What theory will provide quantitative predictions for various experimentally measurable quantities?

Any theory that attempts to explain the FQHE must answer these basic questions. Many subsidiary questions can be added to the above list.

- What is the nature of the state at even denominator fractions? An explanation of the odd-denominator "rule" would be unsatisfactory and surely incomplete in the absence of such understanding.
- A FQHE has been observed at $\nu = \frac{5}{2}$, the only even denominator fraction observed so far (in a single layer system). What is its physics? Why is there FQHE at the half-filled second Landau level, but none at half-filled lowest Landau level?
- In the limit of very high magnetic fields, the spin degree of freedom is completely frozen. The spin does play a role, however, in the experimentally studied parameter regime. What kinds of structures occur when the spin degree of freedom can fluctuate?
- Why is there no FQHE at small filling factors? When does the FQHE disappear and what causes its disappearance?
- The abundance of FQHE in the lowest Landau level is in stark contrast to a scarcity of FQHE in higher Landau levels. Why?
- What is the nature of excitations of the FQHE states? This is a rather complicated question because there are a myriad FQHE states, often with several possible spin polarizations, with their excitations depending on parameters such as the wave vector, the Zeeman energy, etc.

The list of questions appears daunting at first sight, but, encouragingly, they are not all independent and are not going to be answered one at a time. The physical origin of the fractions must be the same, the appreciation of which should tell us why they belong to certain sequences of fractions that exclude even denominator fractions. An understanding of what causes a gap must surely offer an insight into the nature of excitations, and knowing the physics of the fully polarized FQHE will likely give a clue into the role of spin. The questions listed above are so inextricably intertwined that the correct physical principle must answer many of them at once, and show the way for the rest.

A new principle invariably has implications beyond the questions that initially inspire it. New questions arise that could not have been asked before, and new phenomena become apparent that could not have been envisioned previously. So, finally:

- What other phenomena does this state exhibit?

5.2 The Hamiltonian

The goal is to identify the solutions to the Schrödinger equation *and* the physics they signify. The relevant Schrödinger equation is given by

$$H\Psi = E\Psi, \tag{5.1}$$

where

$$H = \sum_j \frac{1}{2m_\mathrm{b}} \left[\frac{\hbar}{i}\nabla_j + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}_j) \right]^2 + \frac{e^2}{\epsilon} \sum_{j<k} \frac{1}{|\boldsymbol{r}_j - \boldsymbol{r}_k|} + \sum_j U(\boldsymbol{r}_j) + g\mu \boldsymbol{B}\cdot\boldsymbol{S}. \tag{5.2}$$

The first term on the right hand side is the kinetic energy in the presence of a constant external magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, the second term is the Coulomb interaction energy,

the third term is a one-body potential incorporating the effects of the uniform positive background and disorder, and the last term is the Zeeman energy.

To get a feel for the relative importance of the various terms, we consider GaAs–AlGaAs heterostructures, on which most quantum Hall experiments have been performed. We use the following values: band mass of electrons $m_b = 0.067 m_e$ (where m_e is the electron mass in vacuum); dielectric constant $\epsilon = 12.6$; Landé g factor $g = -0.44$. Relevant parameters are the cyclotron energy

$$\hbar\omega_c = \hbar \frac{eB}{m_b c} \approx 20B[\mathrm{T}]\,\mathrm{K}, \tag{5.3}$$

the typical Coulomb energy

$$V_C \equiv \frac{e^2}{\epsilon \ell} \approx 50\sqrt{B[\mathrm{T}]}\,\mathrm{K}, \tag{5.4}$$

the Zeeman splitting

$$E_Z = 2g\mu_B \boldsymbol{B} \cdot \boldsymbol{S} = \frac{g}{2}\frac{m_b}{m_e}\hbar\omega_c \approx 0.3B[\mathrm{T}]\,\mathrm{K}, \tag{5.5}$$

and the magnetic length

$$\ell = \left(\frac{\hbar c}{eB}\right)^{1/2} \approx \frac{25}{\sqrt{B[\mathrm{T}]}}\,\mathrm{nm}. \tag{5.6}$$

Here, the last terms in Eqs. (5.3), (5.4), and (5.5) give the energy (in kelvin) for parameters appropriate for GaAs, with B[T] in tesla. The last term in Eq. (5.6) quotes the magnetic length in nm. The Zeeman splitting is defined as the energy required to flip a spin. Another possible unit for the Coulomb interaction is $e^2/\epsilon r_0$, where r_0 is defined by $\pi r_0^2 \rho = 1$; this scale is related to $e^2/\epsilon\ell$ through $r_0/\ell = \sqrt{2/\nu}$.

Following the standard practice, we often use ℓ as the unit of length and $e^2/\epsilon\ell$ as the unit of interaction energy. This amounts to setting

$$\ell = 1, \qquad \frac{e^2}{\epsilon\ell} = 1. \tag{5.7}$$

Our first and foremost concern is with conceptual foundations of the FQHE. With that goal in mind, we simplify the problem to the maximum extent possible, without, of course, throwing away the essential physics of the FQHE. We (i) switch off disorder, (ii) take a vanishing transverse width for the electron wave function, (iii) neglect LL mixing, and (iv) assume that electrons are fully polarized, i.e., are effectively spinless. The simplifications (iii) and (iv) are equivalent to taking the limit $B \to \infty$, because that implies

$$\frac{e^2/\epsilon\ell}{\hbar\omega_c} \to 0, \tag{5.8}$$

$$\frac{e^2/\epsilon\ell}{E_Z} \to 0. \tag{5.9}$$

In this limit, the Coulomb interaction is too weak to cause either LL mixing or spin reversal. The limit $B \to \infty$ is taken *at a fixed filling factor*, which also requires $\rho = \nu B/\phi_0 \to \infty$. (If ρ were held fixed instead, the limit $B \to \infty$ would imply $\nu \to 0$, which is a trivial limit with no FQHE; the distance between electrons becomes large compared to their size ($\sim$ the magnetic length), and the system becomes classical.)

The justification for these simplifying approximations is as follows. (i) As in the IQHE, disorder is crucial for establishing plateaus, but is not needed for an understanding of the underlying physics, which has to do with the appearance of gaps. We see in Chapter 7 that the presence of a gap at $\nu = f$ in a disorder-free system leads to a plateau at $R_{\rm H} = h/fe^2$ when a weak disorder is introduced. (ii) Qualitative features of the phenomenon are not affected by the transverse width, provided it is not too large. (iii) The qualitative phenomenology of the FQHE (e.g., the value of the quantized Hall resistance, or which fractions are observed) is independent of the amount of LL mixing, at least when it is sufficiently small, as can be seen from experiments on samples with different parameters (density, band mass, ϵ, etc.). (iv) Equation (5.9) is not satisfied in most experiments, and not all observed FQHE states are fully spin polarized. However, fully spin-polarized FQHE states do occur. We first focus on these states, coming to the role of spin in Chapter 11.

The simplifying assumptions will have to be relaxed before a detailed *quantitative* comparison with experiment becomes feasible. Neglecting LL mixing and nonzero thickness corrections, however, is often not too unrealistic even when it comes to comparing numbers. The idealized model explains major trends seen in experiments and provides zeroth order estimates for experimental numbers for fully spin-polarized FQHE states. We see in Chapter 11 that the electron spin produces much interesting physics at relatively low magnetic fields. Specifically, many FQHE states with different spin polarizations can occur at a given filling factor. These states can be explained by a generalization of the physics of fully polarized FQHE states.

When electrons are confined in the lowest Landau level, with their spin degree of freedom frozen, the kinetic and Zeeman energies are irrelevant constants which we throw away and forget. They are the same for all eigenstates, and do not contribute to any energy differences. We thus end up with the idealized model of "effectively spinless" electrons in the lowest Landau level, restricted to an ideal two-dimensional plane, with the Hamiltonian given by

$$H = \mathcal{P}_{\rm LLL} \frac{e^2}{\epsilon} \sum_{j<k} \frac{1}{|\boldsymbol{r}_j - \boldsymbol{r}_k|} \mathcal{P}_{\rm LLL}, \tag{5.10}$$

which must be solved with the LLL restriction, as explicitly indicated by the LLL projection operator $\mathcal{P}_{\rm LLL}$. This is the simplest and the cleanest model containing the essential physics of the FQHE, stripped of all inessential features. Study of this model is sufficient for establishing the physical principles of the FQHE.

Note: Without the LLL restriction, the right hand side of Eq. (5.10) is classical, as it contains no noncommuting operators. The LLL projection turns it into a nontrivial quantum mechanical problem. The Coulomb interaction can be formally projected into the lowest

Landau level using the method discussed in Section 3.7. For that purpose, working with the Fourier transform is convenient [192]:

$$\sum_{j<k} V(|\boldsymbol{r}_j - \boldsymbol{r}_k|) = \int \frac{\mathrm{d}^2\boldsymbol{q}}{(2\pi)^2} V(q) \sum_{j<k} \exp\left[\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_j - \boldsymbol{r}_k)\right]$$
$$= \int \frac{\mathrm{d}^2\boldsymbol{q}}{(2\pi)^2} V(q) \sum_{j<k} \exp\left[\frac{\mathrm{i}}{2} q(\bar{z}_j - \bar{z}_k)\right] \exp\left[\frac{\mathrm{i}}{2}\bar{q}(z_j - z_k)\right],$$

where $q = q_x - \mathrm{i}q_y$. We have already normal-ordered the expression on the right, with $\bar{z}$'s preceding z's. The LLL projection is obtained by making the replacement $\bar{z}_j \to 2\partial/\partial z_j$, which gives

$$\mathcal{P}_{\mathrm{LLL}} \sum_{j<k} V(|\boldsymbol{r}_j - \boldsymbol{r}_k|) \mathcal{P}_{\mathrm{LLL}} = \int \frac{\mathrm{d}^2\boldsymbol{q}}{(2\pi)^2} V(q)$$
$$\times \sum_{j<k} \exp\left[\mathrm{i}q\left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_k}\right)\right] \exp\left[\frac{\mathrm{i}}{2}\bar{q}(z_j - z_k)\right]$$

with the proviso that derivatives do not act on the Gaussian part of the wave function. While it brings out the quantum mechanical nature of the problem, this form has not proved to be amenable to further manipulations. In practice, one implements the LLL restriction on wave functions. Once the wave functions are restricted to the LLL sector, projection of the Hamiltonian becomes unnecessary, because, by definition, the projected and the unprojected interactions have identical matrix elements in the LLL subspace.

5.3 Why the problem is hard

The simplicity of the statement of the problem is deceptive. Equation (5.10) already reveals the fundamental difficulty. In many-body problems, the starting point for a theoretical investigation of a phenomenon is often obtained by switching off the interaction altogether. But the interaction is not small compared to anything else in our problem, because there is nothing else. The interaction is the only energy in our problem, and therefore cannot be neglected. Equation (5.10) contains no small parameter. Taking the magnetic length ℓ as the unit of length, and $e^2/\epsilon\ell$ as the unit of energy, we can rewrite Eq. (5.10) as

$$H = \mathcal{P}_{\mathrm{LLL}} \sum_{j<k} \frac{1}{|\boldsymbol{r}_j - \boldsymbol{r}_k|} \mathcal{P}_{\mathrm{LLL}}, \tag{5.11}$$

where all quantities are dimensionless. This form shows that the problem has no real parameters whatsoever.[1] The usual quantity characterizing the strength of correlations,

[1] The eigenfunctions of this problem do not contain any sample specific parameters.

namely the ratio of interaction energy to kinetic energy, is infinite, because the latter is absent. The FQHE state is one of the most strongly correlated systems in the world.

Let us ask what happens if, just for a moment, we set the interaction to zero. That produces a large number of ground states, because the number of available single particle orbitals greatly exceeds the number of electrons. A simple calculation gives a glimpse into the enormous complexity of the problem. A typical 1mm $\times$ 1mm sample contains 10^9 electrons. Assuming 2.5×10^9 single particle orbitals in the lowest Landau level (which corresponds to a filling factor $\nu = 0.4$), the number of distinct ground states is $10^{7\times10^8}$, which is an unimaginably large number. Even a toy system containing only 100 electrons in 250 single particle orbitals has 10^{72} distinct ground state configurations, which is roughly equal to the number of atoms in the Universe.

In the absence of interaction, no single state is picked out at any fractional filling factor. When the interaction is switched back on, the degeneracy is lifted and one linear combination of the allowed states becomes the nondegenerate ground state. The interaction causes a spectacular reorganization of the low-energy Hilbert space, thereby producing remarkable phenomena. The difficulty is that, on purely theoretical grounds, with no small parameter to guide our thinking, we have no clue where to start looking for the ground state.[2] Standard perturbative methods are of no use and the problem appears hopelessly intractable.

In other words: The FQHE has no "normal state." The FQHE cannot be understood as an instability of a familiar state, caused by turning on a weak interaction of the appropriate kind. In the absence of such a reference state, computing deviations from a known state is not possible, and one must calculate *full* answers. That would be akin to solving the Schrödinger equation to explain superconductivity without assuming a normal state with weakly interacting quasiparticles.

It is thus not possible to *first* solve the Schrödinger equation and *then* try to understand what physics the solution signifies. Our approach will be first to identify the physics, with help from experiment, which then will lead us to the solution of the Schrödinger equation.

5.4 Condensed matter theory: solid or squalid?

Solid state physics was famously ridiculed, allegedly by Murray Gell-Mann, as the "squalid state physics." It is worth understanding why that might appear to be the case, and why that is not true (any more than for any other branch of physics).

The reader is surely aware of the crucial difference between mathematics and physics: A mathematical theorem, once proven, is forever true. A physical theory, in contrast, is always subject to revision in light of new information. Examples abound where a once widely accepted theory is discarded or superseded by a better theory. Not that the old theory does not work any more. It continues to explain whatever it explained before. But another theory, based on a fundamentally new approach, explains a wider body of phenomena while

[2] *A priori*, a crystalline state might appear an attractive starting point, but experiments rule it out in the region where FQHE is seen.

obtaining the previous theory as a limiting case. Progress in physics can be viewed as such "paradigm shifts" at various levels (Kuhn [355]).

While no physical theory can have the tidiness or the finality of a mathematical theorem, the situation can be especially confusing in condensed matter physics. An advantage here is that we know exactly what problem needs to be solved: the nonrelativsitic Schrödinger equation for the many-particle system of interacting electrons and ions. (As far as condensed matter physics is concerned, this equation can be taken as the "exact" starting point; relativistic effects are often of no consequence.) The difficulty is that finding its exact solution is impossible. We do not know how to solve the general problem containing even three particles, let alone a macroscopic number. No condensed matter theory, therefore, can be one hundred percent correct. We attempt either to find an approximate solution of the exact problem defined by the aforementioned Schrödinger equation, or to solve exactly an approximate model which, hopefully, contains the essential physics of the phenomenon in question. Most of the time, we have to be content with approximate solutions of approximate models.

How, then, do we come to accept a condensed matter theory? The answer appears easy: It must explain nature. An approximate model or an approximate solution may appear compellingly elegant to its protagonists, but experiments are the ultimate judge for distinguishing a physical theory from a mere mathematical model. What makes this criterion difficult to employ in practice is that, being approximate, no theory of a many-body system will ever explain *all* experimental facts. Every theory will have aspects that deviate from experiments, and the question, therefore, boils down to what one considers the "essential" features. That is the reason why the validity of a given theory is often a topic of an intense and bitter debate; the situation is like the proverbial blind men and the elephant, with scientists focusing on different aspects drawing dissimilar conclusions.

Condensed matter theories are – and should be – viewed with great skepticism. They can go wrong in many places. The effective model being considered may not contain the physics we are seeking to describe. Some of the approximations made in deriving consequences of the model may be invalid. And even if the theory describes some phenomena, it will fail in other aspects. When certain consequences of a theory are inconsistent with experiment, it is not clear if the error lies in our solution of the effective model, in the model itself, or both, and whether the inconsistency is substantial. This explains why theoretical condensed matter physics might appear to be a singularly messy enterprise. No new principle seems to be at stake (we know the starting point precisely), the exact solution is hopelessly complicated, and arriving at any kind of consensus is difficult.

Lest the reader should become overly pessimistic, we hasten to add that the above stated views are flawed in many respects. While condensed matter physics surely has its own share of dubious theoretical attempts, the most successful condensed matter theories are highly nontrivial, stunningly elegant, and so well confirmed by experiment that their essential correctness is beyond doubt.

Perhaps counterintuitively, the exact solution of the exact problem is not what we are after. Suppose we are able to calculate the exact eigenfunctions and eigenvalues of the

Schrödinger equation for interacting electrons in the lowest Landau level by a brute force diagonalization on a huge computer. Of course, given the dimension of the matrix to be diagonalized, it would be impossible for a computer to produce the answer for a realistic system. Let us ignore that problem for a moment and imagine that we have a "quantum computer" that can calculate the exact eigenstates. The computer would disgorge a long list of numbers, which are projections of the ground and excited state vectors along various directions of the very large dimensional Hilbert space. Given all eigenfunctions, we can insert them into the Kubo formula to calculate magnetoresistances, which would surely reproduce the FQHE in its full glory. Is the problem solved? Not really. The computer does not tell us *what* the numbers mean, and *why* nature is behaving the way it is. The computer does not supply an insight into the physics. That is hardly a satisfying state of affairs.

So, what are we after? As enunciated in a landmark 1972 article by P. W. Anderson entitled "More is different" [8] (also see a delightful recent book by Laughlin [373] on this topic), even though we may know the defining theory, many-body systems collectively behave in remarkably surprising ways, an understanding of which requires entirely new sets of concepts. The goal of condensed matter theory is to identify simple "emergent" principles that enable us to unify, explain, predict, and calculate. These principles should facilitate an exact account of the universal properties, and a reasonably accurate quantitative description of nonuniversal observables. While numbers are important, physical insights are more so, especially in condensed matter physics, where our goal is not to test the starting point (we believe in the Schrödinger equation) but to understand beautiful structures that emerge when many particles interact. To that end, an approximate solution that reveals the physics of the problem plays a more important role than the exact solution.

A marvelous example is the Bardeen–Cooper–Schrieffer theory of ordinary superconductors, based on the principle of pairing of electrons. It explained the zero resistance, the Meissner effect, the isotope effect, flux quantization, and many other mysterious phenomena. It led to the prediction of the Josephson effect. And, its quantitative consequences have been verified in great detail for myriad phenomena for weakly coupled superconductors.

How do we determine the emergent principle? Answer: The only method, really, is to make an intelligent *guess* guided by experiment. We then build that principle into a theoretical formulation and deduce its consequences to the best of our ability, which then are put to the test against experiment. A theoretical treatment may often give the misleading impression of "deriving" the principle, but the reader must beware that the principle can only be motivated, not derived. Certain essential assumptions are made at the very first step of every condensed matter theory about the qualitative nature of the state; the rest of the theoretical effort is toward gaining a better quantitative description. For example, in the Landau Fermi liquid theory one may calculate many complicated Feynman diagrams, but the nature of the state is assumed as soon as one decides to perturb around the noninteracting solution. The theory does not *prove* that the actual state is a Landau Fermi liquid, but only gives an approximate description of it were that to be the case. It is sometimes possible to deduce, theoretically,

that a hypothesized phase is unstable, but it is never possible to prove that it is not. Our intention here is not to minimize the importance of quantitative calculations, which play an all important role in the validation of the basic principle, but to urge the reader to make an effort to distinguish between what is being assumed and what is being derived, and not to confuse mathematical formalism with physics.

5.5 Laughlin's theory

A crucial clue into the physics of the FQHE problem came soon after the observation of $f = 1/3$, when Laughlin [369] made a brilliant ansatz for the ground state wave function at $\nu = 1/3$, which we present in this section.

Single particle states in the lowest Landau level are given by

$$\eta_l(z) = (2\pi 2^l l!)^{-1/2} z^l \, \mathrm{e}^{-\frac{1}{4}|z|^2}, \tag{5.12}$$

where $z = x - \mathrm{i}y$ and l is the angular momentum (not to be confused with the magnetic length, which has been set to unity). A LLL wave function for many electrons must necessarily have the form

$$\Psi = F_{\mathrm{A}}[\{z_j\}] \exp\left[-\frac{1}{4}\sum_i |z_i|^2\right], \tag{5.13}$$

where $F_{\mathrm{A}}[\{z_j\}]$ is a polynomial of z's antisymmetric under exchange of two coordinates. (The spin part of the wave function, which is fully symmetric under exchange of particles, is not shown explicitly.) The task of theory is to determine the polynomial $F_{\mathrm{A}}[\{z_j\}]$ for low-energy many-body states of interacting electrons as a function of the filling factor. Laughlin proceeded as follows:

(i) Motivated by the success of Jastrow-type variational wave functions [31, 289] for superfluid helium, which have pairwise correlations, let us assume the form

$$F_{\mathrm{A}}[\{z_j\}] = \prod_{j<k} f(z_j - z_k). \tag{5.14}$$

(ii) In order for the full wave function to be an eigenstate of the total angular momentum, which commutes with the Coulomb interaction, the product $\prod_{j<k} f(z_j - z_k)$ must be a polynomial of $z_1, z_2, \ldots, z_N$ of degree L, i.e., the replacement $z_i \to z_i \mathrm{e}^{-\mathrm{i}\theta}$ is equivalent to multiplication by $\mathrm{e}^{-\mathrm{i}L\theta}$. This is possible only if $f(z_j - z_k)$ itself has a definite angular momentum.

(iii) At the same time, antisymmetry under exchange requires that $f(-z) = -f(z)$. The only form that satisfies these two criteria (and also analyticity in z as required by the LLL constraint) is

$$f(z) = z^m, \tag{5.15}$$

where m is an odd integer. This produces what is known as the Laughlin wave function (see also Ref. [45]):

$$\Psi_{1/m} = \prod_{j<k} (z_j - z_k)^m \exp\left[-\frac{1}{4}\sum_i |z_i|^2\right]. \tag{5.16}$$

As seen later in this chapter, and again in Chapter 12, this wave function describes a uniform density state at filling factor $\nu = 1/m$. In particular, for $m = 3$, it describes a state at $\nu = 1/3$, the fraction that had been observed at the time of Laughlin's work. Detailed calculations (e.g., Haldane and Rezayi [222]) confirm that the Laughlin wave function is a valid and accurate representation of the actual ground states at $f = 1/3$ and $f = 1/5$. The fractions $f = 1/m$ are referred to as the Laughlin fractions.

Had only the fractions $f = 1/m$ and their particle–hole symmetric counterparts $f = 1 - 1/m$ been observed, this would have been the end of the story; it would only remain to evaluate the properties of the Laughlin wave function. The subsequent appearance of a large number of other fractions, to which Laughlin's theory does not apply,[3] tells us that this theory is incomplete. For example, referring to Fig. 1.3, the Laughlin wave function is pertinent to 1/3 and 2/3 but not to the ∼30 other fractions seen on this figure in the range $\nu < 1$.[4]

The remainder of this chapter is concerned with a new principle: the formation of topological particles called composite fermions (to be defined below). They bring out the general physics of the FQHE, giving a physical understanding of why gaps open at fractional fillings. The CF theory explains all fractions in a unified fashion, and also reveals an underlying structure that encompasses phenomena other than the FQHE. While the CF theory also leads to wave functions, the essential physics is independent of wave functions (Section 5.8). Composite fermions represent a paradigm that is more general than a particular theoretical model or a wave function. (Short, non-technical reviews on composite fermions can be found in Refs. [283, 284].)

What about the Laughlin wave function? The CF theory recovers it as a special case (Section 5.8.6). The CF "derivation" of the Laughlin wave function proceeds along a different logical route from the one described above, enriching this wave function with a new physical interpretation and making manifest that it belongs to a larger conceptual structure. In contrast, composite fermions are neither a part of, nor can be derived from, the Laughlin wave function.

Laughlin also demonstrated the striking property, using a general flux insertion argument, that incompressibility at a fractional filling factor implies the existence of fractionally charged excitations.[5] That is one of the topics addressed in Chapter 9.

