LECTURES ON SEIFERT MANIFOLDS

MARK JANKINS

AND

WALTER D. NEUMANN

BRANDEIS LECTURE NOTES 2, MARCH 1983

Preface

This volume comprises notes of a course given by the second author at Brandeis University in Spring, 1981. The notes were written by the first author. The course ended with a discussion of geometric structures on Seifert manifolds and relations to quasihomogeneous complex surface singularities. This material here appears as an appendix which is a reprint, by permission of the American Mathematical Society, of an article in Proceedings of Symposia in Pure Mathematics, volume 40.

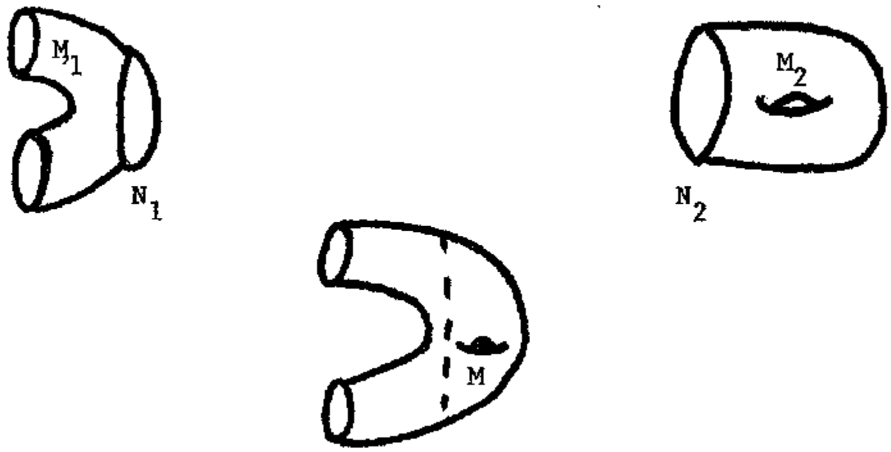
Mark Jankins
Walter D. Neumann
University of Maryland
Department of Mathematics
College Park, Maryland 20742

Contents

Introduction

In this chapter we give, mostly without proof, some necessary topological preliminaries. Proofs of these theorems can be found in most books on differential topology (e.g. M. Hirsch: Differential Topology).

The basic construction used is cutting and pasting of manifolds. Given two smooth manifolds M_1 and M_2 with $N_i \subseteq \partial M_i$ as a component $(i = 1,2)$ and $h: N_1 \to N_2$ a diffeomorphism, we can form $M = M_1 \cup_h M_2 = (M_1 + M_2)/(x) \equiv h(x)$ (see figure) where $+$ denotes disjoint union.



The following theorem shows this construction can be done in the smooth category.

Theorem 0.1.

a) M can be given a smooth structure such that M_1 and M_2 are smooth submanifolds.

b) Given two such smooth structures on M, say $\mathcal{A}_1$ and $\mathcal{A}_2$, there exists a diffeomorphism $h: (M,\mathscr{A}_1) \to (M,\mathscr{A}_2)$ (where $M,\mathscr{A}_i$) is M with smooth structure $\mathscr{A}_i$, $i = 1,2$) such that

i) h is arbitrarily close to the id.

ii) $h = \text{id}$ outside an arbitrarily small neighborhood of N_1

iii) h is isotopic to the id through homeomorphisms satisfying i) and ii)

iv) In i) - iii) one can assume $h|M_1 = \text{id}$ or $h|M_2 = \text{id}$.

c) If $h' = f_2 \circ h \circ f_1$ where $f_i: N_i \to N_i$ $(i = 1,2)$ is a diffeomorphism which extends to a diffeomorphism: $M_i \to M_i$ $(i = 1,2)$. Then $M_1 \cup_h M_2 \underset{C^\infty}{\approx} M_1 \cup_{h'} M_2$.

<u>Proposition 0.2.</u> Let $M_1, M_2, N_1, N_2, h: N_1 \to N_2$ be as in the previous theorem. If $h': N_1 \to N_2$ is isotopic to h then $M_1 \cup_h M_2 \underset{C^\infty}{\approx} M_1 \cup_{h'} M_2$.

We have the following example of pasting:

Let M_1 and M_2 be connected (oriented) n-manifolds and $f_i: D^n \hookrightarrow M_i$ $(i=1,2)$ be embeddings (f_1 orientation preserving, f_2 orientation reversing). Define

$$M_1 \# M_2 = (M_1 - \text{int } f_1(D^n)) \bigcup_{f_2 \circ f_1^{-1}|S^{n-1}} (M_2 - \text{int } f_2(D^n)).$$

$M_1 \# M_2$ is called the connected sum of M_1 and M_2. In the case M_1 and M_2 are oriented f_1 must preserve orientation and f_2 must reverse orientation to have a consistent orientation on M.

Theorem 0.3. Any two embeddings of D^n into the interior of a connected manifold M^n are isotopic (possibly after reversing orientations of one if necessary).

This theorem shows # is a well defined operation: i.e. # is independent of the choice of f_i.

Let N be a smooth manifold and define:

$\text{Diff}(N)$ = diffeomorphism group of N

$\text{Diff}_o(N)$ = identity component of Diff (N)

= diffeomorphisms isotopic to the id

$\text{Diff}(N)/\text{Diff}_o(N)$ = isotopy classes of diffeomorphisms of N

$\text{Diff}^+(N)$ = orientation preserving diffeomorphisms of N.

Theorem 0.4. $\text{Diff}(T^2)/\text{Diff}_o(T^2) \cong GL(2,\mathbb{Z})$

$$\text{Diff}^+(T^2)/\text{Diff}_o(T^2) \cong SL(2,\mathbb{Z})$$

A related result is

Theorem 0.5. An oriented simple closed curve in T^2 is uniquely determined (up to isotopy) by its homology class. Any class of the form $p(S^1 \times \{1\}) + q(\{1\} \times S^1)$ with $\gcd(p,q) = 1$ occurs. Hence up to automorphisms of T^2 there is only one curve.

Dehn Surgery.

Definition 0.6. Given M^3 a 3-manifold such that $T^2 \subseteq \partial M^3$, c a simple closed curve in T^2, define $M^3(c) = M^3 \cup_h (D^2 \times S^1)$ where h is a diffeomorphism of T^2 onto $\partial D^2 \times S^1$ which takes c onto a meridian of $D^2 \times S^1$. (i.e. a curve in $T^2 = \partial D^2 \times S^1$ which is null homotopic in $D^2 \times S^1$.) $M(c)$ is said to be obtained from M by Dehn surgery.

__Proposition 0.7.__ $M^3(c)$ is well defined (up to diffeomorphism).

__Proof:__ We can parametrize T^2 such that $c = S^1 \times \{1\} \subseteq S^1 \times S^1 = T^2$. Then $h(t,1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(t,1) = (t^a 1^b, t^c 1^d) = (t^a, t^c)$. $c = 0$ since $(t,1)$ gets mapped to $(t,1)$. Thus $h = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$ which extends over the solid torus. By a previous theorem $M^3(c) \underset{C^\infty}{\cong} M^3 \cup_{id} (D^2 \times S^1)$ and hence is well defined.

More typically Dehn surgery is the following:

M^3 is the complement of a tubular neighborhood of a closed curve γ in some 3-manifold N^3. $c = p$ (longitude) $+ q$ (meridian). This is called (p,q)-Dehn surgery on γ in N^3. Note that in general there are infinitely many possible choices of longitude in the boundary of a tubular neighborhood of γ. Therefore q is well defined only after making such a choice.

I. Definitions and Examples.

In this chapter we define and classify according to their Seifert invariant, Seifert and Generalized Seifert fibrations. A Seifert fibration over a orientable surface can be viewed also as the orbit space projection of an S^1-action on a 3-manifold. This is discussed in section 2. We extend the definition of the Euler number of an S^1-bundle to include Generalized Seifert fibrations. In section 4 the examples of lens spaces are described and used in Section 5 to describe (with proofs postponed) the classification of Seifert and Generalized Seifert fiberable (as opposed to fibered) manifolds. The final section 6 describes the basic algebraic topology of these manifolds.

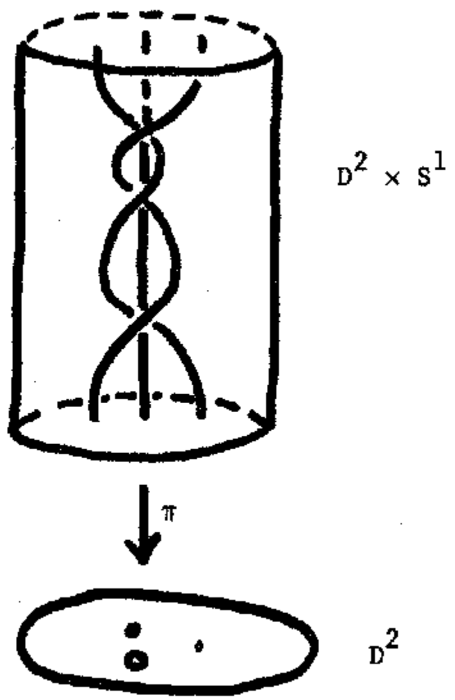
1. Seifert and Generalized Seifert Fibrations.

Definition 1.1. A G.S. (Generalized Seifert) fibration is a triple (M,F,π) (also denoted $M \overset{\pi}{\to} F$) where M is an oriented 3-manifold, F is a surface, oriented or unoriented, and $\pi\colon M \to F$ such that (M,F,π) is "almost" a locally trivial S^1-bundle. To be precise:
For every $x \in F$, there exists a D^2 neighborhood of x such that $\pi^{-1}(D^2) \cong D^2 \times S^1$ and

$$\pi\colon D^2 \times S^1 \to D^2 \text{ is defined by } (rt_1,t_2) \mapsto rt_1^p t_2^q$$

where $t_i \in S^1 = \{t \in \mathbb{C} \mid |t| = 1\}$, $r \in [0,1]$, $p,q \in \mathbb{Z}$ and the $\gcd(p,q) = 1$. Here the values of p and q depend on x. If for every $x \in F$, $p \neq 0$ then (M,F,π) is called a Seifert fibration.

To understand the local structure of a G.S. fibration we look at the above "local model" $\pi: D^2 \times S^1 \to D^2$. If $p \neq 0$ we can parametrize a typical fiber by $\pi^{-1}(rs) = (rs^{1/p}t^q, t^{-p})$, $s,t \in S^1$, $r \in (0,1]$. If we consider $D^2 \times S^1$ as $D^2 \times I$ with ends identified we get the following picture:



The center of the disc 0 lifts to the core circle of the solid torus and points in $D^2 - \{0\}$ lift to fibers that wrap p times around the core in the longitudinal direction and $-q$ times in the meridianal direction.

An alternative description is to consider $D^2 \times I$ fibered by lines $\{x\} \times I$. Form a solid torus $D^2 \times S^1$ by identifying the ends of the solid cylinder with a $2\pi q/p$ twist.

<u>Definition 1.2.</u> We call a fiber singular or exceptional if the value of p associated to this fiber is not equal to ± 1.

Proposition 1.3. If (M,F,π) is a G.S. fibration with F compact, the number of exceptional fibers is finite. (Note F is compact iff M is.)

Proof: For every $x \in F$ there exists a neighborhood D^2x such that $\pi^{-1}(D^2x)$ contains at most one exceptional fiber, namely $\pi^{-1}(x)$. Since F is compact we can cover it by finitely many such neighborhoods.

We now consider the case of a G.S. fibering (M,F,π) where F is closed oriented and connected. If we are given such a fibering, we can remove solid torus neighborhoods of suitable fibers of M and corresponding disc neighborhoods of F, to leave a genuine S^1-bundle over a connected orientable surface with boundary. Any such S^1-bundle is trivial, thus M is the result of Dehn surgery on some fibers of a trivial bundle $F \times S^1 \to F$. As already remarked, each Dehn surgery is determined by a suitable coprime integer pair.

To see exactly how this construction can be done, assume we are given data $(g; (\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ where $g \geq 0$, $\alpha_i,\beta_i \in \mathbb{Z}$, $\alpha_i \geq 0$, and $\gcd(\alpha_i,\beta_i) = 1$ $i= 1,\ldots,n$. Let F_o = oriented surface of genus g with n punctures: $F_o = F - (D_1^2 \cup \ldots \cup D_n^2)$ (F = closed surface of genus g). Define

$$M_o = F_o \times S^1$$
$$\partial M_o = S_1^1 \times S^1 \cup S_2^1 \times S^1 \cup \ldots \cup S_n^1 \times S^1$$

Let
$$R = F_o \times \{1\}$$
$$Q_i = R \cap (S_i^1 \times S^1) = S_i^1 \times \{1\} \quad \text{(oriented as a component of } -\partial R\text{)}$$
$$H_i = \{1\} \times S^1 \subseteq S_i^1 \times S^1$$

We have a trivial S^1-bundle (M_o, F_o, π). To construct a G.S. fibration from this bundle, paste a solid torus $T_i = D^2 \times S^1$ into the i^{th} boundary component $S_i^1 \times S^1$ in such a way that a meridian $M_i = S_i^1 \times \{1\} \subseteq \partial T_i$ satisfies the homology relation $M_i \sim \alpha_i Q_i + \beta_i H_i$ in the homology of ∂T_i.

