9

Topology and quantizations

Topology is the study of properties of geometric configurations that remain invariant under continuous deformations. It is of relevance in graph theory, network theory, and knot theory. Many physical phenomena have topological content. Perhaps the best known is the Aharonov–Bohm phase for a closed path of a charged particle around a localized magnetic flux, which is independent of the shape and the length of the path so long as it is outside the region containing the magnetic flux. It is a special case of a more general concept, known as the Berry phase. The latter is the phase acquired when the parameters of a Hamiltonian execute a closed loop in the parameter space; the Berry phase does not depend on the rate at which the loop is executed, provided it is sufficiently slow. (There is no speed limit for the Aharonov–Bohm phase. In this case the adiabatic approximation of Berry becomes exact.)

The most dramatic examples of topology lead to macroscopic quantizations, such as the quantization of flux (in units of $h/2e$; $2e$ because of pairing) through a superconducting loop or through a vortex in a type-II superconductor. The quantization of flux results from the topological property that the phase of the order parameter can only change by an integral number times 2π around the flux.

Topology enters into the physics of the fractional quantum Hall effect through the formation of composite fermions, one constituent of which, namely the vortex, is topological. The vorticity of a composite fermion ($2p$) is quantized to be an even integer due to single-valuedness and antisymmetry properties of the wave function. Such a quantization is implicit in that we *count* the number of vortices bound to the composite fermion. Because the composite fermion is a topological particle, all phenomenology of the CF quantum fluid has a topological origin. The most direct, robust and well-confirmed manifestation of the topological order of the CF state is the effective magnetic field, numerous consequences of which are discussed elsewhere in the book. This chapter focuses on three other effects arising from it: fractional "local charge" and fractional "braiding statistics" of the FQHE quasiparticles, and the fractional quantization of Hall resistance.

9.1 Charge charge, statistics statistics

Some confusion exists in the literature regarding the issues of charge and statistics in the FQHE. For example, the oft-heard phrase, "FQHE quasiparticles have fractional charge and obey fractional statistics," appears, on the face of it, inconsistent with the fact that they are excited composite *fermions* (Section 8.5). The confusion arises because, in reality, *two* distinct kinds of charges ("intrinsic" and "local") and statistics ("exchange" and "braiding") have been considered in the literature, but often without explicit qualification. Which one is relevant depends on the experimental measurement in question.

One aspect in which the two charges and statistics differ is in the choice of "vacuum," i.e., the reference state, relative to which they are defined. The two natural choices for vacuum are:

- the "null" vacuum: the state containing *no* particles,
- the FQHE vacuum: a (nearby) FQHE state with uniform density.

The charge depends on which vacuum is chosen as the reference state. For example, let us ask how many particles we have in the incompressible ground state at $\nu = n/(2pn + 1)$. From the second perspective the state has no particles, being itself the vacuum. But from the perspective of the null vacuum, it is teeming with composite fermions. A state at a filling factor ν in the range

$$\frac{n+1}{2n+3} > \nu > \frac{n}{2n+1}\,, \qquad \text{i.e.,}\;\; n+1 > \nu^* > n \tag{9.1}$$

can be viewed with reference to (i) the null vacuum, (ii) the $\nu = n/(2n + 1)$ FQHE state, or (iii) the $\nu = (n + 1)/(2n + 3)$ FQHE state, which all result in apparently different descriptions.

Furthermore, the concept of "braiding statistics" is distinct from the ordinary exchange statistics that we encounter in our elementary quantum mechanics course, as explained in more detail in Section 9.8.1.

9.2 Intrinsic charge and exchange statistics of composite fermions

"Intrinsic charge" and the ordinary "exchange statistics" are defined relative to the null vacuum containing no particles. The number of composite fermions is equal to the number of electrons. Increasing the number of composite fermions by one unit amounts to adding a net charge $-e$ to the system. This gives the "intrinsic charge" of composite fermions as $-e$. Furthermore, an exchange of two composite fermion coordinates in the quantum mechanical wave function produces a negative sign, because the CF coordinates are the same as the electron coordinates. Therefore, the exchange statistics of composite fermions is fermionic. That is the reason for calling them *fermions.*

This definition treats *all* composite fermions equivalently. Because a CF-quasiparticle is also a composite fermion, its intrinsic charge is $-e$ and exchange statistics fermionic. The intrinsic charge and the exchange statistics remain sharply defined quantities for composite fermions independent of whether they are in a compressible or an incompressible state. (In contrast, as seen below, the values of "local charge" and "braiding statistics" for

CF-quasiparticles or CF-quasiholes depend on the reference FQHE state, and cannot be defined for compressible states.)

9.3 Local charge

Next we take a uniform FQHE state as the vacuum and define various quantities through their deviations from it. (This is in the spirit of Laudau's Fermi liquid theory.) The deviations from the FQHE vacuum occur in the form of CF-quasiparticles or CF-quasiholes. As seen in Chapter 8, a CF-quasiparticle or a CF-quasihole is a localized defect on top of the uniform density incompressible state. We define the charge *excess* or *deficiency* associated with this defect as the "local charge" of the CF-quasiparticle or the CF-quasihole. If we draw a circle large enough to enclose fully a single CF-quasiparticle or CF-quasihole, then the local charge is equal to the total charge inside the circle minus what the charge would have been without the CF-quasiparticle or CF-quasihole inside it.

The local charge is clearly not $-e$. When an electron gets dressed into a composite fermion through capture of vortices, the vortices push nearby particles outward to create a correlation hole. The local charge is equal to the intrinsic charge of the composite fermion plus the charge deficiency associated with the correlation hole. In other words, the "vacuum" partially screens the charge. Such screening is a necessary consequence of the physics that each composite fermion sees $2p$ vortices on every other composite fermion, including the ones that we have decided to count as belonging to the vacuum.

Remarkably, the local charge of a CF-quasiparticle is a precisely quantized fraction of the electron charge, the value of which depends only on the filling factor of the background FQHE state. Several derivations of fractional charge are given below. With one exception, they use only the general principles of the CF theory but no microscopic details, as appropriate for a universal quantity.

Various authors do not always qualify the charge as "intrinsic" or "local." Which charge is being discussed should be clear from the context. Very simply: A fractional charge always refers to the local charge, and the intrinsic charge is always $-e$.

9.3.1 An electron equals 2pn + 1 CF-quasiparticles

The clearest proof of fractional charge is based on a simple counting argument. It proceeds by asking what happens to an incompressible FQHE system when an *electron* is added to it. Let us consider the ground state at $\nu = n/(2pn + 1)$ in the spherical geometry, which maps into n filled Λ levels of composite fermions. That is, the effective flux

$$Q^* = Q - p(N - 1) \tag{9.2}$$

is such that the total number of single particle states in the lowest n Λ levels is precisely N. Now add one electron to this state *without changing the real external flux Q*. The new state of $N' = N + 1$ particles corresponds to a modified effective flux

$$Q'^* = Q - p(N' - 1) \qquad (9.3)$$
$$= Q^* - p\,.$$

At Q'^* the degeneracy of each Λ level is reduced by $2p$ compared to Q^*. Thus $2p$ composite fermions from each of the n Λ levels must be pushed up to the $(n+1)$th Λ level. Including the added particle, the $(n+1)$th Λ level has $2pn+1$ CF-quasiparticles. These are identical, can move independently, and can be removed far from one another. Since a net charge $-e$ was added, each CF-quasiparticle must carry an excess charge of precisely

$$-e^* = -\frac{e}{2pn+1}\,. \qquad (9.4)$$

In short, the conversion of the additional electron into a composite fermion creates a correlation hole around the electron, which forces $2pn$ additional CF-quasiparticles out of the FQHE vacuum (Fig. 9.1). When viewed in terms of electrons, the system rearranges itself in a fantastically complicated manner under the addition (or removal) of electrons.

Although we do not need to use the actual wave functions, the underlying microscopic theory gives credence to the derivation, because it tells us that the words and concepts we are using are not pure fiction. It must be stressed, however, that the accuracy of the charge is not related to the accuracy of the microscopic wave functions. So long as the number of CF-quasiparticles created upon the insertion of a single electron is $2pn + 1$, Eq. (9.4) follows exactly, independent of the accuracy of the wave function. The actual wave function depends on the interaction between electrons or the amount of LL mixing, but e^* remains unaffected as long as the CF physics is valid. That is the reason why this counting argument provides the most powerful and convincing derivation of the fractional charge. In particular, the result does not depend on whether or not the wave function is projected into the lowest Landau level, although the derivation is certainly valid for the projected wave function of Eq. (5.32).

The local charge of a CF-quasihole can be obtained similarly (Exercise 9.1). Alternatively, we may deduce its charge to be $+e/(2pn+1)$ by noting that a pair of CF-quasiparticle and CF-quasihole has no net charge, because it can be created without changing the values of N and $2Q$.

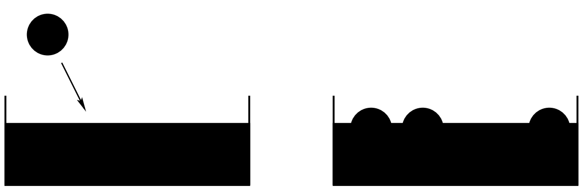

Fig. 9.1. Adding an electron to a FQHE state excites several CF-quasiparticles out of the FQHE state.

9.3.2 A CF-quasiparticle equals an electron + 2p vortices

A vortex at $\eta = R\mathrm{e}^{-\mathrm{i}\theta}$ is defined by the wave function

$$\Psi_\eta = N_R \prod_j (z_j - \eta)\Psi \ . \tag{9.5}$$

Avoidance of the position η creates a charge deficiency there. The amount of the charge deficiency can be determined following Arovas, Schrieffer, and Wilczek [13]. The Berry phase associated with a closed loop (for counterclockwise traversal) of the vortex is calculated in Section 5.11.4, where we see that each particle inside the loop contributes 2π to the Berry phase, giving a total phase of $2\pi N_{\mathrm{enc}}$, where N_{enc} is the number of particles inside the loop. Equating it to the Aharonov–Bohm phase of a particle of charge e_{V} gives

$$\begin{aligned} 2\pi N_{\mathrm{enc}} &= 2\pi \frac{BA\, e_{\mathrm{V}}}{hc} \\ &= 2\pi \frac{N_{\mathrm{enc}}}{\nu} \frac{e_{\mathrm{V}}}{e} \ , \end{aligned} \tag{9.6}$$

where we have used $\nu = N_{\mathrm{enc}}\phi_0/BA$ (BA/ϕ_0 is the number of single particle states in area A, and neglected order-one corrections. The charge of the vortex therefore is

$$e_{\mathrm{V}} = \nu e \ . \tag{9.7}$$

The local charge of a CF-quasiparticle at $\nu = n/(2pn+1)$ can now be ascertained by noting that it is the bound state of an electron and $2p$ vortices:

$$-e^* = -e + 2pe_{\mathrm{V}} = -\frac{e}{2pn+1} \ . \tag{9.8}$$

The notion of binding of electrons and vortices is transparent in the unprojected wave functions of Eq. (5.31), but becomes obscure after projection into the lowest Landau level. It is expected, however, that while the act of LLL projection (or any other perturbation) causes a rearrangement of the charge within a CF-quasiparticle, it does not create any *new* CF-quasiparticles, and hence leaves the local charge intact.