[3] The exponent m in the Laughlin wave function must be an odd integer. An even m gives a wave function for bosons, whereas a fractional m fails to produce a single-valued wave function.

[4] Each FQHE state represents a distinct state that cannot be continuously deformed into another, as evident from the fact that the Hall resistance, being quantized at *discrete* values, cannot be modified *continuously* from one plateau to another.

[5] The possibility of fractionally charged quasiparticles was first mentioned in the experimental discovery paper (Tsui, Stormer, and Gossard [650]). In the context of the quantized Hall resistance at $3h/e^2$, it states: "If we attribute it to the presence of a gap at E_F when 1/3 of the lowest Landau level is occupied, [Laughlin's] argument will lead to quasiparticles with fractional

5.6 The analogy

Experience has taught us that, in physics, life invariably becomes more complicated before it becomes simple. As more and more facts are gathered, the situation grows more confusing at first, but eventually new patterns become apparent that give a clue into the deeper structure, leading to a simplification. For example, the Lyman–Balmer–Paschen spectral lines of the hydrogen atom motivated the simple Bohr model. As another example, the discovery of a large number of hadrons in the 1950s led to an identification of structures such as the SU(3) flavor octet, which took us one level deeper into the realm of quarks, enabling a more fundamental description of hadrons as bound states of quarks. Experimental facts, through patterns they reveal, provide a hint into an underlying connection between the mysterious phenomenon at hand and some other familiar problem. The trick is to identify patterns and the correct analogy.

With a rapid growth in the number of fractions, the situation became, at first, similarly bewildering in FQHE, but eventually two crucial patterns became apparent:

- Fractions do not appear in isolation but are members of certain sequences.
- Even denominator fractions are absent.

Thus, rather than trying to understand one fraction at a time, we should search for a principle that explains whole sequences at once. The correct explanation should produce the observed sequences of fractions, clarifying, at the same time, why they have odd denominators.

What is the key analogy here? Let us again look at Fig. 2.7 and, to avoid distraction by too much detail, do the mental exercise of erasing all the numbers. What then jumps out is that telling the fractional plateaus from the integral ones is impossible. The search for an analogy thus brings us to the IQHE. Can the well-understood integral quantum Hall effect teach us something about the mysterious fractional quantum Hall effect? As we see below, the answer is in the affirmative, and the relation between the two is profound and far-reaching, forming the very basis of our understanding of the FQHE. In hindsight, that is not surprising at all. The history of physics is replete with examples where analogies crop up in unexpected places. Here, the two phenomena, the integral and the fractional Hall effects, are so similar that the possibility of their being unrelated would be inconceivable.

In view of the remarkable parallel between the FQHE and the IQHE, it is tempting to postulate that the FQHE is really an IQHE in disguise. That raises the question: Integral quantum Hall effect of *what*? What objects in the FQHE are analogous to the electrons of the IQHE? These objects must be some kind of fermion, because, after all, the fermionic nature of the electrons is what makes integral fillings special. The pursuit of the analogy between the FQHE and the IQHE thus points inescapably to a new kind of fermionic particles in the FQHE.

electronic charge of 1/3." As recounted in Ref. [627], D. C. Tsui exclaimed "Quarks!" upon first noticing (Fig. 1.2) that the 1/3 plateau occurs at roughly three times the magnetic field as the $\nu = 1$ plateau.

5.7 Particles of condensed matter

The appearance of new particles in condensed matter physics may seem surprising at first, but it should not. A short digression is worthwhile to discuss what we mean by a "particle."[6] Of course, we know all the particles that go into the Hamiltonian of a condensed matter system, namely electrons and ions (only electrons for our problem), but of concern here are the particles that come out. A most profound fact of nature – indeed the very reason why physics can make progress at many different levels – is that strongly interacting particles reorganize themselves to become more weakly coupled particles of a new kind. These new particles are, in a deep sense, the "true" particles of the system in question, because it is reasonable to reserve the title "particle" for nearly independent objects. The old particles were the particles of the problem, and the new particles are the particles of the solution.

Identifying the relevant particles is equivalent to mapping the strongly correlated system of one kind of particles into a weakly coupled system of a different kind of particles. Mathematically, we have expressed the initial Hamiltonian as

$$H = H_0^{\text{new}} + V^{\text{new}}, \tag{5.17}$$

where H_0^{new} describes a system of certain independent particles (not necessarily the particles that define H). V^{new}, the interaction between these particles, can be treated perturbatively; it makes quantitative corrections, but does not change the qualitative behavior, i.e., does not cause a phase transition. Because theorists know what to do with weakly interacting particles – perhaps the only thing they really know – the problem is, in essence, solved as soon as such objects are identified. The qualitative phenomenology can be understood by switching off V^{new} altogether and working with a system of free particles. A quantitative understanding can be obtained by working sufficiently hard on perturbation theory. The primary goal is thus to identify the relevant particles.

At times, what objects are weakly interacting is intuitively obvious, and the choice of true particles is made without much fanfare. We know when to start with quarks and when with ^{4}He atoms. At other times, the identification of true particles may be subtle and nontrivial. Their rigorous derivation from first principles is generally not possible, especially when their formation entails nonperturbative physics. The clue often lies in the experimental phenomenology.

It is magical when the mysterious properties of a system suddenly become obvious, even inevitable, when viewed through the frame of reference of the new particles. Phenomena that were difficult or impossible to understand in terms of the original particles become simply comprehensible as properties of almost free particles.

In many familiar examples outside condensed matter physics, the new particles are simple bound states of old ones. The emergent particles of condensed matter, however, can be fantastically complex objects. Sometimes, only with a great deal of work can their existence be recognized and established. We cite here a few well-known examples.

[6] The views discussed in this section are widely appreciated in the condensed matter community. See, for example, Refs. [282,372].

- **Phonons** Viewing a crystal lattice as a system of almost free atoms makes no sense. The Einstein model considers a system of independent atoms, but with each atom trapped in a harmonic oscillator potential created by the other atoms. This model is also inadequate, for example, for the low-temperature specific heat, because all excitations have the same energy. The coupling between neighboring atoms cannot be neglected. If only terms that are of second order in displacement are kept, the problem can be solved by Fourier transformation, which brings us to the basis of phonons. Of course, phonons are also interacting, due to higher-order terms in the coupling, but the interaction between them is much weaker than that between atoms, and is negligible for most purposes.
- **Magnons** These are the particles of a magnet. Instead of individual spin flips, which are coupled strongly to nearby spin flips, we discuss the physics in terms of spin waves, or magnons, which are weakly interacting.
- **Landau quasiparticles** Electrons in a normal metal interact via the Coulomb interaction, which is strong. Nonetheless, a neglect of the interaction is a valid approximation for many quantities, e.g., the specific heat. The origin of such behavior remained a puzzle for many years, until Landau clarified that the weakly interacting objects are not electrons but "Landau quasiparticles," which are, roughly, screened electrons. Landau quasiparticles share their quantum numbers – charge, spin, and statistics – with electrons, and, in fact, are perturbatively connected to electrons. Here, in principle, it *is* possible to treat the electrons as the "particles," and to extract the correct physics in a perturbative treatment of the interaction. Nonetheless, Landau quasiparticles are better particles because they are more weakly interacting.
- **Cooper pairs** In superconductors, the phonon-mediated attractive interaction between electrons causes a nonperturbative rearrangement of the system. The particles of a superconductor are Cooper pairs. An understanding of the basic physics is obtained in terms of noninteracting, i.e., uncorrelated, Cooper pairs. In this case, the true particles (Cooper pairs) bear no resemblance to the constituent particles, namely electrons. The charge of a Cooper pair is $2e$, its spin is zero, and it is (crudely) a boson. This kind of qualitative distinction between the constituent and the true particles is perhaps the most direct sign that nonperturbative physics is in play.

One characteristic that makes the particles of condensed matter nontrivial is that they live only within the condensed matter background. We cannot pull a phonon out of a crystal, a magnon out of a magnet, or a Cooper pair out of a superconductor. (In the last case, we end up with two electrons, not a Cooper pair.) In that sense, the particles of condensed matter are more akin to quarks: they cannot live in isolation, but are perfectly legitimate particles by virtue of being weakly interacting (in some limit).

Another difference is that the particles of condensed matter physics are never *completely* noninteracting. That is not surprising. What is surprising is that out of the complexity emerge particles which are weakly interacting to the extent they often are.

What exactly is weakly interacting depends on the experimental phenomenon in question, and the energy with which the system is probed. For example, while it is appropriate to view the O_2 molecule as a particle when one has the properties of oxygen gas in mind, one must worry about electrons and nuclei separately in chemical reactions, and even about nucleons inside the the nucleus in high-energy collisions. A hierarchy of phenomena exists, with new particles associated with each level.

5.8 Composite fermion theory

We now return to the FQHE problem. What are the relevant fermions here? They are certainly not electrons, because a system of noninteracting electrons would only exhibit the IQHE. Motivated by the Cooper pairs of superconductivity, we might envision new fermions that are bound states of an odd number of electrons. However, the occurrence of such bound states appears unlikely because the interaction between electrons is strongly repulsive. A little further thought shows this scenario to be grossly inconsistent with the known phenomenology, thus ruling it out as a viable possibility.

The new fermions, which have the unfortunate[7] name "composite fermions" [272], turn out to be unlike any other particles known in nature. They are complex entities from the vantage point of electrons, yet they behave as more or less independent particles in many situations, with well-defined charge, spin, statistics, and other properties that we attribute to particles. This section elucidates the fundamental concepts of the composite fermion theory, and also alerts the reader to some common misinterpretations and misconceptions.

5.8.1 The definition of composite fermion

Topological objects known as "vortices" occur in many contexts in condensed matter physics. An order j vortex is defined by the property that a complete loop around it produces a change of $2j\pi$ in the phase of the wave function or the order parameter. Perhaps the best known example is the Abrikosov vortex [3], which occurs in type-II superconductors in the form of a vortex line. The order parameter of a superconductor is a complex function

$$\Delta(\boldsymbol{r}) = |\Delta(\boldsymbol{r})|\mathrm{e}^{\mathrm{i}\chi(\boldsymbol{r})}. \tag{5.18}$$

For type-I superconductors the phase $\chi(\boldsymbol{r})$ is single valued. For type-II superconductors, on the other hand, a vortex line solution becomes relevant for which the phase of the order parameter changes by 2π around the vortex.

In the context of the quantum Hall liquid the term "vortex" is used in a slightly different sense. Because the magnetic field breaks time reversal invariance, the wave function is not real. In very high magnetic fields, when electrons are confined in the lowest Landau level, the wave function is a polynomial of $z_j = x_j - \mathrm{i}y_j$, apart from a real Gaussian factor. We know from the property of complex functions that $z = r\mathrm{e}^{-\mathrm{i}\theta}$ has a vortex at the origin, because a complete loop around the origin changes θ by -2π. Similarly, $z - z_0$ has a vortex at z_0. This is what is meant by the term vortex in the FQHE. We encounter below factors like $(z_1 - z_2)^{2p}$, in which particle one sees $2p$ vortices on particle two, and vice versa.

The vortices in superconductors and the FQHE are not identical. (i) The former refer to a structure in the order parameter, whereas the FQHE vortices occur in the microscopic wave function. (ii) The phase gradient around a vortex in a superconductor induces a macroscopic supercurrent circulating around it, which, in turn, produces an experimentally detectable

[7] The name "composite fermion" is used in a generic sense in atomic or high energy physics, e.g., for a proton or a ^{3}He atom. It now has acquired a definite meaning in condensed matter physics.

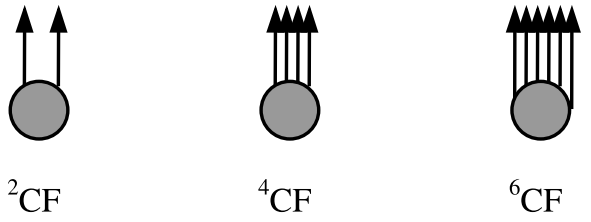

Fig. 5.1. Schematic view of three flavors of composite fermion carrying (a) two, (b) four and (c) six vortices. These are denoted by ^{2}CF, ^{4}CF and ^{6}CF, respectively. Each arrow represents a quantized vortex. Vortices are modeled as flux quanta in a mean-field description.

magnetic "flux tube" carrying a quantized magnetic flux of magnitude $hc/2e$ ($2e$ is the charge of a Cooper pair). In contrast, no real flux is associated with the FQHE vortex; the magnetic field does not bunch near it, nor does any current circulate around it.

Now we are ready to define the composite fermion:

Definition *A composite fermion is the bound state of an electron and an even number of quantized vortices.*

A pictorial representation of several flavors of composite fermions is shown in Fig. 5.1, where each arrow represents a vortex. Sometimes composite fermions are pictured as bound states of electrons and an even number of magnetic flux quanta (a flux quantum is defined as $\phi_0 = hc/e$), which is how they were first introduced [272].[8] This model for the composite fermion derives its justification from the property that a point flux quantum and a vortex are "topologically" similar, in the sense that they both produce the same winding phases. A vortex, by definition, produces a phase of 2π for a closed path around it, which is also the Aharonov–Bohm phase produced by a flux quantum for a closed electron loop encircling it. The electron–flux bound state, however, is only a crude model for the "true" composite fermion. It is intuitively useful because it provides a nice picture, but it must not be taken literally. No real fluxes are bound to electrons, and the physical magnetic field is uniform.[9]

5.8.2 From IQHE to FQHE: the mean-field approximation

This subsection walks us through the golden path connecting the integral quantum Hall effect to the fractional quantum Hall effect, following Ref. [272]. For illustration, we begin by considering the special filling factors $\nu^* = n$. The connection is established through the following steps:

Step I Let us consider noninteracting electrons at $\nu^* = n$. The system is incompressible, i.e., the ground state is nondegenerate, separated from the other eigenstates by a gap (equal

[8] This is reminiscent of, and was inspired by, the ingenious idea of Leinaas–Myrheim–Wilczek braiding statistics [386, 677] (Section 9.8), wherein particles obeying fractional braiding statistics are modeled as fermions or bosons carrying a fictitious flux of magnitude $\alpha\phi_0$. As α is increased to an even integer, the statistics comes back to being fermionic or bosonic.

[9] The name "Chern–Simons fermion" is sometimes used for the electron–flux composite. We refer to both electron–flux and electron–vortex composites as composite fermions to avoid unnecessary proliferation of names.

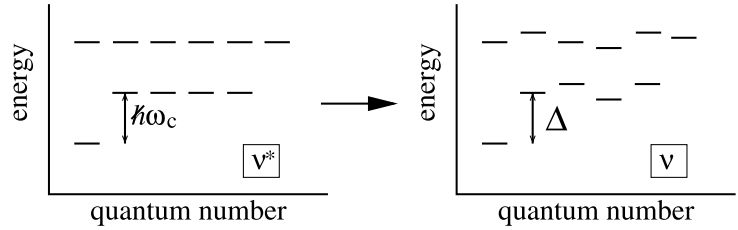

Fig. 5.2. The general structure of the energy spectrum of the many-body system at an integral filling, $\nu^* = n$. The ground state is nondegenerate, containing n Landau levels fully occupied (shown in Fig. 5.3). The excited states form bands separated by the cyclotron energy. The x-axis label is a convenient quantum number (wave vector in the periodic geometry, or the orbital angular momentum in the spherical geometry). The CF *mean-field* theory predicts that the low-energy spectrum at fractional fillings $\nu = n/(2pn \pm 1)$ has identical structure, except that states at ν are only quasi-degenerate and the cyclotron gap evolves into a gap Δ (the determination of which requires a microscopic theory). Explicit calculations beyond the mean-field theory show that the one-to-one correspondence displayed in this figure is valid for the lowest band, but the mean-field theory predicts spurious states in higher bands at ν.

to the cyclotron energy). The *many-particle* energy spectrum is shown in Fig. 5.2. The ground state has n full Landau levels, shown schematically in the left column of Fig. 5.3. The lowest energy excited state is a particle–hole pair, or an exciton, shown in the left column of Fig. 5.4(d) for $\nu^* = 3$. These diagrams have precise wave functions associated with them. We denote the magnetic field by B^*, which can be either positive or negative. It is related to the filling factor by $\nu^* = \rho\phi_0/|B^*| = n$.

The long-range rigidity in the system at an integral filling factor (manifested by the presence of the gap) is caused solely by the Fermi statistics. Thinking in the standard Feynman path-integral language [163, 575] is useful. The partition function gets contributions from all closed paths in the configuration space for which the initial and the final positions of electrons are identical, although the paths may involve fermion exchanges. Some examples are: a path in which one electron moves in a loop while others are held fixed; or a cooperative ring exchange path [342] in which the closed loop involves many electrons, with each electron moving to the position earlier occupied by the next one. The phase associated with each closed path has two contributions: the Aharonov–Bohm phase, which depends on the flux enclosed by the loop; and the statistical phase, which is $+1$ or -1, depending on whether the final state is related to the initial one by an even or an odd number of pairwise electron exchanges. The incompressibility at integral fillings is presumably caused by some special correlations built into the phase factors of various paths. These correlations are not easily identifiable in the path integral language, but we know they are present.

Step II Now we attach to each electron a massless, infinitely thin magnetic solenoid carrying $2p$ flux quanta pointing in the $+z$ direction. This converts electrons into composite fermions. The flux added in this manner is unobservable, because it does not alter the

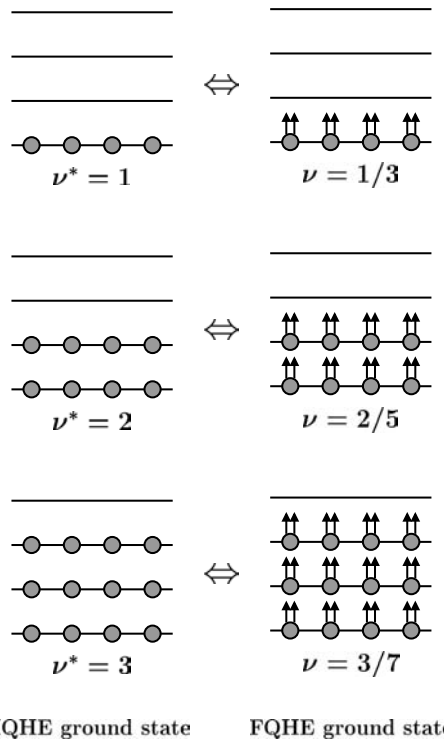

Fig. 5.3. Schematic view of the electron ground state at $\nu^* = n$ (left column). The columns on the right show the CF view of the ground states at $\nu = n/(2n + 1)$, as n filled Λ levels of composite fermions. (The Λ levels of composite fermions are analogous to Landau levels of electrons at ν^*, but lie within the lowest Landau level of electrons at ν.) Horizontal lines in the left column depict Landau levels of electrons, and those on the right depict Λ levels of composite fermions.

phase factors associated with any closed Feynman paths. The excess or deficit of an integral number of flux quanta through any closed path is physically unobservable, and the fermionic nature of particles guarantees that the phase factors of paths involving particle exchanges also remain intact. In other words, the new problem defined in terms of composite fermions is identical to the original problem of noninteracting electrons at B^*. We have thus transformed an incompressible state of electrons into an incompressible state of composite fermions.

Step III This exact reformulation prepares the problem for a mean-field approximation that was not available in the original language. Let us adiabatically (i.e., slowly compared to $\hbar/\Delta$, where Δ is the gap) spread the flux attached to each electron until it becomes a part of the uniform magnetic field.[10] (Because the initial state has a uniform electron density,

[10] This innovative mean-field theory was first introduced by Zhang, Hansson, and Kivelson [743] to map the FQHE at $\nu = 1/m$ into bosons at zero magnetic field, and also employed by Laughlin and collaborators [159, 371] to map a system of anyons (particles obeying fractional statistics; more in Section 9.8) into fermions or bosons in a magnetic field.

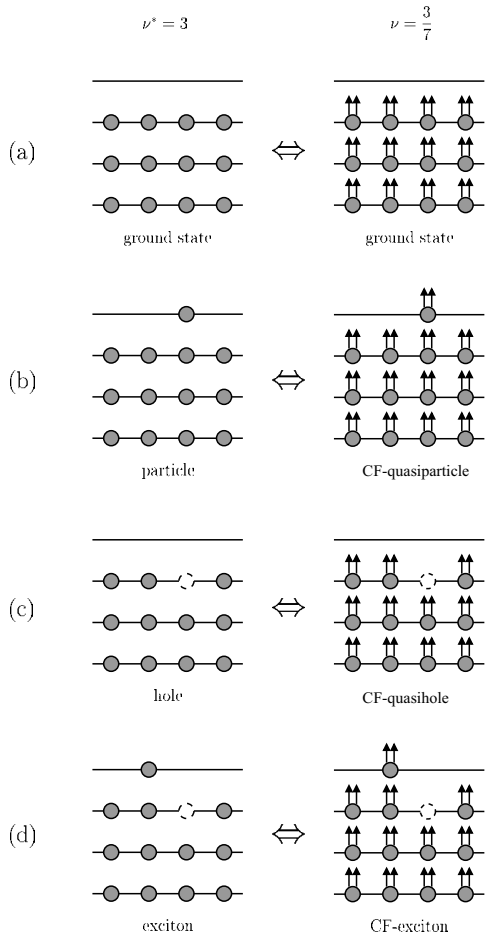

Fig. 5.4. Schematic view of the correspondence between the ground state and excitations at $\nu = 3/7$, viewed as the $\nu^* = 3$ of composite fermions, with the analogous states of electrons at $\nu^* = 3$. Shown on the right are: (a) the CF ground state; (b) a CF-quasiparticle; (c) a CF-quasihole; and (d) a CF-exciton. Horizontal lines in the left column depict Landau levels of electrons, and those on the right depict Λ levels of composite fermions.

the additional flux, tied to the density, produces a uniform magnetic field.) At the end, we obtain particles moving in an enhanced magnetic field B, given by

$$B = B^* + 2p\rho\phi_0. \tag{5.19}$$

The relation $|B^*| = \rho\phi_0/n$ implies that B is always positive (pointing in the $+z$ direction). The corresponding filling factor is given by

$$\nu = \frac{n}{2pn \pm 1}, \tag{5.20}$$

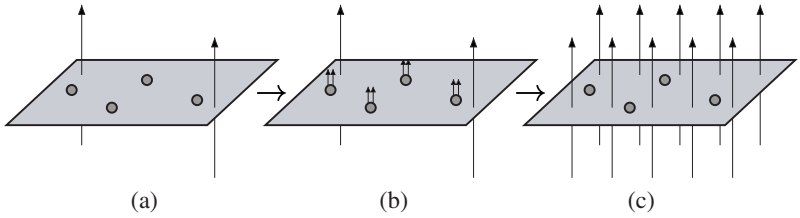

Fig. 5.5. The golden path from the IQHE to the FQHE. We begin with an IQHE state (a); attach to each electron two magnetic flux quanta to convert it into a composite fermion (b); and spread out the attached flux to obtain electrons in a higher magnetic field, which is a FQHE state (c).

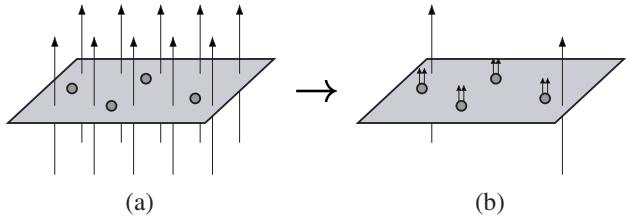

Fig. 5.6. Each electron captures two flux quanta to turn into a composite fermion. Composite fermions sense the residual magnetic field, which is much smaller than the applied magnetic field.

which follows from the relations $\nu = \rho\phi_0/B$ and $\nu^* = n = \rho\phi_0/|B^*|$. The $+$ $(-)$ sign in the denominator corresponds to B^* pointing in the $+z$ $(-z)$ direction.[11]

Let us now make the crucial assumption that the gap does not close during the flux diffusion process, i.e., there is no phase transition. To be sure, *quantitative* changes will occur. The gap and the wave functions will undergo a complex evolution. Nonetheless, if our assumption is correct, then Fig. 5.2 also represents, *qualitatively*, the spectrum at B.