If we let $L_i = \{1\} \times S^1 \subseteq \partial T_i$ and $M_i \sim \alpha_i Q_i + \beta_i H_i$ then $L_i \sim \alpha_i' Q_i + \beta_i' H_i$ for some $\begin{pmatrix} \alpha_i & \beta_i \\ \alpha_i' & \beta_i' \end{pmatrix} \varepsilon\ SL(2, \mathbb{Z})$. Therefore we can solve for H_i and Q_i in terms of M_i and L_i to get

$$-\alpha_i' M_i + \alpha_i L_i = H_i$$

$$\beta_i' M_i - \beta_i L_i = Q_i.$$

In T_i, $M_i \sim 0$, thus we have

$$\left.\begin{array}{l} H_i \sim \alpha_i L_i \\ Q_i \sim -\beta_i L_i \end{array}\right\} \text{ in the homology of } T_i.$$

This gives an alternate description of what α_i and β_i signify, i.e. α_i is the number of times H_i wraps around T_i and $-\beta_i$ the number of times Q_i wraps around T_i.

We denote the G.S. fibration constructed above by $M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ and call $\{g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)\}$ the Seifert invariant.

Definition 1.4. We say two G.S. fibrations (M,F,π), (M',F',π') are isomorphic iff there exist diffeomorphisms $f: F \to F'$ and $\tilde{f}: M \to M'$, with orientation preserving such that

$$\begin{array}{ccc} M & \xrightarrow{\tilde{f}} & M' \\ \pi \downarrow & & \downarrow \pi' \\ F & \xrightarrow{f} & F' \end{array} \quad \text{commutes}$$

We shall see that different Seifert invariants can result in isomorphic G.S. fibrations. The second part of the next theorem gives necessary and sufficient conditions on the Seifert invariant to yield isomorphic G.S. fibrations. The first part shows every G.S. fibration can be obtained by the above method (provided F is oriented).

Theorem 1.5. Let $M \xrightarrow{\pi} F$ be a G.S. fibration with F closed connected and oriented. Then

a) $(M \xrightarrow{\pi} F) \cong M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ for some $g,\alpha_i,\beta_i \in \mathbb{Z}$

b) $M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)) \cong M(g';(\alpha_1',\beta_1'),\ldots,(\alpha_m',\beta_m'))$ iff:

i) $g = g'$

ii) Disregarding any β_i/α_i and β_j'/α_j' which are integers ($\neq \infty$), the remaining β_i/α_i (mod 1) are a permutation of the remaining β_j'/α_j' (mod 1)

iii) $\sum_{i=1}^n \beta_i/\alpha_i = \sum_{j=1}^m \beta_j'/\alpha_j'$ where here we use the convention $1/0 = -1/0 = \infty$, $\infty + x = \infty$ for every $x \in \mathbb{R} \cup \{\infty\}$.

Equivalently, the following collection of operations can be done to a Seifert invariant without changing the corresponding G.S. fibration up to isomorphism:

I) Add or delete any Seifert pair $(\alpha,\beta) = (1,0)$

II) Replace any $(0,\pm 1)$ by $(0,\mp 1)$

III) Replace each (α_i,β_i) by $(\alpha_i,\beta_i + K_i\alpha_i)$ provided $\sum K_i = 0$.

Example. $M(0;(2,1),(3,2),(5,-6)) \cong M(0;(2,-1),(3,2),(5,-1))$

$\cong M(0;(1,0),(2,1),(3,2),(5,-6))$

$\cong M(0;(1,-2),(2,1),(3,2),(5,4))$.

Proof of Theorem 1.5. a) Choose points $p_1,\dots,p_n \in F$ such that $\{\pi^{-1}(p_1),\dots,\pi^{-1}(p_n)\}$ includes all exceptional fibers. Let $D_1,\dots,D_n$ be disjoint disc neighborhoods of the p_i. Then $T_i = \pi^{-1}(D_i)$ are disjoint solid torus neighborhoods of the $\pi^{-1}(p_i)$. Define $M_o = M - \text{int}(T_1 \cup \dots$ and $F_o = F - \text{int}(D_1 \cup \dots \cup D_n)$. $(M_o,F_o,\pi|_{F_o})$ is a genuine S^1-bundle and therefore trivial. Hence we can find a section $s\colon F_o \to M_o$. Define $R = s(F_o) \subset M_o$, $Q_i = R \cap \partial T_i$ and let H_i be a non-singular fiber in ∂T_i. Let α_i be the multiple of the generator of $H_1(T_i)$ that H_i represents and $-\beta_i$ the multiple of the generator of $H_1(T_i)$ that Q_i represents. Then $(M,F,\pi) \cong M(g;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ where $g = \text{genus}(F)$ thus proving a).

b) By part a), we can assume that given any G.S. fibration $M(g;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ that (α_i,β_i) were obtained by the method used in the proof of part a). If we had choosen extra points p_i, i.e. points whose fibers are not exceptional, the result would have been to introduce pairs $(1,0)$ into the Seifert invariant. We also made an arbitrary choice when defining the section $s\colon F_o \to M_o$. With respect to a suitable trivialization $M_o \cong F_o \times S^1$, the section s is given by $s(x) = (x,1)$, and then any other section s' has the form $s'\colon x \mapsto (x,\phi(x))$ where $\phi\colon F_o \to S^1$. We can change s' by an isotopy without changing the corresponding values of (α_i,β_i). Thus we are concerned only with the homotopy class of ϕ. We claim:

> If we denote $\phi|\partial F_o \in [\partial F_o, S^1] = \mathbb{Z}^n$ by $(q_1,\ldots,q_n)$ where $q_i = \deg \phi|s_i'$, then $(q_1,\ldots,q_n)$ occurs for some ϕ iff $q_1 + \ldots + q_n = 0$.

To prove this recall $H^1(X,\mathbb{Z}) = [X,S^1]$ and note the following exact sequences:

$$
\begin{array}{ccccc}
H^1(F_o) & \longrightarrow & H^1(\partial F_o) & \xrightarrow{\alpha} & H^2(F_o,\partial F_o) \\
\| & & \| & & \| \\
H_1(F_o,\partial F_o) & \longrightarrow & H_o(\partial F_o) & \longrightarrow & H_o(F_o) \\
 & & \| & & \| \\
 & & \mathbb{Z}^n & & \mathbb{Z}
\end{array}
$$

α is the map $\alpha(Z_1,\ldots,Z_n) = Z_1 + \ldots + Z_n$. Therefore $(q_1,\ldots,q_n)$ occurs iff it pulls back to $H_1(F_o)$ iff $q_1 + \ldots + q_n = 0$.

We see from the claim that in choosing s' insead of s, Q_j is replaced by $Q_j + q_jH$ which winds $-\beta_j + q_j\alpha$ times around the solid torus T_j. Hence we can replace each β_j by $\beta_j - q_j\alpha_j$ provided $\sum q_j = 0$.

Note the non-uniqueness of the Seifert invariant is due to the arbitrary choice of a section $s: F_o \to M_o$. If we calculate the Seifert invariant with respect to a fixed section, two pairs, consisting each of a G.S. fibration plus a section to the fibration outside a finite collection of fibers, are isomorphic iff their Seifert invariants are equal (up to permutation of pairs).

Corollary 1.6. The Seifert invariant of a Seifert fibration has unique normal form (up to permutation of indices)

$$M(g;(1,\beta_o),(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)), \quad 0 < \beta_i < \alpha_i, \quad i = 1,\ldots,n.$$

If (M,F,π) is a G.S. fibration which is not a Seifert fibration, we can uniquely represent it (up to permutation of indices) as:

$$M(g;(0,1),\ldots,(0,1),(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m)), \quad 0 < \beta_i < \alpha_i, \ i = 1,\ldots,m.$$

Definition 1.7. $e(M \to F) = -\sum \alpha_i/\beta_i$ is called the Euler number of the G.S. fibration $M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$. $e(M \to F) \neq \infty$ iff (M,F,π) is a Seifert fibration.

Note that the Seifert invariant is an invariant of the oriented manifold M with its fibered structure, it does not depend on the orientation of the base F. For if we reverse the orientation of F, we must reverse the orientation of the fibers also, to keep the orientation of M fixed. Thus both Q_i and H_i are reversed, and the homology relation $\alpha_i Q_i + \beta_i H_i \sim 0$ in T_i, which determines (α_i,β_i), is unchanged. This can also be interpreted as saying that there exists a fiber preserving diffeomorphism $f\colon M \to M$, preserving orientation of M, such that the induced map $F \to F$ reverses orientation.

Exercise: Show f can be chosen even as an involution ($f^2 = \mathrm{id}$).

Note also that reversing the orientation of M reverses the sign of either Q_i or H_i, so β_i/α_i gets replaced by $-\beta_i/\alpha_i$. Thus we have

Corollary 1.7. If $M = M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ then $-M = M(g;(\alpha_1,-\beta_1),\ldots,(\alpha_n,-\beta_n))$. In particular $e(M \to F) = -e(-M \to F)$ ($-M$ means M with reversed orientation).

We now consider the case where F is non-orientable. Then $F = F_1 \# F_2$ where F_1 is an orientable surface and $F_2 = \mathbb{R}P^2$ or $F_2 = \mathbb{R}P^2 \# \mathbb{R}P^2$. By homogeneity of manifolds, we can assume the singular fibers of (M,F,π) lie only over points of F_1. Therefore, over F_2 we have a genuine S^1-bundle with oriented total space.

We now introduce Seifert invariants as before:

a) Remove tubular neighborhoods of the singular fibers (and possibly some non-singular fibers). This gives $M_o \to F_o \# F_2$, a genuine S^1-bundle where $F_o = F_1 - (D_1^2 \cup \ldots \cup D_n^2)$, $M_o = M - (T_1 \cup \ldots \cup T_n)$.

b) Choose a section $R \subset M_o$ to the fibration and use this to compute the Seifert pairs (α_i,β_i). This gives a Seifert invariant $(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ where $g < 0$ is the genus of F (we use negative genus for nonorientable surfaces, i.e. $F = \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$, $|g|$ times). As before we have

<u>Theorem 1.8</u>. Let $M \overset{\pi}{\to} F$ be a G.S. fibration with F closed connected and unorientable. Then

a) $(M \overset{\pi}{\to} F) \cong M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ for some $g,\alpha_i,\beta_i \in \mathbb{Z}$ with $g < 0$

b) same as the orientable case, i.e. we can change any β_i/α_i by an integer provided we keep $-\sum \beta_i/\alpha_i = e(M \to F)$ fixed. We can add or delete pairs $(\alpha_i,\beta_i) = (1,0)$.

<u>Proof</u>: Let $F = F' \# \mathbb{RP}^2$ where $F' = F_1$ or $F' = F_1 \# \mathbb{RP}^2$. Then

$$\begin{aligned} F &= (F' - \mathrm{int}(D^2)) \cup_{S^1} (\mathbb{RP}^2 - \mathrm{int}(D^2)) \\ &= (F' - \mathrm{int}(D^2)) \cup_{S^1} (Mb) \qquad (Mb = \text{Moebuis band}). \end{aligned}$$

We need:

<u>Lemma 1.9</u>. Suppose $E \overset{\pi}{\to} Mb$ is a fibration with fiber S^1 and oriented total space. Then

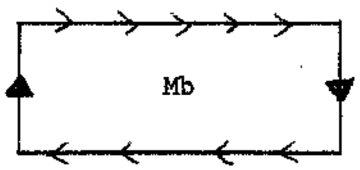
i) There is only one such E up to isomorphism namely $E = T^1Mb$ = unit tangent bundle.

ii) There is exactly one section up to isotopy of $E|\partial(Mb)$ which extends over E.

Given this lemma, there is a canonical way of cutting out $\pi^{-1}(Mb)$ in M and replacing it by $D^2 \times S^1$, to get a G.S. fibration over F'. The proof thus reduces to the case of F orientable.

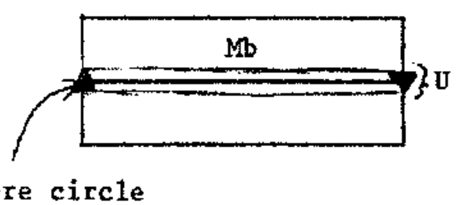
<u>Proof of Lemma 1.9</u>. i) Classifying bundles $E \xrightarrow{\pi} Mb$ with S^1 fiber and orientable total space is equivalent to classifying, up to homotopy, orientation reversing diffeomorphisms $h: S^1 \to S^1$. There is only one such h. Thus $E \to Mb$ is unique, and $E \cong T^1Mb$ since T^1Mb is such a bundle. ii) A section of T^1Mb is a unit vector field on Mb. Call the section on ∂Mb that is parallel to ∂Mb the trivial section. We claim any section $r: Mb \to T^1Mb$ is such that $r|\partial Mb$ is isotopic to the trivial section. To see this, choose a very narrow Möebius band neighborhood U of the core circle. As you traverse the boundary of U, any "rotation" of the vector field gets canceled as you "come around the second time" and therefore $r|\partial U$ is isotopic to the trivial section. Since $Mb - U =$ collar $= \partial Mb \times I$, this gives an isotopy of $r|\partial Mb$ to the trivial section (see below).

This is the Moebius band.



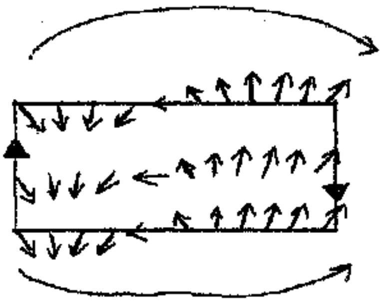
This is the trivial section on ∂Mb.