9.3.3 A vortex equals n CF-quasiholes

Further insight into the meaning of a vortex can be gained through the CF theory. For illustration, consider a vortex at the origin in the $\nu = 2/5$ state. Its wave function, disregarding LLL projection for now, is given by

$$\begin{aligned} \Psi^{(2/5)}_{\eta=0} &= N_0 \left(\prod_j z_j\right) \Psi_{2/5} \\ &= N_0 \prod_{k<l} (z_j - z_l)^2 \left(\prod_j z_j\right) \Phi_2 \ , \end{aligned} \tag{9.9}$$

where Φ_2 is the wave function of two filled Landau levels, given explicitly in Eq. (3.51). The above equation indicates that the vortex at 2/5 is related to the vortex at filling factor two. Referring to Eq. (3.51), the latter has the wave function

$$\left(\prod_j z_j\right)\Phi_2 = \begin{vmatrix} z_1 & z_2 & z_3 & \cdot & \cdot \\ z_1^2 & z_2^2 & z_3^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z_1^{N/2} & z_2^{N/2} & z_3^{N/2} & \cdot & \cdot \\ \bar{z}_1 z_1 & \bar{z}_2 z_2 & \bar{z}_3 z_3 & \cdot & \cdot \\ \bar{z}_1 z_1^2 & \bar{z}_2 z_2^2 & \bar{z}_3 z_3^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar{z}_1 z_1^{N/2} & \bar{z}_2 z_2^{N/2} & \bar{z}_3 z_3^{N/2} & \cdot & \cdot \end{vmatrix} \exp\left[-\frac{1}{4}\sum_i |z_i|^2\right] , \tag{9.10}$$

which has two holes at the origin, one in each of the two occupied Landau levels. The vortex $\Psi_{\eta=0}^{(2/5)}$ is thus not a fundamental excitation but a combination of two CF-quasiholes at the origin, one in each Λ level. Analogously, a vortex at η in the $n/(2pn \pm 1)$ state is a collection of n CF-quasiholes at η. The n CF-quasiholes, which happen to be coincident in the vortex, can split apart and move independently (their wave function is complicated but can, in principle, be written down with the help of the CF theory). The charge of a single CF-quasihole, therefore, is

$$e^* = \frac{e_{\rm V}}{n} = \frac{e}{2pn \pm 1} . \tag{9.11}$$

While this derivation is presented in terms of the unprojected wave function, the number of CF-quasiholes required to make one vortex, and hence also the charge of a single CF-quasihole, is expected to be robust to mild perturbations.

9.3.4 Adiabatic flux insertion

This subsection uses only one consequence of the CF theory, namely incompressibility at certain fractional fillings, to demonstrate fractionally charged excitations. This is how Laughlin first demonstrated fractional charge [369]. With certain additional plausible assumptions, the value of the *elementary* fractional charge is determined. This derivation has the advantage of illustrating that fractional charge is a general consequence of incompressibility, but it is too general to provide a microscopic or intuitive understanding of what the fractionally charged objects really are.

Let us assume that the state at ν is incompressible. An excitation of this state can be obtained by Laughlin's trick of adiabatic flux insertion [369] as follows. We consider, in addition to the uniform magnetic field, a point flux of strength $\alpha\phi_0$ threading the plane at the origin in the normal direction, produced by the vector potential (Appendix C)

$$\boldsymbol{A}_\alpha = \frac{\alpha}{2\pi}\phi_0 \boldsymbol{\nabla}\theta . \tag{9.12}$$

Beginning with the *exact* (though unknown) ground state at $\alpha\phi_0 = 0$, we change the flux slowly from 0 to ϕ_0 adiabatically. The ground state evolves into a new state during this process. At the end of the process, the flux is equal to ϕ_0, which can be gauged away to obtain a new exact eigenstate of the Hamiltonian. This new state has a charge deficiency or excess at the origin, which we denote by e_ϕ.

Keeping track of how the interacting ground state evolves under the above process is, in practice, impossible, because that would require the knowledge of the exact ground state at each possible value of α. Nevertheless, e_ϕ can be determined as follows: As shown in Appendix D, when the flux is adiabatically changed from 0 to ϕ_0, each single particle state moves to the next one. In the presence of a gap, each orbital must carry its charge with it. Far from the insertion point the behavior is trivial: the amount of charge that moves radially outward is precisely equal to the charge contained in one single particle orbital, i.e., νe for the incompressible state at ν. This is equal to the charge excess or deficiency at the origin:

$$e_\phi = \nu e \ . \tag{9.13}$$

The presence of a gap is crucial to this argument; while each single particle orbital always moves to the next one, only for gapped states does it carry its charge along.

A related derivation of the charge uses Faraday's law:

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\phi}{\mathrm{d}t} \ , \tag{9.14}$$

which gives

$$E_\phi = -\frac{1}{2\pi r}\frac{\mathrm{d}\phi}{\mathrm{d}t} \ . \tag{9.15}$$

The current density is $j_r = \sigma_{\mathrm{H}} E_\phi$, where $\sigma_{\mathrm{H}} = \nu e^2/h$ is the Hall conductivity. The charge leaving the area defined by a circle of radius r per unit time is $2\pi r j_r$. The total charge leaving this area in the adiabatic process is

$$Q = \int 2\pi r j_r \,\mathrm{d}t = -\sigma_{\mathrm{H}}\phi_0 = -\nu e. \tag{9.16}$$

This yields for the charge of the excitation

$$e_\phi = \nu e. \tag{9.17}$$

Given that the object produced by adiabatic flux insertion has the same charge as a vortex, it is tempting to equate the two. This is precisely what was done by Laughlin [369]. The correspondence is not microscopically exact, however, and is not needed for the purpose of determining the charge.

The flux insertion process is guaranteed to produce an exact eigenstate. Because e_ϕ is fractional, the above argument establishes a fractionally charged excitation. However, this excitation can in general be a combination of several elementary excitations, called quasiholes (we pretend ignorance of composite fermions in this section). We obtain the charge e^* of the quasihole following an argument put forth by Su [629]. Let us specialize to $\nu = n/(2pn \pm 1)$ and assume that e_ϕ is a collection of an integral number (r_1) of elementary excitations. In other words, we have

$$e_\phi = \frac{n}{2pn \pm 1} e = r_1 e^* . \tag{9.18}$$

As an additional input, let us ask what happens if an electron is removed from the system. That would create, in general, several quasiholes. Assuming only one kind of quasihole, an integral number (r_2) of them must have the same total charge as one missing electron:

$$e = r_2 e^* . \tag{9.19}$$

That gives

$$\frac{r_1}{r_2} = \frac{n}{2pn \pm 1} . \tag{9.20}$$

Because the numerator and the denominator of the filling factor are relatively prime, the only solution is $r_1 = rn$ and $r_2 = r(2pn \pm 1)$, where r is an arbitrary integer, giving $e^* = e/[r(2pn \pm 1)]$. If we further assume that the elementary quasihole has the largest charge allowed by above constraints, we recover the familiar result $e^* = e/(2pn \pm 1)$.

Thus, the presence of a gap at a fractional filling factor not only implies fractional charge, but is a sufficiently powerful restriction to fix the value of the charge uniquely, provided certain plausible assumptions are made. This derivation emphasizes that the fractional charge has nothing to do with the microscopic origin of the FQHE, but is a consequence of incompressibility in a partially filled Landau level. It also provides additional insight into why the value of e^* is robust, insensitive to the detailed form of the wave function. Further, the value of charge is closely related to the filling factor, and it is no accident that $e^* = e/(2pn + 1)$ at $\nu = n/(2pn + 1)$.

The correctness of the above assumptions is established by the CF theory, which confirms that there is only one kind of elementary excitations (CF-quasiparticles), and that $e/(2pn + 1)$ is indeed the smallest charge (otherwise, the CF theory would not give a complete description of the low-energy spectrum). The CF theory further tells us that quasiparticles and quasiholes really are CF-quasiparticles (electrons dressed by vortices) and CF-quasiholes (their absence), and that the "vortex" state produced by adiabatic flux insertion is, in general, a highly excited state containing one CF-quasihole in each occupied Λ level.

9.3.5 Direct calculation from wave function

The excess charge can be obtained directly from the wave functions of Eqs. (5.60) and (5.61), by evaluating the integral

$$-e^* = -e \int d^2\boldsymbol{r} [\rho(\boldsymbol{r}) - \rho] , \tag{9.21}$$

where $\rho(\boldsymbol{r})$ is the density in the presence of the CF-quasiparticle and ρ is the density of the incompressible state. Figures 8.4 and 8.5 show the charge densities associated with CF-quasiparticles and CF-quasiholes at $\nu = 1/3$, 2/5, and 3/7. The above integral has been performed and gives the expected fractional charge within numerical error.

Strictly speaking, this is not a derivation of e^*, but rather a check on the consistency of the trial wave function. This method does not make it obvious that the local charge is robust against perturbations that modify the wave functions. The counting arguments presented in the previous subsections are more satisfactory for demonstrating the sharp fractional quantization of the local charge.

The above proofs are generally applicable to the quasiparticle or quasihole of any FQHE state at $\nu = n/(2pn \pm 1)$. The quasihole at a Laughlin fraction is a simple vortex [369], the charge of which was deduced by Laughlin by mapping it into a two-dimensional one-component plasma (Section 12.4), and by Arovas, Schrieffer, and Wilczek [13] by the Berry phase calculation described above. These methods are special to the quasihole at $\nu = 1/(2p+1)$.

As seen in all of the derivations, the fractional quantization of the local charge is critically dependent on the incompressibility of the state. For compressible states of composite fermions, the local charge of composite fermions is not a meaningful quantum number, although their intrinsic charge continues to be sharply defined.

9.3.6 A quasihole $\neq$ a vortex; a quasiparticle $\neq$ an antivortex

It is sometimes stated, erroneously, that quasiholes and quasiparticles are vortices and antivortices (and the neutral excitations are vortex–antivortex pairs). The misconception arises because the first excitation to be understood (Laughlin [369]) was the quasihole at a Laughlin fraction, which happens to be a vortex (Section 12.5). The Laughlin quasihole is an exception, however. Other quasiholes and quasiparticles are not simple vortices and antivortices but have a much more intricate structure. They are to be understood in terms of composite fermions.