The absence of a phase transition is an assumption that remains to be verified, and will surely not be valid for all n and p. If it is valid for some parameters, however, then the above construction gives a possible way of seeing how a gap can result at the fractions of Eq. (5.20). Three remarkable features already provide a strong hint that we are on the right track. First, these fractions are precisely the observed fractions. Second, they have odd denominators. Third, we naturally obtain sequences of fractions.

The three steps are depicted schematically in Fig. 5.5. The net effect, in a manner of speaking, is that each electron has absorbed $2p$ flux quanta from the external magnetic field to transform into a composite fermion. Composite fermions experience the residual magnetic field B^*. This is shown in Fig. 5.6. See Fig. 5.7 for a humorous portrayal of composite fermions.

Step IV Quantitative theory The CF physics described above is sufficient for an explanation of much of the phenomenology of the FQHE. Can it also help us write microscopic wave functions for the FQHE state? As depicted in Fig. 5.8, solving directly for

[11] The fractions in Eq. (5.20) have been referred to as the Jain sequences or the Jain fractions in the FQHE literature.

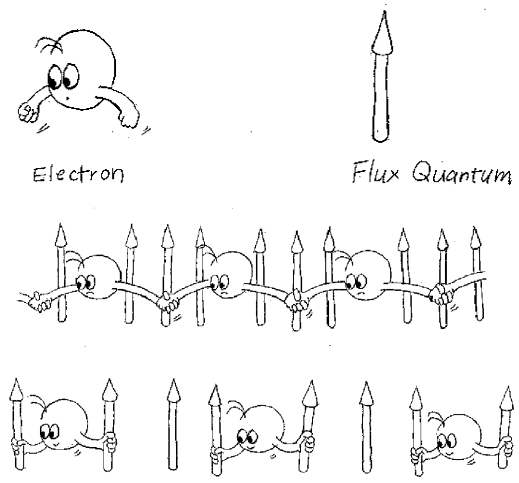

Fig. 5.7. A humorous view of composite fermions. Source: Kwon Park.

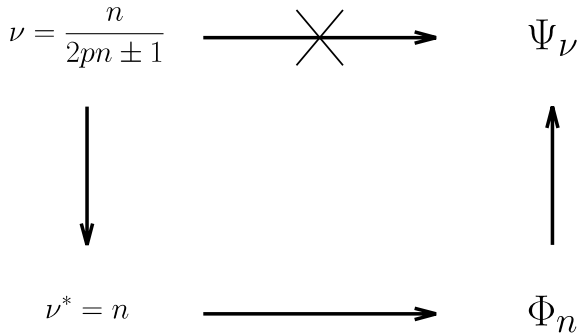

Fig. 5.8. The composite fermion route to the FQHE wave function Ψ_ν.

the wave function Ψ_ν of the FQHE state (the horizontal link) is not possible. However, the CF theory provides a new possible route. We have now mapped the problem of interacting electrons at $\nu = n/(2pn+1)$ into that of weakly interacting fermions at $\nu^* = n$, the ground state of which is known (Φ_n). We now complete, following the reasoning of Ref. [272], the last leg of the route (from Φ_n to Ψ_ν in Fig. 5.8). What follows is not a rigorous derivation but an attempt to construct variational wave functions based on a physical picture. The wave functions obtained at the end must be confirmed by independent means. Some readers may find the alternative treatment in the next section more satisfying, where we first arrive at the wave functions by postulating composite fermions, and then derive from them B^* and ν^*.

Let us consider the mean-field description indicated in Fig. 5.5(b), in which particles sense a uniform external magnetic field B^* and also have $2p$ flux quanta tied to them. The

vector potential $\boldsymbol{a}$, which binds flux quanta to electrons, is given by

$$\boldsymbol{a}(\boldsymbol{r}_i) = \frac{2p}{2\pi}\phi_0 \sum_j{}' \nabla_i \theta_{ij}, \tag{5.21}$$

where the prime denotes the condition $j \neq i$, and

$$\theta_{jk} = i \ln \frac{z_j - z_k}{|z_j - z_k|} \tag{5.22}$$

is the relative angle between particles j and k. Note that $\theta_{jk} = \theta_{kj} \pm (2m+1)\pi$, m integer, but below we only need the gradient of this angle, which satisfies

$$\nabla_j \theta_{jk} = \nabla_j \theta_{kj}. \tag{5.23}$$

As shown in Appendix C, $\boldsymbol{a}(\boldsymbol{r}_i)$ generates a magnetic field

$$\boldsymbol{b}_i = \nabla_i \times \boldsymbol{a}(\boldsymbol{r}_i) = 2p\phi_0 \sum_l{}' \delta^{(2)}(\boldsymbol{r}_i - \boldsymbol{r}_l), \tag{5.24}$$

i.e., each electron sees a flux tube of strength $2p\phi_0$ on every other electron. This suggests the following mean-field Hamiltonian for composite fermions in B^*:

$$H_{\rm MF} = \frac{1}{2m_{\rm b}} \sum_i \left(\boldsymbol{p}_i + \frac{e}{c}\boldsymbol{A}^*(\boldsymbol{r}_i) + \frac{e}{c}\boldsymbol{a}(\boldsymbol{r}_i)\right)^2, \tag{5.25}$$

where $\boldsymbol{A}^*$ produces a uniform magnetic field B^*. We assume that composite fermions are free; the current discussion is too crude for an explicit treatment of the interaction between particles.

Let us now consider the Schrödinger equation

$$H_{\rm MF}\Psi^{\rm MF} = E\Psi^{\rm MF}. \tag{5.26}$$

The vector potential $\boldsymbol{a}$ can be eliminated by making the gauge transformation

$$\Psi^{\rm MF} = \Phi\, {\rm e}^{-{\rm i}2p\sum_{j<k}\theta_{jk}} = \Phi \prod_{j<k} \left(\frac{z_j - z_k}{|z_j - z_k|}\right)^{2p}, \tag{5.27}$$

with Φ determined by

$$\frac{1}{2m_{\rm b}} \sum_i \left(\boldsymbol{p}_i + \frac{e}{c}\boldsymbol{A}^*(\boldsymbol{r}_i)\right)^2 \Phi = E\Phi. \tag{5.28}$$

This is the Schrödinger equation for noninteracting electrons at $\nu^* = n$, for which we know the solution. In particular, the ground state wave function, denoted by $\Phi_{\pm n}(B^*)$, is a single

Slater determinant with the lowest n Landau levels fully occupied, where the sign $\pm$ refers to the direction $\pm z$ for B^*. Because switching the direction of the magnetic field is equivalent to complex conjugation, we have, in general,

$$\Phi_{-\nu^*}(B^*) = \left[\Phi_{\nu^*}(B^*)\right]^* . \tag{5.29}$$

Thus, we obtain for the ground state at $\nu = n/(2pn \pm 1)$:

$$\Psi^{\rm MF}_{\frac{n}{2pn\pm 1}}(B) = \Phi_{\pm n}(B^*) \prod_{j<k} \left(\frac{z_j - z_k}{|z_j - z_k|} \right)^{2p} . \tag{5.30}$$

This "mean-field" wave function is unsatisfactory for the following reasons [272]:

(i) It does not build favorable correlations between electrons. In fact, $\Psi^{\rm MF}_\nu$ has the same probability amplitude as the uncorrelated IQHE wave function $\Phi_{\pm n}$. As seen in Table 5.1, the interaction energy for the mean-field wave function is significantly higher than that of the LLL Hartree–Fock Wigner crystal, which itself is not an especially good state in the filling factor region where the FQHE is seen.
(ii) $\Psi^{\rm MF}_\nu$ also involves significant mixing with higher Landau levels. The wave function consists of large powers of $\bar{z}_j$'s through both the factors $\Phi_{\pm n}(B^*)$ and $|z_j - z_k|^{2p} = (z_j - z_k)^p (\bar{z}_j - \bar{z}_k)^p$. The kinetic energy per particle, shown in Table 5.1, is a measure of the amount of LL mixing.
(iii) At $\nu = 1/m$, m odd, $\Psi^{\rm MF}_\nu$ does not reduce to the Laughlin wave function.

Reference [272] notes that these problems can be remedied, to a great extent, by throwing away the factor $|z_j - z_k|^{2p}$ in the denominator, which yields

$$\Psi_{\frac{n}{2pn\pm 1}}(B) = \Phi_{\pm n}(B) \prod_{j<k} (z_j - z_k)^{2p} . \tag{5.31}$$

(As explained in Section 5.8.13, the wave function $\Phi_{\pm n}$ must now be evaluated at B in Eq. (5.31) to ensure the correct filling factor for Ψ.) The Jastrow factor now explicitly builds repulsive correlations between electrons, which results in a substantially lower interaction energy, as seen in Table 5.1. At the same time, admixture with higher Landau levels is substantially reduced (Table 5.1), although not completely eliminated. For many quantitative purposes, a wave function strictly confined to the lowest Landau level is very useful. Such a wave function is obtained by an explicit projection of Eq. (5.31) into the lowest Landau level:

$$\Psi_{\frac{n}{2pn\pm 1}} = \mathcal{P}_{\rm LLL} \Phi_{\pm n} \prod_{j<k} (z_j - z_k)^{2p} , \tag{5.32}$$

where $\mathcal{P}_{\rm LLL}$ denotes projection into the lowest Landau level (more details in Section 5.14). These wave functions[12] do not contain any adjustable parameters (for a given filling factor),

[12] Wave functions of the form given in Eqs. (5.31) and (5.32) have been referred to in the literature as the Jain wave functions.

Table 5.1. *Interaction and kinetic energies for various trial wave functions*

ν	1/3	2/5	3/7	4/9
V^{MF}	−0.3619(9)	−0.3848(16)	−0.3947(15)	−0.4007(16)
$E_{\mathrm{K}}^{\mathrm{MF}}$	0.335(4)	0.408(9)	0.430(7)	0.459(17)
V^{up}	−0.4098	−0.4489(10)	−0.4644(20)	−0.4734(15)
$E_{\mathrm{K}}^{\mathrm{up}}$	0	0.0403(6)	0.0575(15)	0.0701(36)
V^{p}	−0.4098	−0.4328	−0.4423	−0.4474
$E_{\mathrm{K}}^{\mathrm{p}}$	0	0	0	0
V^{WC}	−0.3885	−0.4130	−0.4225	−0.4275
$E_{\mathrm{K}}^{\mathrm{WC}}$	0	0	0	0

Notes: V and E_K per particle for the mean-field wave function, the Laughlin wave function (Eq. 5.16) at $\nu = 1/3$, and unprojected and projected wave functions of Eqs. (5.31) and (5.32) at $\nu = 2/5$, 3/7 and 4/9.

The energies of the mean-field, unprojected, and projected wave functions are labeled by superscripts "MF," "up" and "p," respectively.

As a reference, the interaction energy of the LLL Hartree–Fock Wigner crystal state, V^{WC}, is also given.

The interaction energy V is quoted in units of $e^2/\epsilon\ell$ and includes the contribution from the positively charged background. The kinetic energy E_{K} is expressed in units of the cyclotron energy.

The zero point energy $\hbar\omega_{\mathrm{c}}/2$ has been subtracted from the kinetic energy, so E_{K} is zero for a strictly LLL wave function. E_{K} is a measure of LL mixing in the wave function.

We note that the kinetic energy per particle for Φ_n is $E_{\mathrm{K}} = (n-1)/2$. All energies represent thermodynamic limits; the uncertainty in the last digit(s) from Monte Carlo sampling and extrapolation to $N^{-1} \to 0$ is shown in parentheses.

Source: Liquid energies are from Kamilla and Jain [314,313]; some of the numbers are improvements on the earlier work by Trivedi and Jain [644], Levesque *et al.* [389] and Morf and Halperin [455]. The interaction energy of the Hartree–Fock Wigner crystal state is taken from Levesque, Weiss and MacDonald [389] (using the interpolation formula given in Ref. [363]).

because $\Phi_{\pm n}$ are parameter free.[13] They can be generalized to excited states (Eqs. 5.60, 5.61, 5.62, 12.35, 12.37), to partially spin polarized FQHE (Eq. 11.59), to arbitrary fractions (Eq. 5.36), to bilayer systems (Eq. 13.7), and to quantum dots (Eq. 6.4). Additionally, wave functions have also been constructed for paired states of composite fermions (Eq. 7.11) and composite fermion crystals (Eq. 15.8).

We see in Section 5.14 that the unprojected and projected wave functions of Eqs. (5.31) and (5.32) differ in quantitative detail, but describe the same qualitative physics. Many essential features of the unprojected wave function survive projection into the LLL. The

[13] These wave functions are sometimes called "variational" wave functions, which might appear to be a misnomer in view of the lack of adjustable parameters. The variational freedom here is the form of the wave function. Once the form is fixed by the CF physics, the wave function is fully determined.

projected wave function is used for quantitative calculations, but the unprojected wave function is useful for a derivation of the universal properties of the CF state.

Composite-fermionization Constructing a wave function Ψ from a given wave function Φ according to Eq. (5.32) is called composite-fermionization. It involves multiplication by an appropriate Jastrow factor, followed by projection into the lowest Landau level.

Vortex vs. flux tube We earlier defined composite fermions in two somewhat different ways: as bound states of electrons and flux quanta, and as bound states of electrons and quantized vortices. In going from Eq. (5.30) to Eq. (5.31), we have gone from the former to the latter. The factor

$$\prod_{j<k}\left(\frac{z_j - z_k}{|z_j - z_k|}\right)^{2p} \tag{5.33}$$

binds, by construction, a *point* flux of strength $2p\phi_0$ to each electron. On the other hand, the Jastrow factor

$$\prod_{j<k}(z_j - z_k)^{2p} \tag{5.34}$$

binds $2p$ *vortices* to each electron. (More precisely, each electron sees $2p$ vortices on every other electron.) Throwing away the denominator converts the flux tubes into vortices.[14] In other words, we are postulating that flux quanta turn into vortices during the adiabatic process in going from Fig. 5.5(b) to 5.5(c), and electron–flux composites evolve into electron–vortex composites.

5.8.3 Generalization to arbitrary fillings

The "law of corresponding states" [276] motivated by the above mean-field approximation can be generalized to an arbitrary filling ν^* as follows [115, 272, 276]:

(i) We begin with noninteracting electrons at a magnetic field B^* at a filling factor $n+1 > \nu^* > n$. The energy spectrum now looks like that shown in Fig. 5.9. It contains many degenerate many-body ground states corresponding to distinct configurations of electrons in the partly occupied $(n+1)$th Landau level. These are denoted $\Phi^{\alpha}_{\pm\nu^*}$, where α labels different states. Excited states are separated by at least one unit of cyclotron energy.

(ii) We take *all* degenerate ground states $\Phi^{\alpha}_{\pm\nu^*}$, and attach to each electron a massless, infinitely thin magnetic solenoid carrying $2p$ flux quanta, pointing in the $+z$ direction.

(iii) The attached flux is slowly spread until it turns into a uniform magnetic field. States will now mix with one another, if allowed by symmetry, and the degeneracy will be lifted in some complicated

[14] Strictly speaking, the factor in Eq. (5.33) also produces $2p$ vortices, in that it produces a phase of $2p \times 2\pi$ for a closed loop around it. It is customary in the FQHE literature to reserve the term "vortex" for factors of the type given in Eq. (5.33), and refer to Eq. (5.33) as attaching (point) "flux quanta."

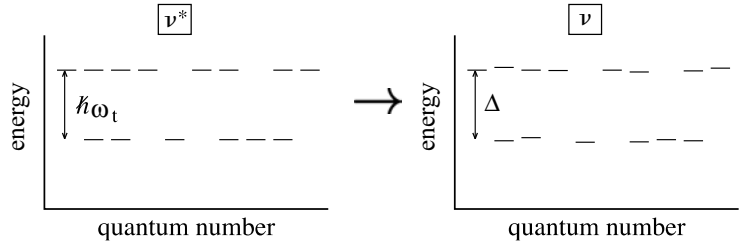

Fig. 5.9. Schematic structure of the energy spectrum of the many-body system at a nonintegral filling, $\nu^* \neq n$; it contains many degenerate ground states (separated from excited states by the cyclotron energy). The CF theory predicts that the low-energy spectrum at fractional fillings $\nu = \nu^*/(2p\nu^* \pm 1)$ has a band of *quasi*-degenerate ground states that has a one-to-one correspondence with the degenerate ground states at ν^*. The last statement in the caption of Fig. 5.2 applies here as well.

fashion. But it is possible that the gap separating these states from the rest will not close. At the end we obtain particles moving in a larger magnetic field $B = B^* + 2p\rho\phi_0$, corresponding to

$$\nu = \frac{\nu^*}{2p\nu^* \pm 1}. \tag{5.35}$$

As before, if no phase transition occurs (i.e., if the gap does not close), then the "band structure" at ν^* carries over to ν, implying, as depicted in Fig. 5.9, a one-to-one correspondence between the low-energy states at ν^* and ν.

(iv) The wave function for each final state is constructed by composite-fermionization (cf. Eq. 5.32):

$$\Psi^{\alpha}_{\frac{\nu^*}{2p\nu^*\pm 1}} = \mathcal{P}_{\rm LLL}\Phi^{\alpha}_{\pm\nu^*}\prod_{j<k}(z_j - z_k)^{2p}. \tag{5.36}$$

Now we end up with many wave functions. These constitute a correlated basis for the low-energy band of quasi-degenerate "ground states" at ν. A determination of the fine structure within the ground band requires a diagonalization of the full Hamiltonian in this basis, referred to as "CF diagonalization."

5.8.4 CF diagonalization

The wave functions and energies for low-energy states at any given ν are obtained by the method outlined in Fig. 5.10. We choose the value $2p$ that gives the largest ν^*. The distinct degenerate ground states at ν^* (for noninteracting electrons) form a low-energy band spanned by a basis $\{\Phi^{\alpha}_{\pm\nu^*}\}$ (with magnetic field pointing in $\pm z$ direction). A basis for low-energy states at ν, $\{\Psi^{\alpha}_{\nu}\}$, is constructed by composite-fermionization of $\{\Phi^{\alpha}_{\pm\nu^*}\}$, as in Eq. (5.36). These states are in general not orthogonal, but an orthogonal basis can be generated through the standard Gram–Schmidt method. The spectrum at ν is obtained by diagonalizing the *full* Hamiltonian in the space defined by correlated basis functions $\{\Psi^{\alpha}_{\nu}\}$. This entire set of steps is termed "composite fermion diagonalization."

The LLL projection of basis functions and the evaluation of the Hamiltonian matrix are technically challenging, but mathematically well defined, and CF diagonalization has been

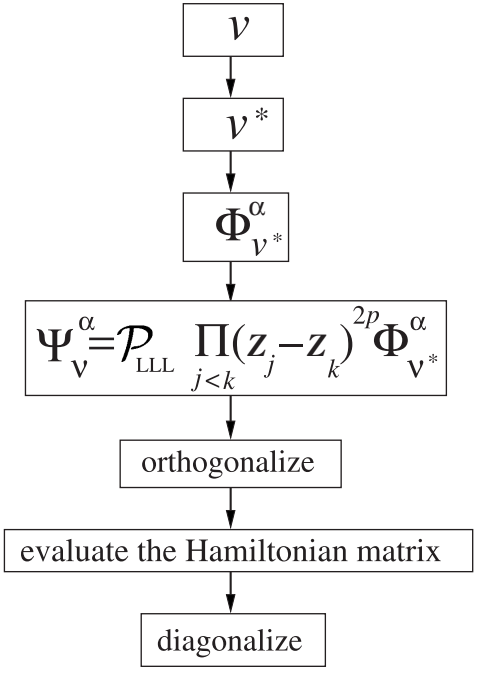

Fig. 5.10. CF diagonalization.

explicitly carried through for many nontrivial cases. Various multi-dimensional integrals required for this purpose do not reduce to known functions catalogued in standard books. But that is merely a technical point and should not be taken to indicate a gap in our understanding. The integrals are well defined, and can be evaluated numerically, usually by the Metropolis Monte Carlo method, to produce energies to desired accuracy (typically four or five significant figures) without making any approximations. Systems with as many as 100 composite fermions have been studied. See Appendix L for further technical details.

5.8.5 Λ levels and energy level diagrams

For each initial state at ν^* in the left column of Figs. 5.3 and 5.4, the corresponding final state at ν is depicted by the diagram in the right column. Each diagram on the right has a precise wave function associated with it, obtained by composite-fermionization of the wave function of the corresponding state in the left column.

The CF theory suggests the following interpretation: We picture that composite fermions form Landau-*like* levels in the reduced magnetic field B^*. These are called "Λ levels."[15] While Λ levels are analogous to Landau levels of electrons at B^*, the two are not the

[15] They have also been called "composite fermion Landau levels," "pseudo-Landau levels," or "quasi-Landau levels." These names have been a source of confusion, however, because the term "Landau level" has the universally accepted meaning as

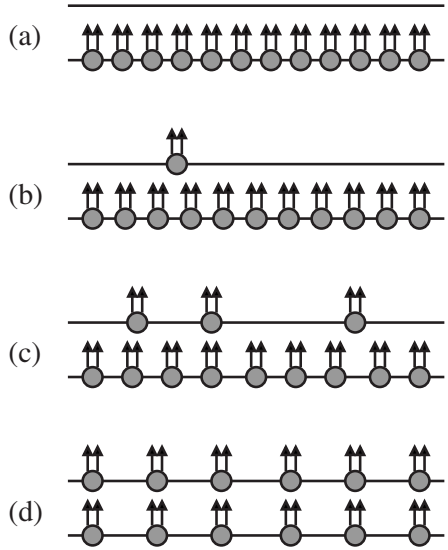

Fig. 5.11. Schematic view of the evolution of the state from $\nu = 1/3$ to $\nu = 2/5$. (The filling factor is changed by varying the magnetic field, altering the degeneracy of each Λ level.) (a) The 1/3 state, i.e., one filled Λ level of composite fermions. (b) A CF-quasiparticle, i.e., a composite fermion in an otherwise empty Λ level. (c) An intermediate state. (d) The $2/5$ ground state, which is equivalent to two filled Λ levels. While the ground states at $1/3$ and $2/5$ are nondegenerate, many quasi-degenerate ground states exist at intermediate fillings, which correspond to different configurations of composite fermions in the second Λ level in (b) and (c); for example, the degeneracy of the CF-quasiparticle is equal to the degeneracy of the second Λ level.

same; the Λ levels reside within the lowest Landau level. In other words, when interacting electrons at ν transform into composite fermions, the lowest Landau level splits into Λ levels. In particular, the fractions $\nu = n/(2pn \pm 1)$ of Eq. (5.20) map into integral fillings of composite fermions; the ground state here fills an integral number of Λ levels (Fig. 5.3) and excited states are CF excitons (Fig. 5.4). Figure 5.11 depicts how the system evolves as the filling factor changes from $\nu = 1/3$ to $\nu = 2/5$.

The right hand side of the projected wave function in Eq. (5.32) or (5.36) is interpreted as describing composite fermions at filling ν^*. At the same time, it is also a wave function for interacting electrons at ν. There is no inconsistency. Such a "dual" interpretation is not dissimilar to that for the BCS wave function for superconductors: The BCS wave function describes electrons, but, since electrons always appear paired, it is also interpreted as a pair wave function. Similarly, because an even number of vortices are bound to each electron in

the quantized kinetic energy level of an *electron* in a magnetic field. Because the kinetic energy levels of composite fermions are a crucial and nontrivial concept in the theory of the FQHE, they deserve their own name and we call them "Λ levels." The term "Landau level" refers exclusively to the familiar electronic Landau level in this book.

the wave function of Eq. (5.32) or (5.36), it can be viewed as a wave function for composite fermions.