Given any section r of Mb, choose U.



On the core circle the section looks like;

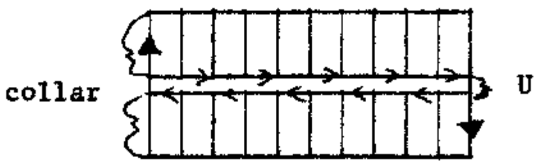
By continuity of r, on U the section looks like:



as you go around here the rotation is α

as you go around here the rotation is $-\alpha$.

Thus this section is isotopic to the trivial section



Now using the collar extend this isotopy to give an isotopy of $r|\partial Mb$ to the trivial section.

2. Seifert Fibrations as S^1-Actions.

Observe, if (M,F,π) is a Seifert fibration with M closed and oriented and F oriented, by the way M is constructed, we can put an S^1 action on M. The orbits of this action are the fibers of π. If (M,F,π) is a G.S. fibration with F oriented, then there is an S^1 action on M such that each fiber of π is either

a) an orbit if the fiber is non-singular or singular with $\alpha \neq 0$

b) a component of a fixed point set if it is a singular fiber with $\alpha = 0$.

The converse is also true.

Theorem 2.1. The classification of G.S. fibrations with F orientable is equivalent to the classification of effective S^1 actions on closed oriented 3-manifolds.

Before proving this theorem we need some definitions and results from the theory group actions on a manifold. A reference for these results is G. Bredon: Introduction to Compact Transformation Groups.

Definition. Let M be a smooth manifold and G a compact Lie group. A smooth G-action on M is a C^∞ map $G \times M \to M$, $(g,x) \mapsto gx$ satisfying

i) $1x = x$ for every $x \in M$

ii) $g_1(g_2x) = (g_1g_2)x$ for every $g_1, g_2 \in G$, for every $x \in M$.

This action is effective if $gx = x$ for every $x \in M$, then $g = 1$. This action is fixed point free if for every $x \in M$, there exists $g \in G$ such that $gx \neq x$. This action is free if for every $x \in M$ and for every $g \neq 1$, $gx \neq x$. We define the orbit space $M/G = M/\{x \equiv gx\}$, the orbit $Gx = \{gx \mid g \in G\}$ and we define the isotropy subgroup $G_x = \{g \in G \mid gx = x\}$.

Lemma 2.2. G_x is a closed subgroup of G. $G/G_x \cong Gx$ where this diffeomorphism is G-equivariant and given by $gG_x \mapsto gx$.

Example: 1) Let H be a compact Lie group and $\rho: H \to GL(n)$ be a representation. Then H acts on $\mathbb{R}^n$ by $h \cdot x = \rho(h)x$, $x \in \mathbb{R}^n$, $h \in H$.

2) Suppose $H \subseteq G$ is a closed subgroup and ρ is as above. We define $G \times_H \mathbb{R}^n = G \times \mathbb{R}^n/H$ where H acts on $G \times \mathbb{R}^n$ by $h(g,x) = (gh^{-1}, \rho(h)x))$.

Theorem 2.3. Let G,H be as in example 2). Then $G \times_H \mathbb{R}^n \to G/H$ given by $[g,v] \mapsto gH$ is a vector bundle with fiber $\mathbb{R}^n$. It has a natural G action given by $g_1[g,v] = [g_1 g,v]$.

Theorem 2.4. (Slice Theorem) Let G be a compact Lie group and $G \times M \to M$ a smooth action of G. Then

1) $Gx \subseteq M$ is a smooth submanifold

2) G_x acts on $V_x = \nu_x(Gx)$ by a representation $\rho_x: G_x \to GL(V_x)$ ($\nu_x(Gx)$ = normal bundle of Gx in M at x), called the "slice representation"

3) $G \times_{G_x} V_x$ is G-equivariantly diffeomorphic to a neighborhood of $Gx \subseteq M$ by a diffeomorphism which takes the zero section

$G/G_x = \{[g,0]\}$ to Gx.

4) After choosing an invariant Euclidean metric on V_x we can assume $\rho_x: G_x \to O(V_x)$.

We now return to Theorem 2.1. To prove this theorem, it suffices to show any effective S^1 action on a closed oriented 3-manifold M yields a G.S. fibration. By part 3 of the Slice theorem, if we know all the possible slice representations of the isotropy subgroups of S^1, we know what the orbits look like locally. We must show that locally these orbits look like a local model as in the definition of a G.S. fibration.

In our case $G = S^1$ and the possible isotropy subgroups are $G_x = \{1\}, \mathbb{Z}/n, S^1$.

Case 1. $G_x = \{1\}$

Then $G \times_{G_x} V = G \times V = S^1 \times \mathbb{R}^2$ since $\dim V = 2$.

Case 2. $G_x = \mathbb{Z}/n$

As in case 1, since $\dim G \times_{G_x} V = 3$ and G_x is discrete, we have $\dim V = 2$. For the action of G to be effective $\rho: G_x \to O(V)$ must be injective. If $n > 2$ the only possibility for ρ is a generator of $\mathbb{Z}/n$ goes to a rotation by $2\pi q/n$ where $\gcd(q,n) = 1$. If $n = 2$ we have the additional possibility, a generator goes to $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. In this case $G \times_{G_x} V$ is non-orientable and thus cannot occur. Thus $G \times_{G_x} V \cong (S^1 \times \mathbb{R}^2)/(\mathbb{Z}/n)$ which is again a standard model.

Case 3. $G_x = S^1$

In this case $\dim V = 3$. Again, for the action to be effective

$\rho: S^1 \to O(V)$ must be injective. There is only one such representation, namely writing $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}^1$, S^1 rotates $\mathbb{R}^2$ and fixes $\mathbb{R}^1$. Thus $G \times_{G_x} V = S^1 \times_{S^1} \mathbb{R}^3 \cong \mathbb{R}^3$ as S^1 manifolds. The fixed point set is thus a closed one dimensional submanifold of M so a component of the fixed point set looks like a $(0,1)$ fiber in a G.S. fibration.

Thus we have shown that given any orientable closed 3 manifold M with an effective S^1 action and orbit space M/S^1, then the non-fixed orbits and the fixed point components induce a G.S. fibration on M.

<u>Proposition 2.5.</u> Let M be a closed orientable effective S^1 manifold with Seifert invariant $(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ and $\mathbb{Z}/a \subseteq S^1$. Then $M/(\mathbb{Z}/a)$ is a $(S^1/(\mathbb{Z}/a))$ manifold and its Seifert invariant can be written $(g;(\alpha_1',\beta_1'),\ldots,(\alpha_n',\beta_n'))$ where $\beta_j'/\alpha_j' = a\beta_j/\alpha_j$. In particular $e(M/(\mathbb{Z}/a) \to M/S^1) = a \cdot e(M \to M/S^1)$.

<u>Proof:</u> By checking the local structure, $M/(\mathbb{Z}/a)$ is easily seen to be a 3-manifold. Remove from M tubular neighborhoods of a suitable collection of orbits. In M − {tubular neighborhoods} choose a section R which gives the stated Seifert invariant for M. The image of R in $M/(\mathbb{Z}/a)$ is still a section and this section gives the desired Seifert invariant for $M/(\mathbb{Z}/a)$.

3. Euler Number.

There are several equivalent ways of defining the Euler number for a genuine S^1-bundle. We list three:

1) An obstruction to finding a cross section.

2) A bundle is classified by an element of $[F,BS^1] = [F,K(\mathbb{Z},2)] = H^2(F;\mathbb{Z})$. If F is a closed surface $H^2(F;\mathbb{Z}) = \mathbb{Z}$ and the $\alpha \in H^2(F,Z)$ classifying the S^1 bundle is the Euler number

3) "Fill in the circle fibers" to get a D^2 bundle $E \to F$ with $\partial E = M$ $(E = M \times_{S^1} D^2 = (M \times D^2)/S^1)$. We have the zero section $F \hookrightarrow E$. Then $e(M \to F) = [F]\cdot[F]$ (self intersection number).

One can show that each of these ways has an extension to Seifert fibrations and they all give the same result. Our definition of $e(M \to F) = -\sum \alpha_i/\beta_i$ corresponds to the first definition. In this section we will give a definition of $e(M \to F)$ corresponding to 3), and it will be used in the proof of a theorem. To 2) we remark without proof that if $S^1_{(0)}$ is the rationalized circle, then to any Seifert bundle $M \to F$ is associated a genuine fibration over F with fiber $S^1_{(0)}$, classified by an element of $[F,BS^1_{(0)}] = [F,K(\mathbb{Q},2)] = H^2(F;\mathbb{Q})$, and this too is our Euler number.

Before stating and proving the main theorem of this section we need a proposition and in particular the corollary following.

Proposition 3.1. Let $M \overset{\Pi}{\to} F$ be a G.S. fibration and $p\colon F' \to F$ a covering with degree $(p) = d$. We can form the pullback bundle $p^*M \to F'$ where this diagram commutes:

$$\begin{array}{ccc} p^*M & \xrightarrow{\bar{p}} & M \\ \pi' \downarrow & & \downarrow \pi \\ F' & \xrightarrow{p} & F \end{array}$$

Then

i) $p^*M \to F'$ is a G.S. fibration

ii) If $M \overset{\pi}{\to} F \cong M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ then $p^*M \overset{\pi'}{\to} F' \cong M(g';d(\alpha_1,\beta_1),\ldots,d(\alpha_n,\beta_n))$ where $d(\alpha_i,\beta_i) = (\alpha_i,\beta_i),\ldots,(\alpha_i,\beta_i)$ (d times) and g' = genus F'.

<u>Proof</u>. $p^*M = \{(x,y) \in M \times F' \mid p(y) = \pi(x)\}$. Choose a section R to $M \overset{\pi}{\to} F$ outside of a collection of tubular neighborhoods of suitable fibers. Then $\{(x,y) \in R \times F' \mid \pi(x) = p(y)\}$ is a similar section in p^*M leading to the desired Seifert invariant.

<u>Corollary 3.2</u>. Let $M \overset{\pi}{\to} F$ be a G.S. fibration with Seifert invariant $(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ with $g < 0$, i.e. F is unorientable. Let $\bar{F} \overset{p}{\to} F$ be the orientation double cover of F. We form the pullback bundle $\bar{M} \overset{\bar{\pi}}{\to} \bar{F}$. Then:

i) $\bar{M} \overset{\bar{\pi}}{\to} \bar{F}$ is a G.S. fibration.

ii) $\bar{M} \overset{\bar{\pi}}{\to} \bar{F} \cong M(|g| - 1;\ (\alpha_1,\beta_1),(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n),(\alpha_n,\beta_n))$.

In particular $e(\bar{M} \to \bar{F}) = 2e(M \to F)$.

We now state the main theorem of this section.

<u>Theorem 3.3</u>. Let $M_1 \overset{\pi_1}{\to} F_1$ and $M_2 \overset{\pi_2}{\to} F_2$ be two Seifert fibrations. Assume there exists a map $\tilde{g}: M_1 \to M_2$ such that the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{\tilde{g}} & M_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ F_1 & \xrightarrow{g} & F_2 \end{array}$$

commutes and degree $(g) = b$, degree $(\tilde{g}|_{fiber}) = f$ and thus degree $(\tilde{g}) =$ Then $e(M_1 \to F_1) = (b/f)e(M_2 \to F_2)$. (Note: The pair (b,f) is determined only up to sign, but b/f is well defined.)

We leave as an exercise the

Remark: The theorem is valid also for G.S. fibrations.

In proving this theorem, we shall also show that $e(M \to F)$ is equal to the self intersection number of the zero section of the corresponding "disc bundle," which we now define.

Given a Seifert fibration (M,F,π), let $C(\pi)$ be the mapping cylinder, i.e. $C(\pi) = M \times [0,1] \cup_t F$ where $t: M \times \{1\} \to F$ is $t(x,1) = \pi(x)$. π induces a mapping $\bar{\pi}: C(\pi) \to F$ whose fibers are the cones over the fibers of π. This is a "Seifert disc bundle" over F, and $M = \partial(C(\pi))$ is the corresponding circle bundle. We have $F \hookrightarrow C(\pi)$ as the zero section. $C(\pi)$ is a 4-manifold except at points $p \in F$ over which the singular orbits of M lie.

Definition 3.4. The pair (X,Y) is an R-homology manifold pair of dimension n iff (X,Y) is a relative C.W. complex and

$$\tilde{H}_i(X, X - \{p\};R) = \begin{cases} R & i = n \\ 0 & i \neq n \end{cases}$$

for all $p \in X - Y$.

Claim. $(C(\pi),M)$ is a $\mathbb{Q}$ homology manifold pair.

Exercise. If G is a finite group with an orientation preserving action on a manifold N, then N/G is a $\mathbb{Q}$ homology manifold.