9.3.7 Quasiparticle size: a crude estimate

A reliable determination of the size of a CF-quasiparticle or a CF-quasihole requires a detailed calculation of its density profile. A crude estimate can be obtained by assuming that it spreads itself out into a disk of uniform charge density. Consider an island of $\nu_{n+1} = (n+1)/(2pn+2p+1)$ state in the background of the $\nu_n = n/(2pn+1)$ state. When we add a composite fermion to the island, the excess charge in the island increases by $e^*/e = 1/(2pn+1)$, increasing the size of the ν_{n+1} island. From the result that the excess charge per flux quantum is $\nu_{n+1} - \nu_n$, it follows that the island size increases by an area that encloses

$$\frac{(e^*/e)}{\nu_{n+1} - \nu_n} = 2pn + 2p + 1 \tag{9.22}$$

flux quanta. Identifying that increase with the size of a single CF-quasiparticle, πa^2, its radius is given as

$$a = \sqrt{2(2pn + 2p + 1)}\ \ell\ . \tag{9.23}$$

The area enclosing $2pn+2p+1$ flux quanta contains $(2pn+2p+1)\nu_{n+1} = n+1$ electrons, which gives a rough estimate for the number of electrons in the region containing a single CF-quasiparticle. Detailed calculation shows that the quasiparticle size obtained from this simple argument is an underestimate.

9.3.8 Charge addition to an inhomogeneous FQHE state

We saw (Fig. 9.1) that the addition of an electron to a uniform density compact state with a fixed area (for example, a FQHE state on a sphere) results in the creation of many CF-quasiparticles. For an inhomogeneous state, consisting of regions of several fillings and edges, the situation is somewhat more complicated, because charge can also accumulate at the boundaries. From the null vacuum perspective, adding a unit charge is equivalent to adding a composite fermion, which, however, causes nonlocal changes in the electronic state; the nonlocality arises because the addition of a composite fermion also involves the addition of $2p$ vortices. We illustrate this with two examples.

We first consider a $1/3$ droplet with an edge, denoted by $[N_0]$. (We use the notation $[N_0, N_1, \ldots]$ where N_j denotes the number of composite fermions in the jth Λ level.) Adding a composite fermion to the edge of the lowest Λ level simply expands the size of the $1/3$ state by three flux quanta to give $[N_0 + 1]$. If the new composite fermion is placed in the second Λ level in the interior of the sample, the resulting state $[N_0, 1]$ has an excess charge of $1/3$ in the interior and $2/3$ at the edge. The latter arises because the addition of two vortices pushes the edge out by an area enclosing two flux quanta. The net additional charge, of course, is one unit.

Let us next take an island of 2/5 state surrounded by the $1/3$ state, denoted by $[N_0, N_1]$. Adding a composite fermion at the boundary of the island produces $[N_0, N_1 + 1]$, which amounts to an excess charge of $1/3$ at the edge of the 2/5 island (increasing its area by $5\phi_0$), and of $2/3$ to the outer edge of the $1/3$ island. What happens when the new composite fermion is placed in the third Λ level, producing the state $[N_0, N_1, 1]$? Naively, there seems to be an ambiguity. From the perspective of the $1/3$ vacuum, the excess charge of the CF-quasiparticle is $1/3$, but from the perspective of the 2/5 vacuum the charge excess is $1/5$. Which one is correct? Both are. The excess charge sticking out of the 2/5 state is indeed 1/5. However, the 2/5 island itself also expands by an area that encloses two more flux quanta, which produces an additional excess charge of 2/15 at the edge of the island (using results from the previous subsection), giving a total excess charge of $1/3$ relative to the $1/3$ vacuum. Of course, the larger $1/3$ island also expands by two flux quanta, equivalent to a charge of $2/3$, consistent with a unit increase in the total number of electrons relative to the null vacuum. As shown in Fig. 9.2, in the transition $[N_0, N_1] \rightarrow [N_0, N_1, 1]$, which is rather simply stated in the CF description, the additional charge distributes itself in three

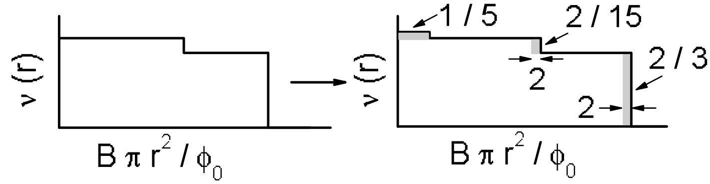

Fig. 9.2. The figure on the left shows a 2/5 island on top of a $1/3$ state. This state is denoted as $[N_0, N_1]$, with the two entries being the number of composite fermions in the lowest two Λ levels. The figure on the right shows how the state changes when a composite fermion is added at the center of the 2/5 island (which corresponds to the state $[N_0, N_1, 1]$). The y-axis shows the local filling factor at a distance r from the center; the x-axis is the flux enclosed by the disk of radius r. The charge of the added composite fermion appears in three places, indicated by the shaded regions, with the charges in each region being 1/5, 2/15, and $2/3$, as shown on the figure. Both the 2/5 and the $1/3$ islands expand to enclose two more flux quanta. The charge density fluctuations at the edges as well as for the charge 1/5 quasiparticle have been suppressed for simplicity.

places. This example also demonstrates how the charge of a CF-quasiparticle depends on the choice of the vacuum state.

9.3.9 "Electron fractionalization"

Treating a uniform density FQHE state as the vacuum, it appears as though an added electron "fractionalizes" into several fractionally charged CF-quasiparticles, as shown in Figs. 9.1 and 9.2. The fractional quantum Hall liquid is the prototypical example of a "fractionalized" quantum fluid. Of course, the system is still made up of integrally charged objects (by shifting to the null vacuum, one can describe the physics perfectly well in terms of unit charge composite fermions plus a uniform neutralizing background); it is just that the charge density is nonuniform, and the nonuniformity manifests through several localized, fractionally charged, charge deficiencies or excesses relative to the uniform density ground state. While the fractional charge in the FQHE is not as revolutionary as a real breaking up of an electron would be, it does constitute an example where nature produces a sharply defined, fractional value for excess charge. (An individual atom inside a molecule also has a fraction of an electron charge on it, but the charge is not sharply quantized.)

9.4 Quantized screening

The binding of vortices to an electron creates a correlation hole around it, and therefore can be viewed as a screening of the electron (Goldhaber and Jain [201]). The composite fermion can thus be thought of as a screened electron. The word screening, however, is used here in a somewhat nonstandard sense. Many familiar examples of screening, such as that occurring in a conventional insulating medium, are perturbative, and often investigated theoretically in perturbative treatments. The screening of an electron into a composite fermion is a fundamentally nonperturbative effect. It occurs through the binding of *exactly* an even number of quantized vortices to each electron. Being quantized, vortices cannot be attached or removed adiabatically (akin to our inability to untie a knot while holding the ends of a rope fixed). That is the reason why the local charge of the CF-quasiparticle, the sum of the electron charge and the charge of the correlation hole, is precisely quantized, the value of which depends only on the filling factor, but not on details of interaction or wave function.

Quantization of screening is not unprecedented. The nonperturbative screening of a charge in a superconductor is also quantized, producing a local charge that is exactly zero. The quantization of screening in the composite fermion fluid, however, is the first example in physics where sharp, nonzero values for the local charge are produced.

9.5 Fractionally quantized Hall resistance

We saw in Chapter 4 that gaps at integral fillings combined with Anderson localization result in quantized Hall resistance plateaus at $R_{\rm H}=h/ne^2$. The gap at $\nu=f$ in a pure system similarly produces quantized plateaus at $R_{\rm H}=h/fe^2$ when a weak disorder potential is introduced. (A strong disorder will destroy the gap and the FQHE.) The explanations of Chapter 4 carry over to the FQHE problem almost verbatim, with "electron" replaced by "composite fermion" and "Landau level" by "Λ level." We go over the basic ideas briefly.

9.5.1 Plateaus as a consequence of incompressibility

Let us assume that we know, from other considerations, that the state at $\nu=f$ has a gap in the absence of disorder. We consider Fig. 4.6 in which ideal leads are attached on both ends of the disordered sample. The chemical potential is adjusted to lie in the bulk gap of the ideal regions, so the filling factor in these regions is $\nu=f$. The Hall resistance in the ideal regions is

$$R_{\rm H}=\frac{h}{fe^2}\ . \tag{9.24}$$

In the disordered part the filling factor is different from $\nu=f$ due to the presence of disorder-induced localized defects. (We do not need to know that the defects are fractionally charged quasiparticles or quasiholes.) We assume that: (i) all defects are localized; (ii) the edge states are extended along the entire length of the sample; and (iii) backscattering is absent. These assumptions guarantee that the current is the same everywhere, as also is the Hall voltage. Consequently, the Hall resistance has the quantized value given in Eq. (9.24) also for the disordered region.

This explanation demonstrates that the fractional Hall quantization at $R_{\rm H}=h/fe^2$ is a direct consequence of incompressibility at the fractional filling $\nu = f$, combined with disorder-induced Anderson localization. The only role of composite fermions is to produce incompressibility at a fractional filling.

No use is made above of the fractional local charge of a quasiparticle or quasihole. The quantized Hall resistance is a property of the incompressible "condensate"; the quasiparticles are localized and do not contribute to transport.

9.5.2 Laughlin's explanation

Referring to Fig. 4.12, let us extend the ideal region at the inner edge all the way to the origin. The azimuthal current is given by

$$I = c\frac{\mathrm{d}U(\phi_\mathrm{t})}{\mathrm{d}\phi_\mathrm{t}} = c\frac{\Delta U}{\phi_0}\ , \tag{9.25}$$

where ϕ_t is an additional test point flux piercing the origin, U is the energy of the system, and ΔU is the change in the energy for $\Delta\phi_\mathrm{t} = \phi_0$.

It was shown in Section 9.3.4 that piercing an ideal FQHE state adiabatically by one flux quantum produces an object of charge $e_\phi = fe$, where f is the filling factor of the *ideal* region. This charge comes in from the edge of the system. Although now the ideal region is connected to a disordered sample, the extended states (that go around the origin) are everywhere gapped, and therefore the charge e_ϕ must still come from the outer edge. In other words, a charge e_ϕ has moved from the outer edge to the inner edge as the flux is varied adiabatically from 0 to ϕ_0, implying $\Delta U = e_\phi V_\mathrm{H}$. Thus,

$$R_\mathrm{H} = \frac{V_\mathrm{H}}{I} = \frac{h}{e_\phi e} = \frac{h}{fe^2}\ . \tag{9.26}$$

This argument again did not require anything beyond the incompressibility of the state at $\nu = f$. In particular, we did not need to know that for $\nu = n/(2pn \pm 1)$, n composite fermions, one in each Λ level, are transported from the inner edge to the outer edge under adiabatic insertion of one flux quantum, each carrying its local charge $e^* = e/(2pn \pm 1)$ with it.

9.5.3 Exactness of the Hall quantization

Why is the Hall resistance so accurately quantized? Its value is tied to the value of the filling factor where a gap appears in the absence of disorder. That, in turn, is precisely given by $f = n/(2pn \pm 1)$, with no correction, because the right hand side contains whole numbers, and, therefore, is not susceptible to small perturbations. The topological quantization of the vorticity of composite fermions ($2p$) lies at the root of the exactness of the FQHE.