5.8.6 A new derivation of the Laughlin wave function

We now show how the Laughlin wave function can be "derived" from the CF theory. For the ground state at $\nu = \frac{1}{2p+1}$, the projected wave function of Eq. (5.32) reduces to

$$\Psi_{\frac{1}{2p+1}} = \mathcal{P}_{\rm LLL} \prod_{j<k} (z_j - z_k)^{2p} \Phi_1. \tag{5.37}$$

With

$$\Phi_1 = \prod_{j<k} (z_j - z_k) \exp\left[-\frac{1}{4} \sum_i |z_i|^2 \right], \tag{5.38}$$

it reduces to

$$\Psi_{\frac{1}{2p+1}} = \mathcal{P}_{\rm LLL} \prod_{j<k} (z_j - z_k)^{2p+1} \exp\left[-\frac{1}{4} \sum_i |z_i|^2 \right], \tag{5.39}$$

which is identical to the Laughlin wave function of Eq. (5.16) with $m = 2p + 1$. (The projection operator $\mathcal{P}_{\rm LLL}$ does not do anything because the wave function is already in the lowest Landau level.) This was part of the motivation for throwing away the denominator in the mean-field wave function of Eq. (5.30), leading to the unprojected wave function of Eq. (5.31). Written as in Eq. (5.37), the wave function acquires the physical interpretation of one filled Λ level of composite fermions. The form in Eq. (5.37) reveals a close connection between the FQHE and the IQHE, gives new insight into the excitations at $\nu = 1/(2p+1)$ by analogy to the known excitations of the one filled Landau level state (Sections 12.5 and 12.6), and shows that this wave function is part of a more general structure.[16] The ground state wave function at $\nu = 1/(2p + 1)$ happens to have a particularly simple form because of the fortunate coincidence that Φ_1 has a simple Jastrow form. Although written in a compact form, the wave functions of Eq. (5.32) are, in general, extremely complicated after projection into the lowest Landau level, and could not have been arrived at without guidance from the composite fermion principle.

5.8.7 Remarks

(i) No rigorous theoretical method is known at present for implementing the adiabatic flux delocalization process to test directly whether or not the gap closes during the process. Several facts suggest that this is likely to be a complicated issue. Whether the final state is incompressible or not depends on the detailed form of the interaction. At any given filling factor, say $\nu = 2/5$, the system can be incompressible for some interactions (say,

[16] The simple equation $m = 2p + 1$ contains much physics.

the interaction corresponding to electrons in the lowest Landau level) but compressible for others (e.g., for the interaction that simulates electrons in the 17th Landau level). Evolution of the Landau level gap (the cyclotron energy) at integral fillings into the Λ level gap (sometimes called the "CF-cyclotron energy") is also bound to be a messy problem, because the two have different parametric dependences. The former is proportional to B and inversely proportional to the band mass of the electron. The latter is determined entirely by the Coulomb interaction, the only energy scale in the LLL problem; it is proportional to $e^2/\ell \sim \sqrt{B}$ and contains no electron mass.

Our philosophy will be not to worry about the adiabatic *process* of going from ν^* to ν, but directly to make statements at ν that can be tested by independent means. Various energies and energy gaps can be calculated directly at ν with the help of projected wave functions of Eqs. (5.32) and (5.62) for ground and excited states. Because these are restricted, by construction, to the lowest Landau level, their energies are guaranteed to have the correct units of e^2/ℓ.

(ii) The CF vortex-attachment construction can produce new *possible* incompressible QHE states from any given incompressible QHE state. Its most important use lies in generating new FQHE states from the known *integral* QHE states. Starting from the *fractional* QHE states, we can produce yet more incompressible states.

(iii) We have described above the original version of the composite fermion mean-field approximation [272]. A more powerful formulation, pioneered by Lopez and Fradkin [401], is described in Section 5.16.

5.8.8 The composite fermion principle

Although the mean-field approach of the previous subsection serves as inspiration for writing wave functions, and also illustrates much of the physics, some of its aspects are not literally correct. It is instructive to see how the entire theory follows from the assumption of composite fermions. The logical order in this subsection is approximately the reverse of what we had in the previous subsection: the wave functions appear first, from which B^* and ν^* are deduced.

The fundamental postulate *Strongly interacting electrons turn into weakly interacting composite fermions, where a composite fermion is a bound state of an electron and an even number of quantized vortices.*

The microscopic meaning of the formation of composite fermions is that the wave function of interacting electrons at ν has the form

$$\Psi_\nu = \Phi \prod_{j<k} (z_j - z_k)^{2p}, \tag{5.40}$$

where Φ is an antisymmetric wave function for electrons. The Jastrow factor, $\prod_{j<k}(z_j - z_k)^{2p}$, binds $2p$ vortices to each electron to convert it into a composite fermion.

Equation (5.40) has no content as it stands, because *any* wave function can be written in this form at the expense of making Φ sufficiently complicated. We further demand that Φ

be the wave function for *weakly interacting* electrons. That is the precise meaning of the statement that composite fermions are weakly interacting. In other words, the interaction enters only through the Jastrow factor, that is, only through the formation of composite fermions.

One more observation completes the basic conceptual story. Section 5.11 derives by several methods,[17] starting from the form in Eq. (5.40), that Φ is a wave function for electrons at ν^* given by Eq. (5.35). With Φ replaced by $\Phi_{\pm\nu^*}$, Eq. (5.40) reduces to the earlier unprojected wave function. In Section 5.11 we also see a better way of understanding why the magnetic field sensed by composite fermions is B^* rather than B: the Berry phase due to the bound vortices partly cancels the Aharonov–Bohm phase due to the external magnetic field, with the result that composite fermions experience a reduced magnetic field

$$B^* = B - 2p\rho\phi_0. \tag{5.41}$$

This also implies the filling factor ν^* given by Eq. (5.35). We thus arrive at the following corollary of the fundamental postulate:

Corollary *Composite fermions experience a reduced magnetic field B^* and fill ν^* Landau-like levels.*

5.8.9 CF theory in a nutshell

The next three paragraphs summarize the defining statements of the CF theory. The first two list the *exact* consequences of the CF theory; these are sufficient for an understanding of the universal phenomenology of the FQHE and many other phenomena. The third paragraph gives a mathematical formulation of composite fermions, which enables quantitative calculations.

The physics The CF theory postulates the formation of composite fermions. This is a nonperturbative consequence of the repulsive interactions, because the number of vortices, quantized to be an even integer, cannot be changed continuously. (Wave functions with nonintegral values of p are not valid wave functions for electrons.) The CF theory further asserts that composite fermions are weakly interacting, and that their interaction can be neglected for many qualitative purposes. The most important outcome of the formation of composite fermions is that the vortices bound to them produce Berry phases that partly cancel the Aharonov–Bohm phase due to the external magnetic field. Consequently, composite fermions experience a reduced magnetic field $B^* = B - 2p\rho\phi_0$. They form Landau-like levels, called Λ levels, in the reduced magnetic field, and their filling factor ν^* is given by $\nu = \nu^*/(2p\nu^* \pm 1)$.

Qualitative consequences for the energy spectrum The mapping of strongly interacting electrons at B (ν) into weakly interacting composite fermions at B^* (ν^*) implies a one-to-one correspondence between the low-energy spectra of the two. The CF theory

[17] The reason for postponing the proofs to Section 5.11 is that some of the derivations use wave functions in the spherical geometry, introduced in Section 5.9.

thus predicts formation of bands in the energy spectrum of interacting electrons at ν, and further asserts that the lowest band resembles the lowest band of noninteracting fermions at ν^*. That implies incompressibility at the the fractions $\nu = \frac{n}{2pn\pm1}$.

Quantitative theory The low-energy spectrum is obtained quantitatively by the method of CF diagonalization, outlined in Section 5.8.4 and Fig. 5.10.

The CF theory thus tells us how nature takes advantage of the enormous degeneracy to produce new nonperturbative structures. The fundamental effect of the repulsive interaction between electrons is to produce composite fermions; the interaction between composite fermions themselves is much weaker and can be neglected altogether to a good first approximation.[18] The formation of bands and incompressibility at certain special fractions are nonperturbative consequences of the formation of composite fermions. All LLL states are degenerate in the absence of interaction, and a gap opens as soon as the interaction is switched on, no matter how small its strength. (One may manipulate the interaction strength by varying the dielectric function of the host material, but that merely affects the overall energy scale.)

The correspondence between interacting electrons at ν and weakly interacting composite fermions at ν^* is immensely powerful and beautifully constraining. For the important special cases of the fractions $\nu = n/(2pn \pm 1)$, it not only predicts incompressibility but picks out unique, parameter-free wave functions for the ground states out of the astronomically large number of possibilities. The lack of adjustable parameters may appear to be a serious deficiency at first sight, raising doubts regarding the applicability of these wave functions to the actual states. It, however, turns out to be one of the strengths of the CF theory. Aside from the obvious aesthetic appeal of a parameter-free theory, it enables an unbiased testing of the CF theory (Chapter 6). Substantial simplification is achieved away from the special fractions also, because the dimension of the CF basis is exponentially small compared to the dimension of the full LLL Fock space at ν. To take an example, the number of degenerate ground states for 100 particles at $\nu^* = 5/2$ is smaller by a factor of 2×10^{58} compared to that at $\nu = 5/12$ (with spin neglected and the interaction switched off).

5.8.10 Why composite fermions form

Why electrons *might* want to capture vortices to turn into composite fermions is easy to see. A typical wave function satisfying the Pauli principle vanishes as r when two particles approach one another, r being the distance separating them, but the unprojected wave functions $\Phi\Phi_1^{2p}$ vanish as r^{2p+1}, with the Jastrow factor contributing $2p$ to the exponent and Φ contributing the rest. The Jastrow factor is very effective in keeping particles apart from one another, producing favorable correlations. These correlations should survive,

[18] (a) This is analogous to the BCS theory of superconductivity. There, Cooper pairs are produced due to an attractive interaction between electrons. Cooper pairs themselves are taken as noninteracting, however, as seen from the fact that the BCS wave function has *all* Cooper pairs in the same state, as appropriate for uncorrelated bosons.
(b) Bound states are always more weakly interacting than the constituent particles, because much of the interaction has been "spent" in making the bound states. Some familiar examples are: atoms, neutrons, Cooper pairs.

to some extent, projection into the lowest Landau level, given that the unprojected wave functions are already predominantly in the lowest Landau level.

5.8.11 1 = 1 + 1

A consequence of the formation of composite fermions is a dynamical generation of a new energy scale. Two energy scales emerge out of one. The Coulomb interaction energy of electrons splits into a "large" effective cyclotron energy and a "small" residual inter-CF interaction. Both have the same units ($e^2/\epsilon\ell$), however, and their relative strengths cannot be changed for a given filling factor.[19]

5.8.12 Caveats

While the systems of composite fermions at ν^* and noninteracting electrons at ν^* share many qualitative features, they are not identical in every respect, and one must be careful not to overinterpret their correspondence. We note two instances where a naive application of the CF theory leads to a wrong conclusion.

The energy of a particle–hole pair of *electrons* at $\nu^* = n$ is equal to the cyclotron energy $\hbar eB^*/m_b c = \hbar eB/(2pn+1)m_b c$. (We have used $B^* = B/(2pn+1)$ for $\nu = n/(2pn+1)$.) That is *not* the energy of a particle–hole pair of composite fermions at $\nu^* = n$. The latter must be proportional to $e^2/\ell \sim \sqrt{B}$, and also independent of the electron band mass. The analogy between electrons at ν^* and composite fermions at ν^* extends to the *structure* of the low-energy spectrum, but not to the energies. The energies are to be determined by CF diagonalization.

The second example is that of the Hall resistance. The mapping of interacting electrons at ν into free composite fermions at ν^* might suggest that the Hall resistance is $R_H = h/\nu^* e^2$. That is obviously incorrect; for a disorder-free system at ν, one can derive $R_H = h/\nu e^2$, independent of interactions, by the general trick of boosting to a moving frame of reference of velocity $v = cE/B$. The resolution of this apparent paradox is given in Section 5.16.3.

5.8.13 Another representation of wave functions

In Eq. (5.31), (5.32), or (5.36) all of the Gaussian factors are incorporated into the definition of $\Phi^\alpha_{\nu^*}$ by evaluating it at the *external* magnetic field *but at the effective filling factor*. Multiplication by $\prod_{j<k}(z_j - z_k)^{2p}$ expands the size of the disk containing electrons, reducing the density and producing the filling factor ν.

An alternative form for the wave functions of Eq. (5.36) is given by

$$\Psi^\alpha_\nu = \mathcal{P}_{\rm LLL}\Phi^\alpha_{\pm\nu^*}\Phi_1^{2p}. \tag{5.42}$$

[19] That should be contrasted with the IQHE, where the Coulomb and the cyclotron energies can be varied independently relative to one another by controlling various parameters (e.g., the electron mass, the dielectric constant, or the density).

Here Φ_1 is the wave function of one filled Landau level:

$$\Phi_1 = \prod_{j<k}(z_j - z_k)\exp\left[-\frac{1}{4\ell_1^2}\sum_i |z_i|^2\right], \tag{5.43}$$

where ℓ_1 is the magnetic length at $\nu = 1$. The wave function $\Phi^\alpha_{\pm\nu^*}$ in Eq. (5.42) is constructed at the effective magnetic field B^*, i.e., has the Gaussian factor $\exp[-\frac{1}{4\ell^{*2}}\sum_i |z_i|^2]$. The Gaussian factors in Φ_1^{2m} and $\Phi^\alpha_{\nu^*}$ combine to produce a Gaussian factor corresponding to the external magnetic field B, since

$$\frac{2p}{\ell_1^2} + \frac{1}{\ell^{*2}} = \frac{1}{\ell^2}. \tag{5.44}$$

(This is nothing but the formula $2p\rho\phi_0 + B^* = B$.) In Eq. (5.42), multiplication by Φ_1^{2p} leaves the size of the disk unchanged (Exercise 5.2), but increases the magnetic field, thereby decreasing the filling factor.

Of course, this is only a difference of representation; the wave function in Eq. (5.42) is identical to that in Eq. (5.36). We use either form below; it should be clear from the context whether $\Phi^\alpha_{\pm\nu^*}$ is to be evaluated at the effective or the real magnetic field. The form in Eq. (5.42) will be more natural when we generalize the wave function to the spherical geometry.

5.8.14 Simplifying notation

In Eqs. (5.35) and (5.36) we have defined ν^* to be positive. Remembering that complex conjugation is equivalent to the reversal of magnetic field direction, we have

$$\Phi^\alpha_{-\nu^*} = [\Phi^\alpha_{+\nu^*}]^*. \tag{5.45}$$

We sometimes use a convention in which ν^* can take either positive or negative values, with the understanding that negative values refer to magnetic field in the $-z$ direction. That removes the $\pm$ signs in Eqs. (5.35) and (5.36). We sometimes also suppress the superscript α, and the subscripts ν and ν^*, remembering that Ψ refers to interacting electrons at ν and Φ to noninteracting electrons at ν^*. In the simplest form, the equations reduce to

$$\Psi = \mathcal{P}_{\rm LLL}\Phi\prod_{j<k}(z_j - z_k)^{2p} = \mathcal{P}_{\rm LLL}\Phi\Phi_1^{2p} \tag{5.46}$$

$$\nu = \frac{\nu^*}{2p\nu^* + 1}. \tag{5.47}$$

5.9 Wave functions in the spherical geometry

Generalizing to the spherical geometry, the wave function for interacting electrons at Q, Ψ_Q, is obtained from that of noninteracting electrons at Q^*, Φ_{Q^*}, as

$$\Psi_Q = \mathcal{P}_{\rm LLL}\Phi_{Q^*}\Phi_1^{2p}. \tag{5.48}$$

The Jastrow factor is given by

$$\Phi_1^{2p} = \prod_{j<k}(u_j v_k - v_j u_k)^{2p}, \tag{5.49}$$

where Φ_1 is the wave function of the lowest filled Landau level (Eq. 3.158).

5.9.1 Relationship between Q and Q^*

The monopole strength Q is related to Q^* by the property that the monopole strength of the product is the sum of monopole strengths. The monopole strength for Φ_1 is $Q_1 = (N-1)/2$, because the LLL degeneracy here is $2Q_1 + 1 = N$. Adding monopole strengths for factors on the right hand side of Eq. (5.48) gives

$$Q = Q^* + p(N-1). \tag{5.50}$$

5.9.2 Quantum numbers

In the spherical geometry, eigenstates of the many-particle system are also eigenstates of the orbital angular momentum, with the angular momentum quantum number denoted by L. One of the most important properties of the wave function $\Psi_Q = \Phi_{Q^*}\Phi_1^{2p}$, with or without LLL projection, is that it has the same angular momentum quantum numbers as Φ_{Q^*}. (The same is true for spin quantum numbers, as we see in Chapter 11.) In other words, composite-fermionization of a wave function does not alter its angular momentum quantum numbers.

Proof Let us assume that Φ_{Q^*} is an eigenstate of the total angular momentum operators $\boldsymbol{L}^2$ and L_z with eigenvalues $L(L+1)$ and M. Write

$$\boldsymbol{L}^2 = L_z^2 + \frac{1}{2}(L_+^2 + L_-^2) \tag{5.51}$$

where $L_+ = L_x + \mathrm{i}L_y$ and $L_- = L_x - \mathrm{i}L_y$. The wave function Φ_1 has $L = 0$, so it satisfies

$$L_z\Phi_1 = L_+\Phi_1 = L_-\Phi_1 = 0. \tag{5.52}$$

Noting that all L_z, L_+, and L_- involve at most first-order derivatives, we can commute them through the factor Φ_1. Thus,

$$L_z\Phi_1^2\Phi_{Q^*} = \Phi_1^2 L_z\Phi_{Q^*} = M\,\Phi_1^2\Phi_{Q^*} \tag{5.53}$$

and

$$\boldsymbol{L}^2\Phi_1^2\Phi_{Q^*} = \Phi_1^2\boldsymbol{L}^2\Phi_{Q^*} = L(L+1)\Phi_1^2\Phi_{Q^*}. \tag{5.54}$$

To prove that the angular momentum is not altered upon projection into the lowest Landau level, we write the projection operator, following Rezayi and MacDonald [548], as

$$\mathcal{P}_{\rm LLL} = \prod_{i=1}^{N} \mathcal{P}^i_{\rm LLL} \tag{5.55}$$

$$\mathcal{P}^i_{\rm LLL} = \prod_{l=Q+1}^{\infty} \frac{l(l+1) - \boldsymbol{L}_i^2}{l(l+1) - Q(Q+1)} \tag{5.56}$$

where $\boldsymbol{L}_i^2$ is the angular momentum operator for the ith electron. That $\mathcal{P}_{\rm LLL}$ is the projection operator is seen by noting that $\mathcal{P}^i_{\rm LLL}$ produces a zero when applied to any single particle state in a higher Landau level, and one when applied to any state in the lowest Landau level. Given that the total angular momentum operators $\boldsymbol{L}^2$ and L_z commute with the $\boldsymbol{L}_i^2$ of an individual electron, they also commute with the projection operator $\mathcal{P}_{\rm LLL}$. □

5.9.3 Wave functions for filled Λ levels

The filling factor $\nu = n/(2pn+1)$ of electrons corresponds to $\nu^* = n$ of composite fermions. The wave function for the state containing n filled Λ levels (Fig. 5.3) is given by

$$\Psi_{\frac{n}{2pn+1}} = \mathcal{P}_{\rm LLL}\Phi_n\Phi_1^{2p}, \tag{5.57}$$

where Φ_n is the Slater determinant wave function of n filled Landau levels of electrons. In the spherical geometry, n Landau levels are fully occupied when $2Q^* = N/n - n$. The corresponding state $\Psi_{n/(2pn+1)}$ occurs at the real monopole strength

$$2Q = \frac{(2pn+1)}{n}N - (2p+n). \tag{5.58}$$

(In general, this equation does not have a solution for all N. Only those integral N values are meaningful for which Q is either an integer or half-integer.) In the thermodynamic limit, $N \to \infty$, the expected filling factor is obtained:

$$\nu \equiv \frac{N}{2Q} = \frac{n}{2pn+1}. \tag{5.59}$$

5.9.4 Wave functions for CF-quasiparticle and CF-quasihole

Excitations of the FQHE state at $\nu = n/(2pn \pm 1)$ are understood by analogy to excitations of the IQHE state at $\nu^* = n$. For the latter, adding an electron to the lowest unoccupied Landau level produces a "particle," with charge equal to the charge of an electron. Similarly,

the removal of an electron from the topmost occupied Landau level creates a "hole." Let us denote the wave functions of these states by $\Phi_n^{\rm qp}$ and $\Phi_n^{\rm qh}$. The left column of Fig. 5.4 depicts (a) the ground state, (b) a particle, and (c) a hole for $\nu^* = 3$.

The images of IQHE particle and hole are called CF-quasiparticle and CF-quasihole, shown schematically in the right column of Fig. 5.4 for $\nu^* = 3$ of composite fermions ($\nu = 3/7$). A CF-quasiparticle is a solitary composite fermion in an otherwise empty Λ level, and a CF-quasihole is the state in which a single composite fermion is missing from an otherwise full Λ level. Their wave functions are given by

$$\Psi^{\text{CF-qp}}_{\frac{n}{2pn+1}} = \mathcal{P}_{\rm LLL}\Phi_n^{\rm qp}\Phi_1^{2p} \tag{5.60}$$

and

$$\Psi^{\text{CF-qh}}_{\frac{n}{2pn+1}} = \mathcal{P}_{\rm LLL}\Phi_n^{\rm qh}\Phi_1^{2p}. \tag{5.61}$$

These do not contain any adjustable parameters.

5.9.5 Wave function for CF-exciton

A CF-exciton is a particle–hole pair of composite fermions, also shown schematically in Fig. 5.4. Microscopic wave function for a CF-exciton is constructed by analogy to the wave function $\Phi_n^{\rm ex}$ of the exciton of the IQHE state:

$$\Psi^{\text{CF-ex}}_{\frac{n}{2pn+1}} = \mathcal{P}_{\rm LLL}\Phi_n^{\rm ex}\Phi_1^{2p}. \tag{5.62}$$

In the spherical geometry, $\Phi_n^{\rm ex}$ is taken to be an eigenstate of the total orbital angular momentum. Its wave function can be written straightforwardly, because it is effectively a two-particle problem. Let us denote the orbital angular momentum quantum numbers of an electron by $(\bar{l}, m)$, with $\bar{l} = q + n - 1$ for the topmost occupied LL shell, q being the monopole strength and n being the number of filled shells. Let us denote by $\Phi^{(m,m')}$ the Slater determinant for the state in which an electron is removed from the orbital $(\bar{l}, m)$, creating a hole with quantum numbers $(\bar{l}, -m)$, and placed in the orbital $(\bar{l}+1, m')$. The z component of the total angular momentum can be chosen to be zero with no loss of generality, so we set $m = m'$. The exciton state with a definite total angular momentum L is then given by

$$\Phi^{\rm ex}_{q,L} = \sum_m \langle \bar{l}, -m; \bar{l}+1, m | L, 0\rangle \Phi^{(m,m)}, \tag{5.63}$$

with $m = -\bar{l}, \ldots, \bar{l}$. The wave function for the CF-exciton with a well defined L is then constructed as

$$\Psi^{\text{CF-ex}}_{Q,L} = \mathcal{P}_{\rm LLL}\Phi^{\rm ex}_{q,L}\Phi_1^{2p}, \tag{5.64}$$

with $Q = q + p(N-1)$. $\Psi^{\text{CF-ex}}_{Q,L}$ contains no adjustable parameters. The relative amplitudes of various Slater determinants ("coherence factors") remain unchanged in going from the electron-exciton at q to the CF-exciton at Q according to the preceding displayed equation.

5.10 Uniform density for incompressible states

The filled Λ level states have uniform electron density. That is most easily proven in the spherical geometry. Translational invariance of the planar geometry is equivalent to rotational invariance in the spherical geometry. Consequently, the states with angular momentum quantum number $L = 0$ have uniform density, independent of microscopic details of the wave function. The state Φ with n filled Landau levels has $L = 0$. It follows that the n filled Λ level state also has $L = 0$, and, therefore, a uniform density.