Proof of Claim. Using the above exercise, and since being an R-homology manifold is a local condition, it suffices to show that if $p \in F$ in $C(\pi)$ is a singular point, then p has a neighborhood homeomorphic to $\mathbb{R}^4/G$ for some finite G. Let p be a singular point and U a neighborhood of p. $\pi^{-1}(U) \overset{\pi}{\to} U$ "looks like" $S^1 \times_{\mathbb{Z}_\alpha} \mathbb{R}^2 \to \mathbb{R}^2/(\mathbb{Z}/\alpha)$ where $(\mathbb{Z}/\alpha)$ acts diagonally on $S^1 \times \mathbb{R}^2$, i.e. by right multiplication on S^1 and by some rotation on $\mathbb{R}^2$. The corresponding neighborhood of $\pi^{-1}(p)$ in $C(\pi)$ is $D^2 \times_{(\mathbb{Z}/\alpha)} \mathbb{R}^2 \cong (D^2 \times \mathbb{R}^2) / (\mathbb{Z}/\alpha)$.

Now the standard treatment of R-orientation, fundamental classes and Poincaré-Lefschetz duality (as for instance in Spanier: Algebraic Topology) carries through for R-homology manifold pairs (X,Y) with X compact and Y closed. Therefore $(C(\pi),M)$ satisfies Poincaré-Lefschetz duality with $\mathbb{Q}$ coefficients. Precisely, the sum of the top dimensional simplices in a subdivision of $C(\pi)$ defines a fundamental class $[C(\pi)] \in H_4(C(\pi),M;\mathbb{Q})$, and the maps

$$D: H^q(C(\pi);\mathbb{Q}) \to H_{4-q}(C(\pi),M;\mathbb{Q}) \quad \text{and}$$
$$D: H^q(C(\pi),M;\mathbb{Q}) \to H_{4-q}(C(\pi);\mathbb{Q})$$

defined by $D(\alpha) = \alpha \cap [C(\pi)]$ are isomorphisms.

In analogy to the case of a genuine S^1-bundle we now define

$e'(M \to F) = D(D^{-1}([F]) \cup D^{-1}([F]))$ where $[F] \in H_2(C(\pi);\mathbb{Q})$ is the homology class represented by $F \subseteq C(\pi)$. Denoting $D^{-1}(F) = \eta \in H^2(C(\pi);\mathbb{Q})$ we have

$$\begin{aligned} e'(M \to F) &= (\eta \cup \eta) \cap [C(\pi)] \\ &= \eta \cap (\eta \cap [C(\pi)]) \\ &= \eta \cap [F]. \end{aligned}$$

The proof of Theorem 3.3 is divided into two steps. The first step is to show $e'(M_1 \to F_1) = (b/f)\,e'(M_2 \to F_2)$. The second step is to show $e'(M \to F) = e(M \to F)$.

It may appear that we are implicitly assuming F is orientable above. However, if we take our coefficients $\mathbb{Q}$ in $H_2(C(\pi);\mathbb{Q})$ and $H^2(C(\pi),M;\mathbb{Q})$ to be the local coefficient system on $C(\pi)$ which pulls back from the orientation system on F (but take untwisted coefficients for H_4 and H^4), then our definition of $e'(M \to F)$, and the subsequent analysis, applies also for F unorientable. In the following proof we therefore implicitly assume these local coefficients are being used where necessary. The reader who prefers to avoid local coefficients can instead deduce the theorem in general from the special case of oriented base surfaces using corollary 3.2.

<u>Proof of Theorem 3.3</u>. (Step 1) We have

$$\begin{array}{ccc} M_1 & \xrightarrow{\tilde{g}} & M_2 \\ \downarrow & & \downarrow \\ F_1 & \xrightarrow{g} & F_2 \end{array} ,$$

this induces $G\colon C(\pi_1) \to C(\pi_2)$ with $G|M_1 = \tilde{g}$ and $G|F_1 = g$. Then degree (G) = degree $(\tilde{g}) = bf$. Let $\eta_i \in H^2(C(\pi_i);\mathbb{Q})$ $(i = 1,2)$ be as above. Then $e'(M_i \to F_i) = \eta_i \cap [F_i]$. $H^2(C(\pi_i);\mathbb{Q}) = H^2(F_i;\mathbb{Q}) = \mathbb{Q}$, so $G^*\eta_2$ is some multiple $k\eta_1$ of η_1. Thus

$$G^*\eta_2 \cap [C(\pi_1)] = k\eta_1 \cap [C(\pi_1)] = k[F_1]$$

$$G_*((G^*\eta_2) \cap [C(\pi_1)]) = G_*k[F_1] = bk[F_2].$$

On the other hand

$$\begin{aligned} G_*((G^*\eta_2) \cap [C(\pi_1)]) &= \eta_2 \cap G_*[C(\pi_1)] \\ &= bf\, \eta_2 \cap [C(\pi_2)] \\ &= bf[F_2] \end{aligned}$$

and hence $k = f$. Therefore $G^*\eta_2 = f\eta_1$, so

$$\begin{aligned} e'(M_1 \to F_1) &= \eta_1 \cap [F_1] \\ &= (1/f)G^*(\eta_2) \cap [F_1] \\ &= (1/f)G_*(G^*\eta_2 \cap [F_1]) \\ &= (1/f)(\eta_2 \cap G_*[F_1]) \\ &= (b/f)(\eta_2 \cap [F_2]) \\ &= (b/f)e'(M_2 \to F_2). \end{aligned}$$

(Step 2) We want to show that for any Seifert manifold $e(M \to F) = e'(M \to F)$. First assume F is oriented. Then M is a fixed point free

S^1-manifold and $M \cong M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$. Let a be a multiple of $\mathrm{lcm}(\alpha_1,\ldots,\alpha_n)$ and define $M' = M/(\mathbb{Z}/a)$. Then $M' = M(g;\ (1,a\beta_1/\alpha_1),\ldots,(1,a\beta_n/\alpha_n)) = M(g;\ (1,\ \sum a\beta_i/\alpha_i))$. Thus M' is a genuine S^1-bundle.

<u>Exercise</u>: For a genuine S^1-bundle $e' = e$.

With this exercise we have

$$\begin{aligned}(1/a)e(M \to F) &= e(M' \to F) && \text{(Proposition 3.1)}\\ &= e'(M' \to F) && \text{(exercise)}\\ &= (1/a)e'(M \to F) && \text{(Step 1).}\end{aligned}$$

Now if F is not oriented, let $\bar{M} \to \bar{F}$ be as constructed in Corollary 3.2 Then

$$\begin{aligned}e(M \to F) &= (1/2)e(\bar{M} \to \bar{F}) && \text{(Corollary 3.2)}\\ &= (1/2)e'(\bar{M} \to \bar{F}) && \text{(above)}\\ &= e'(M \to F) && \text{(Step 1).}\end{aligned}$$

4. Lens Spaces as Seifert Manifolds.

In this section we define lens spaces and show they are examples of Seifert fibrations. As we shall see in the next section, this allows us to give a classification of Seifert structures on a 3-manifold.

Definition 4.1. Let $S^3 = \{(Z_1,Z_2) \in \mathbb{C}^2 \mid |Z_1|^2 + |Z_2|^2 = 1\}$. $\mathbb{Z}/p$ acts freely on S^3 by $e^{2\pi ip}(Z_1,Z_2) = (e^{2\pi i/p}Z_1, e^{2\pi iq/p}Z_2)$ where $\gcd(p,q) = 1$ Define the lens space $L(p,q) = S^3/(\mathbb{Z}/p)$.

$L(p,q)$ has the following properties:

1) $S^3 \to L(p,q)$ is the universal cover.

2) Since the covering transformation group is $\mathbb{Z}/p$

 $\pi_1(L(p,q)) = \mathbb{Z}/p$

3) By elementary algebraic topology we get

 $H_1(L(p,q)) = \mathbb{Z}/p$

 $H_2(L(p,q)) = 0$

 $H_3(L(p,q)) = \mathbb{Z}.$

4) $L(p,q) \cong L(p,q')$ if $q \equiv q' \pmod p$

5) $L(p,q) \cong L(p,q')$ if $qq' \equiv 1 \pmod p$

 Proof: $e^{2\pi iq'/p}$ is a generator of $\mathbb{Z}/p$ and in $L(p,q)$ it takes $(Z_1,Z_2) \mapsto (e^{2\pi iq'/p}Z_1, e^{2\pi iqq'/p}Z_2) = (e^{2\pi iq'/p}Z_1, e^{2\pi i/p}Z_2$

 Thus by exchanging Z_1 and Z_2 the (p,q) action becomes the (p,q') action. Therefore $L(p,q) \cong L(p,q')$.

6) $L(p,q) \cong -L(p,-q)$

 Proof: The map $(Z_1,Z_2) \mapsto (Z_1,\bar{Z}_2)$ induces $L(p,q) \xrightarrow{\cong} -L(p,-q)$.

7) $L(p,q) \simeq L(p,q')$ (homotopy equivalent preserving orientation) if qq' is a square mod p.

Proof. See exercise 1.

__Theorem 4.2__. $L(p,q) \cong L(p,q')$ iff $q \equiv (q')^{\pm 1} \pmod p$

$L(p,q) \simeq L(p,q')$ iff $qq' \equiv$ square (mod p).

Here both the diffeomorphism and the homotopy equivalence are orientation preserving.

__Proof:__ See J. H. C. Whitehead, "On incidence matrices, nuclei and homotopy types," __Ann. of Math.__ vol. 42, 1941.

$T^2 = S^1 \times S^1$ acts on S^3 by $(t_1,t_2)(Z_1,Z_2) = (t_1Z_1,t_2Z_2)$. The above $\mathbb{Z}/p$ action is a subaction of T^2 on S^3. Thus $T^2/(\mathbb{Z}/p)$ acts on $S^3/(\mathbb{Z}/p) \simeq L(p,q)$. $T^2/(\mathbb{Z}/p) \cong T^2$. There are many S^1 subgroups of $T^2/(\mathbb{Z}/p)$ giving effective actions on $L(p,q)$, hence there are many Seifert and G.S. fibrations on $L(p,q)$.

__Theorem 4.3__. $L(p,q) \cong D^2 \times S^1 \underset{\begin{pmatrix} -q & r \\ p & s \end{pmatrix}}{\cup} D^2 \times S^1$ if $\det\begin{pmatrix} -q & r \\ p & s \end{pmatrix} = -1$.

__Remark.__ If we define $L(p,q) = L(-p,-q)$ if $p < 0$

$$L(1,0) = S^3$$

$$L(0,1) = S^1 \times S^2,$$

then the theorem remains true with these conventions.

__Proof of Theorem 4.3__. Let $S^3 = \{(Z_1,Z_2) \mid |Z_1|^2 + |Z_2|^2 = 2\}$. Then $S^3 = U_1 \cup U_2$ where $U_1 = \{(Z_1,Z_2) \in S^3 \mid |Z_2|^2 \geq 1\}$ and

$U_2 = \{(Z_1,Z_2) \in S^3 \mid |Z_2|^2 \leq 1\}$. $U_1 \cong (D^2 \times S^1)_1$ where the diffeomorphism is given by $(Z_1,Z_2) \mapsto (Z_1, Z_2/\|Z_2\|)$. Similarly $U_2 \cong (D^2 \times S^1)_2$ by $(Z_1,Z_2) \mapsto (Z_2,Z_1/\|Z_1\|)$. Thus the pasting map $\partial(D^2 \times S^1)_1 = S^1 \times S^1 \to S^1 \times S^1 = \partial(D^2 \times S^1)_2$ is $(Z_1,Z_2) \mapsto (Z_2,Z_1)$ i.e. given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Now if we factor by the $\mathbb{Z}/p$ action we have $(D^2 \times S^1)_1/(\mathbb{Z}/p) \overset{\cong}{\to} D^2 \times S^1$ by $[Z_1,Z_2] \mapsto (Z_2^{-s}Z_1, Z_2^p)$. Here s is chosen such that $qs \equiv 1 \pmod p$. Thus a matrix $\begin{pmatrix} -q & r \\ p & s \end{pmatrix}$ of determinant -1 exists. We also have $(D^2 \times S^1)_2/(\mathbb{Z}/p) \overset{\cong}{\to} D^2 \times S^1$ by $[Z_2,Z_1] \mapsto (Z_1^{-q}Z_2, Z_1^p)$. Thus $L(p,q) = S^3/(\mathbb{Z}/p) \cong D^2 \times S^1 \cup_f D^2 \times S^1$ where $f: S^1 \times S^1 \to S^1 \times S^1$ and is given by $f(Z_2^{-s}Z_1, Z_2^p) = (Z_1^{-q}Z_2, Z_1^p)$. To put f in a more convenient form let $t_1 = Z_2^{-s}Z_1$, $t_2 = Z_2^p$. Then $Z_1^{-q}Z_2 = t_1^{-q}t_2^r$, $Z_1^p = t_1^p t_2^s$. Therefore $f: S^1 \times S^1 \to S^1 \times S^1$ is given by $f(t_1,t_2) = (t_1^{-q}t_2^r, t_1^p t_2^s)$.

<u>Remark.</u> We can choose M_i, L_i a meridian and longitude in $\partial(D^2 \times S^1)_i$ so $M_i = \{(t,1) \in \partial(D^2 \times S^1)_i\}$, $L_i = \{(1,t) \in \partial(D^2 \times S^1)_i\}$. Then under the pasting map $M_1 \mapsto \{(t^{-q},t^p) \in \partial(D^2 \times S^1)_2\}$, so $M_1 \sim -qM_2 + pL_2$. Thus this homology relation determines which lens space we are in.

We will apply this to find the Seifert invariants of the various G.S. fibered structures on $L(p,q)$.