9.5.4 Anderson localization of composite fermions

Anderson localization of a single electron was considered in Section 4.6. The problem becomes notoriously complicated when both disorder and interaction are present, especially because interactions play a nonperturbative role in the FQHE problem. Some progress can be made within the CF theory. To the extent that composite fermions may be treated as noninteracting, the problem is identical to that of localization of free electrons, discussed in Section 4.6. That view [276] is consistent with experimental studies of scaling behavior in the FQHE regime. Engel *et al.* [142] ascertain that the width of the transition region for the $1/3 \to 2/5$ vanishes as $\Delta B^* \sim T^{\kappa^*}$ with $\kappa^* \approx 0.42$; Koch *et al.* [348] measure an exponent in the range $\kappa^* = 0.56 - 0.77$ for the $1 \to 2/3$ transition (which is the $\nu^* = 0$ to $\nu^* = 1$ transition of composite fermions made of holes in the lowest Landau level [277]). These exponents are of roughly the same order as those for the IQHE transitions (Section 4.6), and also nonuniversal, presumably as a result of the residual interaction between composite fermions.

9.6 Evidence for fractional local charge

The appearance of e_ϕ in Eq. (9.26) in Laughlin's explanation of Hall quantization might suggest that the fractional Hall plateau itself is a measurement of, or proof for, the fractional quantization of local charge. But the fractionally charged quasiparticles, being *localized* on the quantum Hall plateau, do not contribute to transport. The Hall current is carried by the background incompressible state containing no quasiparticles or quasiholes. Both the fractional local charge and the fractional quantization of the Hall resistance are consequences of incompressibility of the pure state at a fractional filling. The measurement of the fractional local charge of the quasiparticles must rely on the movement of these quasiparticles, i.e., on *deviations* from the perfect quantum Hall effect. A number of experiments have reported evidence for fractional charge [114, 172, 209, 432, 551, 565, 590].

Simmons *et al.* [590] observe quasiperiodic resistance fluctuations, near the resistance minima, in narrow Hall bars as a function of the magnetic field. These are naturally interpreted as arising from resonant tunneling through quasi-bound states localized on equipotential contours around potential hills or valleys along the Hall bar [270], created by unintentional impurity potentials. The observed quasi-period ΔB in the $\nu = 1/3$ minimum is approximately three times larger than that in the $\nu = 1, 2, 3, 4$ minima, which is taken as an indication of fractional charge. A detailed understanding of the experimental result is lacking. A ϕ_0 period follows generally from the Byers–Yang argument [46], which shows that, independent of the correlations in the electronic state, the ordering of various eigenstates does not change due to the addition of an integral number of flux quanta through a region devoid of electrons. Kivelson [343] argues that the period should be $\phi_0^* = hc/e^*$ for a fixed particle number (as opposed to a period of $\phi_0 = hc/e$ for a fixed chemical potential); the experiment can possibly be in that regime due to energetic considerations involving Coulomb blockade.

Goldman and Su [209] study resonant tunneling through a quantum antidot (a potential hill) etched into the 2D layer, both as a function of B, which alters the filling factor at a fixed density, and as a function of a backgate voltage, which changes the density at a fixed B through a capacitive coupling to the 2D electron system. The area A of the closed loop around the antidot is determined from the quasi-period ΔB, assuming that the two successive resonant tunneling peaks are separated by $A\Delta B = \phi_0$. From the knowledge of the dependence of the average charge density on the backgate voltage, it is surmised that the change in the charge in an area A between two resonant tunneling peaks (as a function of the backgate voltage) is $e^* \approx e/3$ for filling factor $\nu = 1/3$.

Franklin *et al.* [172] also observe quasi-periodic peaks associated with resonant tunneling through a quantum antidot. They conclude that the periods ΔB are the same at $\nu = 1/3$ and in the integral quantum Hall regime. This experiment, as the one in the previous paragraph, provides a demonstration of Byers and Yang theorem [46] in the FQHE regime.

Saminadayar *et al.* [565] and de Picciotto *et al.* [114] measure shot noise in the current due to tunneling from one edge to another in a narrow part of the sample (i.e., a quantum point contact), from which they deduce the charge of the tunneling objects. "Shot noise" refers to time-dependent fluctuations in the current arising from the property that the current flow is not continuous but occurs through discrete pulses in time, due to the discreteness of the particles carrying the current. (This is analogous to the noise created by raindrops falling on a tin roof.) The shot noise is conveniently characterized by the frequency-dependent noise power, $S(\omega)$, defined as $S = 2\int dt\langle\delta I(0)\delta I(t)\rangle \cos(\omega t)$, which is the Fourier transform of the correlator of the time-dependent fluctuations in the current at a given voltage and temperature. When the tunneling of particles is completely random, the noise spectral density is given by $S = 2qI$, q being the charge of the object that is tunneling. Any correlations between successive tunnelings turn it into an inequality $S \leq 2qI$. In devices like tunnel junctions, the maximum value of the noise power indeed occurs when $q = e$.

For the FQHE experiments, in the limit of $k_{\rm B}T/qV \to 0$ and weak coupling between edges (so the tunneling probability is small), one expects $S = 2qI$ for uncorrelated tunnelings of quasiparticles (i.e., composite fermions). More general expressions for S including the T and V dependences have been derived (Chamon, Freed, and Wen [59]; Fendley, Ludwig, and Saleur [153]; Kane and Fisher [318,319]) using Wen's chiral Tomonaga–Luttinger (TL) liquid model for the edge states [673] (Chapter 14). The value of effective shot-noise charge thus determined in the experiments of Saminadayar *et al.* [565] and de Picciotto *et al.* [114] is consistent with $q = e/3$ at $\nu = 1/3$. The local charge of quasiparticles is not quantized at the edge, due to the presence of massless excitations; it may be argued, however, that tunneling through the bulk filters out a sharp fractional charge.

Subsequent shot noise experiments have produced somewhat puzzling results. Chung *et al.* [80] find that at $\nu = 2/5$ and 3/7, the shot-noise charge has the expected value of approximately 1/5 or 1/7 at certain filling-factor-dependent "high" temperatures (>50 mK 2/5), but changes continuously to $q = \nu e$ at low temperatures (~ 9 mK). In the same experiment, the transmission through the quantum point contact is seen to increase with decreasing temperature for certain parameters, contrary to the expectation based on the chiral TL liquid model [319], which predicts the transmission to be completely suppressed at $T = 0$. Further observations contradicting the predictions of the chiral TL liquid model have been reported by Comforti *et al.* [93] and Chung *et al.* [81] for situations when the beam of "edge quasiparticles" impinging on the quantum point contact is very dilute, as can be achieved by passing the current through an additional weak backscatterer. For example, the former experiment [93] finds that in this situation a $1/3$ charge can tunnel through a nearly opaque barrier. These observations are not understood. Wen's TL model used in the analysis of the shot-noise charge has been questioned by experiments probing tunneling from an ordinary Landau Fermi liquid into the FQHE edge (Chapter 14). These experiments determine the TL exponent (defined in Chapter 14), which governs the long-distance, low-energy behavior of various correlation functions, to be nonuniversal and filling-factor dependent. This may have implications for the shot-noise experiments, because a single parameter in the TL model determines both the TL exponent and the shot-noise charge. A measurement of the filling-factor dependence of the shot-noise charge across the $1/3$ plateau should further confirm the interpretation of the experiments.

Martin *et al.* [432] investigate the microscopic nature of localization in the FQHE regime using a scanning single-electron transistor, which allows them to identify the charging of a localized state as a function of the overall density of the system. They observe approximately three times as many charging lines at $\nu = 1/3$ and $2/3$ as at $\nu = 1$ and 2, which is consistent with localization of fractionally charged quasiparticles.

9.7 Observations of the fermionic statistics of composite fermions

How does one measure the particle statistics? The proof of statistics stems from the inability to explain experimental results without recourse to the concept of particle statistics, both in condensed matter physics and in quantum chemistry. In the former, particle statistics announces itself loudly through the formation of collective quantum states, for example a Bose–Einstein condensate or a Fermi sea, the properties of which are impossible to understand without appealing to statistics. In the latter, the statistics plays a crucial role in quantitative predictions of the energy levels of atoms and molecules. An explanation of the energy levels of the H_2 molecule or the He atom would not be possible if we did not know that electrons are fermions, and the extremely precise agreement between theory and experiment on atomic and molecular spectra is a "quantum chemistry" proof of the fermionic statistics of electrons. It is only satisfying that a concept as novel as statistics should have decisive and dramatic manifestations.

The fermionic exchange statistics of composite fermions is firmly established through a variety of facts, both of quantum chemistry and condensed matter varieties. Listing the most prominent of these is worthwhile:

- The low-energy spectra exhibit a one-to-one correspondence with the energy levels of weakly interacting fermions. The description in terms of fermions neither misses any states at low energies nor predicts spurious states.
- The eigenenergies and eigenfunctions are predicted accurately by the CF theory.
- The FQHE is explained as the IQHE of composite fermions. The appearance of the sequences of fractions given in Eq. (5.20) is a direct consequence of the fermionic nature of composite fermions.

- The actual incompressible states are well described, quantitatively, as n filled Λ levels of composite fermions.
- The excited states are explained, in great detail, as excitations of composite fermions across Λ levels.
- The state at $\nu = 1/2$ is a Fermi sea of composite fermions.
- The FQHE state at $\nu = 5/2$ is currently believed to be a BCS-like paired state of composite fermions.

We note that the exchange statistics of neither ordinary particles nor composite fermions bears any relation to their charge. Should the reader be wondering why we bring up this rather obvious point at all, the reason is that the same is not true for the braiding statistics. The braiding statistics of the CF-quasiparticles is so intimately connected with their local charge that disentangling the two in a practical measurement is nontrivial.

9.8 Leinaas–Myrheim–Wilczek braiding statistics

Indistinguishability of particles imposes a fundamental constraint on the space of allowed quantum mechanical wave functions. They must be either even or odd under an exchange of two indistinguishable particles, giving bosonic or fermionic statistics. This property is independent of the Hamiltonian describing the particles, and the relevant symmetry group is the permutation group.

Leinaas and Myrheim [386] and Wilczek [677] (also see Goldin, Menikoff, and Sharp [202, 203]) introduced the fascinating theoretical concept of "braiding statistics" in which particle exchange is viewed as a continuous process. The corresponding symmetry group is Artin's braid group rather than the permutation group, and gives new possible structure in two space dimensions (but not in higher dimensions where the braidings can be disentangled). The Leinaas–Myrheim–Wilczek (LMW) braiding statistics is distinct from the ordinary exchange statistics in that it is an emergent consequence of interactions between the constituent particles, and it is defined through a dynamical Berry phase associated with the braiding of particles around one another. The exchange statistics of CF-quasiparticles is fermionic, but they are seen below to have a fractional braiding statistics.

Although it is not always specified in the literature which statistics – exchange or braiding – is being discussed, that should be clear from the context. Fractional statistics always refers to the braiding statistics. Particles obeying fractional braiding statistics are called "anyons" ("any" statistics).