No similarly simple proof is available for the planar geometry. One reason is that for any finite N, the state, strictly speaking, does *not* have uniform density, because it represents electrons inside a disk of finite radius, with the density vanishing outside. Here, we need to show that the density is uniform inside the disk. For the Laughlin wave function, a mapping into a one-component plasma allows demonstration of uniform density (Section 12.4). The density has also been calculated for the wave functions in Eq. (5.32) numerically, explicitly demonstrating that they describe a state with a constant density in the interior. Of course, that is hardly surprising in light of the result from the spherical geometry.

5.11 Derivation of ν^* and B^*

We now derive, as promised, the relation between ν and ν^* (Eq. 5.35) and that between B and B^* (Eq. 5.41) directly from the wave function of Eq. (5.36). The two relations are equivalent; one can be derived from the other with the help of

$$\nu^* = \frac{\rho\phi_0}{|B^*|} \tag{5.65}$$

and

$$\nu = \frac{\rho\phi_0}{B}. \tag{5.66}$$

Here, B is always assumed to be positive; B^* can be either parallel (for the $+$ sign) or antiparallel (for the $-$ sign) to B.

5.11.1 Derivation I

In the spherical geometry, the CF and electron filling factors are given by

$$\nu^* = \lim_{N\to\infty} \frac{N}{2|Q^*|}, \tag{5.67}$$

$$\nu = \lim_{N\to\infty} \frac{N}{2|Q|}. \tag{5.68}$$

Substituting in $Q = Q^* + p(N-1)$ produces Eq. (5.35). The relation between B^* and B can be derived by noting that

$$B = \frac{2Q\phi_0}{A}, \tag{5.69}$$

where $2Q\phi_0$ is the flux through the sphere and A is the surface area. Similarly,

$$B^* = \frac{2Q^*\phi_0}{A}. \tag{5.70}$$

Substituting in Eq. (5.50) and taking the limit $N \to \infty$ gives Eq. (5.41), with $\rho = N/A$.

5.11.2 Derivation II

In planar geometry the simplest way of determining the filling factor of the wave function on the right hand side of $\Psi_\nu = \Phi_{\nu^*} \prod_{j<k}(z_j - z_k)^{2p}$ is to ask what is the total number of single particle orbitals with nonzero occupation. Neglecting $O(1)$ corrections, it is given by the largest power of one of the coordinates, say z_1, in the polynomial part of the wave function. In Φ_{ν^*} this power is N/ν^*, as required to give the filling factor ν^*. In the Jastrow factor, the largest power of z_1 is $2p(N-1)$. The largest power of z_1 in Ψ, therefore, is $N\nu^{*-1} + 2p(N-1)$, which yields the filling factor

$$\nu = \frac{N}{N\nu^{*-1} + 2p(N-1)} = \frac{\nu^*}{2p\nu^* + 1}. \tag{5.71}$$

5.11.3 Derivation III

In planar geometry, the filling factor of *uniform* density states is related to the total angular momentum. Remembering that each factor of z_j in the polynomial in front of the Gaussian factor contributes $+1$ to the total angular momentum, and each factor of $\bar{z}_j$ contributes -1, the total angular momentum is given by the difference between the total power of z's and the total power of $\bar{z}$'s. We neglect $O(1)$ corrections below, which do not affect the thermodynamic value of the filling factor.

In a uniform state at ν, each orbital is occupied with a probability ν, with a total of N/ν orbitals occupied within a disk defined by the outermost orbital with angular momentum $\nu^{-1}N - 1$. The total angular momentum is given by

$$\begin{aligned} L &= \sum_{m=0}^{\nu^{-1}N-1} f_m m \\ &= \frac{N^2}{2\nu}, \end{aligned} \tag{5.72}$$

where $f_m = \nu$ is the average occupation of the angular momentum m orbital, and subdominant terms have been dropped in the last step. This gives the relation $\nu = N^2/2L$.

The total angular momentum of $\Psi_\nu = \mathcal{P}_{\mathrm{LLL}} \prod_{j<k}(z_j - z_k)^{2p} \Phi_{\pm\nu^*}$ can be calculated before projection, because projection into the lowest Landau level conserves the angular

momentum. Φ_{ν^*} contributes $N^2/2\nu^*$, as required to produce the correct filling, and the Jastrow factor contributes $pN(N-1)$. Thus, the total angular momentum of Ψ_ν is

$$L = pN(N-1) \pm \frac{N^2}{2\nu^*}, \tag{5.73}$$

where we have used that complex conjugation changes the sign of the angular momentum. (Φ_{ν^*} also has some powers of $\bar{z}$ but the largest power is finite, limited by the number of occupied Landau levels, and, therefore, does not affect the filling factor in the thermodynamic limit.) The filling factor of Ψ_ν is given by

$$\nu = \frac{N^2}{2L} = \frac{\nu^*}{2p\nu^* \pm 1}. \tag{5.74}$$

5.11.4 Derivation IV

We now derive the effective magnetic field experienced by a composite fermion from a Berry phase calculation. For this purpose, we first show that when a composite fermion, i.e., an electron along with its vortices, is taken in a closed loop enclosing an area A (in the counterclockwise direction), it acquires a Berry phase

$$\Phi^* = -2\pi \frac{BA}{\phi_0} + 2\pi\, 2pN_{\rm enc}, \tag{5.75}$$

where $N_{\rm enc}$ is the number of composite fermions inside the loop. The first term is the familiar Aharonov–Bohm (AB) phase due to a charge going around in a loop. The second, as proven in the following paragraph, is the Berry phase due to the $2p$ vortices going around $N_{\rm enc}$ particles, with each particle producing a phase of 2π.

The Berry phase of a vortex is derived following Arovas, Schrieffer, and Wilczek [13]. A vortex at

$$\eta = R\,\mathrm{e}^{-\mathrm{i}\theta} \tag{5.76}$$

is defined by the wave function

$$\Psi_\eta = N_R \prod_j (z_j - \eta)\Psi, \tag{5.77}$$

where Ψ is the wave function for the incompressible ground state in question. Electrons avoid the point η, creating a hole there, which has a positive charge relative to the incompressible state. We have explicitly included the normalization factor N_R, which depends on the amplitude of η, but can be chosen to be independent of the angle θ. Let us now take η in a circular loop of radius R by slowly changing θ from 0 to 2π, while holding

R constant. The Berry phase (Appendix E) associated with this path is given by

$$\begin{aligned}
\gamma &= \oint \mathrm{d}t \left\langle \Psi_\eta | \mathrm{i} \frac{\mathrm{d}}{\mathrm{d}t} \Psi_\eta \right\rangle \\
&= \oint \mathrm{d}\theta \left\langle \Psi_\eta | \mathrm{i} \frac{\mathrm{d}}{\mathrm{d}\theta} \Psi_\eta \right\rangle \\
&= \oint \mathrm{d}\theta (-\mathrm{i}) \frac{\mathrm{d}\eta}{\mathrm{d}\theta} \left\langle \Psi_\eta | \sum_j \frac{1}{z_j - \eta} | \Psi_\eta \right\rangle \\
&= \oint (-\mathrm{i}) \mathrm{d}\eta \int \mathrm{d}^2 \boldsymbol{r} \frac{1}{z - \eta} \left\langle \Psi_\eta | \hat{\rho}(\boldsymbol{r}) | \Psi_\eta \right\rangle \\
&= 2\pi \int_{r<R} \mathrm{d}^2 \boldsymbol{r}\, \rho_\eta(\boldsymbol{r}) \\
&= 2\pi N_{\mathrm{enc}}.
\end{aligned} \tag{5.78}$$

In the above, we have used: $z = x - \mathrm{i}y$, $\boldsymbol{r} = (x, y)$, $\hat{\rho}(\boldsymbol{r}) = \sum_j \delta^{(2)}(\boldsymbol{r}_j - \boldsymbol{r})$, $\rho_\eta(\boldsymbol{r}) = \left\langle \Psi_\eta | \hat{\rho}(\boldsymbol{r}) | \Psi_\eta \right\rangle$, and N_{enc} is the number of particles inside the closed loop. Because of the definition $z = x - \mathrm{i}y$, the residue for a contour integral differs from the usual by a sign; this can be seen, for example, by evaluating $\oint \mathrm{d}\eta / 2\pi \mathrm{i}\eta = -1$ for a conterclockwise contour. The Berry phase of a vortex thus simply counts the number of particles inside the loop, with each particle contributing 2π.

For uniform density states, we replace N_{enc} in Eq. (5.75) by its average value ρA, where ρ is the electron or the CF density (this is a mean-field approximation), and equate the entire phase to the AB phase due to an effective magnetic field, B^*:

$$\Phi^* = -2\pi \left(\frac{BA}{\phi_0} - 2p\rho A \right) \equiv -2\pi \frac{B^* A}{\phi_0}. \tag{5.79}$$

Composite fermions thus experience an effective magnetic field given by

$$B^* = B - 2p\rho\phi_0. \tag{5.80}$$

In essence, *the Berry phases originating from the vortices bound to electrons partly cancel the AB phase of the external magnetic field*, and as a consequence, composite fermions behave as though they were in a much smaller effective magnetic field. The field B^* can be antiparallel to B.

The relative sign in Eq. (5.75) The two terms on the right hand side of Eq. (5.75) come with opposite signs, leading to a partial *cancellation* of the magnetic field. In other words, the phases due to vortices are equivalent to an effective magnetic field pointing in the $-z$ direction. This can also be seen by eliminating the phases of the Jastrow factor in favor of a vector potential. Let us consider the Schrödinger equation

$$\left[\frac{1}{2m_{\mathrm{b}}} \sum_i \left(\boldsymbol{p}_i + \frac{e}{c} \boldsymbol{A}(\boldsymbol{r}_i) \right)^2 + V \right] \prod_{j<k} (z_j - z_k)^{2p} \Phi_{\nu^*} = E \prod_{j<k} (z_j - z_k)^{2p} \Phi_{\nu^*}, \tag{5.81}$$

where V is the interaction. Our focus in the following is on the kinetic energy term. It is convenient for our purpose to display the phases due to the Jastrow factor explicitly:

$$\prod_{j<k}(z_j - z_k)^{2p} = \mathrm{e}^{-\mathrm{i}2p\sum_{j<k}\theta_{jk}} \prod_{j<k}|z_j - z_k|^{2p}, \tag{5.82}$$

where θ_{jk} was defined in Eq. (5.22). We have been careful to keep track of the definition $z = r\mathrm{e}^{-\mathrm{i}\theta}$, as appropriate for an external magnetic field in the $+z$ direction. Using

$$\boldsymbol{p}_i \mathrm{e}^{-\mathrm{i}2p\sum_{j<k}\theta_{jk}} = \mathrm{e}^{-\mathrm{i}2p\sum_{j<k}\theta_{jk}} \left(\boldsymbol{p}_i - 2p\hbar \sum_j{}' \nabla_i \theta_{ij}\right), \tag{5.83}$$

where the prime denotes the condition $j \neq i$, we write

$$\begin{aligned}\left[\frac{1}{2m_\mathrm{b}}\sum_i \left(\boldsymbol{p}_i + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}_i) - \frac{e}{c}\boldsymbol{a}(\boldsymbol{r}_i)\right)^2 + V\right] \prod_{j<k}|z_j - z_k|^{2p}\Phi_{\nu^*} \\ = E\prod_{j<k}|z_j - z_k|^{2p}\Phi_{\nu^*},\end{aligned} \tag{5.84}$$

with $\boldsymbol{a}$ defined in Eq. (5.21). The additional vector potential $(-\boldsymbol{a})$ simulates the effect of the phases of the Jastrow factor. Using the formulas in Appendix C, the corresponding magnetic field is

$$\boldsymbol{b}_i = \nabla_i \times (-\boldsymbol{a}_i) = -2p\phi_0 \sum_l{}' \delta^{(2)}(\boldsymbol{r}_i - \boldsymbol{r}_l). \tag{5.85}$$

Thus, the phase of the Jastrow factor is equivalent to each electron seeing a flux tube of strength $-2p\phi_0$ on every other electron; the minus sign indicates that the flux tube points in the $-z$ direction, opposite to the direction of the external field $\boldsymbol{B}$.

The singular vector potential does not take care of *all* of the phases in the wave function, as Φ_{ν^*} has additional vortices and antivortices. What about their effect? That is incorporated in the CF theory by saying that composite fermions fill ν^* Λ levels. Also, the Jastrow factor does more than produce flux tubes. Flux tubes are only a result of the phase part of the Jastrow factor; the amplitude $\prod_{j<k}|z_j - z_k|^{2p}$ builds repulsive correlations between electrons.

5.12 Reality of the effective magnetic field

The picture in Fig. 5.6 ought not to be taken literally. No real magnetic flux binds to electrons. The external magnetic field does not bundle into individual flux quanta as it passes through the 2D surface (as it does in type-II superconductors). The currents required to create a localized flux would incur a prohibitively high kinetic energy cost. An external

magnetometer will measure a uniform magnetic field of magnitude B. The effective magnetic field is a purely quantum mechanical effect, produced because the Berry phases due to vortices partly cancel the Aharonov–Bohm phases. The effective magnetic field is fully internal to composite fermions.

If the effective magnetic field could not be measured, a legitimate question would be: "Is it real or a mere mathematical construct?" Fortunately, it *can* be measured, except that composite fermions themselves must be used to detect it. Experiments to be described in subsequent chapters have shown that composite fermions indeed experience the magnetic field B^* rather than B, a fact that has numerous striking observable consequences. It has also been confirmed extensively (Chapter 6) that the low-energy spectrum of interacting electrons in the lowest Landau level resembles, in a model independent manner, the low-energy spectrum of noninteracting fermions at B^*.

5.13 Reality of the Λ levels

Intimately connected to the reality of B^* is the formation of Λ levels. How do we know that the lowest Landau level really splits into Λ levels of composite fermions, which are analogous to the electronic Landau levels in the effective magnetic field?

We stress a fundamental theoretical distinction between Λ levels and Landau levels. Landau levels are the solutions of the single electron problem, starting from which we build many-particle states. The Λ levels are more subtle and complex because they are an emergent structure, and provide a single particle-like interpretation for the inherently many-particle states in the lowest Landau level. A derivation of Λ levels for a single composite fermion is not possible (a single composite fermion is itself a nonsensical notion); the only way of convincing ourselves of the existence of Λ levels is to postulate them, and then to put their consequences to the test.

Subsequent chapters show that, in spite of the "theoretical" difference, Λ levels manifest themselves in experiments with amazing similarity to Landau levels. The FQHE is understood as a direct consequence of the formation of Λ levels just as the IQHE is of Landau levels. Various excitations across Λ levels have been observed. Computer experiments show higher LL-like structures appearing within the lowest Landau level, and also establish a direct connection between the LLL wave functions at ν with wave functions involving many Landau levels at ν^*.

5.14 Lowest Landau level projection

This section has three parts. It begins with a brief discussion of the need for LLL projection. The subsequent part addresses in what sense the CF physics survives projection into the lowest Landau level. Then we learn how the projection is performed.

5.14.1 Why projection?

In the limit of $B \to \infty$, only the lowest Landau level is relevant. However, the FQHE can, and indeed does, occur at finite B. Since some LL mixing is always present at a finite B, this

proves that the quantization of Hall resistance is invariant under at least a small amount of admixture with higher Landau levels.

Although not necessary for the FQHE, the limit $B \to \infty$ is very convenient for several reasons. First, it enormously simplifies computer experimentation; the dimension of the relevant Fock space is finite because of the LLL constraint, allowing us to obtain *exact* results, which, in turn, enable a rigorous and unbiased testing of any theoretical ideas. Second, a reliable treatment of LL mixing is an extra complication that distracts from the physics of the FQHE, and is, therefore, best dispensed with when the primary concern is to establish the basic principle of the phenomenon. Stated differently, the $B \to \infty$ limit simplifies the problem by eliminating one parameter (the electron band mass) from the problem. Finally, nonuniversal quantities (e.g., the excitation gaps) calculated in the limit $B \to \infty$ often provide decent approximations for the experimentally measured quantities.

As seen above, the unprojected wave functions of Eq. (5.31) have some amplitude in higher Landau levels. We wish to adiabatically deform them to produce wave functions that reside entirely in the lowest electronic Landau level. How can that be accomplished? Let us note that these wave functions are already predominantly in the lowest Landau level. A measure of LL mixing in the unprojected wave function is the kinetic energy per particle, which is in general small, as shown in Table 5.1. Eliminating the part that does not reside in the lowest electronic Landau level is called "LLL projection." Two methods for LLL projection are outlined below, with further details in Appendix J.

The unprojected wave functions of Eq. (5.31) do not describe the physics of LL mixing. They have a fixed amount of LL mixing, whereas the LL mixing of the actual state is a function of the magnetic field. Even for an investigation of the effects of LL mixing, it is appropriate to begin with the projected wave functions and then incorporate admixture with higher Landau levels through an additional Jastrow factor [219,434,435,529,567]. Melik-Alaverdian and Bonesteel [434] have considered a mixture of projected and unprojected wave functions to treat LL mixing, which amounts to a partial projection.[20] Güçlü *et al.* [218,219] have shown that a moderate amount of hybridization with higher Landau levels leaves the phases of the LLL wave function unchanged to a very good approximation; therefore, the fixed phase diffusion Monte Carlo offers a reliable method for dealing with LL mixing, provided we have in our possession an accurate LLL wave function.

5.14.2 Composite fermions in the lowest Landau level

The physics of the formation of composite fermions and the effective magnetic field is most transparent in the unprojected wave function $\Phi_{\nu^*}\Phi_1^{2p}$, but becomes obscure after LLL projection. This is best illustrated by specializing to the filling factor range $\nu > 1/3$. To satisfy antisymmetry, a wave function confined to the lowest Landau level must have the form (apart from the Gaussian) $\prod_{j<k}(z_j - z_k)^{2p+1} F_S[\{z_i\}]$, where the polynomial $F_S[\{z_i\}]$ is symmetric under an exchange of any two coordinates. This wave function has $\nu \leq 1/(2p+1)$. Therefore, for $\nu > 1/3$, the wave function contains only a single power of

[20] This is akin to the well-known Gutzwiller projection in studies of the Hubbard model.

the factor $\prod_{j<k}(z_j - z_k)$, implying that each electron binds *one and only one* vortex. That seems, naively, to exclude the formation of composite fermions in the lowest Landau level in this filling factor range.

Yet, the CF physics undoubtedly survives in the lowest Landau level. The defining property of composite fermions, namely that they experience an effective magnetic field, has been established within the lowest Landau level, both in real and computer experiments (subsequent chapters), and forms the basis for the explanation of many phenomena.

How do we reconcile the two preceding paragraphs? By appealing to the powerful principle of adiabatic continuity of physics. Let us ask what happens to the vortices of the unprojected wave function when it is projected into the lowest Landau level. The *two* essential pieces in the unprojected wave function, the Jastrow factor and Φ_{ν^*}, are, *together*, interpreted as composite fermions at filling ν^*. It should be stressed, however, that the factor Φ_{ν^*} itself contains many vortices *and* antivortices. When the full wave function is projected, the vortices from the Jastrow factor and the vortices and the antivortices from Φ_{ν^*} get all mixed up (with all antivortices annihilated by vortices). The two pieces of physics thus get entangled beyond recognition. Nonetheless, unless a phase transition (or a level crossing) takes place during the act of projection, the CF physics is guaranteed to survive projection into the lowest Landau level.

This, in fact, brings out the beauty of the CF theory. The LLL wave functions of the FQHE states are, in general, so complicated that understanding their physics is a formidable challenge. The CF theory recognizes that they are LLL projections of certain simpler wave functions, from which the CF physics can be "read off." While composite fermions become effectively disguised when we project the wave functions into the lowest Landau level, the qualitative consequences of composite fermions' existence remain sharply recognizable.

It may help the reader to recall other, well-known examples where the physics of a complex system is understood by its adiabatic continuity to a simpler system. When the interaction between fermions is switched on in a Landau Fermi liquid, free fermions are renormalized into quasiparticles that are complex, many-body objects; their mass can sometimes be modified by two orders of magnitude, as in heavy fermion systems. Nonetheless, so long as no phase boundary is crossed, the physics of the interacting system is qualitatively similar to that of the free system, apart from a renormalization of parameters. Another example is that of superconductivity. In the limit of strong attractive interaction, electrons form tightly bound Cooper pairs, which are viewed as bosons, and superconductivity as their Bose–Einstein condensation. As the interaction strength is slowly reduced, the BEC state evolves into the Bardeen–Cooper–Schrieffer state in the other limit of weak attractive interactions. Spatially bound pairs of electrons are no longer identifiable, but the qualitative physics of the strong coupling limit continues to hold.

5.14.3 Use of higher Landau levels in the LLL physics

We should not be distressed by the use, at intermediate steps, of higher Landau levels, even if we were only interested in the $B = \infty$ physics. An accepted strategy in condensed matter

physics is to enlarge the Hilbert space (sometimes into unphysical directions, as done in slave-boson, large-N, large-S and large-d approaches), hope that the physics is more transparent there, obtain an approximate solution, project back into the original space, and pray that the essential physics survives. To be sure, there is nothing wrong in using higher Landau levels – LL mixing is at least not unphysical. We can also directly ascertain the validity of the final solution against exact results; after all, that is what ultimately counts.

Most importantly, the use of higher Landau levels is not just a technical tool but reveals the emergence of Λ level structure within the lowest Landau level, which has numerous experimental manifestations.

5.14.4 Projection methods

The term "LLL projection" refers to the step which produces a LLL wave function starting from an unprojected wave function of the form $\Psi_\nu = \Phi_{\nu^*}\Phi_1^{2p}$. The factor Φ_{ν^*} is in general a linear superposition of Slater determinants. The following theorems are useful for LLL projection.

Theorem 5.1 The product of two LLL wave functions is also in the lowest Landau level (although at a different magnetic field).

The reader is asked to verify this statement using explicit wave functions for the disk and spherical geometries. A simple corollary is that Φ_1^{2p} lies in the lowest Landau level. Therefore, the basic term that we need to project into the lowest Landau level is the product of a higher LL single particle wave function (from the expansion of Φ_{ν^*}) and a LLL wave function (from Φ_1^{2p}).

Theorem 5.2 In spherical geometry, the LLL projection of the product of a LLL wave function and a higher LL wave function can be written as

$$\mathcal{P}_{\rm LLL} Y_{Qlm}(\mathbf{\Omega})\Psi_{Q'Q'm'}(\mathbf{\Omega}) = \hat{Y}^{Q'}_{Qlm}(\mathbf{\Omega})\Psi_{Q'Q'm'}(\mathbf{\Omega}), \tag{5.86}$$

where $\Psi_{Q'Q'm'}$ is an arbitrary LLL wave function at monopole strength Q' and $\hat{Y}^{Q'}_{Qlm}(\mathbf{\Omega})$ is an m' independent operator. Similarly, in the disk geometry, the LLL projection is given by

$$\mathcal{P}_{\rm LLL}\eta_{nm}(\boldsymbol{r})\eta_{0,m'} = \hat{\eta}_{nm}(\boldsymbol{r})\eta_{0,m'}, \tag{5.87}$$

where $\eta_{0,m'}$ is an arbitrary LLL wave function.

We can, of course, project any single particle wave function into the lowest Landau level. The point of this theorem is that the operator $\hat{Y}^{Q'}_{Qlm}$ or $\hat{\eta}_{nm}$ accomplishes LLL projection independent of the LLL state it acts upon.[21]

[21] $\hat{Y}^{Q'}_{Qlm}$ depends on Q' but that is not a problem because, for the LLL projections of our interest, it acts upon Φ_1^{2p}, which has a fixed $Q' = p(N-1)$. The superscript is often omitted.