<u>Theorem 4.4.</u> $L(p,q) \cong M(0;(\alpha_1,\beta_1),(\alpha_2,\beta_2))$ if

$$p = \det\begin{pmatrix} \alpha_1 & \alpha_2 \\ -\beta_1 & \beta_2 \end{pmatrix} = \alpha_1\beta_2 + \alpha_1\beta_2$$

$$q = \det\begin{pmatrix} \alpha_1 & \alpha_2' \\ -\beta_1 & \beta_2' \end{pmatrix} = \alpha_1\beta_2' + \beta_1\alpha_2'$$

where $\det\begin{pmatrix}\alpha_2 & \alpha_2'\\ \beta_2 & \beta_2'\end{pmatrix} = \alpha_2\beta_2' - \beta_2\alpha_2' = 1.$

<u>Proof</u>: $M(0;(\alpha_1,\beta_1),(\alpha_2,\beta_2)) \cong ((D^2 \times S^1) \cup (\text{annulus} \times S^1)) \cup (D^2 \times S^1)$

$$= (D^2 \times S^1) \cup (D^2 \times S^1).$$

We have homology relations $M_1 \sim \alpha_1 Q_1 + \beta_1 H_1 \quad M_2 \sim \alpha_2 Q_2 + \beta_2 H_2$

$$L_1 \sim \alpha_1' Q_1 + \beta_1' H_1 \quad L_2 \sim \alpha_2' Q_2 + \beta_2' H_2$$

where $\det\begin{pmatrix}\alpha_1 & \alpha_1'\\ \beta_1 & \beta_1'\end{pmatrix} = \det\begin{pmatrix}\alpha_2 & \alpha_2'\\ \beta_2 & \beta_2'\end{pmatrix} = 1.$ $Q_1 + Q_2 = \partial$ (section in annulus $\times S^1$) ~ 0

in homology, thus $Q_1 \sim -Q_2$. Also $H_1 \sim H_2$. Therefore

$$\begin{pmatrix}M_1\\ L_1\end{pmatrix} = \begin{pmatrix}\alpha_1 & \beta_1\\ \alpha_1' & \beta_1'\end{pmatrix}\begin{pmatrix}Q_1\\ H_1\end{pmatrix} = \begin{pmatrix}-\alpha_1 & \beta_1\\ -\alpha_1' & \beta_1'\end{pmatrix}\begin{pmatrix}Q_2\\ H_2\end{pmatrix}$$

$$= \begin{pmatrix}-\alpha_1 & \beta_1\\ -\alpha_1' & \beta_1'\end{pmatrix}\begin{pmatrix}\beta_2' & -\beta_2\\ -\alpha_2' & \alpha_2\end{pmatrix}\begin{pmatrix}M_2\\ L_2\end{pmatrix}$$

$$= \begin{pmatrix}-(\alpha_1\beta_2' + \beta_1\alpha_2') & \alpha_1\beta_2 + \beta_1\alpha_2\\ -(\alpha_1'\beta_2' + \beta_1'\alpha_2') & \alpha_1'\beta_2 + \beta_1'\alpha_2\end{pmatrix}\begin{pmatrix}M_2\\ L_2\end{pmatrix},$$

so by the remark preceding the theorem, Theorem 4.4 is proved.

<u>Example</u>: Let S^1 act on S^3 by $t(Z_1,Z_2) = (t^a Z_1, t^b Z_2)$ where $\gcd(a,b) = 1$. The isotropy subgroups are $\mathbb{Z}/a$ and $\mathbb{Z}/b$. By the above theorem, we get $e(S^3 \to S^3/S^1) \cong M(0; (a,a'),(b,b'))$ where $ab' + ba' = \pm 1$.

<u>Exercise</u>: The correct sign here is $ab' + ba' = +1$, so $e(S^3 \to S^3/S^1) = -(a'/a) - (b'/b) = -(1/ab)$.

5. Classification of Seifert Fiberable Manifolds.

With the previous section as preparation, we can now state the classification, up to (not necessarily fiber preserving) orientation preserving homeomorphism of Seifert and G.S. fiberable 3-manifolds.

Theorem 5.1.

1) $M(-1;(\alpha,\beta)) \cong M(0;\ (2,1),(2,-1),(-\beta,\alpha))$

   $M(-2;(1,0)) \cong M(0;(2,1),(2,1),(2,-1),(2,-1))$.

2) The diffeomorphisms in 1) and Seifert fibered structures on lens spaces give the only examples of a 3-manifold having two non isomorphic Seifert fibrations.

3) $$M(g;(0,1),(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n)) \cong \mathop{\#}_{i=1}^{2g} (S^1\times S^2) \# \mathop{\#}_{i=1}^{n} -L(\alpha_i,\beta_i) \quad \text{if } g \geq 0$$
   $$\cong \mathop{\#}_{i=1}^{|g|} (S^1\times S^2) \# \mathop{\#}_{i=1}^{n} -L(\alpha_i,\beta_i) \quad \text{if } g < 0.$$

4) The only Seifert fibered manifold which is not connected sum prime is $M(-1;(1,0)) \cong \mathbb{R}P^3 \# \mathbb{R}P^3$.

Proof. For 1) and 3) see exercise 2. 2) and 4) will follow from our analysis of $\pi_1(M)$. See [O-V-Z], [O-R] In fact much more is true:

Theorem 5.2. (Waldhausen [Wa 1,2]) Let M_1, M_2 be Seifert fibered and not in the following list:

i) lens spaces

ii) $M(0;(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3))$

iii) $M(1;(1,0)) = T^3$

iv) As in part 1 of Theorem 5.1.

Then any homomorphism $M_1 \to M_2$ is isotopic to a fiber preserving homeomorphism.

6. The Fundamental Group of a Seifert Manifold.

Theorem 6.1. Let $M = M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$, then:

$$\pi_1(M) = \langle a_i,b_i,q_j,h \mid [h,a_i] = [h,b_i] = [h,q_j] = 1,$$
$$q_j^{\alpha_j}h^{\beta_j} = 1,\ q_1q_2,\ldots,q_m[a_1,b_1],\ldots,[a_n,b_n] = 1\rangle \quad \text{if} \quad g \geq 0$$

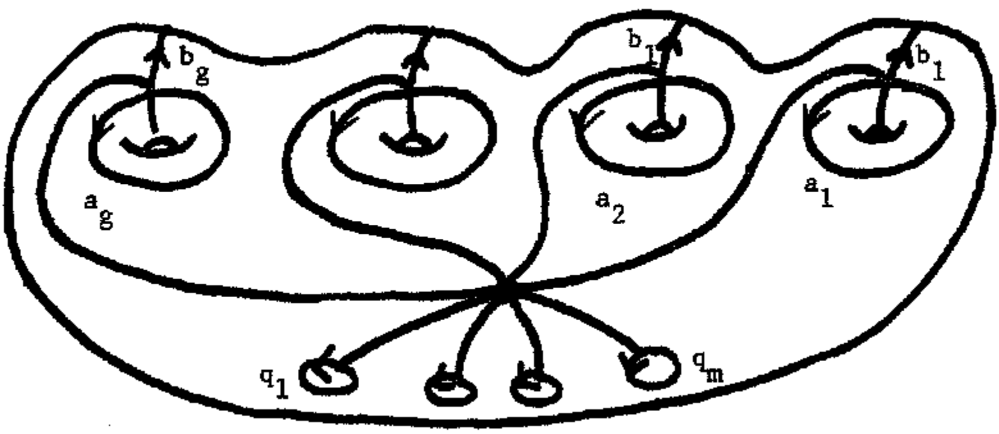
$$\pi_1(M) = \langle a_i,q_j,h \mid a_i^{-1}ha_i = h^{-1},\ [h,q_j] = 1,$$
$$q_j^{\alpha_j}h^{\beta_j} = 1,\ q_1,\ldots,q_m a_1^2,\ldots,a_{|g|}^2 = 1\rangle \quad \text{if} \quad g < 0,$$
$$(i=1,\ldots,|g|;\ j=1,\ldots,m).$$

Proof. We prove only the case $g \geq 0$. The proof for $g < 0$ is analogous We shall apply Van Kampen's theorem to the representation

$$M = (F - mD^2) \times S^1 \cup T_1 \cup \ldots \cup T_m \qquad T_i = D^2 \times S^1.$$

$$\pi_1(F - mD^2) = \langle a_1,b_1,\ldots,a_g,b_g,q_1,\ldots,q_m \mid$$
$$q_1\ldots q_m[a_1,b_1]\ldots[a_g,b_g] = 1\rangle$$

where the a_i,b_i,q_j are represented in schematically in the figure below.



Then $\pi_1(F - mD^2 \times S^1) = \langle a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_m, h \mid \prod_{j=1}^{m} q_j \prod_{i=1}^{g} [a$

$$[h, a_i] = [h, b_j] = [h, q_j] = 1\rangle.$$

__Claim.__ Pasting in T_j adds the relation $q_j^{\alpha_j} h^{\beta_j} = 1$.

__Proof.__ By Van Kampen's theorem pasting in T_j adds a new generator and two new relations

i) $q_j^{\alpha_j} h^{\beta_j} = 1$

ii) $q_j^{\alpha_j'} h^{\beta_j'} = t.$

The new generator t and relation ii) can be deleted by a Tietze transformation.

__Corollary 6.2.__ If $g \geq 0$

$$H_1(M;\mathbb{Z}) = \langle A_i, B_i, Q_j, H \mid \alpha_j Q_j + \beta_j H = 0,\ Q_1 + \ldots + Q_m = 0\rangle$$

$$= \mathbb{Z}^{2g} \oplus \langle Q_j, H \mid \alpha_j Q_j + \beta_j H = 0,\ Q_1 + \ldots + Q_m = 0\rangle$$

$$= \mathbb{Z}^{2g} \oplus \operatorname{cok} \begin{pmatrix} 1 & 0 & \cdots & 1 & 0 \\ \alpha_1 & 0 & \cdots & 0 & \beta_1 \\ 0 & \alpha_2 & \cdots & 0 & \beta_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \alpha_m & \beta_m \end{pmatrix}$$

In particular: If $e(M \to F) \neq 0$ then $H_1(M;\mathbb{Z}) = \mathbb{Z}^{2g} \oplus T$ with $|T| = \alpha_1, \ldots, \alpha_m\ |e(M \to F)|$.

If $e(M \to F) = 0$ and (M, F, π) is a Seifert fibration then

$H_1(M;\mathbb{Z})$ has free rank $2g+1$.

<u>Proof</u>: Cok A has order equal to $|\det A|$ if $\det A \neq 0$. Here $\det A = (-1)^m\alpha_1\cdots\alpha_m e(M\to F)$ by a simple induction. If $\det A = 0$ then rank $A = m$ if no α_i is 0.

<u>Corollary 6.3</u>. If $g \geq 0$ and (M,F,π) is a Seifert fibration, the following is a short exact sequence:

$$1 \to C \to \pi_1(M) \to \Gamma(g,\alpha_1,\ldots,\alpha_m) \to 1$$

where C is the central cyclic subgroup of π generated by h and $\Gamma(g;\alpha_1,\ldots,\alpha_n) = \langle a_1,b_1,\ldots,a_g,b_g,q_1,\ldots,q_j \mid q_j^{\alpha_j} = 1, \prod_j q_j \prod_i [a_i,b_i] = 1\rangle$

<u>Remark</u>. $\Gamma(g;\alpha_1,\ldots,\alpha_n)$ is a spherical, Euclidean, or hyperbolic crystallographic group according as $(2g-2) + \sum(\alpha_i - 1)/\alpha_i <, =, > 0$ respectively. We will see what this means and its significance for Seifert manifolds later.

We are now in a position to determine which Seifert manifolds are homology spheres. Assume $M = M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ is a homology sphere. We can immediately conclude $e(M\to F) \neq 0$ and $g = 0$. For if $e(M\to F) = 0$ or $g > 0$, then by Corollary 6.2 $H_1(M;\mathbb{Z})$ is infinite. Also $g < 0$ cannot occur, for otherwise M would admit a connected 2-fold cover, implying $H^1(M;\mathbb{Z}/2) \neq \{0\}$. We have

$$1 = |H_1(M)| = \alpha_1,\ldots\alpha_n \mid \sum \beta_i/\alpha_i|$$

$$= |\beta_1\alpha_2 \ldots\alpha_n + \beta\alpha_1\hat{\alpha}_2\ldots\alpha_n + \ldots + \beta_1\alpha_1\ldots\alpha_{n-1}|.$$

By reversing orientation if necessary, we can assume $\sum_{i=1}^{n} \beta_i\alpha_1\ldots\hat{\alpha}_i\ldots\alpha_n=1$ Therefore the α_i's must be pairwise coprime, since if $d|\alpha_i$ and $d|\alpha_j$ then $d| \sum \beta_i\alpha_1\ldots\hat{\alpha}_i\ldots\alpha_n$ and Hence $d|1$.

Moreover β_i is determined modulo α_i by

$$\beta_i\alpha_1\ldots\hat{\alpha}_i\ldots\alpha_n \equiv 1 \bmod {}_i.$$

This completely determines the Seifert manifold since we know $e(M \to F)$ and β_i mod α_i for each i.