9.8.1 General concept: ideal anyons

To illustrate the basic idea, we consider two indistinguishable particles in two space dimensions. Switching to the center of mass and relative coordinates, and denoting the latter by (r, θ), the part of the wave function depending on the azimuthal coordinate θ is

$$e^{im\theta}\ . \tag{9.27}$$

The parameter m must be an integer to ensure that the wave function is single valued. Furthermore, because an exchange of particles corresponds to $\theta \to \theta + \pi$, odd (even) integral values of m imply an antisymmetric (symmetric) wave function under exchange. The parameter m is the eigenvalue of the canonical angular momentum operator $L_\theta = -i\partial/\partial\theta$. Thus we see that the relative angular momentum quantum number is an an even integer for bosons, and an odd integer for fermions. (For simplicity, we assume that the particles are spinless; that leads to no inconsistency in nonrelativistic quantum mechanics.)

Now consider two particles for which the θ dependent part of the wave function is

$$e^{i(m+\alpha)\theta}\ , \tag{9.28}$$

where α is an arbitrary number and m is taken to be an even integer. An exchange of particles now produces a phase factor

$$e^{i\alpha\pi}\ , \tag{9.29}$$

which is neither $+1$ nor -1 but a complex quantity. Such particles are said to obey fractional statistics, i.e., are anyons. The relative angular momentum of anyons contains a fractional piece (α), although the different allowed values continue to be spaced by two units, as for bosons or fermions.

Consider a single particle on a ring. Its wave function is given by $e^{im\theta}$, where m is an integer to ensure continuity under a full traversal. Wave functions of Eq. (9.28) are solutions with "twisted" boundary conditions; they can be obtained by inserting a magnetic flux of appropriate strength through the ring. Analogously, anyons can be modeled as bosons (or fermions) each carrying a flux of an appropriate amount with it. To see how it works mathematically, consider the Schrödinger equation

$$H\Psi = E\Psi, \tag{9.30}$$

$$H = \sum_i \frac{\boldsymbol{p}_i^2}{2m_b}, \tag{9.31}$$

for noninteracting anyons of mass m_b obeying fractional braiding statistics α. This can be seen to be equivalent to the bosonic problem:

$$H_B\Psi_B = E\Psi_B, \tag{9.32}$$

$$\Psi_B = e^{-i\alpha\sum_{j<k}\theta_{jk}}\Psi = \prod_{j<k}\left[\frac{z_j - z_k}{|z_j - z_k|}\right]^\alpha \Psi, \tag{9.33}$$

$$H_B = \sum_i \frac{1}{2m_b}\left(\boldsymbol{p}_i + \frac{e}{c}\boldsymbol{a}_i\right)^2, \tag{9.34}$$

$$\boldsymbol{a}_i = \frac{\alpha\phi_0}{2\pi}\sum_j{}' \nabla_i\theta_{ij}, \tag{9.35}$$

$$\nabla_i \times \boldsymbol{a}_i = \alpha\phi_0 \sum_j \delta^{(2)}(\boldsymbol{r}_j), \tag{9.36}$$

$$\theta_{ij} = \arg(\bar{z}_i - \bar{z}_j) = -\mathrm{i}\ln\frac{\bar{z}_i - \bar{z}_j}{|\bar{z}_i - \bar{z}_j|}, \tag{9.37}$$

$$\theta_{ji} = \pi + \theta_{ij}. \tag{9.38}$$

Here, Ψ_B is a wave function for bosons, i.e., is single valued and symmetric under particle exchange. The wave function Ψ,

$$\Psi = \mathrm{e}^{+\mathrm{i}\alpha\sum_{j<k}\theta_{jk}}\Psi_\mathrm{B}, \tag{9.39}$$

satisfies the desired statistics under the exchange of particles. Each boson sees a flux tube of strength $\alpha\phi_0$ on every other boson due to the vector potential $\boldsymbol{a}(\boldsymbol{r})$. The prime on the sum denotes the condition $j \neq i$. The angle θ_{ij} is the angle that the vector $\boldsymbol{r}_i - \boldsymbol{r}_j$ makes relative to the x-axis, and is only defined modulo 2π. Recall that $z = x - \mathrm{i}y = r\mathrm{e}^{-\mathrm{i}\theta}$. We have thus shown that the problem of anyons, with nonanalytic, non-single-valued wave functions, can be recast into a more familiar language of bosons, but at the cost of introducing a singular vector potential that ties to each boson a point flux tube of appropriate strength. (This is an example of how a singular gauge transformation can shift statistics from wave function to the Hamiltonian.) These flux tubes have no mass, no independent dynamics, and do not refer to a physical magnetic field that requires currents and stores energy. Their *raison d'être* is to impose appropriate boundary conditions.

We close this subsection with the following remarks.

- Equation (9.28) implies that a full circuit of one anyon around another produces a phase factor $\mathrm{e}^{\mathrm{i}\alpha}$ *independent* of the path. Fractional braiding statistics thus entails a topological distinction between the loops in which a particle goes around another and those in which it does not (paths labeled A and B in Fig. 9.3). It can be consistently defined only in two dimensions, because, in

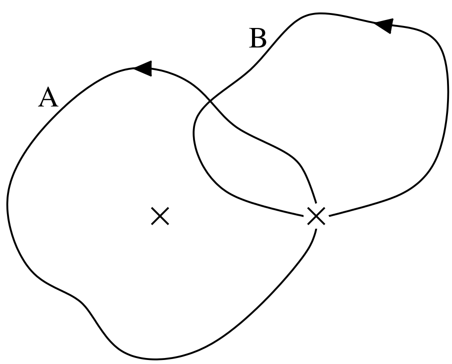

Fig. 9.3. Fractional braiding statistics implies a topological difference between classes of closed loops that enclose another particle and those that do not (marked A and B, respectively). Each cross represents a particle.

higher dimensions, one kind of loop can be continuously deformed into the other kind by peeling it off the plane of the page containing the two particles. Even in two dimensions, removal of the coincident points from the configuration space is necessary (for each particle the surface then has a complex topology with punctures at the positions of the other particles) to ensure that the loops enclosing different numbers of particles cannot be deformed into one another, i.e., are topologically distinct. This can be achieved by adding to the Hamiltonian an infinitely strong short-range repulsive interaction.

- Just because fractional braiding statistics can be consistently defined theoretically does not mean that it exists in the real world. It is an experimental fact that no elementary particles in nature have fractional braiding statistics. Why, then, do we talk of fractional braiding statistics at all? That would indeed be a valid question from an elementary particle physicist's perspective. No principle precludes, however, the possibility that certain *emergent* quasiparticles of a condensed matter system might obey fractional braiding statistics. It would obviously take a highly nontrivial state of matter to produce anyons, and the fractional quantum Hall state is currently the only viable candidate for such physics.
- A fundamental operational distinction between ordinary particles and anyons is that while we know the solution for noninteracting bosons or fermions, the problem of noninteracting anyons is nontrivial, analytically intractable, and remains unsolved. The bosonic reformulation of noninteracting anyons does not lend itself to an exact solution. The Hamiltonian $H_{\rm B}$ appears to be a sum of single particle terms due to bad notation, but it describes an interacting problem, because the vector potential depends on the coordinates of *all* particles. An expansion of the Hamiltonian

$$2m_{\rm b}H_{\rm B} = \sum_i \left[\boldsymbol{p}_i^2 + \left(\frac{e}{c}\boldsymbol{a}_i\right)^2 + \left(\frac{e}{c}\right)\boldsymbol{p}_i \cdot \boldsymbol{a}_i + \left(\frac{e}{c}\right)\boldsymbol{a}_i \cdot \boldsymbol{p}_i \right] \tag{9.40}$$

shows that the Hamiltonian contains two- and three-body interactions. The flux attachment thus produces complex interactions that cannot be treated perturbatively. Because they cannot be considered weakly interacting, anyons are not true particles in the sense defined in Section 5.7.
- Unlike the ordinary statistics, which imposes a kinematic constraint on allowed wave functions, fractional braiding statistics is an intrinsically dynamical concept. That is perhaps seen most clearly when anyons are mapped into bosons; the fractional statistics is then equivalent to a long-range interaction mediated by the gauge potential.
- The many-body theory of bosons or fermions rests crucially on the fact that the many-particle wave functions can be built from single particle wave functions. That is not true for anyons.
- Another manner in which fractional braiding statistics differs from ordinary exchange statistics is that even distinguishable particles can have a well-defined relative braiding statistics. We see an example below.

9.8.2 FQHE quasiparticles: nonideal anyons

In view of Section 9.7, the reader may be wondering why we discuss fractional braiding statistics. Everything to date is well explained in terms of composite fermions, without any mention of fractional braiding statistics.

As a motivation, let us now consider the state in which an integral number of Λ levels are full, and the lowest unoccupied Λ level contains two additional composite fermions, i.e., CF-quasiparticles. Each CF-quasiparticle has an excess of precisely $1/(2pn+1)$ electrons associated with it. We know how to describe this state in great detail; we learned in Chapter 6 how to write explicit wave functions for states containing a single or many CF-quasiparticles.

Let us now switch our vantage point from the null vacuum to the FQHE vacuum. In other words, we only wish to keep track of the *deviations* from the uniform incompressible FQHE state. We then have only *two* quasiparticles, each with a fractional local charge. This seems a reasonable thing to do. We have gone from a large number of composite fermions to two quasiparticles, and can, therefore, hope for simplification.

We cannot simply forget about the composite fermions in the FQHE vacuum, however. They must be included indirectly through the effect they have on the two quasiparticles. Composite fermions in the filled Λ levels mediate complicated interactions between the two quasiparticles due to the strongly coupled nature of the state; after all, each composite fermion sees $2p$ vortices on all other composite fermions, independent of which Λ level they inhabit, and of whether or not they belong to what we have decided to call vacuum. We now demonstrate that the *long-distance* character of this induced interaction is precisely equivalent to fractional braiding statistics.

In our preceding discussion of braiding statistics, we assumed charge neutral particles. CF-quasiparticles are electrically charged objects in a magnetic field. As a result, a closed loop has two contributions to the phase: the Aharonov–Bohm phase (which occurs even when the loop does not enclose another quasiparticle), and the braiding phase (due to the enclosed quasiparticles). The latter is deduced by subtracting the former from the total phase (Fig 9.4). The braiding statistics is thus given by the *change* in the Berry phase when a new CF-quasiparticle is inserted inside a closed loop.