The explicit expression for $\hat{Y}_{Qlm}$ is given in Appendix J; for now it will be sufficient to know that it exists. The operator $\hat{\eta}$ for disk geometry is simpler, and we show here that it is given by

$$\hat{\eta}_{nm}(\bar{z}, z) =: \eta_{nm}\left(\bar{z} \to 2\frac{\partial}{\partial z}, z\right) :, \tag{5.88}$$

where the normal ordering of a function (denoted by two colons ::) amounts to moving $\bar{z}$ to the left of z. (We also follow the convention that the derivatives do not act upon the Gaussian factor.) This follows because

$$\mathcal{P}_{\rm LLL}\bar{z}^n z^m \mathrm{e}^{-\frac{1}{4}z\bar{z}} = \mathrm{e}^{-\frac{1}{4}z\bar{z}}\left(2\frac{\partial}{\partial z}\right)^n z^m. \tag{5.89}$$

Proof One way to proceed is to write

$$\bar{z}^n z^m\, \mathrm{e}^{-\frac{1}{4}z\bar{z}} = a_0 z^{m-n}\, \mathrm{e}^{-\frac{1}{4}z\bar{z}} + \text{higher LL terms}, \tag{5.90}$$

and determine a_0 by multiplying both sides by $\bar{z}^{m-n}\mathrm{e}^{-\frac{1}{4}z\bar{z}}$ and integrating over the position. The answer can be obtained without performing Gaussian integrals:

$$\int \mathrm{d}^2\boldsymbol{r}\, \bar{z}^{m-n}\bar{z}^n z^m \mathrm{e}^{-\frac{1}{2}z\bar{z}} = \int \mathrm{d}^2\boldsymbol{r}\, \bar{z}^{m-n} z^m \left(-2\frac{\partial}{\partial z}\right)^n \mathrm{e}^{-\frac{1}{2}z\bar{z}} \tag{5.91}$$

$$= \int \mathrm{d}^2\boldsymbol{r}\, \bar{z}^{m-n}\left[\left(2\frac{\partial}{\partial z}\right)^n z^m\right] \mathrm{e}^{-\frac{1}{2}z\bar{z}}, \tag{5.92}$$

which demonstrates that the projection of $z^{m-n}\,\mathrm{e}^{-\frac{1}{4}z\bar{z}}$ on $\bar{z}^n z^m\,\mathrm{e}^{-\frac{1}{4}z\bar{z}}$ is the same as its projection on $\mathrm{e}^{-\frac{1}{4}z\bar{z}}\left(2\frac{\partial}{\partial z}\right)^n z^m$. □

A wave function strictly in the lowest Landau level has no powers of $\bar{z}$ in the polynomial part. The preceding discussion clarifies that even when the wave function $\Phi\prod_{j<k}(z_j - z_k)^{2p}$ contains powers of $\bar{z}$, it has a nonzero amplitude in the lowest Landau level. In fact, it may be predominantly in the lowest Landau level [272, 275]. To project $\Phi\prod_{j<k}(z_j - z_k)^{2p}$, let us consider a typical term in its expansion, which contains factors of the type

$$\phi = \bar{z}^n z^{n+m}\, \mathrm{e}^{-\frac{1}{4}z\bar{z}} \tag{5.93}$$

for each particle. Typically, n is of order one (bounded by the index of the highest occupied Landau level in Φ), and m is a large number (because, for large N, terms with the highest weights in the expansion of $\prod_{j<k}(z_j - z_k)^{2p}$ contain large powers of z_j's). Thus, the typical factor has $m \gg n$. Some algebra (Exercise 5.3) shows that

$$\frac{\langle\phi_p|\phi\rangle}{\langle\phi|\phi\rangle} = \frac{(m+n)!(m+n)!}{m!(m+2n)!}, \tag{5.94}$$

where ϕ_p is the LLL projection of ϕ. For $m \gg n$, the right hand side differs from unity only by terms of order $1/m$, demonstrating that ϕ is almost the same as its LLL projection ϕ_p.

In other words, ϕ has only a small amplitude outside the lowest Landau level. This single particle argument suggests that the many-body wave function $\Phi \prod_{j<k}(z_j - z_k)^{2p}$ might be mostly in the lowest Landau level even without projection. In fact, one might wonder if the amplitude in higher Landau levels vanishes in the thermodynamic limit. Explicit calculations show that the mixing with higher Landau levels in these wave functions is small but nonzero (See Table 5.1), and an explicit projection is required to obtain strictly LLL wave functions.

Two methods have been used for projecting $\Phi_{\nu^*}\Phi_1^{2p}$ into the lowest Landau level.

5.14.5 Method I

The most natural method for projecting any given wave function into the lowest Landau level is to express it as a linear superposition of Slater determinant basis states (involving higher Landau level orbitals), and then retain only the part that fully resides in the lowest Landau level. Formally, we can express the projected wave function in the spherical geometry as

$$\Psi = \mathcal{P}_{\rm LLL}\Phi(Y)\Phi_1^{2p} = \hat{\Phi}(\hat{Y})\Phi_1^{2p}, \tag{5.95}$$

where $\hat{\Phi}(\hat{Y})$ is an operator obtained by replacing monopole harmonics in $\Phi(Y)$ by the corresponding operators. In the disk geometry, we have

$$\mathcal{P}_{\rm LLL}\mathrm{e}^{-\sum_j |z|_j^2/4}\Phi(\eta)\prod_{j<k}(z_j - z_k)^{2p} = \mathrm{e}^{-\sum_j |z|_j^2/4}\hat{\Phi}(\hat{\eta})\prod_{j<k}(z_j - z_k)^{2p}. \tag{5.96}$$

LLL projection through this procedure is rather cumbersome, as each term of $\hat{\Phi}(\hat{\eta})$ involves a large number of derivatives. The projection has been implemented for up to 10 particles by an expansion into Slater determinant basis states [96, 115, 116, 549, 703], but becomes impractical for larger systems, because the number of LLL basis states quickly grows too large to be stored on a computer.

5.14.6 Method II

An alternative projection method (Jain and Kamilla [280, 281]) allows calculations for much larger systems (systems with up to 100 composite fermions have been studied [571]; larger systems are treatable if needed) and has proved crucial for obtaining quantitative information from the CF theory. The key is to write the projected wave function in a way that can be evaluated without expansion into Slater determinant basis functions.

We first consider the disk geometry, while assuming that Φ is a single Slater determinant. We write (modulo an overall sign)

$$\prod_{j<k}(z_j - z_k)^{2p} = \prod_{j\neq k}(z_j - z_k)^{p} \equiv \prod_j \mathcal{J}_j^p, \tag{5.97}$$

where

$$\mathcal{J}_j = \prod_k{}' (z_j - z_k), \tag{5.98}$$

and the prime denotes the condition $k \neq j$. The Jastrow factor can be subsumed into the Slater determinant:

$$\begin{aligned}\Psi &= \begin{vmatrix} \eta_1(\boldsymbol{r}_1) & \eta_1(\boldsymbol{r}_2) & . & . & . \\ \eta_2(\boldsymbol{r}_1) & \eta_2(\boldsymbol{r}_2) & . & . & . \\ . & . & . & & \\ . & . & & . & \\ . & . & & & . \end{vmatrix} \prod_{j<k} (z_j - z_k)^{2p} \\ &= \begin{vmatrix} \eta_1(\boldsymbol{r}_1)\mathcal{J}_1^p & \eta_1(\boldsymbol{r}_2)\mathcal{J}_2^p & . & . & . \\ \eta_2(\boldsymbol{r}_1)\mathcal{J}_1^p & \eta_2(\boldsymbol{r}_2)\mathcal{J}_2^p & . & . & . \\ . & . & . & & \\ . & . & & . & \\ . & . & & & . \end{vmatrix}. \end{aligned} \tag{5.99}$$

We now project each element individually to *define* the LLL projection as

$$\mathcal{P}_{\mathrm{LLL}}\Psi = \begin{vmatrix} \hat{\eta}_1(\boldsymbol{r}_1)\mathcal{J}_1^p & \hat{\eta}_1(\boldsymbol{r}_2)\mathcal{J}_2^p & . & . & . \\ \hat{\eta}_2(\boldsymbol{r}_1)\mathcal{J}_1^p & \hat{\eta}_2(\boldsymbol{r}_2)\mathcal{J}_2^p & . & . & . \\ . & . & . & & \\ . & . & & . & \\ . & . & & & . \end{vmatrix}. \tag{5.100}$$

An explicit expression for each matrix element can be obtained by the projection method I. The largest power of $\bar{z}$ in $\eta_{n,m}$ is $\bar{z}^n$, so we need to evaluate the nth derivative of $\mathcal{J}_j^p$. Explicit expressions for derivatives can be obtained by Mathematica when n is not too large. Further details can be found in Appendix J.

When Φ is a linear superposition of many Slater determinants, the corresponding wave function Ψ is obtained as

$$\begin{aligned}\Psi &= \mathcal{P}_{\mathrm{LLL}}\Phi\Phi_1^{2p} \\ &= \Phi[\eta_\alpha(\boldsymbol{r}_j) \to \hat{\eta}_\alpha(\boldsymbol{r}_j)\mathcal{J}_j^p], \end{aligned} \tag{5.101}$$

where α denotes the quantum numbers n, m. The projected wave function Ψ is thus obtained from Φ by replacing each single particle state by a complicated expression. The coefficients of superposition remain unchanged.

This trick also works for the spherical geometry, with

$$\prod_{j<k}(u_j v_k - v_j u_k)^{2p} = \prod_{j\neq k}(u_j v_k - v_j u_k)^{p} \equiv \prod_j \mathcal{J}_j^p, \tag{5.102}$$

$$\mathcal{J}_j = \prod_k{}'(u_j v_k - v_j u_k). \tag{5.103}$$

When Φ is a single Slater determinant,

$$\begin{aligned}\mathcal{P}_{\rm LLL}{\rm Det}[Y_\alpha(\mathbf{\Omega}_j)]\prod_{k<l}(u_k v_l - v_k u_l)^{2p} &= \mathcal{P}_{\rm LLL}{\rm Det}[Y_\alpha(\mathbf{\Omega}_j)]\prod_k \mathcal{J}_k^p \\ &= \mathcal{P}_{\rm LLL}{\rm Det}[Y_\alpha(\mathbf{\Omega}_j)\mathcal{J}_j^p] \\ &\equiv {\rm Det}[\mathcal{P}_{\rm LLL}Y_\alpha(\mathbf{\Omega}_j)\mathcal{J}_j^p] \\ &= {\rm Det}[\hat{Y}_\alpha(\mathbf{\Omega}_j)\mathcal{J}_j^p].\end{aligned} \tag{5.104}$$

For an arbitrary Φ, which is a linear superposition of Slater determinants, the corresponding wave function Ψ is obtained as

$$\Psi = \mathcal{P}_{\rm LLL}\Phi\Phi_1^{2p} = \Phi[Y_\alpha(\mathbf{\Omega}_j) \to \hat{Y}_\alpha(\mathbf{\Omega}_j)\mathcal{J}_j^p]. \tag{5.105}$$

The coefficients of superposition remain unchanged in going from Φ to Ψ.

The LLL projected wave functions from methods I and II are not identical. We should remember, however, that our aim is to obtain a LLL wave function starting from certain unprojected wave functions, and, therefore, we have no a-priori preference between different schemes that accomplish that. Each scheme must be tested individually against exact results to establish the level of its accuracy. We should not be surprised if more than one projection method turns out be valid; because the unprojected wave functions are mostly in the lowest Landau level to begin with, it should be unimportant how the higher LL part is eliminated.

5.15 Need for other formulations

So far, we have dealt with the physics of composite fermions and its quantitative implementation through microscopic wave functions. An advantage of this implementation is that its statements can be tested rigorously against exact results known for systems with up to 12–15 particles (Chapter 6). Extracting numbers from wave functions requires evaluation of $2N$-dimensional integrals, which can be accomplished numerically for fairly large N with the help of the Metropolis Monte Carlo method. The wave functions are suitable for the calculation of energy gaps and equal-time, zero-temperature correlation functions (which require only the ground state wave function). CF diagonalization can also tell us, in principle, if a state is not incompressible. Microscopic wave functions, however, are not useful for compressible states, because a proper account of the thermodynamic limit requires

systems that are larger than what can currently be handled, and because the absence of a gap precludes a systematic truncation of the basis for CF diagonalization.[22] Wave functions are also impractical for time, wave vector, frequency, or temperature dependent response functions.

That constitutes part of the motivation for seeking other theoretical formulations of composite fermions, to which the remainder of the chapter is devoted. We note that none of the approaches discussed below *derives* composite fermions. They all assume composite fermions at the very outset, sometimes through an exact transformation. Also, resorting to uncontrolled approximations is necessary in all of the formulations, just as it was in writing wave functions. This fact, although hardly surprising in view of the complexity of the problem and the lack of a small parameter, necessitates independent tests of the "final" statements of each approach. Because the qualitative phenomenology of the FQHE and the CF Fermi sea follows from the CF principle without recourse to any specific theoretical scheme, the validity of any formulation is to be ascertained through detailed comparisons of its *quantitative* consequences against theoretical and experimental facts.

5.16 Composite fermion Chern–Simons theory

The Chern–Simons field theory of FQHE was pioneered by Zhang, Hansson, and Kilvelson [743], who mapped the problem at Laughlin fractions into a boson superfluid, following ideas of Girvin and MacDonald [193]. The composite fermion Chern–Simons (CFCS) theory (also called the "fermion CS theory") was introduced by Lopez and Fradkin [401], and further extended by Halperin, Lee, and Read [231], Murthy and Shankar [581], and several other groups. A thorough treatment of the CFCS approach is outside the scope of this book. We present here only a basic introduction, and refer the reader to several excellent review articles (Halperin [232, 233]; Lopez and Fradkin [404]; Murthy and Shankar [468]; and Simon [594]) for further details and successes of this approach, as well as for many relevant references.

5.16.1 Why field theory?

Some condensed matter physicists hold the view (subconsciously) that a field theory is more fundamental than a theory based on wave functions. This notion perhaps has its roots either in the success of field theory in high-energy physics[23], or, more likely, in the tendency to equate the term "theory" with complicated mathematical and technical formalism.

[22] Certain quantities for the compressible states at $\nu = 1/2p$ have been estimated by calculating them for incompressible states belonging to the sequence $\nu = n/(2pn \pm 1)$, followed by an extrapolation to the limit $n \to \infty$. Such a procedure, however, does not test the validity of the CF description at $\nu = 1/2p$; rather, it makes the assumptions that (i) composite fermions remain relevant in that limit, and (ii) the residual interaction between them does not cause an instability with increasing n. Both these assumptions are known to fail in special cases: In higher Landau levels, the CF description is invalid altogether, whereas in the second Landau level composite fermions exist at half filling but, apparently, form a paired FQHE state rather than a CF Fermi sea.

[23] Ironically, attempts are underway to move away from field theory in high-energy physics; the string theory program is based on quantum mechanics rather than field theory.

The possibility of creation and annihilation of particles necessitates field theory in high-energy physics. In nonrelativistic quantum many-body physics, with a fixed number of particles, the wave function and field theoretical approaches are merely different calculational tools for deducing quantitative information from the Schrödinger equation. The question is not which is more fundamental, but which is more reliable, which takes us further, and which offers new insights. *In principle*, they are equivalent, in the sense that the starting point of a field theory in condensed matter physics is an exact reformulation of the many-particle Schrödinger equation. In practice, however, the two approaches can be remarkably different, and offer distinct intuitions. To be able to identify what an approximation in one language means in the other is rare. Thinking of wave functions has facilitated initial breakthroughs in several important problems, for example, the BCS theory for superconductivity (wave functions for the ground and excited states) and Feynman's theory of superfluidity (wave function for the collective mode). On the other hand, field theory is an essential tool for a description of quantum or classical phase transitions.

Why, then, field theory? On general grounds, having in our possession as many theoretical formulations as possible of a new physical concept is desirable. While one may be more useful for one question, another may take us further in a different context. The power of a field theory lies in its treatment of spontaneous symmetry breaking and order parameter, and in organizing a perturbation theory through Feynman diagrams. The theory of superconductivity provides an ideal example. The Gorkov–Eliashberg field theoretical formulation obtains the Bardeen–Cooper–Schrieffer results in one limit, but takes us farther by enabling a more complete treatment of several features (electron–phonon coupling, effect of disorder), and even allows a first-principles determination of quantities such as the critical temperature.

It is worth stressing that the essential physics must be already known before a perturbation program can be implemented through a field theory. By definition, perturbation theory keeps us within the phase we started with (phase transitions involve nonperturbative reorganizations), and is of no use unless we begin in the correct phase. For example, with the noninteracting Fermi sea as the starting point, only the properties of a Landau Fermi liquid will be accessible in perturbative calculations; the theory of superconductivity must incorporate, at the very outset, the physics of pairing, which was discovered by other means.

5.16.2 Lopez–Fradkin theory

We started in Section 5.8.2 with electrons at B^*, added $2p$ flux quanta to each electron, and then made a mean-field approximation to end up with fermions at B. The Lopez–Fradkin [401] construction begins with electrons at B, attaches $2p$ flux quanta pointing in the direction opposite to B, and then performs a mean-field approximation to cancel part of the external field, producing fermions at B^* in the end. The basic idea is shown in Fig. 5.12.

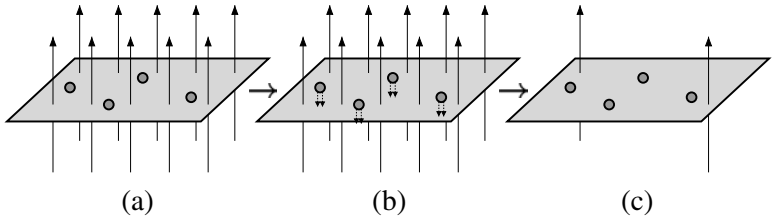

Fig. 5.12. The physical idea of the Lopez–Fradkin theory. The FQHE system is shown in (a). Two magnetic flux quanta are attached to each electron pointing in the downward direction to convert electrons into composite fermions (b). Composite fermions respond to a combination of the real and attached fields, and, consequently, behave as though they were in a smaller magnetic field (c).

We consider the Schrödinger equation

$$\left[\frac{1}{2m_{\rm b}}\sum_i\left(\boldsymbol{p}_i+\frac{e}{c}\boldsymbol{A}(\boldsymbol{r}_i)\right)^2+V\right]\Psi=E\Psi, \tag{5.106}$$

where V is the interaction. Through an exact singular gauge transformation defined by

$$\Psi=\prod_{j<k}\left(\frac{z_j-z_k}{|z_j-z_k|}\right)^{2p}\Psi_{\rm CS}, \tag{5.107}$$

known as the CFCS transformation, the eigenvalue problem can be expressed as

$$H'\Psi_{\rm CS}=E\Psi_{\rm CS}, \tag{5.108}$$

$$H'=\left[\frac{1}{2m_{\rm b}}\sum_i\left(\boldsymbol{p}_i+\frac{e}{c}\boldsymbol{A}(\boldsymbol{r}_i)-\frac{e}{c}\boldsymbol{a}(\boldsymbol{r}_i)\right)^2+V\right], \tag{5.109}$$

$$\boldsymbol{a}(\boldsymbol{r}_i)=\frac{2p}{2\pi}\phi_0{\sum_j}'\boldsymbol{\nabla}_i\theta_{ij}, \tag{5.110}$$

following algebra similar to that in Section 5.8.2. The vector potential $-\boldsymbol{a}$ amounts to attaching a point flux of strength $-2p\phi_0$ to each electron. This is the starting model for a composite fermion in the Lopez–Fradkin theory.

Further progress is not possible without making approximations. The usual approach is to make a "mean-field" approximation, which amounts to spreading the CS flux on each composite fermion into a uniform CS magnetic field, and assuming that, to zeroth order, the particles respond to the sum of CS and external fields. Formally, we write

$$\boldsymbol{A}-\boldsymbol{a}\equiv\boldsymbol{A}^*+\delta\boldsymbol{A}, \tag{5.111}$$

$$\boldsymbol{\nabla}\times\boldsymbol{A}^*=B^*\hat{\boldsymbol{z}}, \tag{5.112}$$

$$B^*=B-2p\rho\phi_0, \tag{5.113}$$

where B^* is the effective magnetic field experienced by composite fermions. The transformed Hamiltonian can now be written as

$$H' = \frac{1}{2m_\mathrm{b}} \sum_i \left(\boldsymbol{p}_i + \frac{e}{c} \boldsymbol{A}^*(\boldsymbol{r}_i) \right)^2 + V + V' = H_0' + V + V', \tag{5.114}$$

where V' contains terms proportional to $\delta \boldsymbol{A}$. The solution to H_0' is trivial, describing free fermions in an effective magnetic field B^*. We have thus decomposed the Hamiltonian into two parts: H_0' can be solved exactly and the remainder, $V + V'$, is to be treated perturbatively.

The mean-field approximation of expanding around H_0' is a crucial, nonperturbative step. A mathematically exact CS transformation can be made to attach any amount of flux, integral or nonintegral, to each electron. But because solving the transformed problem exactly is no more possible than it was to solve the original problem, the mathematical exactness of the CS transformation is of little value. The question is what one can do with it. With the mean-field approximation, CS transformations with different values of attached flux become nonequivalent, producing distinct physics. The CS transformation must be physically motivated. The goal is to transform into particles that are weakly interacting, which can only be ascertained by comparing the consequences of the noninteracting model of the CS-transformed particles (H_0') with the observed phenomenology.

Lopez and Fradkin recast the problem in the language of functional integrals, which is suitable for studying corrections to the mean-field theory. For readers who are familiar with this method, the zero-temperature quantum partition function is written as

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\psi^* \mathcal{D}\boldsymbol{a} \, \exp\left(\frac{\mathrm{i}}{\hbar} \mathcal{S} \right), \tag{5.115}$$

$$\mathcal{S} = \int \mathrm{d}^2 \boldsymbol{r} \int \mathrm{d}t \, \mathcal{L}, \tag{5.116}$$

$$\begin{aligned} \mathcal{L} = {} & \psi^* (\mathrm{i}\partial_t - a_0) \psi + \frac{1}{2m_\mathrm{b}} \left| \left(-\mathrm{i}\hbar \boldsymbol{\nabla} + \frac{e}{c} \boldsymbol{A} - \frac{e}{c} \boldsymbol{a} \right) \psi \right|^2 \\ & + \frac{1}{2p\phi_0} a_0 \boldsymbol{\nabla} \times \boldsymbol{a} + \int \mathrm{d}^2 \boldsymbol{r}' (\rho(\boldsymbol{r}) - \bar{\rho}) V(\boldsymbol{r} - \boldsymbol{r}') (\rho(\boldsymbol{r}') - \bar{\rho}), \end{aligned} \tag{5.117}$$

where ψ and ψ^* are anticommuting Grassmann variables, $\bar{\rho}$ is the average density, and $V(\boldsymbol{r})$ is the interaction between electrons. Flux attachment is introduced through a Lagrange multiplier a_0. Because a_0 enters linearly in the action, it can be integrated out to produce a delta function that imposes the constraint

$$\boldsymbol{\nabla} \times \boldsymbol{a}(\boldsymbol{r}) = 2p\phi_0 \rho(\boldsymbol{r}) = 2p\phi_0 \psi^*(\boldsymbol{r}) \psi(\boldsymbol{r}). \tag{5.118}$$

What is the connection with the topological Chern–Simons theory familiar to particle theorists? The Lagrangian for the electromagnetic field is given by the Lorentz scalar $\mathcal{L} = (1/4) F^{\mu\nu} F_{\mu\nu}$, with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{5.119}$$

which is invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda(\boldsymbol{r}, t)$. (The standard summation convention for repeated indices is assumed.) The Chern–Simons Langrangian has the form

$$\mathcal{L}_{\rm CS} \sim \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} = 2\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \tag{5.120}$$

Here $\epsilon^{\mu\nu\lambda}$ is the antisymmetric Levy–Civita tensor, with $\epsilon^{012} = 1$. The index μ takes values $0, 1, 2$, the first being the time component and the rest space components. The CS action is invariant, up to surface terms, under a gauge transformation, because the functional variation $\delta A_\mu = \partial_\mu \Lambda$ causes a change in $\mathcal{L}_{\rm CS}$ that is a total derivative:

$$\begin{aligned} \delta\mathcal{L}_{\rm CS} &= \epsilon^{\mu\nu\lambda} \delta A_\mu \partial_\nu A_\lambda + \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu \delta A_\lambda \\ &= 2\epsilon^{\mu\nu\lambda} \delta A_\mu \partial_\nu A_\lambda \\ &= -2\epsilon^{\mu\nu\lambda} \partial_\nu (A_\lambda \partial_\mu \Lambda). \end{aligned} \tag{5.121}$$

Zhang, Hansson, and Kivelson [743, 744] note that the term proportional to $a_0 \nabla \times \boldsymbol{a}$ in Eq. (5.117), which enforces flux attachment, is precisely equal to the Chern–Simons Langrangian in the Coulomb gauge. This is clarified by writing

$$\begin{aligned} \mathcal{L}_{\rm CS} &= \frac{1}{4p\phi_0} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \\ &= \frac{1}{2p\phi_0} \epsilon^{ij} a_0 \partial_i a_j - \frac{1}{4p\phi_0} \epsilon^{ij} a_i \partial_0 a_j, \end{aligned} \tag{5.122}$$

where i, j represent the spatial components ($i, j = 1, 2$); the time components have been displayed explicitly in the second step ($\partial_0 = \partial_t$). The first term on the right hand side of Eq. (5.122) is identical to the third term on the right hand side of Eq. (5.117). What about the last term in Eq. (5.122)? In Fourier space it is proportional to

$$\epsilon^{ij} a_i(\boldsymbol{q}, \omega)(-\mathrm{i}\omega) a_j(-\boldsymbol{q}, -\omega). \tag{5.123}$$

By choosing the x-axis along $\boldsymbol{q}$, the Coulomb gauge condition $\boldsymbol{q} \cdot \boldsymbol{a} = 0$ implies $a_2(\boldsymbol{q}, \omega) = 0$, guaranteeing that the last term in Eq. (5.122) is identically zero.