Conversely, if we have pairwise coprime α_i's we can find β_i satisfying $\sum \beta_i\alpha_i\ldots\hat{\alpha}_i\ldots\alpha_n = 1$. This proves:

<u>Theorem 6.4</u>. Given pairwise co-prime $\alpha_1,\ldots,\alpha_n$, there exists a unique Seifert manifold $M = \sum(\alpha_1,\ldots,\alpha_n)$ with the α_i's representing exceptional fibers and $e(\sum(\alpha_1,\ldots,\alpha_n) \to S^2) = -(1/(\alpha_1\ldots\alpha_n)$ is a homology sphere.

<u>Example</u>. $\sum(2,3,5)$. Here $e(\sum(2,3,5) \to S^2) = -1/30$. Thus $\beta_1/2 + \beta_2/3 + \beta_3/5 = 1/30$ and we have

$$\sum(2,3,5) = M(0;(2,1),(3,1),(5,-4))$$

$$= M(0;(1,-1),(2,1),(3,1),(5,1)).$$

EXERCISES TO CHAPTER I

Exercise 1 (Homotopy classification of Lens spaces)

1) Show that a degree 1 map $L(p,q) \to L(p,q')$ is an orientation preserving homotopy equivalence.

2) Show that if $\phi: L(p,q) \to L(p,q')$ has degree d, then by suitable connected summing with the covering projection $S^3 \to L(p,q')$ you can get a map $\phi': L(p,q) \to L(p,q')$ of degree $d \pm p$. Thus if $\phi: L(p,q) \to L(p,q')$ exists of degree congruent to 1 mod p, then $L(p,q) \simeq L(p,q')$ (preserving orientation).

3) If $ab \equiv 1 \bmod p$ then the map $(Z_1,Z_2) \mapsto (Z_1^a,Z_2^b)/\|(Z_1^a,Z_2^b)\|$ of S^3 induces a map $L(p,q) \to L(p,b^2q)$ of degree ab, so by 1) and 2) $L(p,q) \simeq L(p,b^2q)$.

4) Conversely, show q is determined up to squares by the homotopy type of $L(p,q)$ as follows: Let $\beta: H^1(X;\mathbb{Q}/\mathbb{Z}) \to H^2(X:\mathbb{Z})$ be the connecting homomorphism for the coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. For $g_1,g_2 \in H^2(L(p,q);\mathbb{Z})$ define $\ell(g_1,g_2) = g_1 \cup \beta^{-1}(g_2) \in H^3(L(p,q);\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$. Show $\ell(-,-)$ is well defined, and ℓ determines q up to squares as follows: for any generator $g \in H^2(L(p,q);\mathbb{Z})$ one has $\ell(g,g) = qx^2/p$ for some x prime to p.

__Remark.__ Via Poincaré duality ℓ becomes the "torsion linking form," which is more generally defined for any closed oriented $2n+1$ manifold as

$$\ell: \begin{array}{ccc} \mathrm{Tor}\, H^{p+1}(M;\mathbb{Z}) \times \mathrm{Tor}\, H^{2n+1-p}(M;\mathbb{Z}) & & \\ \| \qquad\qquad\qquad \| & \to & \mathbb{Q}/\mathbb{Z}\,. \\ \mathrm{Tor}\, H_{2n-p}(M;\mathbb{Z}) \times \mathrm{Tor}\, H_p(M;\mathbb{Z}) & & \end{array}$$

The above is essentially the original approach, due to Rueff (<u>Compositio Math</u>. 6 (1938)), to classify $L(p,q)$'s up to homotopy.

*5) Use a similar approach to homotopy classify higher dimensional lens spaces $L^{2n+1}(p;q_1,\ldots,q_n)$, by replacing ℓ of 4) by an n-linear map $\ell(g_1,\ldots,g_n) = g_1 \cup \ldots \cup g_{n-1} \cup \beta^{-1}(g_n)$.

<u>Exercise 2</u>

1) Prove the following diffeomorphisms of G.S. fibered manifolds (preserving orientation but, of course, not preserving G.S. fibration), As stated in Section 5, these examples plus lens spaces give the only examples of non-equivalent G.S. fibrations of the same manifold.

a) $M(-1;(\alpha,\beta)) \cong M(0;(2,1),(2,-1),(-\beta,\alpha))$

b) $M(-2;(1,0)) \cong M(0;(2,1),(2,1),(2,-1),(2,-1))$

c) $M(g;(0,1),(\alpha_j,\beta_j),\ j = 1,\ldots,n) \cong \mathop{\#}\limits_{i=1}^{k} S^1 \times S^2 \# \mathop{\#}\limits_{j=1}^{n} L(\alpha_j,-\beta_j)$ where $k = 2g$ $(g \geq 0)$ or $|g|$ $(g < 0)$.

<u>Hints</u> a) T^1Mb (the unit tangent bundle of the Möbius band) has two natural Seifert fibrations. The one is the projection $T^1Mb \to Mb$ and the other is given by the S^1-action on T^1Mb induced by an effective S^1-action on Mb. This gives two G.S. fibrations on any manifold of the form $T^1Mb \cup_{T^2} D^2 \times S^1$.

b) $T^1K\ell$ has two Seifert fibrations for the same reason T^1Mb does.

Define an operation of "connected sum along $(0,1)$-orbits" to show how to build up a G.S.-fibration on a connected sum of simpler non-Seifert G.S.-fibered manifolds. You know c) for $g = 0$ and $n = 1$ by classification of GS-fibrations on Seifert manifolds. Hence you know it also for $g = -1$, $n = 0$, by a). You only need it then for $g = 1$, $n = 0$. Find a suitable G.S. fibration on $((S^1 \times S^2) - 2D^2) \cup (S^2 \times I) \cong S^1 \times S^2 \# S^1 \times S^2$.

II. Further Examples

In this chapter we give two less basic examples of Seifert manifolds: Breiskorn complete intersections and the universal abelian cover of certain Seifert manifolds.

7. Breiskorn Complete Intersections.

Let $V(a_1,a_2,a_3) = \{(Z_1,Z_2,Z_3) \in C^3 \mid Z_1^{a_1} + Z_2^{a_2} + Z_3^{a_3} = 0\}$ where $a_1,a_2,a_3 \geq 2$. V has an isolated singular point at 0. We define the link of the singularity as $V(a_1,a_2,a_3) \cap S^5$ and denote it by $\sum(a_1,a_2,a_3)$. $\sum(a_1,a_2,a_3)$ has a natural S^1 action given by:

$$t(Z_1,Z_2,Z_3) = (t^{a/a_1}Z_1, t^{a/a_2}Z_2, t^{a/a_3}Z_3) \quad \text{where} \quad a = \operatorname{lcm}(a_1,a_2,a_3).$$

This is an effective fixed point free action on $\sum(a_1,a_2,a_3)$. Therefore $\sum(a_1,a_2,a_3) \to \sum(a_1,a_2,a_3)/S^1$ is a Seifert fibration.

We can generalize this example by letting $A = (\alpha_{ij})^{i=1,\ldots,n-2}_{j=1,\ldots,n}$ be an $(n-2) \times n$ dimensional complex matrix and defining:

$$V_A(a_1,\ldots,a_n) = \{(Z_1,\ldots,Z_n) \in \mathbb{C}^n \mid \alpha_{i1}Z_1^{a_1} + \alpha_{i2}Z_2^{a_2} + \ldots + \alpha_{in}Z_n^{a_n} = 0,\ i = 1,\ldots,n-2\}$$

<u>Proposition 7.1.</u> V_A is a 2-dimensional complex variety which is non-singular except at 0 iff each maximal $(n-2) \times (n-2)$ submatrix of A is non-singular.

<u>Proof</u>: Let $f_i(Z_1,\dots,Z_n) = \sum_j \alpha_{ij}Z_j^{a_j}$ and $f = (f_1,\dots,f_{n-2})$, $f\colon \mathbb{C}^n \to \mathbb{C}^{n-2}$ and $V_A = f^{-1}(0)$. We want to show: Df has rank $n-2$ at each point of $V_A - \{0\}$ iff each $(n-2)\times(n-2)$ submatrix of A is non-singular.

$$Df = (\partial f_i/\partial Z_j) = (\alpha_{ij}a_jZ_j^{a_j-1}) = A\begin{pmatrix} a_1Z_1^{a_1-1} & & 0 \\ & \ddots & \\ 0 & & a_nZ_n^{a_n-1}\end{pmatrix}.$$

Assume there is an $(n-2)\times(n-2)$ submatrix of Df that is singular. Without loss of generality, we can assume the first $n-2$ columns of A are linearly dependent. By a change of coordinates in $\mathbb{C}^{n-2}$, f is equivalent to $(\bar{f}_1,\bar{f}_2,\dots,\bar{f}_{n-2})$ where $\bar{f}_1(Z_1,\dots,Z_n) = \bar{\alpha}_{1n-1}Z_{n-1}^{a_{n-1}} + \bar{\alpha}_{1n}Z_n^{a_n}$. Then $\{\bar{f}_2 = \bar{f}_3 = \dots = \bar{f}_{n-2} = Z_{n-1} = Z_{n-2} = 0\}$ is at least 1-dimensional (and in particular not zero) and contained in V_A. On this set Df has rank less than $n-2$, which is a contradiction.

Conversely, if every $(n-2)\times(n-2)$ submatrix is non-singular then Df has rank $n-2$ if at least $n-2$ Z_i's are non-zero. On $V_A - \{0\}$ if $Z_i = Z_j = 0$ then $Z_k^{a_k} = 0$ for the remaining $n-2$ indices k, which is a contradiction.

We call a matrix that satisfies the conditions of proposition 7.1 "good," and assume from now on A satisfies these conditions.

Definition $\sum_A(a_1,\dots,a_n) = V_A \cap S^{2n-1}$. $\mathbb{C}^* = \mathbb{C} - \{0\}$ acts on $\mathbb{C}^n$ by $t(Z_1,\dots,Z_n) = (t^{a/a_1}Z_1,\dots,t^{a/a_n}Z_n)$ for all $t \in \mathbb{C}^*$ where $a = \mathrm{lcm}(a_i)$. This action preserves V_A. Therefore $S^1 \subseteq \mathbb{C}^*$ acts on $\sum_A(a_1,\dots,a_n)$ making $\sum_A(a_1,\dots,a_n) \to \sum_A(a_1,\dots,a_n)/S^1$ a Seifert fibration.

<u>Remark</u>. $\sum_A(a_1,\dots,a_n)$ is (as a Seifert manifold) independent of the

choice of A with A good. This follows from the fact that $\{A \mid A \text{ is good}\}$ is connected. We denote $\sum_A(a_1,\ldots,a_n)$ by $\sum(a_1,\ldots,a_n)$. Of course the complex structure on V_A <u>does</u> depend on A.

<u>Theorem 7.2</u>. $\sum(a_1,\ldots,a_n) = M(g;s_1(t_1,\beta_1),\ldots,s_n(t_n,\beta_n))$ where

$$t_i = a/\operatorname{lcm}_{j\neq i}(a_j)$$

$$s_i = (\prod_{j\neq i} a_j)/\operatorname{lcm}_{j\neq i}(a_j)$$

$$g = (1/2)(2 + (n-2)(\Pi a_i)/a - \sum_{j=1}^{n} s_j)$$

$$e(\textstyle\sum(a_1,\ldots,a_n) \to \sum(a_1,\ldots,a_n)/S^1) = -((\Pi a_i)/a^2).$$

Note that these four equations determine $\beta_i \pmod{t_i}$. The last equation can be written $\sum_j s_j\beta_j/t_j = \Pi(a_i)/a^2$. Dividing by the right side gives $\sum_j (a/a_j)\beta_j = 1$. Since $t_i = \gcd_{j\neq i}(a/a_j)$, t_i divides a/a_j if $i \neq j$, so $(a/a_i)\beta_i \equiv 1 \pmod{t_i}$.

<u>Example</u>. Assume the a_i are pairwise coprime. Then

$$t_i = a_i$$
$$s_i = 1$$
$$g = (1/2)(2 + (n-2)\cdot 1 - n) = 0$$
$$e = -1/(a_1\ldots a_n).$$

This is the Seifert homology sphere $\sum(a_1,\ldots,a_n)$. Thus our notation is consistent with that of the previous section.

<u>Proof of Theorem 7.2.</u> We require two facts from basic number theory which we state without proof.

i) if $\mathbb{Z}/m_i \subseteq S^1$, $i = 1,\dots,k$ then $\bigcap_{i=1}^{k} \mathbb{Z}/m_i = \mathbb{Z}/\gcd(m_i)$

ii) $\gcd(m/m_i) = m/\operatorname{lcm}(m_i)$

$\operatorname{lcm}(m/m_i) = m/\gcd(m_i)$.

Note if $Z = (Z_1,\dots,Z_n) \in V_A$ and $Z_i \neq 0$ for all i, then the isotropy subgroup $S^1_Z = \{1\}$. This follows from

$$S^1_Z = \mathbb{Z}/(a/a_1) \cap \mathbb{Z}/(a/a_2) \cap \dots \cap \mathbb{Z}/(a/a_n)$$

$$= \mathbb{Z}/\gcd(a/a_i) = \mathbb{Z}/(a/\operatorname{lcm}(a_i)) = \{1\}.$$

Similarly, if $Z = (Z_1,\dots,Z_n) \in V_A$ and $Z_i = 0$, $Z_j \neq 0$ for all $i \neq j$ then

$$S^1_Z = \mathbb{Z}/(a/a_1) \cap \dots \cap \widehat{\mathbb{Z}/(a/a_i)} \cap \dots \cap \mathbb{Z}/(a/a_n)$$

$$= \mathbb{Z}/\gcd_{j\neq i}(a/a_j) = \mathbb{Z}/(a/\operatorname{lcm}_{j\neq i}(a_j)) = \mathbb{Z}/t_i .$$

We also have, if $Z = (Z_1,\dots,Z_n) \in V_A$ and $Z_i = Z_j = 0$ for $i \neq j$ then $Z = 0$, so $Z \notin \sum(a_1,\dots,a_n)$.