The appearance of fractional braiding statistics in the fractional quantum Hall effect was first suspected by Halperin [230], and demonstrated in a microscopic calculation for the

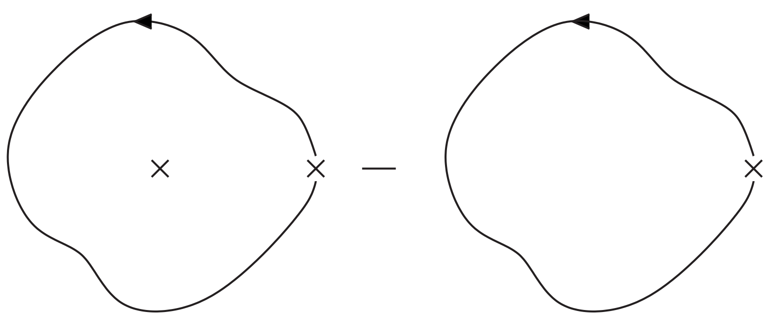

Fig. 9.4. The phase from braiding statistics is given by the change in the phase associated with a closed loop caused by the addition of a quasiparticle inside it.

quasiholes of $\nu = 1/m$ by Arovas, Schrieffer, and Wilczek [13] (ASW). It can be deduced as a general consequence of incompressibility at a fractional filling using the same kind of arguments as those leading to fractional charge [629].

The CF theory provides a simple and general derivation for fractional braiding statistics (Goldhaber and Jain [201]). Here, we go back to the "full" description, i.e., to the null vacuum, from where we see *all* composite fermions. The Berry phase associated with a closed loop of a CF-quasiparticle (or any other composite fermion for that matter) encircling an area A is given by Eq. (5.75), reproduced here for convenience:

$$\Phi^* = -2\pi \left(\frac{BA}{\phi_0} - 2pN_{\rm enc} \right) . \tag{9.41}$$

The two terms on the right hand side are due to the two constituents of the composite fermion, charge $-e$ electron and $2p$ vortices, going around a loop enclosing BA flux and $N_{\rm enc}$ particles. The *average* change in the phase due to the insertion of another CF-quasiparticle inside the loop is given by

$$\Delta\Phi^* = 2\pi\, 2p \Delta \langle N_{\rm enc} \rangle = 2\pi \cdot 2p \cdot \frac{e^*}{e} = 2\pi \frac{2p}{2pn+1} . \tag{9.42}$$

We have used that the change in the average number of electrons due to an extra CF-quasiparticle is $\Delta\langle N_{\rm enc}\rangle = e^*/e = 1/(2pn+1)$, which is valid under the assumption that the two CF-quasiparticles do not overlap at any point of the closed trajectory; otherwise a part of the interior CF-quasiparticle would spill out of the loop. (The change in the Berry phase can also be evaluated when the two quasiparticles are overlapping, from the knowledge of the microscopic wave functions, but only with substantially greater effort.) With $\Delta\Phi^* = 2\pi\alpha$, the braiding statistics parameter is obtained as the product of the local charge and the number of vortices bound to a CF-quasiparticle:

$$\alpha_{\rm qp} = 2p \cdot \frac{e^*}{e} = \frac{2p}{2pn+1} . \tag{9.43}$$

This completes the derivation of the braiding statistics of the CF-quasiparticles. It is a straightforward corollary of Eq. (5.75) or Eq. (9.41), which is also the equation that embodies the physics of the effective magnetic field. Essentially, even though each electron carries an even number of vortices, we have changed the charge inside the loop by a fraction of an electron charge, which amounts to the addition of a fractional number of vortices, thus producing a fractional braiding statistics. The fractional braiding statistics is a direct descendant of the fractional charge.

The long-distance limit of fractional braiding statistics of the excitations of the FQHE states at $\nu = n/(2pn \pm 1)$ can also be derived in the CFCS framework. Lopez and Fradkin [404] obtain it by considering fluctuations in the gauge field around the mean field at the Gaussian level, which, they argue, gives the exact long-distance behavior.

How about the LMW braiding statistics of the CF-quasihole? Let us begin by noting that the relative braiding statistics of the CF-quasiparticle and the CF-quasihole is given by

(reader, please derive this)

$$\alpha_{\mathrm{qh-qp}} = -\frac{2p}{2pn+1} \; . \tag{9.44}$$

The braiding statistics of a CF-exciton requires taking a quasiparticle quasihole pair going around another, which produces four terms:

$$\alpha_{\mathrm{qp}} + 2\alpha_{\mathrm{qh-qp}} + \alpha_{\mathrm{qh}} = 0 \; . \tag{9.45}$$

The zero on the right hand side follows because the CF-exciton is a boson. This gives

$$\alpha_{\mathrm{qh}} = \frac{2p}{2pn+1} \; . \tag{9.46}$$

(Various braiding statistics parameters α are only defined mod 2.)

Equations (9.43) and (9.46) have been confirmed for $\nu = 1/3$ and $2/5$ in detailed microscopic calculations (Jeon, Graham, and Jain [293,294]; Kjønsberg and Myrheim [345]; Kjønsberg and Leinaas [346]), with the Berry phase integral evaluated by Monte Carlo for the appropriate wave function of composite fermions. Some results are shown in Figs. 9.5 and 9.6. The answer is independent of the path, provided CF-quasiparticles or CF-quasiholes are sufficiently far separated (typically more than 10 magnetic lengths), and also of whether the projected or unprojected wave function is used. The values for the braiding statistics parameter quoted in the literature (for example, Halperin [230]; Su [629]) differ from that in Eq. (9.43) by an integer, and sometimes by a sign; some of these differences can be traced to different conventions used in the definition of the braiding statistics.

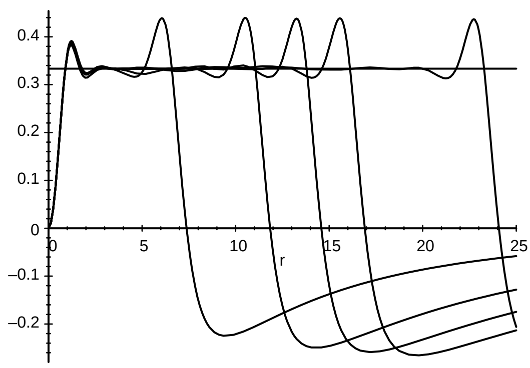

Fig. 9.5. The braiding statistics parameter (y-axis) for the quasihole at $\nu = 1/3$, determined from a Monte Carlo evaluation of the Berry phase for systems with up to 200 electrons. The parameter r is defined as $r = d/(2\sqrt{2})$, where d is the distance between the centers of the quasiholes in units of the magnetic length. The curves are, from left to right, for 20, 50, 75, 100, and 200 electrons, and the horizontal line marks $1/3$. Source: H. Kjønsberg and J. Myrheim, *Int. J. Mod. Phys. A* **14**, 537 (1999). (Reprinted with permission.)

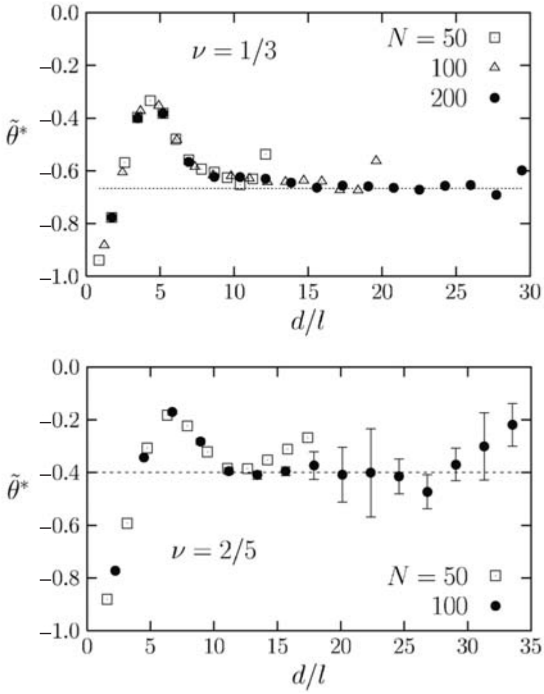

Fig. 9.6. The braiding statistics parameter (denoted $\tilde{\theta}^*$ in this figure) for the quasiparticles at $\nu = 1/3$ (upper panel) and $\nu = 2/5$ (lower panel) as a function of their separation d, calculated using the CF theory. N is the total number of composite fermions, and l is the magnetic length. The error bars from Monte Carlo sampling are not shown explicitly when they are smaller than the symbol size. The deviation at the largest d/ℓ for each N is due to proximity to the edge. The projected (unprojected) wave function is used for the CF-quasiparticles at $\nu = 1/3$ ($\nu = 2/5$). (The computation at $\nu = 2/5$ becomes prohibitively expensive if the projected wave function is used.) Source: G. S. Jeon, K. L. Graham, J. K. Jain, *Phys. Rev. Lett.* **91**, 036801 (2003). (Reprinted with permission.)

Two points need clarification. For the Laughlin fractions, Arovas, Schrieffer, and Wilczek considered the wave function

$$\Psi^{\rm ASW}_{\eta,\eta'} = \prod_j (z_j - \eta)(z_j - \eta')\Psi_{\frac{1}{2p+1}} \tag{9.47}$$

for two quasiholes at η and η'. The quasiholes are vortices at these fractions. As shown in Section 5.11.4, the Berry phase associated with a closed loop of a vortex is $\Phi^* = 2\pi N_{\rm e}$, with $N_{\rm e}$ being the number of electrons inside the loop. The difference in the Berry phase with or without the other quasihole is $\Delta\Phi^* = 2\pi\,\Delta N_{\rm e} = -2\pi\nu$, equating which to $2\pi\alpha^{\rm ASW}_{\rm qh}$ gives the braiding statistics of the quasiholes at $\nu = 1/(2p+1)$:

$$\alpha^{\rm ASW}_{\rm qh} = -\frac{1}{2p+1} \; . \tag{9.48}$$

Equation (9.46), on the other hand, gives $\alpha_{\mathrm{qh}} = 2p/(2p+1)$ for the quasiholes at $\nu = 1/(2p+1)$. The difference is merely that of convention. In the CF theory, the wave function that naturally appears is

$$\Psi_{\eta,\eta'} = (\eta - \eta') \prod_j (z_j - \eta)(z_j - \eta')\Psi_{\frac{1}{2p+1}} , \tag{9.49}$$

because the wave function for two holes at $\nu = 1$ is $(\eta - \eta') \prod_j (z_j - \eta)(z_j - \eta')\Psi_1$ – that is what we get by applying the destruction operators $\hat{\psi}(\eta)\hat{\psi}(\eta')$ to the $\nu = 1$ state, which amounts to replacing two of the particle coordinates in Φ_1 by η and η'. The reader can verify by an explicit ASW calculation that $\Psi_{\eta,\eta'}$ gives $\alpha_{\mathrm{qh}} = -1/(2p+1) + 1$, the last term originating from the factor $(\eta - \eta')$, making the result consistent with Eq. (9.46).

Second, let us now check the result for consistency with the notion that $(2pn + 1)$ CF-quasiparticles (CF-quasiholes) make one electron (hole). The braiding statistics of a bound complex of $(2pn + 1)$ CF-quasiholes is given by

$$\alpha_{\mathrm{h}} = (2pn + 1)^2 \alpha_{\mathrm{qh}} = 2p(2pn + 1) \bmod 2 = 0 , \tag{9.50}$$

which is the result of $(2pn + 1)^2$ equal terms originating from $(2pn + 1)$ CF-quasiholes going around as many CF-quasiholes. The same is true of the braiding statistics of a bound complex of $(2pn + 1)$ CF-quasiparticles. This appears to contradict the expectation that the fermionic nature of the holes or electrons ought to produce an odd integer. Exercises 9.7 and 9.9 show, however, that the Berry phase calculation for two electrons or two holes in the lowest Landau level also produces $\alpha_{\mathrm{h}} = \alpha_{\mathrm{e}} = 0$. (In fact, the result was already derived in the previous paragraph for two holes.) The origin of the apparent inconsistency is explained in Exercise 9.7.