The constraint of Eq. (5.118) is used to eliminate two factors of density in the last term of Eq. (5.117) in favor of $(2p\phi_0)^{-1} \nabla \times \boldsymbol{a}$. The action is then quadratic in the fermion field, which can be integrated out. Various response functions can be expressed as correlation functions of the vector potential field and their averages over the CS field configurations are evaluated perturbatively by standard diagrammatic methods.

The solutions of the unperturbed Hamiltonian H_0' are far from the actual solution, as can be seen both from the energy eigenvalues and the eigenfunctions. Specializing to $\nu = n/(2pn + 1)$, the "unperturbed" energy gap is given by

$$\Delta = \hbar\omega_{\rm c}^* = \hbar \frac{eB^*}{m_{\rm b} c} = \frac{\hbar}{(2pn + 1)} \frac{eB}{m_{\rm b} c}, \tag{5.124}$$

with $B^* = B/(2pn+1)$. The actual energy gap in the lowest Landau level problem, in contrast, is proportional to $e^2/\epsilon\ell \sim \sqrt{B}$, the only energy scale in the LLL problem, and is independent of the electron mass. Coming to the "unperturbed" eigenfunctions, let us denote by $\Phi_n^\alpha(B^*)$ the eigenfunctions of H_0' at magnetic field B^*. Undoing the CFCS transformation gives the mean-field electron wave functions (using Eq. 5.107):

$$\Psi_\nu^\alpha(B) = \prod_{j<k}\left(\frac{z_j - z_k}{|z_j - z_k|}\right)^{2p} \Phi_n^\alpha(B^*), \tag{5.125}$$

which are nothing but the wave functions encountered previously in Eq. (5.30). As explained in the discussion following Eq. (5.30), they are not satisfactory (also see Sitko [595, 597]; Ciftja and Wexler [86, 88]). The perturbation theory program faces the formidable challenge of producing repulsive correlations and getting rid of the electron mass.[24] What classes of Feynman diagrams will accomplish that is not known, precluding, so far, a first principles determination of various quantities in the CFCS approach. The difficulties confronted here are related to what would be faced in attempting a derivation of the quantized Landau levels in a perturbative treatment of the magnetic field.

Some comments are in order.

(i) Exorcising the electron mass from the problem requires a proper treatment of the LLL projection. For electrons, LLL wave functions have the form

$$\Psi = \prod_{j<k}(z_j - z_k) F_{\rm S}[\{z_i\}] \exp\left[-\frac{1}{4}\sum_l |z_l|^2\right], \tag{5.126}$$

where $F_{\rm S}[\{z_i\}]$ is a symmetric polynomial. These wave functions translate into CS wave functions with the form:

$$\Psi_{\rm CS} = \prod_{j<k}\frac{|z_j - z_k|^{2p}}{(z_j - z_k)^{2p-1}} F_{\rm S}[\{z_i\}] \exp\left[-\frac{1}{4}\sum_l |z_l|^2\right]. \tag{5.127}$$

Restricting solutions of the Chern–Simons problem to the space defined by such wave functions is difficult to implement, given that the constraint cannot be imposed on single particle wave functions.

(ii) The mean-field theory does not describe all qualitative features correctly, as illustrated by two examples. (a) The Hall resistance of the mean-field theory is $R_{\rm H} = h/ne^2$. We see below how the correct value, $R_{\rm H} = h/[n/(2pn \pm 1)]e^2$, is recovered. (b) The "local charge" of the quasiparticle (defined in Section 9.3) is equal to the electron charge and not a fraction of it. This follows because, at the mean-field level, the charge excess associated with a quasiparticle at $\nu = n/(2pn \pm 1)$ is the same as that at $\nu^* = n$. (The "correct" fractional charge is produced by the random phase approximation [401].)

(iii) One may encounter phrases to the effect: "composite fermions interact through the Chern–Simons gauge potential." That is indeed the way it appears in the CFCS formulation, but we prefer not to

[24] A renormalization of the the electron mass is not sufficient – it must be eliminated entirely.

use such language, because it does not give a quantitative account of the interaction between the "physical" composite fermions. The CFCS field theory is designed to capture the long-distance physics, and does not account for the short-distance part of the interaction between composite fermions. At the present, the most accurate determination of the inter-CF interaction is based on a theory that makes no reference to the CS gauge potential. Section 6.7 contains further discussion on this issue, as well as the actual form of the inter-CF interaction.

5.16.3 Halperin–Lee–Read theory

Halperin, Lee, and Read [231] (also Kalmeyer and Zhang [309]) make the remarkable prediction that composite fermions form a Fermi sea at $\nu = 1/2$, motivated by the observation that composite fermions, should they exist at $\nu = 1/2$, experience no effective magnetic field here (i.e., $B^* = 0$). The CF Fermi sea has been beautifully confirmed experimentally (see Chapter 10); the explanation of the physics of the compressible half-filled Landau level state has been one of the triumphs of the CF approach.

Halperin, Lee, and Read treat the Lopez–Fradkin theory as a starting point to develop an effective theory of the CF Fermi sea, following the spirit of the Landau theory of ordinary Fermi liquids, which expresses various observables in the low-temperature limit in terms of a finite number of parameters. The "energy scale problem" is avoided by replacing the electron mass m_b with a free parameter m^*, interpreted as the mass of composite fermions, to be fixed empirically. We list here the principal results of this approach, often without proof, closely following the reviews by Simon [594] and Halperin [232, 233].

Magnetoresistances With composite fermions viewed as electrons bound to CS flux quanta, their charge current has associated with it a CS flux current. That, in turn, induces, through the Faraday effect, a CS electric field, given by (Exercise 5.4)

$$\boldsymbol{e}_{\rm CS} = 2p\frac{h}{e^2}(\hat{\boldsymbol{z}} \times \boldsymbol{j}), \tag{5.128}$$

where $\boldsymbol{j} = -\rho e \boldsymbol{v}$ is the current density, $\boldsymbol{v}$ being the local *average* drift velocity. The effect of $\boldsymbol{e}_{\rm CS}$ must be included in the calculation of various magnetoresistances. Just as with the real and CS magnetic fields, composite fermions respond to the combination of real and induced CS electric fields. The current density is given by (switching to matrix notation)

$$j = \sigma_{\rm CF}(E + e_{\rm CS}(\boldsymbol{j})) \tag{5.129}$$

which defines the CF conductivity. Here, j, E and $e_{\rm CS}$ represent two-dimensional vectors as column vectors, and $\sigma_{\rm CF}$ is a 2×2 matrix. Experiments do not measure $\sigma_{\rm CF}$, however. Because the CS electric field is "internal" to the CF system, it is felt by composite fermions but not detected by a voltmeter in the laboratory. The physical resistivity matrix ρ (not to be confused with the density) is given by

$$E = \rho j, \tag{5.130}$$

which can be evaluated with the help of above equations. Equation (5.128) is conveniently rewritten as

$$e_{\rm CS} = -\rho_{\rm CS} j, \tag{5.131}$$

with

$$\rho_{\rm CS} = \frac{h}{e^2}\begin{bmatrix} 0 & 2p \\ -2p & 0 \end{bmatrix}. \tag{5.132}$$

Equation (5.129) then becomes

$$E + e_{\rm CS} = \rho_{\rm CF} j, \tag{5.133}$$

with

$$\rho_{\rm CF} = (\sigma_{\rm CF})^{-1}. \tag{5.134}$$

With the help of

$$\rho = \rho_{\rm CS} + \rho_{\rm CF}, \tag{5.135}$$

the physical resistivity matrix can be obtained from the CF resistivity ($\rho_{\rm CS}$ is a fixed matrix).

The random-phase approximation (RPA) of the CS theory amounts to using the mean-field result for $\rho_{\rm CF}$. Physical resistivities can thus be obtained from those of noninteracting fermions at an effective magnetic field. Let us see how this yields the correct Hall resistance at $\nu = n/(2pn \pm 1)$. In the absence of disorder, we have (at the mean-field level)

$$\rho_{\rm CF} = \frac{h}{e^2}\begin{bmatrix} 0 & \pm\frac{1}{n} \\ \mp\frac{1}{n} & 0 \end{bmatrix}, \tag{5.136}$$

which produces, using Eqs. (5.132) and (5.135), the physical resistivity matrix

$$\rho = \frac{h}{e^2}\begin{bmatrix} 0 & \frac{2pn\pm1}{n} \\ -\frac{2pn\pm1}{n} & 0 \end{bmatrix}. \tag{5.137}$$

The point is this: At the mean-field level, the Hall resistance appears to be $R_{\rm H} = h/\nu^* e^2$. The correct value of Hall resistance, $R_{\rm H} = h/\nu e^2$, is obtained, however, once it is recognized that while composite fermions sense the aggregate of the real and (fictitious) CS electric fields, only the former is measured by a voltmeter. Consideration of the special case of $\nu = 1/2$, where $B^* = 0$, is instructive. A current of composite fermions also implies a CS flux flow, which induces a transverse CS electric field. However, for composite fermions to move straight ahead and carry current, they must not feel any transverse electric field. Therefore, a real transverse electric field must build up to cancel the effect of the internal Faraday electric field. While composite fermions sense a sum of the internal and external electric fields (which add to zero) only the latter is measured by the voltmeter, producing a nonzero Hall resistance.

Another way of understanding this result (Kirczenow and Johnson [338, 341]) is to note that Eq. (5.128) can be expressed as

$$\boldsymbol{e}_{\rm CS} = -\frac{1}{c}\boldsymbol{v} \times \boldsymbol{b}, \tag{5.138}$$

in terms of the CS magnetic field $\boldsymbol{b} = -2p\rho\phi_0\hat{\mathrm{z}}$. This produces an *average* force of

$$-e\boldsymbol{e}_{\rm CS} = \frac{e}{c}\boldsymbol{v} \times \boldsymbol{b}, \tag{5.139}$$

which exactly cancels the average force due to the CS magnetic field, $-\frac{e}{c}\boldsymbol{v} \times \boldsymbol{b}$. Thus, the *average* Lorentz force experienced by composite fermions is identical to the average Lorentz force experienced by electrons, producing the Hall resistance $R_{\rm H} = h/\nu e^2$ rather than $R_{\rm H} = h/\nu^* e^2$.[25]

The longitudinal conductivity is obtained by inverting ρ to yield

$$\sigma_{xx} = \frac{\rho_{yy}^{\rm CF}}{{\rm Det}[\rho_{\rm CF} + \rho_{\rm CS}]} \approx \rho_{yy}^{\rm CF}\left(2p\frac{h}{e^2}\right)^{-2}, \tag{5.140}$$

where we have used, in the last step, the result that, for typical parameters, $\rho_{\rm CS}$ dominates in the denominator.

We next consider finite wave vector conductivity for the CF Fermi sea at $\nu = 1/2p$. For $\rho_{xx}^{\rm CF}$ we take the resistivity for noninteracting electrons at zero magnetic field, which, for low frequency, is given by [594]

$$\begin{aligned}\rho_{yy}^{\rm CF}(q) &= \frac{q}{k_{\rm F}}\frac{h}{e^2} \quad \text{for} \quad q \gg \frac{2}{l^*} \\ &= \frac{2}{k_{\rm F} l}\frac{h}{e^2} \quad \text{for} \quad q < \frac{2}{l^*},\end{aligned} \tag{5.141}$$

where $l^* = v_{\rm F}^*\tau$ is the mean-free path due to scattering by disorder. This predicts the following wave vector dependence for the conductivity of the CF Fermi sea:

$$\begin{aligned}\sigma_{xx}(q) &= \frac{q}{k_{\rm F}}\frac{e^2}{(2p)^2 h} \quad \text{for} \quad q \gg \frac{2}{l^*} \\ &= \frac{2}{k_{\rm F} l}\frac{e^2}{(2p)^2 h} \quad \text{for} \quad q < \frac{2}{l^*}.\end{aligned} \tag{5.142}$$

Electromagnetic response function For $\omega \ll q v_{\rm F}$ the density response function $K_{00}(q,\omega)$ is given by

$$[K_{00}(q,\omega)]^{-1} = \frac{2\pi}{m^*}\left(1 + \frac{p^2}{3}\right) + V(q) + {\rm i}\left(\frac{4\pi p}{q}\right)^2\left(\frac{2\hbar\rho}{m^*}\right)\frac{\omega}{q v_{\rm F}}, \tag{5.143}$$

[25] The cancellation applies [338, 341] only to the slow guiding center drift of composite fermions, but not to the cyclotron orbit motion. Consequently, the latter is determined by the effective magnetic field, as confirmed experimentally (Section 10.1).

where $V(q)$ is the Fourier transform of the interaction. The static compressibility is given by $K_{00}(q \to 0, \omega = 0)$; it is nonzero for short-ranged interactions (e.g., for a delta function interaction, for which $V(q) =$ constant), but vanishes as q for the Coulomb interaction. For nonzero ω, $K_{00}(q, \omega)$ has a pole at an imaginary frequency

$$\omega \propto \mathrm{i} q^3 V(q), \tag{5.144}$$

which is referred to as the "overdamped mode." For Coulomb interaction it behaves as $\omega \sim q^2$, i.e., is "diffusive" (the diffusion equation is given by $\partial\psi/\partial t \sim \nabla^2\psi$).

CF mass divergence The mass of a quasiparticle in a condensed matter background is renormalized due to its interaction with other quasiparticles. Mass renormalization can be obtained from a calculation of its self energy. For composite fermions, the most dominant contribution at low energies comes from the transverse gauge field propagator, which is proportional to K_{00} and has a pole at the overdamped mode. That results in a logarithmic divergence to the CF mass at $\nu = 1/2p$:

$$m^* \sim \ln \omega. \tag{5.145}$$

Because the divergence is weak, one assumes that it does not invalidate the Fermi liquid starting point. The low-temperature specific heat of an ordinary Landau Fermi liquid is proportional to m^*T; for the CF Fermi sea, this gives

$$C_v \sim T \ln T. \tag{5.146}$$

Away from $\nu = 1/2$, say at $\nu^* = n$ for large n, one might expect that the singular behavior of the self energy would be cut off by the gap. Curone and Stamp [103] calculate the self energy perturbatively within the CFCS approach to find an unphysical vanishing of the quasiparticle spectral weight at the Λ level energies. They take this to mean either that a nonperturbative calculation is needed, or that the system is unstable.

Landau Fermi liquid modifications As a consequence of Galilean invariance, the density response function of a fully interacting system has a pole at the cyclotron energy $\hbar eB/m_{\mathrm{b}}c$, which exhausts the f-sum rule at small wave vectors (Kohn [351]). In the CFCS-RPA response, the density response function has a pole at the cyclotron frequency with proper oscillator strength provided the electron band mass is retained. The $m_{\mathrm{b}} \to m^*$ replacement results in a violation of Kohn's theorem [231]. Simon and Halperin [592] introduce a modified random phase approximation (MRPA), in the spirit of Silin's extension of the Landau Fermi liquid theory to deal with the long-range part of the interaction, to ensure that the high-frequency response is determined by m_{b}.

Simon, Stern, and Halperin [593] note that the static response to a spatially varying magnetic field is not correctly given within the MRPA framework. To see the problem, consider $B(\boldsymbol{r}) = B_{1/2} + \delta B(\boldsymbol{r})$, where $B_{1/2}$ corresponds to $\nu = 1/2$. This reduces to noninteracting composite fermions with $B^*(\boldsymbol{r}) = \delta B(\boldsymbol{r})$. From kinetic energy considerations, the CS theory would suggest density maxima at positions where $\delta B(\boldsymbol{r}) = 0$, whereas

we know from the original problem that density maxima occur at the minima of $\delta B(\boldsymbol{r})$. (A more complete treatment must solve for the density and the effective magnetic field self-consistently, because $B^*(\boldsymbol{r}) = B(\boldsymbol{r}) - 2\phi_0\rho(\boldsymbol{r})$.) Simon, Stern, and Halperin propose to fix the problem by attaching an orbital magnetization to each composite fermion (not related to its spin) which couples to the total magnetic field $B(\boldsymbol{r})$, resulting in the so-called magnetized MRPA, or M^2RPA.

The most important accomplishment of the CFCS theory is to provide description of the long-wavelength, low-energy behavior of incompressible and compressible states, especially the CF Fermi sea at $\nu = 1/2$, where its consequences are in general agreement with experiments. The CFCS approach, in particular, successfully predicts nontrivial structure in finite wave vector conductivity measured in a surface acoustic wave experiment (Fig. 10.4). It is, however, not expected to produce, from first principles, reliable numbers for various quantities (such as excitation gaps, CF masses, or the interaction between composite fermions) for incompressible FQHE states; the field theory focuses on the long-distance behavior, whereas such numbers are often governed by the short-distance physics. The Chern–Simon and the microscopic wave function approaches thus play complementary roles.

The Landau energy functional for the ordinary Landau Fermi liquids (at $B = 0$) has been justified from renormalization group considerations (Shankar [580]), which demonstrate that the terms left out are "irrelevant" in the low-energy sector. Some progress has been made toward a similar justification for the CFCS theory of the CF sea [474].

5.17 Other CF based approaches

5.17.1 Murthy–Shankar theory

There is a tension between the lowest Landau level physics, which has no knowledge of the kinetic energy for electrons, and CS flux attachment, which relies in a fundamental way on the kinetic energy term in the Hamiltonian. Therein lies the origin of the difficulty in imposing the LLL constraint and getting rid of the electron band mass in the CFCS formulation in a systematic manner.

Murthy and Shankar [462, 468, 581] propose a method to overcome this dilemma by separating inter-LL degrees of freedom that depend on the electron mass and the cyclotron energy, and intra-LL degrees of freedom that are determined by the interaction. They are motivated by the work of Bohm and Pines [32] on the electron gas in three dimensions. In that problem, the relevant objects are collective modes (plasmons) and particle–hole excitations. The plasmons are not independent objects; indeed the Hamiltonian contains no plasmons. Yet they are well-defined "particles" that can be seen as sharp resonances in scattering experiments, and to treat them as independent entities is tempting. Bohm and Pines showed that this can be done by enlarging the Hilbert space by introducing additional oscillator degrees of freedom, but at the cost of introducing certain constraints that eliminate double counting.

Murthy and Shankar (MS) treat "vortices" in an analogous manner. Like the plasmon of the three-dimensional electron gas, a vortex in the lowest Landau level is a collective

excitation of the electron system. Murthy and Shankar introduce new, charge $2p\nu$, vortex operators to enlarge the Hilbert space, and then combine them with electrons to construct a Hamiltonian in terms of composite fermions coordinates. This is accomplished as follows:[26]

The kinetic energy Hamiltonian for a single electron is given by

$$H_0 = \frac{(\boldsymbol{p} + e\boldsymbol{A})^2}{2m_\mathrm{b}} \equiv \frac{\boldsymbol{\pi}^2}{2m_\mathrm{b}} \tag{5.147}$$

where $\boldsymbol{B} = \nabla \times \boldsymbol{A} = -B\hat{\boldsymbol{z}}$, and we have taken $\hbar = 1 = c$. Define the cyclotron coordinate

$$\boldsymbol{\eta} = \ell^2 \hat{\boldsymbol{z}} \times \boldsymbol{\pi}, \tag{5.148}$$

and the guiding center coordinate

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{\eta}. \tag{5.149}$$

It can be shown that

$$[\eta_x, \eta_y] = \mathrm{i}\ell^2 \qquad [R_x, R_y] = -\mathrm{i}\ell^2, \tag{5.150}$$

$$H_0 = \frac{\boldsymbol{\eta}^2}{2m_\mathrm{b}\ell^2}, \tag{5.151}$$

and that $\boldsymbol{\eta}$ commutes with $\boldsymbol{R}$.

Let us consider now the filling $\nu = n/(2pn+1)$ which maps into n filled Λ levels of composite fermions. For each electron coordinate Murthy and Shankar introduce a new guiding center coordinate for a vortex,[27] $\boldsymbol{R}_\mathrm{v}$, defined by

$$[R_{\mathrm{v}x}, R_{\mathrm{v}y}] = \frac{\mathrm{i}\ell^2}{g^2}, \tag{5.152}$$

which represents (compare Eqs. 5.150 and 5.152) an object of charge (in units of e)

$$e_\mathrm{v} = g^2 \equiv 2p\nu = \frac{2pn}{2pn+1}, \tag{5.153}$$

as appropriate for a vortex in the $\nu = n/(2pn+1)$ state. We next combine $\boldsymbol{R}_\mathrm{v}$ with $\boldsymbol{R}$ to define a new set of guiding center and cyclotron coordinates, which are identified with the CF coordinates:

$$\boldsymbol{R}^* = \frac{\boldsymbol{R} - g^2\boldsymbol{R}_\mathrm{v}}{1-g^2} \tag{5.154}$$

and

$$\boldsymbol{\eta}^* = \frac{g}{1-g^2}(\boldsymbol{R}_\mathrm{v} - \boldsymbol{R}). \tag{5.155}$$

[26] We follow closely the development and conventions of Murthy and Shankar [468] in this subsection. The first version of their theory began with the CFCS Hamiltonian, followed by a canonical transformation to separate the magnetoplasmon degrees of freedom; it could be implemented only in the long-wavelength limit, however. The more powerful later version, the one described here, builds composite fermions directly in the lowest Landau level without going through the CS route.

[27] They call it a *pseudo*vortex to stress that it is an unphysical degree of freedom; however, the pseudovortex is expected to turn into a real vortex after projection into the physical sector.