We must next compute the number of orbits with isotropy $\mathbb{Z}/t_i$ or more precisely, the number s_i of orbits in $\sum(a_1,\dots,a_n) \cap \{Z_i = 0\}$. We shall later show that (for fixed i) all these s_i orbits have the same β's.

Each orbit in $\sum(a_1,\dots,a_n) \cap \{Z_1 = 0\}$ contains at least one point of the form $(0,r_2,Z_3,\dots,Z_n)$ where $r_2 \in \mathbb{R}_+$.

a) In fact each orbit contains exactly $a/(a_2 t_1)$ such points since $\mathbb{Z}/(a/a_2)$ maps such a point to a similar point, while $\mathbb{Z}/\gcd_{i \neq 1}(a/a_i) = \mathbb{Z}/t_1$ maps such a point to itself.

b) $\sum(a_1,\ldots,a_n)$ contains exactly $a_3 \ldots a_n$ such points since $\alpha_{i2} r_2^{a_2} + \alpha_{i3} Z_3^{a_3} + \ldots + \alpha_{in} Z_n^{a_n} = 0$, $i = 1,\ldots,n-2$, determines $(0, r_2^{a_2}, Z_3^{a_3}, \ldots, Z_n^{a_n})$ up to a multiple. As we are on S^{2n-1}, r_2 is determined and Z_j is determined up to a_j-th roots of unity.

c) a) and b) imply there are $(a_3 \ldots a_n)/(a/(a_2 t_1)) = (a_2 \ldots a_n t_1)/a = s$ orbits with $Z_1 = 0$.

To complete the proof we must verify the statements concerning g and e, and show the β's for all orbits in $\sum(a_1,\ldots,a_n) \cap \{Z_i = 0\}$ are the same for fixed i.

We have a map $\Phi\colon V_A(a_1,\ldots,a_n) - \{0\} \to V_A(1,\ldots,1) - \{0\}$ given by $(Z_1,\ldots,Z_n) \mapsto (Z_1^{a_1},\ldots,Z_n^{a_n})$. If this map induced a map $\varphi\colon \sum(a,\ldots,a_n) \to \sum(1,\ldots,1)$ we could use φ and theorem 3.3 to compute e. However $\Phi(S^{2n-1}) \not\subseteq S^{2n-1}$ and we must show Φ can be nevertheless used to induce such a φ. We have $\mathbb{R}_+ \subseteq \mathbb{C}^*$ and

$$\begin{array}{ccc} \sum_A(a_1,\ldots,a_n) & \hookrightarrow & V_A(a_1,\ldots,a_n) - \{0\} \\ & \searrow^{\cong} & \downarrow \\ & & (V_A(a_1,\ldots,a_n) - \{0\})/\mathbb{R}_+ \end{array}$$

Denote
$$\begin{aligned} \textstyle\sum_A(a_1,\ldots,a_n)/S^1 &\cong (V_A(a_1,\ldots,a_n) - \{0\})/(\mathbb{R}_+ \times S^1) \\ &\cong (V_A(a_1,\ldots,a_n) - \{0\})/\mathbb{C}^* \end{aligned}$$

by $P_A(a_1,\ldots,a_n)$.

We have:

$$\begin{array}{ccc}
V_A(a_1,\dots,a_n) - \{0\} & \xrightarrow{\Phi} & V_A(1,\dots,1) - \{0\} \\
\downarrow /\mathbb{R}_+ & & \downarrow /\mathbb{R}_+ \\
\sum_A(a_1,\dots,a_n) & \xrightarrow{\varphi} & \sum_A(1,\dots,1) \\
\downarrow /S^1 & & \downarrow /S^1 \\
P_A(a_1,\dots,a_n) & \xrightarrow{\bar{\Phi}} & P_A(1,\dots,1)
\end{array}$$

φ is S^1-equivariant if we let S^1 act non effectively on $\sum(1,\dots,1)$ by $t(Z_1,\dots,Z_n)=(t^aZ_1,\dots,t^aZ_n)$, $a = \mathrm{lcm}(a_i)$. Thus the degree of $\varphi|$(non-singular fiber) is a. The degree of Φ = degree $\Phi = a_1\dots a_n$. Therefore

$$\begin{aligned} e(\textstyle\sum(a_1,\dots,a_n) \to P(a_1,\dots,a_n)) &= ((a_1\dots a_n)/a^2)e(\textstyle\sum(1,\dots,1) \to P(1,\dots,1)) \\ &= -(a_1\dots a_n)/a^2 \end{aligned}$$

by Theorem 3.3 and since $\sum(1,\dots,1) \to P(1,\dots,1)$ is the Hopf fibration.

$P(a_1,\dots,a_n) \to P(1,\dots,1)$ is a $(\Pi a_i)/a$-fold branched covering. The branching occurs over $Z_i = 0$ over which we have s_i points in $P(a_1,\dots,a_n)$ (because points in $P(a_1,\dots,a_n)$ are, by definition, the same thing as orbits in $\sum(a_1,\dots,a_n)$). The standard "Hurewitz formula" for the Euler characteristic of a branched cover thus gives

$$\begin{aligned} \chi(P(a_1,\dots,a_n)) &= ((\Pi a_i)/a)(\chi(P(1,\dots,1)) - n) + \textstyle\sum s_i \\ &= ((\Pi a_i)/a)(2-n) + \textstyle\sum s_i \ . \end{aligned}$$

Since $\chi = 2 - 2g$, this gives the claimed value of g.

The fibers of Φ are the orbits of the natural $H = (\mathbb{Z}/a_1 \times \ldots \times \mathbb{Z}/a_n)$ action on $V_A - \{0\}$. (H acts by multiplication by a_i-th roots of unity in the i-th coordinate). This action on $V_A - \{0\}$ induces actions of H also on $\sum_A(a_1,\ldots,a_n)$ and $P_A(a_1,\ldots,a_n)$, and the fibers of φ and $\bar{\varphi}$ are therefore also the orbits of this action. In particular, for fixed i, $0 \leq i \leq n$, the orbits in $\sum_A(a_1,\ldots,a_n)$ with $Z_i = 0$ correspond to points in $P_A(a_1,\ldots,a_n)$ with $Z_i = 0$ which are all related by this H-action, since $P_A(1,\ldots,1)$ has exactly one point with $Z_i = 0$. Thus the H-action permutes the (t_i,β_i)-orbits of $\sum_A(a_1,\ldots,a_n)$ transitively, so the β's are the same for these orbits.

The whole proof can be expressed a bit more concisely in terms of this H-action, see [N-R], but the elementary nature of the computation is then even more obscured than by the presentation given here.

8. Universal Abelian Covers.

Lemma 8.1. If $M' \xrightarrow{p} M$ is a finite covering of a Seifert fibered manifold M, then M' is Seifert fibered by the components of p^{-1} of fibers of M

Proof. Any finite coverings of one of our "standard models" (Seifert fibering of $D^2 \times S^1$) will be a disjoint union of solid tori $D^2 \times S^1$, each with some standard Seifert fibering induced on it.

As an example of Lemma 8.1, assume $M = M(0;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ with $e(M \to F) \neq 0$. Let $\bar{M} \xrightarrow{p} M$ be the universal abelian cover of M. Therefore the covering transformation group is $H_1(M)$, which has order $\alpha_1 \ldots \alpha_n \, |e(M \to F)|$. By the lemma $\bar{M}$ has an induced Seifert fibering $\bar{M} \to \bar{F}$.

This example and the Breiskorn complete intersections are two examples of Seifert fibrations arising in a "natural" way. As the next theorem shows, even though these two examples arise from very different situations, surprisingly they are the same.

Theorem 8.2. Let $M = M(0;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ with $\hat{e} = e(M \to F) \neq 0$. Reverse the orientation of M if necessary to make $e < 0$. Let $\bar{M} \xrightarrow{p} M$ be the universal abelian cover. Then $\bar{M} = \sum(\alpha_1,\ldots,\alpha_n)$

Proof: We must recall some facts from covering space theory. Let X be a "nice" space (i.e. X has a universal cover) and assume X is connected

The normal coverings of X with covering transformation group F are classified by homomorphisms $\pi_1(X) \to F$. If the covering is $Y \to X$, then the components of Y are in 1-1 correspondence with the cosets of $\operatorname{Im} \pi_1(x) \subseteq F$. Namely, given $Y \to X$, the exact homotopy sequence gives:

$$\pi_1(\text{fiber}) = \{1\} \to \pi_1(Y) \to \pi_1(X) \to \pi_0(\text{fiber}) = F \to \pi_0(Y) \to \{1\}.$$

Given $\pi_1(X) \to F$, one can construct Y as $\tilde{X} \times_{\pi_1(X)} F$ ($\tilde{X}$ = universal cover of X). Then $Y \to \tilde{X}/\pi_1(X) = X$. The universal abelian cover is classified by the homomorphism $\pi_1(X) \to H_1(X)$.

Returning to the situation of the theorem, we have a commutative diagram

$$\begin{array}{ccc} \bar{M} & \xrightarrow{p} & M \\ {\scriptstyle \bar{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \bar{F} & \longrightarrow & F \end{array}$$

We want to compute the fiber degree f and the base degree b to apply Theorem 3.3. Recall

$$H_1(M) = \langle Q_1, \ldots, Q_n, H \mid \alpha_i Q_i + \beta_i H = 0,\ Q_1 + \ldots + Q_n = 0 \rangle$$

has order $\alpha_1 \ldots \alpha_n |e|$ so the total degree bf is $\alpha_1 \ldots \alpha_n |e|$. Let O be a non-exceptional fiber. The induced coverings $p^{-1}(O) \to O$ is classified by $\pi_1(O) \to H_1(M)$. Thus to compute f and b we must compute $|\operatorname{Im} \pi_1(O)|$ and $|H_1(M) \colon \operatorname{Im} \pi_1(O)|$.

As we have already seen $|\mathrm{Im}\ \pi_1(O)| = |\langle H\rangle|$. Thus

$$\begin{aligned}
|H_1(M): \mathrm{Im}\ \pi_1(O)| &= |H_1(M)/\langle H\rangle| \\
&= |\langle Q_1,\ldots,Q_n \mid \alpha_i Q_i = 0,\ Q_1 + \ldots + Q_n = 0\rangle| \\
&= |(\mathbb{Z}/\alpha_1 \times \ldots \times \mathbb{Z}/\alpha_n)/\langle Q_1 + \ldots + Q_n\rangle| \\
&= (\Pi\alpha_i)/\ \mathrm{lcm}\,\alpha_i \\
&= (\Pi\alpha_i)/a\ ,\ \text{where}\ \ a = \mathrm{lcm}(\alpha_i)
\end{aligned}$$

$$|H| = (\Pi\alpha_i))|e|/(\Pi\alpha_i/a) = a|e|.$$

Thus we have $f = a|e|$ and $b = \Pi\alpha_i/a$.

Using Theorem 3.3, we can calculate

$$\begin{aligned}
e(\bar{M} \to \bar{F}) &= b/f\ e(M \to F) \\
&= (((\Pi\alpha_i)/a)/a|e|)e(M \to F) \\
&= -(\Pi\alpha_i)/a^2.
\end{aligned}$$

Now let O_i be the i-th exceptional fiber. If $\det\begin{pmatrix}\alpha_i & \beta_i \\ \alpha_i' & \beta_i'\end{pmatrix} = 1$, then the homology class of O_i is

$$O_i = \alpha_i' Q_i + \beta_i' H\ .$$

Hence $|H_1(M)\colon \operatorname{Im} \pi_1(O_i)| = |H_1(M)/ \langle O_i\rangle|$

$=|\langle Q_1,\ldots,Q_n,H \mid \alpha_j Q_j + \beta_j H = 0,\ j = 1,\ldots,n,\ Q_1 + \cdots + Q_n = 0,\ \alpha_i' Q_i + \beta_i' H = 0\rangle$

$= |\langle Q_1,\ldots,\hat{Q}_i,\ldots,Q_n \mid \alpha_j Q_j = 0,\ j = 1,\ldots,\hat{i},\ldots,n,\ Q_1 + \cdots + \hat{Q}_i + \cdots + Q_n = 0\rangle$

$= (\prod_{j\neq i} \alpha_j)/ \operatorname{lcm}_{j \neq i} \alpha_j = s_i$. Thus $p^{-1}(O_i)$ consists of s_i fibers, each of which covers O_i with degree

$$(\Pi\alpha_j)|e|/s_i = (\Pi\alpha_j)|e|/(\prod_{j\neq i} \alpha_j)/(\operatorname{lcm}_{j\neq i} \alpha_j)$$

$$= \alpha_i|e| \operatorname{lcm}_{j\neq i} \alpha_j$$

$$= (\alpha_i/t_i)a|e| \ .$$

Let $\bar{O}_i$ be one of the s_i fibers in $\bar{M}$ which cover O_i and U be a neighborhood of O_i and $\bar{U}$ a neighborhood of $\bar{O}_i$. Let H and $\bar{H}$ denote non-exceptional fibers in M and $\bar{M}$. $\bar{O}_i$ is a $(\alpha_i/t_i)a|e|$-fold cover of O_i and $\bar{H}$ is an $a|e|$-fold cover of H. In $H_1(U)$, $\alpha_i O_i \sim H$. $\rho_*\colon H_1(\bar{U}) \to H_1(U)$ is injective, hence $\alpha_i a|e|\bar{O}_i \sim (\alpha_i/t_i)a|e|\bar{H}$ in $H_1(\bar{U})$. Therefore in $H_1(\bar{U})$, $t_i\bar{O}_i \sim \bar{H}$, and $\bar{O}_i$ is a (t_i,β_i) fiber for some β_i. (Note that the β's for all s_i of these fibers are equal since these fibers are transitively permuted by the covering transformations.)