Several comments are in order.

- In essence: With the null vacuum as our reference, the state is described in terms of charge $-e$ fermions (in an effective magnetic field). On the other hand, relative to an incompressible FQHE state, the qualitative physics may be described, under appropriate conditions, in terms of a smaller number of quasiparticles, which are to be assigned fractional local charge and fractional braiding statistics.
- The braiding statistics of the FQHE quasiparticles is well defined only when their separation is large compared with their size. When they begin to overlap, the phases become path dependent and the topological notion of braiding statistics ceases to be meaningful. The FQHE quasiparticles are not ideal but "fuzzy anyons" because of their nonzero size, as depicted in Fig. 9.7. The statistical flux tube associated with them is not point-like but spread out.
- The composite fermion and anyon descriptions are not equivalent, the former being more general. (i) While the CF theory is valid for the low-energy physics at all fillings in the FQHE regime, the anyon description is applicable only in regions close to the fillings $\nu = n/(2pn \pm 1)$ where the quasiparticle density is dilute, i.e., when quasiparticles are far apart. (ii) Even when valid, the anyon language does not allow quantitative calculation. (iii) Anyons can be derived from composite fermions, but the reverse is not true.

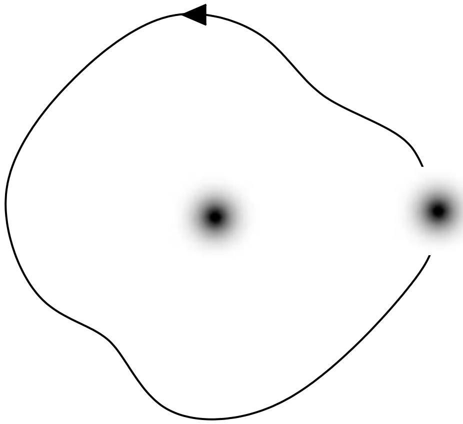

Fig. 9.7. The FQHE quasiparticles are not point objects but have a size of approximately $\geq 10\ell$ each.

- Fractional braiding statistics is not an additional concept but a property of composite fermions that follows from Eq. (9.41). It provides a natural interpretation for how the effective magnetic field changes upon a localized $O(1)$ variation in the particle density. A measurement of fractional braiding statistics will provide a more microscopic test of Eq. (9.41) than the measurement of the effective magnetic field, which is a more robust $O(N)$ quantity.
- The FQHE system is uncommon in one respect. It often pays to lighten the baggage that we must carry by working with only a few quasiparticles near the Fermi energy, because that usually results in a simplification. That is the Landau Fermi liquid theory philosophy. In the FQHE, however, the problem becomes more intractable when we shift our reference from the null vacuum (where we have almost free composite fermions) to the FQHE vacuum (where we end up with anyons). Nonetheless, the fractional braiding statistics is a remarkable manifestation of the correlations in the state, and we may ask how it can be measured.

9.8.3 Experimental situation

So far, neither a quantum chemistry nor a condensed matter type confirmation has been possible for the LMW braiding statistics of CF-quasiparticles. It has been asserted [372] that the non-Laughlin fractions of the type 2/5 and 2/7 are a measurable consequence, and prove the existence, of fractional braiding statistics. This claim is based on the so-called "hierarchy" approach for the FQHE (Section 12.1), which proposes to use the fractional braiding statistics of quasiparticles to explain the non-Laughlin fractions. If the explanation of 2/5 and 2/7 had indeed required fractional braiding statistics in the same essential manner as the bosonic statistics is required for BEC or the fermionic statistics for the Fermi sea, then such a claim would indeed be warranted. However: (i) the notion of fractional statistics becomes ambiguous at high quasiparticle or quasihole densities required for 2/5 and 2/7 (Section 12.1); and (ii) the CF theory explains all fractions (including 2/5 and 2/7) convincingly without any mention of fractional braiding statistics. A consideration of the totality of observed fractions indicates that the FQHE provides an evidence for the fermionic statistics of composite fermions.

The best chance for an observation of fractional braiding statistics may be through an experiment that implements the ASW geometry and measures the Berry phase induced by the insertion of a new quasiparticle inside the loop. The situation is somewhat complicated, in practice, by the fact that the order-one contribution originating from the braiding statistics sits atop a large Aharonov–Bohm phase (the AB phase is proportional to the area of the loop, which must be large enough to avoid overlap between the two quasiparticles in question). The conceptual question of disentangling the local charge, the braiding statistics, and the edge exponent is also relevant for any experimental situation, especially for the Laughlin fractions, for which, in a Wen-type approach for the edge state, all three arise out of a single parameter. In such situations, although interpretation in terms of one of them may seem more natural, it is not unique. Kim *et al.* [337] suggest that it would help to investigate other fractions of the sequences $\nu = n/(2pn \pm 1)$ for which the local charge, the braiding statistics, and the edge exponents have different values.

9.9 Non-Abelian braiding statistics

The excitations of certain FQHE states may show an even more complicated behavior under braiding, which goes under the name non-Abelian braiding statistics. The effective wave functions of such excitations transform as a non-Abelian representation of the braid group. We considered in Section 7.4.2 the Moore–Read Pfaffian wave function at $\nu = 1/2$, $\Psi^{\rm Pf}_{1/2}$ (Eq. 7.11). Moore and Read [454] suggested that its quasiholes obey non-Abelian braiding statistics. An Arovas–Schrieffer–Wilczek-type adiabatic Berry phase calculation has not yet been possible, but we describe the essential physics. The following considerations are valid for a three-body short-range interaction (see Section 12.8.2 for further details) for which the Moore–Read wave function and its quasi-hole excitations are exact (zero energy) ground states.

The wave function for a vortex excitation at η is given by

$$\prod_j (z_j - \eta)\Psi^{\rm Pf}_{1/2} = \prod_j (z_j - \eta){\rm Pf}\left(\frac{1}{z_i - z_j}\right)\Phi_1^2 \,, \tag{9.51}$$

which has a local charge of $e/2$ associated with it (Section 9.3.2). However, as we saw in Section 9.3.3, the vortex is not necessarily an elementary excitation; for the FQHE states at $\nu = n/(2pn \pm 1)$ it is a collection of n quasiholes, due to one missing composite fermion in each Λ level. For the Moore–Read wave function also, the vortex can be split into two elementary excitations. This becomes clear by writing the following wave function for *two* quasiholes at η and η' (Moore and Read [454])

$$\Psi^{\rm Pf}_{1/2}(\eta, \eta') = {\rm Pf}\left(M_{ij}\right)\Phi_1^2 \,, \tag{9.52}$$

where

$$M_{ij} = \frac{(z_i - \eta)(z_j - \eta') + (i \leftrightarrow j)}{(z_i - z_j)} \ . \tag{9.53}$$

For $\eta' = \eta$ it reduces to the charge $e/2$ vortex. Thus, pulling the vortex factor $\prod_j (z_j - \eta)$ into the Pfaffian shows that it actually represents *two* coincident quasiholes, each of which has a local charge $e/4$.

Moore and Read construct the wave function for $2n$ quasiholes at $\eta_1, \ldots, \eta_{2n}$ by generalizing Eq. (9.51) to n vortices, and then splitting them into $2n$ quasiholes by pulling the vortex factors into the Pfaffian. The wave function is given by Eq. (9.52), but with

$$M_{ij} = \frac{\prod_{\alpha=1}^{n} (z_i - \eta_\alpha)(z_j - \eta_{\alpha+n}) + (i \leftrightarrow j)}{(z_i - z_j)} \ . \tag{9.54}$$

Note that quasiholes can be created only in pairs.

To appreciate the origin of non-Abelian braiding statistics, let us first recall the case of several CF-quasiholes at $\nu = n/(2pn \pm 1)$. The lowest energy state is obtained when they are all in the topmost occupied Λ level. Because the wave function is specified uniquely by the positions of CF-quasiholes, when two CF-quasiholes adiabatically braid around one another, the final wave function may differ from the initial at most by a Berry phase factor. Two consecutive operations thus commute (the phases add), and the LMW braiding statistics is Abelian.

In contrast, the Moore–Read wave function for $2n$ quasiholes is not fully specified by their positions. In Eq. (9.54) we associated $\eta_1, \ldots, \eta_n$ with z_i and the remaining quasiholes with z_j, but we could equally well have chosen to associate different sets of η's with z_i and z_j to obtain different wave functions. This might suggest that the number of independent wave functions for a specified set of quasihole positions is equal to $(2n)!/[2(n!)^2]$, which is the number of ways $2n$ quasiholes could be arranged into two groups of n. But only 2^{n-1} of these wave functions turn out to be linearly independent, as can be seen from an explicit consideration of the $(2n)!/[2(n!)^2]$ polynomials (Moore and Read [454]; Nayak and Wilczek [475]). The initial wave function will, therefore, in general, be a linear superposition of 2^{n-1} orthogonal basis functions. An adiabatic interchange of two quasiholes will produce a new linear superposition which can be obtained from the initial one by application of a $2^{n-1} \times 2^{n-1}$ matrix. Matrices associated with consequent exchanges do not commute in general. Hence the name non-Abelian braiding statistics. The numerical Berry phase calculations by Tserkovnyak and Simon [648] support these ideas.

While non-Abelian braiding statistics has been established for a model interaction for which the Moore–Read wave functions are exact, its relevance for the excitations of the actual Coulomb 5/2 state is unclear. Ho has shown [255] that the Moore–Read wave function is adiabatically connected to the Halperin (331) wave function (Section 13.2), the excitations of which are known to obey Abelian braiding statistics. Exact diagonalization studies on finite systems [641, 642] indicate that the Moore–Read Pfaffian model for the quasiholes may not provide a very accurate representation of the Coulomb quasiholes of the 5/2 FQHE state. Further work is needed to ascertain the range of interactions for which the actual 5/2 FQHE state supports excitations with non-Abelian braiding statistics, and whether this range includes the Coulomb interaction.

9.10 Logical order

The logical order for the understanding of various concepts discussed in this chapter is shown in Fig. 9.8, which illustrates the cause-and-effect relation between them. The directions of the arrows are consistent with the following facts: (a) Incompressibility at certain fractions follows from the formation of composite fermions. Composite fermions, on the other hand, cannot be derived from incompressibility. Composite fermions form compressible states as well, and are thus more general than the FQHE. (b) FQHE, fractional local charge, and fractional LMW braiding statistics are fundamentally tied to, and a direct consequence of, incompressibility. The local charge or braiding statistics are not sharply defined concepts for compressible states. (c) The FQHE can be understood without appealing to the charge of the excitations. (d) Fractional local charge is a prerequisite for fractional braiding statistics. (e) Fractional local charge and fractional braiding statistics can be derived from composite fermions, but the reverse is not true.