(The knowledge of $\boldsymbol{R}^*$ and $\boldsymbol{\eta}^*$ is equivalent to knowing $\boldsymbol{r}^*$ and $\boldsymbol{\pi}^*$ for composite fermions.) These definitions imply the commutators

$$[\eta_x^*, \eta_y^*] = \mathrm{i}\ell^{*2} = \frac{\mathrm{i}\ell^2}{1-g^2} = -[R_x^*, R_y^*], \tag{5.156}$$

which are the cyclotron and guiding center coordinates for an object of charge

$$e^* = -(1-g^2) = -\frac{1}{2pn+1}, \tag{5.157}$$

which is the "local charge" of a composite fermion (Section 9.3). Inverse operators are

$$\boldsymbol{R} = \boldsymbol{R}^* + \boldsymbol{\eta} g = \boldsymbol{r}^* - \frac{\ell^2}{(1+g)}\hat{z} \times \boldsymbol{\pi}^* \tag{5.158}$$

$$\boldsymbol{R}_{\mathrm{v}} = \boldsymbol{R}^* + \frac{\boldsymbol{\eta}}{g} = \boldsymbol{r}^* + \frac{\ell^2}{g(1+g)}\hat{z} \times \boldsymbol{\pi}^*, \tag{5.159}$$

where $\boldsymbol{r}^*$ and $\boldsymbol{\pi}^*$ are composite fermion coordinate and velocity operators, respectively. The form of vortex and CF operators is uniquely determined by demanding the correct charge and commutators.

Let us now turn to the many-particle problem. The LLL projected Hamiltonian for many interacting electrons is given by

$$\begin{aligned} H &= \sum_j \frac{\boldsymbol{\eta}_j^2}{2m_{\mathrm{b}}\ell^2} + \frac{1}{2}\sum_{j,k} V(q)\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_j - \boldsymbol{r}_k)} \\ &\to \frac{1}{2}\sum_{j,k} V(q)\mathrm{e}^{-q^2\ell^2/2}\mathrm{e}^{i\boldsymbol{q}\cdot(\boldsymbol{R}_j - \boldsymbol{R}_k)}. \end{aligned} \tag{5.160}$$

The constant kinetic energy has been suppressed. LLL projection of the interaction energy follows by substituting $\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{\eta}$ and then taking the expectation value within the lowest Landau level, which gives

$$\mathcal{P}_{\mathrm{LLL}}\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} = \langle \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{\eta}}\rangle_{\mathrm{LLL}}\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{R}} = \mathrm{e}^{-q^2\ell^2/4}\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{R}}. \tag{5.161}$$

Expressed in terms of composite fermion variables, the Hamiltonian takes the form

$$H_{\mathrm{MS}} = \frac{1}{2}\sum_{j,k} V(q)\mathrm{e}^{-q^2\ell^2/2}\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot[(\boldsymbol{R}_j^* - \boldsymbol{R}_k^*) + g(\boldsymbol{\eta}_j^* - \boldsymbol{\eta}_k^*)]}, \tag{5.162}$$

which is the Murthy–Shankar Hamiltonian for composite fermions. No approximation has been made so far. The advantage of this formulation is that, because of the reduced degeneracy for composite fermions implied by their commutation relations, the IQHE wave

function Φ_n, written in terms of CF coordinates $\boldsymbol{r}^*$, is a natural trial wave function. This wave function can also be shown to be a valid Hartree–Fock solution for the MS Hamiltonian (i.e., $H_{\rm MS}$ does not mix different single particle excitations of a filled Landau level state [468]). That allows variational estimates for various kinds of gaps, which are seen to be in reasonable agreement with those obtained from microscopic wave functions.

The following subtlety deserves mention. $H_{\rm MS}$ depends only on $\boldsymbol{R} = \boldsymbol{R}^* + \boldsymbol{\eta} g$ but not on $\boldsymbol{R}_{\rm v} = \boldsymbol{R}^* + \boldsymbol{\eta}/g$; it commutes with the latter. The eigenfunctions of $H_{\rm MS}$ are thus products of two factors, one dependent on $\boldsymbol{R}$ and the other on $\boldsymbol{R}_{\rm v}$. Because $\boldsymbol{R}_{\rm v}$ is unphysical, no observable may depend on it. This represents a gauge symmetry in the problem. Murthy and Shankar fix the gauge by demanding that the expectation value for "vortex density"

$$\chi(\boldsymbol{q}) = \sum_j \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{R}_{\mathrm{v}j}}, \tag{5.163}$$

which commutes with the Hamiltonian of Eq. (5.162), vanish for all states. This constraint, which projects the enlarged space with unphysical degrees of freedom back onto the "physical" sector, is difficult to implement in practice, but Murthy and Shankar have demonstrated that satisfactory answers are obtained for various observables even in the unconstrained theory.[28]

The qualitative consequences of the Murthy–Shankar theory are generally consistent with those of the Halperin–Lee–Read theory, but provide alternative justifications. The Murthy–Shankar approach has been used to estimate gaps, collective mode dispersions, spin-related phase transitions, time and temperature dependent response of the CF liquid, as well as the effect of disorder on transport properties.

5.17.2 Modified Chern–Simons approaches

Ichinose and Matsui [262–265] use methods similar to those employed in treating the possibility of spin charge separation in the so-called t–J model to decompose the composite fermion into two slave fields, a fermionic "chargon" representing charge degrees of freedom, and a bosonic field "fluxon" representing the correlation hole. This generates a dynamical auxiliary gauge field (distinct from the Chern–Simons gauge field). They identify the low-temperature phase at $\nu = 1/2$ with the "deconfined" phase, in which the particle and flux separate, with the latter undergoing Bose–Einstein condensation.

Rajaraman [534] defines, through an exact nonunitary transformation, composite fermion operators which obey canonical anticommutation rules and carry unit charge. He shows how a mean-field approximation produces the unprojected wave functions of the type in Eq. (5.31) for single- and double-layer quantum Hall systems.

[28] This may seem surprising but has precedent. Already in 1961, Baym and Kadanoff [22] showed that theories violating gauge invariance can sometimes produce valid, gauge invariant response functions.

5.17.3 Semiclassical model for composite fermion dynamics

A semiclassical model for CF dynamics has been employed by several authors (Evers *et al.* [144–146]; Fleischmann *et al.* [166]; Mirlin *et al.* [447–449]). The semiclassical equations of motion for an electron at the Fermi energy are given by [14]

$$\boldsymbol{v}(\boldsymbol{k}) = \frac{1}{\hbar}\nabla_{\mathbf{k}}\epsilon(\boldsymbol{k}), \tag{5.164}$$

$$\hbar\dot{\boldsymbol{k}} = -\frac{e}{c}\boldsymbol{v}(\boldsymbol{k}) \times \boldsymbol{B}(\boldsymbol{r}, t) - \nabla V(\boldsymbol{r}), \tag{5.165}$$

where $\epsilon(\boldsymbol{k})$ is the energy dispersion for the electron and $V(\boldsymbol{r})$ is the potential sensed by it. Analogous equations for a composite fermion at the Fermi energy are as follows (Fleischmann *et al.* [166]):

$$\boldsymbol{v}(\boldsymbol{k}) = \frac{1}{\hbar}\nabla_{\boldsymbol{k}}\epsilon^*(\boldsymbol{k}), \tag{5.166}$$

$$\hbar\dot{\boldsymbol{k}} = -\frac{e}{c}\boldsymbol{v}(\boldsymbol{k}) \times \boldsymbol{B}^*(\boldsymbol{r}, t) - \nabla V^*(\boldsymbol{r}), \tag{5.167}$$

$$B^*(\boldsymbol{r}) = B - 2p\phi_0\rho(\boldsymbol{r}), \tag{5.168}$$

$$V^*(\boldsymbol{r}) = E_{\mathrm{F}}^* - \epsilon^*(\rho(\boldsymbol{r})), \tag{5.169}$$

$$\rho = \frac{|\boldsymbol{k}|^2}{4\pi}. \tag{5.170}$$

The replacement $B \to B^*$ naturally incorporates the effect of inhomogeneity in the effective magnetic field induced by density variations. The naive expectation would be for composite fermions to see the same potential as electrons. However, that would lead to a very different density distribution for composite fermions from that for electrons at $B = 0$, as the two have unrelated Fermi energies, and the density is determined by the condition $E_{\mathrm{F}} = \epsilon(\rho(\boldsymbol{r})) + V(\boldsymbol{r})$ (and a similar relation for composite fermions). A more reasonable assumption is that the spatial density of composite fermions at $B^* \approx 0$ is the same as that of electrons at $B = 0$, which implies that they see an effective potential V^*. By introducing a nonlinear "time" variable $s(t)$ that satisfies $\mathrm{d}s/\mathrm{d}t = (1/\alpha)(\mathrm{d}\epsilon^*/\mathrm{d}\rho)|_{\rho=\rho(\boldsymbol{r}(t))}$ (the arbitrary constant α is introduced for dimensional reasons), and assuming that $\boldsymbol{v}(\boldsymbol{k})$ and $\epsilon(\boldsymbol{k})$ depend only on $|\boldsymbol{k}|$ (hence are functions of ρ), we obtain

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\alpha}{\hbar}\nabla_{\boldsymbol{k}}\rho, \tag{5.171}$$

$$\hbar\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}s} = -\frac{e}{c}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} \times \boldsymbol{B}^* + \alpha\nabla\rho. \tag{5.172}$$

These equations are independent of the CF energy dispersion $\epsilon^*(\boldsymbol{k})$, implying that the trajectories are independent of the CF mass. (We have already encountered that property for the radius of the CF cyclotron orbit.) The semiclassical approximation has been used to address nonlinear dynamics of composite fermions in nanostructures [166].

5.17.4 Boltzmann equation for composite fermions

The semiclassical Boltzmann transport equation is a powerful tool for analyzing transport phenomena. A Boltzmann equation can be written for composite fermions in close analogy to the standard Boltzmann equation for electrons. We begin by assuming a local distribution function $f(\boldsymbol{k},\boldsymbol{r},t)$, which measures the number of composite fermions in momentum state $\boldsymbol{k}$ at position $\boldsymbol{r}$ and time t. (The reason for the qualification "semiclassical" is that such a function is not strictly defined due to the Heisenberg uncertainty principle.) The current density is then given by

$$\boldsymbol{j}(\boldsymbol{r},t) = e\int \mathrm{d}^2\boldsymbol{k}\,\frac{\boldsymbol{k}}{m^*}f(\boldsymbol{k},\boldsymbol{r},t) \tag{5.173}$$

where $\boldsymbol{v} = \boldsymbol{k}/m^*$ is the velocity. We need to determine how the distribution function is affected by the presence of applied electric field and temperature gradients in the presence of various types of scatterings. Besides using m^* and B^* for mass and magnetic field, a new aspect of the CF physics is that any external potential, e.g., created by random ionized impurities or by periodic modulation in a lateral superlattice, translates into a spatially varying $B^*(\boldsymbol{r})$. The Boltzmann equation for composite fermions (Mirlin and collaborators [447, 448]; Simon and Halperin [591]; Stern and Halperin [613]; von Oppen, Stern and Halperin [662]) is given by [447]

$$\mathrm{i}(\omega - \boldsymbol{v}\cdot\boldsymbol{q})f_1 + \frac{e}{c}(\boldsymbol{v}\times\boldsymbol{B}_0^*)\cdot\nabla_{\boldsymbol{k}}f_1 + e\boldsymbol{E}\cdot\nabla_{\boldsymbol{k}}f_0 = C\{f_1\}, \tag{5.174}$$

where f_1, the deviation of the distribution function from its equilibrium value f_0, is caused by the application of an electric field $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}\exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r})$. The collision integral, $C\{f_1\}$, includes the effects of scattering by random impurities as well as fluctuations in the effective magnetic field. The results for CF transport coefficients must be transformed back into electron transport coefficients before a contact with experiment may be made.

As for ordinary metals, the validity of the Boltzmann equation for composite fermions rests upon the existence of well-defined CF "quasiparticles" close to the Fermi energy. This requires a determination of the CF-quasiparticle width, which, in turn, is given by the imaginary part of the CF self energy. In the CFCS approach, the most singular correction to the CF self energy comes from transverse gauge field fluctuations, which gives (Halperin, Lee and Read [231]; Lee and Nagaosa [380])

$$\mathrm{Re}\,\Sigma \sim \omega\ln\omega, \qquad \mathrm{Im}\,\Sigma \sim \omega \tag{5.175}$$

for the Coulomb interaction. As the imaginary part of the self energy is of the same order as the energy itself, it does not satisfy the Landau criterion for the Landau Fermi liquid, namely that the quasiparticle width ($\mathrm{Im}\,\Sigma$) vanish faster than the energy; we have an example of what is called a "marginal" Landau Fermi liquid. Nonetheless, a Boltzmann equation description can be valid, as shown by Kim, Lee, and Wen [334]. They follow the approach of Prange and Kadanoff [525], who had derived transport equations for electrons coupled to phonons and found that they have the same form as from Landau's Fermi liquid theory even when

Landau quasiparticles are not well defined (for example, at temperatures large compared to the Debye temperature). Using a nonequilibrium Green's function approach, Kim, Lee, and Wen discover that the generalized Fermi surface displacement satisfies a closed equation of motion for composite fermions, called the quantum Boltzmann equation; the smooth Fermi surface fluctuations conform to the usual Landau Fermi liquid behavior, whereas the rough ones do not. In other words, the "quasiparticle approximation" is valid for quantities that are dominated by smooth fluctuations, which include the physically relevant density–density and current–current correlation functions, but not for all quantities (e.g., the energy gap away from $\nu = 1/2$).

Parenthetically, the quasiparticle approximation is not valid for interactions that have a shorter range than Coulomb, i.e., for which $V(\boldsymbol{q}) \sim q^{-1+\eta}$ with $1 \geq \eta > 0$ ($\eta = 0$ corresponds to Coulomb). Such interactions are less effective in suppressing gauge field fluctuations, producing real and imaginary parts of the self energy that are proportional to $\omega^{2/(2+\eta)}$, implying a strong violation of the Landau criterion. For interactions shorter ranged than Coulomb, the quasiparticle approximation does not produce the same specific heat for the CF sea as that obtained from the free energy of the gauge field (Kim and Lee [335]).

The Kubo conductivity tensor can also be evaluated in the semiclassical approximation [220,552]. In a path integral representation, the Kubo expression for the conductivity can be written as a double sum over the semiclassical periodic orbits, one from each of two Green's functions. Quantum corrections to conductivity appear through the phase associated with each orbit, given by the classical action of the orbit, which can lead to quantum oscillations or other phase coherence effects as a function of the magnetic field. The semiclassical method is convenient for dealing with spatial fluctuations or modulations in the effective magnetic field, particularly for treating the "snake" trajectories of composite fermions along lines of zero effective magnetic field [75], and has been employed for calculating CF conductivity at a finite effective magnetic field, finite wave vector, and finite frequency [144–146, 166, 447–449].

5.17.5 Fermi hypernetted-chain approximation

The hypernetted-chain approximation is a widely used method for calculating the pair correlation function $g(r)$ for liquids in equilibrium [236]. The pair distribution function is expanded in powers of density to produce a cluster expansion (the Mayer–Montroll expansion), the terms of which are represented by linked irreducible diagrams. The sum of these diagrams can be converted into an integral equation. The exact equation is intractable, but can be solved self consistently by discarding a class of diagrams (those free of "nodal circles"). The resulting self-consistent integral equation is called the hypernetted-chain equation, which is accurate in the limit of low densities. It has been used to deduce $g(r)$ for Jastrow correlated wave functions for boson liquids. Ciftja and collaborators [82, 83, 84, 85, 86, 87] have developed a Fermi hypernetted-chain method to carry out approximate computations of the pair correlation function and energies (which can be obtained from

$g(r)$) for the unprojected wave functions of Eq. (5.31). The method captures the behavior qualitatively and semiquantitatively.

5.17.6 Evolution from one dimension to two dimensions

Bergholtz and Karlhede [27, 28] study the evolution of a FQHE system on a cylinder as a function of the radius of the cylinder. When the radius is very small, the circular single particle orbitals of the Landau gauge are far separated along the length of the cylinder. In that limit, at any rational filling, a crystal state is obtained with a gap to excitations. Bergholtz and Karlhede argue that at $\nu = n/(2pn \pm 1)$ this crystal evolves continuously into the bulk incompressible FQHE state as the radius of the cylinder is increased. At $\nu = 1/2$, on the other hand, it undergoes a transition into a compressible state when the radius of the cylinder is approximately equal to one magnetic length; numerical calculations make a convincing case that the transition is into a finite-size version of the CF Fermi sea. Furthermore, the $\nu = 1/2$ system at small but nonzero radii is well described by a short-range interaction model with nearest neighbor hopping [27, 28], which can be solved exactly with the help of the well-known Jordan–Wigner transformation, and produces a non-interacting Fermi sea. Understanding how this 1D Fermi sea evolves into the CF Fermi sea in the 2D limit should prove interesting.

5.17.7 Conformal field theory

A correspondence has been made between certain FQHE wave functions and certain correlation functions of chiral conformal field theory (Cristofano *et al.* [100]; Fubini [176]; Moore and Read [454]), which is very briefly outlined in this section. We use some standard results from conformal field theory without derivation. (The derivations can be found, for example, in the textbook of Di Francesco, Mathieu, and Senechal [118].)

Consider a free bosonic field in 1+1-dimensional Euclidean spacetime, with its correlator given by

$$\langle \phi(z)\phi(z')\rangle = -\ln(z - z'). \tag{5.176}$$

The so-called vertex operators are defined by

$$\mathcal{V}_\alpha(z) = \mathrm{e}^{\mathrm{i}\alpha\phi(z)}. \tag{5.177}$$

With the help of Wick's theorem, their correlators can be shown to be given by the expression

$$\left\langle \prod_i \mathcal{V}_{\alpha_i}(z_i) \right\rangle = \exp\left[-\sum_{j<k} \alpha_j\alpha_k \langle \phi(z_j)\phi(z_k)\rangle \right] = \prod_{j<k} (z_j - z_k)^{\alpha_j\alpha_k}, \tag{5.178}$$

provided the neutrality condition,

$$\sum_i \alpha_i = 0, \tag{5.179}$$

is satisfied; otherwise the correlator vanishes identically. (The vertex operators are assumed to be normal ordered, so only Wick contractions between boson fields at different z_i are considered.)

Disregarding the neutrality condition, the choice $\alpha_j = \sqrt{m}$ gives precisely the Laughlin wave function on the right hand side of Eq. (5.178). (The neutrality condition can be taken care of by introducing a uniform background charge through a term $\exp[-\mathrm{i}\sqrt{m}\int \mathrm{d}^2 z' \rho\phi(z')]$, where ρ is the charge density. This term produces $\exp(-\sum_j |z_j|^2/4)$, and an additional singular phase factor that is neglected.) Inserting the vertex operator $\mathcal{V}_{1/\sqrt{m}}(\eta)$ produces a Laughlin quasihole (Eq. 12.29) at η.

There is no fundamental reason why the correlation functions (or conformal blocks) of vertex operators in a two-dimensional Euclidean conformal field theory should bear any relation to the quantum mechanical wave functions of electrons in the lowest Landau level interacting via the Coulomb potential. Nonetheless, one can ask if some other correlation functions in conformal field theory may also qualify as legitimate FQHE wave functions. Of course, conformal field theory cannot *derive* FQHE wave functions; any ansatz wave function must be tested and confirmed in a quantum mechanical calculation.

Moore and Read [454] interpret the Pfaffian wave function of Eq. (7.11), originally obtained as a p-wave generalization of the Haldane–Rezayi wave function (Eq. E7.6), using conformal field theory. They introduce Majorana fermions (massless real fermions) χ, which satisfy $\langle \chi(z)\chi(z')\rangle = (z-z')^{-1}$. The Pfaffian wave function can then be expressed as (again suppressing the background term)

$$\left\langle \prod_j \chi(z_j) \mathrm{e}^{\mathrm{i}\sqrt{m}\phi(z_j)} \right\rangle. \tag{5.180}$$

Conformal field theory has also been extended to other FQHE states of the series $\nu = n/(2pn \pm 1)$ (Flohr [167]; Flohr and Osterloh [168]; Hansson *et al.* [239]). Hansson *et al.* have constructed conformal field theory representations for the quasiparticle of $\nu = 1/m$ (Eq. 12.37), as well as for the FQHE states at $\nu = n/(2pn+1)$ (Eq. 5.32). For example, defining

$$\mathcal{V}^{(0)}(z) = \mathrm{e}^{\mathrm{i}\sqrt{3}\phi_0(z)} \qquad \text{and} \qquad \mathcal{V}^{(1)}(z) = \partial_z\, \mathrm{e}^{\mathrm{i}\frac{2}{\sqrt{3}}\phi_0(z)}\, \mathrm{e}^{\mathrm{i}\sqrt{\frac{5}{3}}\phi_1(z)}, \tag{5.181}$$

where ϕ_0 and ϕ_1 are two free scalar fields, the wave function for the 2/5 state is given by

$$\Psi_{2/5} = \sum_{i_1<i_2\cdots<i_n} (-1)^{\sum_k^n i_k} \left\langle \mathcal{V}^{(1)}(z_{i_1}) \ldots \mathcal{V}^{(1)}(z_{i_n}) \prod_{j\notin\{i_1\ldots i_n\}}^{N} \mathcal{V}^{(0)}(z_j) \right\rangle \tag{5.182}$$

with $n = N/2$. This exactly reproduces the projected wave function of Eq. (5.32) with n composite fermions in the second Λ level and the remaining in the lowest [239]. The vertex operators $\mathcal{V}^{(0)}$ and $\mathcal{V}^{(1)}$ create composite fermions in the lowest two Λ levels. Vertex operators corresponding to composite fermions in higher Λ levels can be constructed by introducing one additional bosonic field for each level [239].

Exercises

5.1 Going from the mean-field wave function (Eq. 5.30) to the unprojected wave function (Eq. 5.31) lowers both the interaction and the kinetic energies. When the latter is projected into the lowest Landau level however, the kinetic energy is lowered at the cost of an increase in the interaction energy. The unprojected wave function thus has lower energy than the projected one at sufficiently small cyclotron energies. Taking the numbers in Table 5.1, and parameters appropriate for GaAs, determine the magnetic fields, and also the corresponding densities, below which the unprojected wave function has lower energy than the projected one at $\nu = 2/5$, 3/7 and 4/9. What does the result mean? Does it imply a phase transition as a function of the cyclotron energy?

5.2 Show that the wave functions $\Phi_{\pm\nu^*}$ and Ψ_ν in Eq. (5.42) are confined to a disk of the same radius. (Hint: Determine the outermost occupied orbital.)

5.3 Derive Eq. (5.94).

5.4 This exercise concerns a derivation of Eq. (5.128) for the induced Chern–Simons electric field $\boldsymbol{e}_{\rm CS}$. This electric field is related, through Faraday's law, to the Chern–Simons magnetic field $\boldsymbol{b}$ as

$$\nabla \times \boldsymbol{e}_{\rm CS}(\boldsymbol{r},t) = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{b}(\boldsymbol{r},t), \tag{E5.1}$$

where

$$\boldsymbol{b}(\boldsymbol{r},t) = -2p\phi_0\rho(\boldsymbol{r},t)\hat{z}. \tag{E5.2}$$

Transform these relations, as well as the continuity equation

$$\nabla\cdot\boldsymbol{j} + (-e)\frac{\partial\rho}{\partial t} = 0, \tag{E5.3}$$

to Fourier space by substituting $\boldsymbol{e}_{\rm CS}(\boldsymbol{r},t) = \boldsymbol{e}^0_{\rm CS}\mathrm{e}^{-\mathrm{i}(qx-\omega t)}$ and analogous equations for other space and time dependent variables (the wave vector is chosen along the x direction without any loss of generality). Obtain Eq. (5.128) by eliminating ρ in favor of $\boldsymbol{j}$. Note that Eq. (5.128) is independent of $\boldsymbol{q}$ and ω, and remains valid in the static limit.

5.5 This exercise follows the mean-field physics described in Sections 5.16.3 and 7.1 to obtain the correct Hall resistance for composite fermions at $\nu^* = n$. The apparent Hall resistance is $V_{\rm apparent}/I = \pm h/ne^2$, where $V_{\rm apparent} = V_{\rm external} + V_{\rm induced}$. Show that if one assumes that each electron carries $2p$ flux quanta, then Faraday's law implies that $V_{\rm induced} = -2p\phi_0 I/ec$. Calculate $R_{\rm H} = V_{\rm external}/I$.