Exercises

9.1 This exercise builds on the arguments of Section 9.3.1.
(i) Derive the charge of a CF-quasihole by asking what happens when an electron is removed from an incompressible state.
(ii) Show that the local charge of a CF-quasiparticle or a CF-quasihole is independent of its Λ level index.
(iii) Show that the charge of a CF-quasiparticle is given by $-e^* = -e/(2pn+1)$ even for filling factors away from the special filling factors, when many CF-quasiparticles or CF-quasiholes are already present (localized by disorder).
(iv) Show that, for the incompressible states at $n/(2pn-1)$, a composite fermion in the $(n+1)$th Λ level has an excess charge $+e/(2pn-1)$ (note the sign).

9.2 This exercise seeks to formulate the argument of Section 9.3.1 in the disk geometry. To avoid excitations at the edge, the size of the disk will be kept constant. Begin with the $n/(2pn+1)$ FQHE state, $\Psi = \prod_{j<k}(z_j - z_k)^{2p}\Phi_n$, confined to a disk of a given radius, which fixes the largest allowed power of z_j. Now add an *electron* to this system while insisting that the size of the system not change. The Jastrow factor increases the largest power of z_j, which requires taking some particles from the boundary of each Landau level in Φ and placing them in the $(n+1)$th Landau level. Show that the total

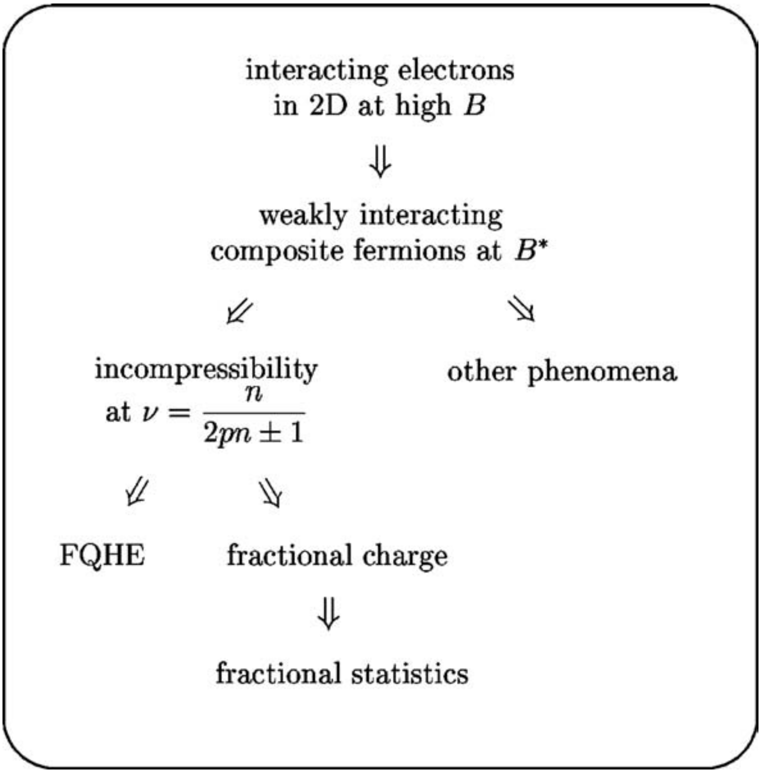

Fig. 9.8. The logical order of the topological quantities discussed in this chapter. Source: G. S. Jeon, K. L. Graham, J. K. Jain, *Phys. Rev. B* **70**, 125316 (2004). (Reprinted with permission.)

number of CF-quasiparticles in the $(n+1)$th Λ level is $2pn+1$, which produces the local charge $-e^* = -e/(2pn+1)$.

9.3 Apply the arguments of Section 9.3.4 to an even denominator fraction. What can you conclude?

9.4 Following Section 9.3.7, estimate the size of a CF-quasihole.

9.5 Consider an island of the $\nu = (n+1)/(2n+3)$ state surrounded by the $\nu = n/(2n+1)$ state. Show schematically, as in Fig. 9.2 (but neglecting the outer edge of the $n/(2n+1)$ state), what happens when:
(i) a composite fermion is added to the boundary of the island in the nth Λ level;
(ii) a composite fermion is added to the center of the island in the $(n+1)$th Λ level;
(iii) a composite fermion is moved from the edge of the nth Λ level to the center of the $(n+1)$th Λ level. (Source: Jain and Shi [288].)

9.6 With the help of a singular gauge transformation, map (i) anyons into fermions; (ii) fermions into bosons.

9.7 The wave function for a localized wave packet centered at η is given by Eq. (E3.2)

$$\chi_\eta(z) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{1}{2}\bar{\eta}z - \frac{1}{4}|z|^2 - \frac{1}{4}|\eta|^2\right] \tag{E9.1}$$

where $\eta \equiv R\mathrm{e}^{-\mathrm{i}\theta}$. The Berry γ for a closed loop in which θ goes from 0 to 2π is shown in Appendix E to be $\gamma = -\pi R^2$. Now let us consider the problem of two electrons at η and η'. Antisymmetrizing and taking one particle centered at the origin ($\eta' = 0$) and the other at $\eta = R\mathrm{e}^{-\mathrm{i}\theta}$, the wave function is given by

$$\chi_{\eta,0}(\boldsymbol{r}_1, \boldsymbol{r}_2) = N\left(\mathrm{e}^{\frac{\bar{\eta}z_1}{2}} - \mathrm{e}^{\frac{\bar{\eta}z_2}{2}}\right)\mathrm{e}^{-\frac{1}{4}(R^2+r_1^2+r_2^2)} . \tag{E9.2}$$

Show the following:

(i) The normalization constant is

$$N = \frac{1}{2\pi\sqrt{2[1-\exp(-R^2/2)]}} . \tag{E9.3}$$

(ii) The Berry phase for the path $\theta = 0 \to 2\pi$ is

$$\gamma' = -\frac{\pi R^2}{1-\exp(-R^2/2)} . \tag{E9.4}$$

(iii) The difference is given by

$$\Delta\gamma = \gamma' - \gamma = -\frac{\pi R^2}{\exp(R^2/2) - 1} . \tag{E9.5}$$

This gives the correction to the Berry phase due to the presence of another electron inside the loop.

(iv) The correction is exponentially small for $R/\ell \gg 1$. What does this imply for the braiding statistics? For what R does $\Delta\gamma$ become equal to 0.05? 0.01? In the other limit $R/\ell \ll 1$, we have

$$\Delta\gamma = -2\pi . \tag{E9.6}$$

That can be understood from the observation that for $R/\ell \ll 1$, the wave function reduces to

$$\chi_{\eta_1,\eta_2}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{2}(z_1 - z_2)(\bar{\eta}_1 - \bar{\eta}_2) , \tag{E9.7}$$

that is, the particle $\boldsymbol{r}_1$ sees a simple vortex at $\boldsymbol{r}_2$, which has a phase of magnitude 2π associated with it.

Hint: Use the identity, derived in Appendix E:

$$\int \frac{\mathrm{d}^2\boldsymbol{r}}{2\pi}\frac{z\bar{\eta}}{2}\exp\left[\frac{1}{2}\left(\bar{\eta}z + \eta\bar{z} - r^2 - R^2\right)\right] = \frac{R^2}{2} . \tag{E9.8}$$

Comment: The above Berry phase calculation produces the statistics parameter $\alpha = 0$, contrary to the expectation of an odd-integer value of α for fermions. The reason is that the adiabatic process considered here amounts to an exchange of two "hydrogen atoms," with the η's playing the role of the proton coordinates. (The wave function is antisymmetric with respect to an exchange of both the electron coordinates and the η's.) The correct electron statistics is obtained after subtracting the statistics of the η's from the Berry phase result. Similar considerations are relevant to Exercise 9.9.

9.8 Show that, for charged *bosons* in the lowest Landau level, the change in the Berry phase of a closed loop due to the insertion of another boson is given by $\Delta\gamma = \pi R^2/(1+e^{R^2/2})$.

9.9 This exercise repeats the calculation for one or two holes in the lowest Landau level. Denote the state in which the lowest Landau level is fully occupied by $|\Phi_1>$. A hole at $\boldsymbol{\eta}$ is created by the application of the field operator as follows:

$$\begin{aligned} |\chi_\eta\rangle &= N_1 \Psi \boldsymbol{\eta} |\Phi_1\rangle \\ &= N_1 \sum_{m=0}^{\infty} \frac{\eta^m}{\sqrt{2^m m!}} \mathrm{e}^{-|\eta|^2/4} c_m |\Phi_1\rangle \ . \end{aligned} \tag{E9.9}$$

The wave function for two holes, one at the origin and the other at $\boldsymbol{\eta}$ is similarly given by

$$|\chi_{\eta,0}\rangle = N_2 \sum_{m=1}^{\infty} \frac{\eta^m}{\sqrt{2^m m!}} \mathrm{e}^{-|\eta|^2/4} c_m c_0 |\Phi_1\rangle \ . \tag{E9.10}$$

Here c_m is annihilates the electron in the angular momentum m orbital. Show the following:

(i) The normalization constants are given by

$$N_1 = 1, \qquad N_2 = [1 - \exp(-R^2/2)]^{-1/2} \ . \tag{E9.11}$$

(ii) For a single hole, the Berry phase for the closed loop defined by $\eta = R\mathrm{e}^{-\mathrm{i}\theta}$, $R = \text{constant}$, $\theta = 0 \to 2\pi$ is given by

$$\gamma = \pi R^2 = 2\pi \frac{\phi}{\phi_0} \ . \tag{E9.12}$$

(iii) When the loop encloses a hole at the origin, the Berry phase is given by

$$\gamma' = \frac{\pi R^2}{1 - \exp(-R^2/2)} \ . \tag{E9.13}$$

(iv) Discuss the behavior of $\gamma' - \gamma$ in the limiting cases $R/\ell \gg 1$ and $R/\ell \ll 1$.

9.10 We made several consistency checks in the text. Here is one more. A vortex at $\nu = n/(2pn + 1)$ is a collection of n CF-quasiholes. Show that the LMW braiding statistics of a vortex agrees with that of n CF-quasiholes (modulo an integer).

9.11 Another candidate for non-Abelian braiding statistics (Wen [672]) is the wave function $\Phi_1\Phi_2^2$ (Jain [274]). Determine its filling factor, the charge of a vortex excitation, and the charge of an elementary excitation. (Hint: Into how many elementary excitations can a vortex be split?) The elementary excitations are CF-quasiholes or CF-quasiparticles in the Φ_2 factors. They obey non-Abelian braiding statistics, because specifying their positions does not uniquely determine the wave function; they can be arranged in many ways among the two Φ_2 factors. (The wave function, so far, is not known to be applicable to a realistic situation.)