$\bar{F}$ is a $(\Pi\alpha_i)/a$-fold branched cover of F. The branching is at n points of F over which we have respectively $s_1,\ldots,s_n$ points in $\bar{F}$. Then

$$X(\bar{F}) = ((\Pi\alpha_i)/a)(X(F) - n) + \sum s_j \ .$$

As $\chi(F) = 2$, this gives the desired g, and as in the proof of Theorem 7.2 the computed data completely determines $\bar{M}$.

III. Crystallographic Groups

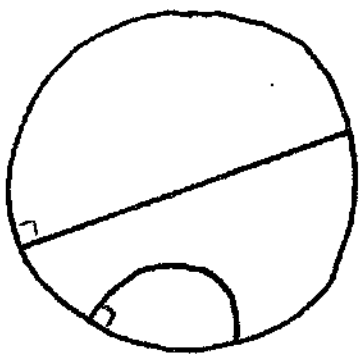
In this chapter we define and characterize 2-dimensional crystallographic groups. In the second section we use this to show that $\pi_1(M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ determines the Seifert invariant of $M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$. We begin this chapter with a short discussion of models for the three basic 2-dimensional geometries, Spherical, Euclidean and Hyperbolic.

1) Spherical (S^2): The sphere S^2 is a conformal model for this geometry. For geodesics we take great circles. If so desired, to avoid having two lines intersect in more than one point, this geometry can be projected onto $\mathbb{R}P^2$.

2) Euclidean ($\mathbb{E}^2$): This is the usual geometry on $\mathbb{R}^2$.

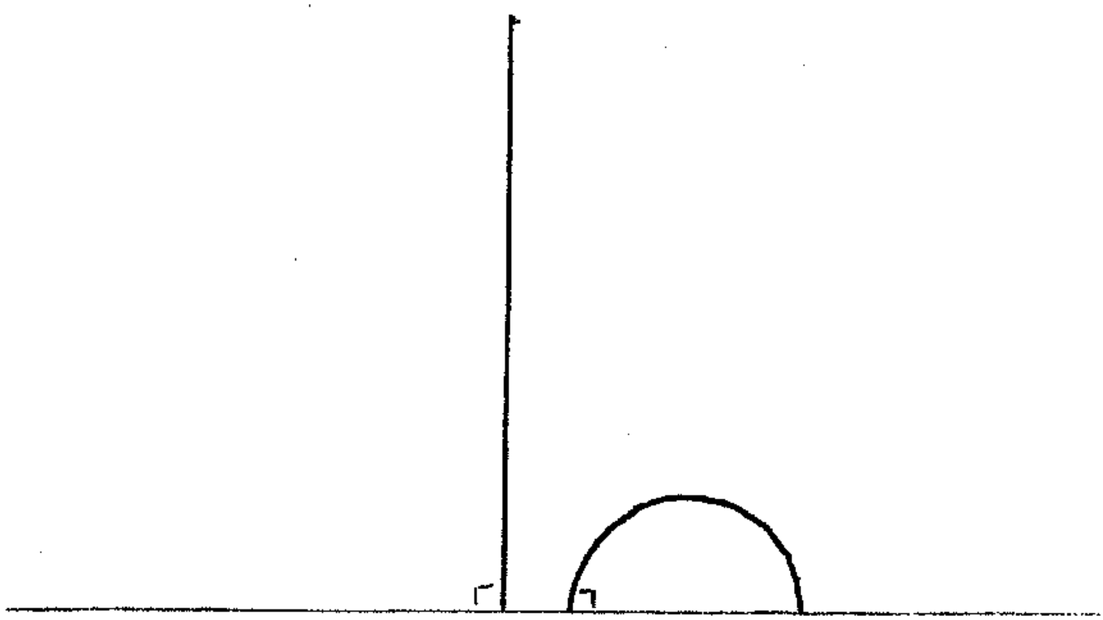
3) Hyperbolic ($\mathbb{H}^2$): There are two commonly used conformal models for $\mathbb{H}^2$: the Poincaré disc model and the upper half-plane model.

   a) Poincaré disc model: Here the underlying space is the interior of the unit disc D^2. Geodesics are either circular arcs that intersect the boundary of D^2 at right angles or diameters. Hyperbolic length is related to Euclidean length by $ds^2 = ds^2_{Eucl}/(1-r^2)$ where ds^2_{Eucl} is the Euclidean metric.



Poincaré disc model of $\mathbb{H}^2$ with examples of geodesics.

b) Upper-half plane model: The underlying space of this model is the upper-half plane in $\mathbb{R}^2$ i.e. $\{(x,y) \varepsilon \mathbb{R}^2 \,|\, x > 0\}$. Geodesics are either circular arcs that intersect the x-axis at right angles or vertical lines. In this model Hyperbolic distance is related to Euclidean distance by $ds^2 = ds^2_{Eucl}/y$



Upper-half plane model of $\mathbb{H}^2$
with examples of geodesics.

Note: $\text{Isom}^+(\mathbb{H}^2)$ = {conformal orientation preserving homeomorphisms of $\mathbb{H}^2$

= {Moebius transformations}

$\cong \text{PSL}(2,\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,|\, a,b,c,d \ \varepsilon\ \mathbb{R},\ ad-bc = \{1\}/\{\pm I\}$.

Here $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon\ \text{PSL}(2,\mathbb{R})$ acts in the upper-half plane model by the Moebius transformation $z \mapsto (az+b)/(cz+d)$.

9. Crystallographic Groups

Definition 9.1. A 2-dimensional (Spherical, Euclidean or Hyperbolic) crystallographic group is a discrete subgroup $\Gamma \subset \mathrm{Isom}^+(X)$ $(X = S^2, \mathbb{E}^2, \mathbb{H}^2)$ such that Γ acts properly discontinuously on X and X/Γ is compact. More precisely this is an orientation preserving crystallographic group, but we drop the extra adjectives for brevity.

Theorem 9.2. As an abstract group, a crystallographic group Γ is either finite cyclic (if $X = S^2$) or is isomorphic to a unique group of the form

$$\Gamma(g,\alpha_1,\ldots,\alpha_n) = \langle a_1,b_1,\ldots,a_g,b_g,q_1,\ldots,q_n \mid q_j^{\alpha_j} = 1 \ j=1,\ldots,n,$$

$$q_1\ldots q_n \prod_{i=1}^{g} [a_i,b_i] = 1\rangle$$

with $g \geq 0$ and $\alpha_i \geq 2$. We assume that if $g = 0$ then $n \geq 3$. Moreover this is a spherical, Euclidean or hyperbolic group according as $\chi = 2 - 2g - \sum_{i=1}^{n} \frac{\alpha_i - 1}{\alpha_i}$ satisfies $\chi > 0$, $\chi = 0$, $\chi < 0$. Furthermore all such groups $\Gamma(g;\alpha_1,\ldots,\alpha_n)$ occur as crystallographic groups. The above condition on χ gives the following possibilities:

Spherical case ($\chi > 0$). We must have $g = 0$, $n = 3$ and $\sum_{i=1}^{n} \frac{1}{\alpha_i} > 1$. This gives possibilities

$$\begin{aligned}(g;\alpha_1,\alpha_2,\alpha_3) &= (0;2,2,n)\\ &= (0;2,3,3)\end{aligned}$$

$= (0;2,3,4)$

$= (0;2,3,5)$

<u>Euclidean Case ($\chi = 0$)</u>. In this case the only possibilities are

$$(g;\alpha_1.,,,.\alpha_n) = (1;\ )$$

$$(0;2,4,4)$$

$$(0;2,3,6)$$

$$(0;3,3,3)$$

$$(0;2,2,2,2).$$

<u>Hyperbolic Case ($\chi < 0$)</u>. This case consists of all possibilities not previously listed.

<u>Notation</u>: If $g = 0$ we abbreviate $\Gamma(0;\alpha_1,\ldots,\alpha_n)$ to $\Gamma(\alpha_1,\ldots,\alpha_n)$.

<u>Example</u>. We can easily realize the spherical and Euclidean cases as orientation preserving isometries of S^2 and $\mathbb{E}^2$. The spherical crystallographic groups can be realized as the regular polyhedral groups, i.e. isometries of regular spherical polyhedra. More precisely we have the following chart:

Regular Polyhedron	Isometry Group	Order of Group
n-gonal dihedron*	$\Gamma(2,2,n) \cong D_{2n}$	2n
tetrahedron	$\Gamma(2,3,3) \cong A_4$	12
cube	$\Gamma(2,3,4) \cong S_4$	24
dodecahedron	$\Gamma(2,3,5) \cong A_5$	60

$\Gamma(2,3,4)$ can also be realized as the isometry group of the octahedron and $\Gamma(2,3,5)$ can also be realized as the isometry group of the icosohedron.

In the Euclidean case let $\overline{\Gamma}(\alpha_1,\alpha_2,\alpha_3)$ be the group generated by reflections in the sides of a triangle with angles π/α_1, π/α_2, π/α_3, e.g. $\overline{\Gamma}(2,4,4)$ is generated by reflections in the sides of a triangle with angles $\pi/2$, $\pi/4$, $\pi/4$. Then $\Gamma(\alpha_1,\alpha_2,\alpha_3) = (\overline{\Gamma}(\alpha_1,\alpha_2,\alpha_3))^+$ i.e. $\Gamma(\alpha_1,\alpha_2,\alpha_3)$ is the orientation preserving subgroup of $\overline{\Gamma}(\alpha_1,\alpha_2,\alpha_3)$. $\overline{\Gamma}(2,2,2,2)$ is the group generated by reflections in the sides of a rectangle. Then $\Gamma(2,2,2,2) = (\overline{\Gamma}(2,2,2,2))^+$.

Exercise. Show that the spherical and Euclidean groups described above have the abstract description claimed in Theorem 9.2.

Remark. The above construction of the Euclidean groups extends to give $\Gamma(\alpha_1,\ldots,\alpha_n)$ as the orientation preserving subgroup of the group $\overline{\Gamma}(\alpha_1,\ldots,\alpha_n)$ generated by reflections in the sides of a spherical, Euclidean or hyperbolic n-gon with angles $\pi/\alpha_1,\ldots,\pi/\alpha_n$.

* This is the regular spherical polyhedron with two regular n-gonal faces. Each face is a hemisphere and the vertices are regularly spaces around the equator.

Proof of Theorem 9.2. Throughout this proof X is one of S^2, $\mathbb{E}^2$, $\mathbb{H}^2$. Let Γ be any discrete group acting properly discontinuously on X with X/Γ a compact surface. Here by properly discontinuous we mean for all $x,y \in X$ one can find neighborhoods N_x, N_y of x and y such that $N_x \cap gN_y = \phi$ for all but finitely many $g \in G$. $X \xrightarrow{\pi} X/\Gamma$ is a branched covering. The branching occurs over points on which Γ acts with non-trivial isotropy. Call the branch points $x_1,\ldots,x_n \in X/\Gamma$. Then $\pi|_{\pi^{-1}(X/\Gamma-\{x_1,\ldots,x_n\})}$ is a genuine covering map. Since a finite group of orientation preserving homeomorphism acting on $\mathbb{R}^2$ must be a rotation (see [ker]) π restricted to each component of $\pi^{-1}(D_i)$ (D_i is a small disc about x_i) "looks like" $D^2 \to D^2$ given in complex coordinates by $z \mapsto z^{\alpha_i}$.

From X/Γ cut out the discs D_i and in X cut out $\pi^{-1}(D_i)$. What remains is a genuine cover. We want to replace D_i and $\pi^{-1}(D_i)$ by complexes in such a way that this new space is a cover with covering transformation group Γ. We do this by replacing D_i by an Eilenberg Maclane complex $K_i = K(\mathbb{Z}/\alpha_i,1)$ for each i, pasting along a generator $S^1 \to K_i$ and correspondingly replacing $\pi^{-1}(D_i)$ by the universal cover $\widetilde{K}_i$ of K_i pasting along the cover of S^1 in K_i. Call the result of this pasting $Y \to Y'$. Clearly Y is a cover of Y' and, since in X we replaced contractible spaces by contractible spaces, $X \simeq Y$. Thus Y is the universal cover of Y'. Γ acts properly discontinuously and freely on Y and by looking at this action on a fiber not lying above a point of D_i we see Γ is the group of covering transformations of Y. Thus $Y' \cong Y/\Gamma$ and $\Gamma \cong \pi_1(Y/\Gamma)$. By Van Kampen's theorem

$$\pi_1(Y/\Gamma) = \langle a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_n \mid q_j^{\alpha_j} = 1, j = 1, \ldots, n,$$

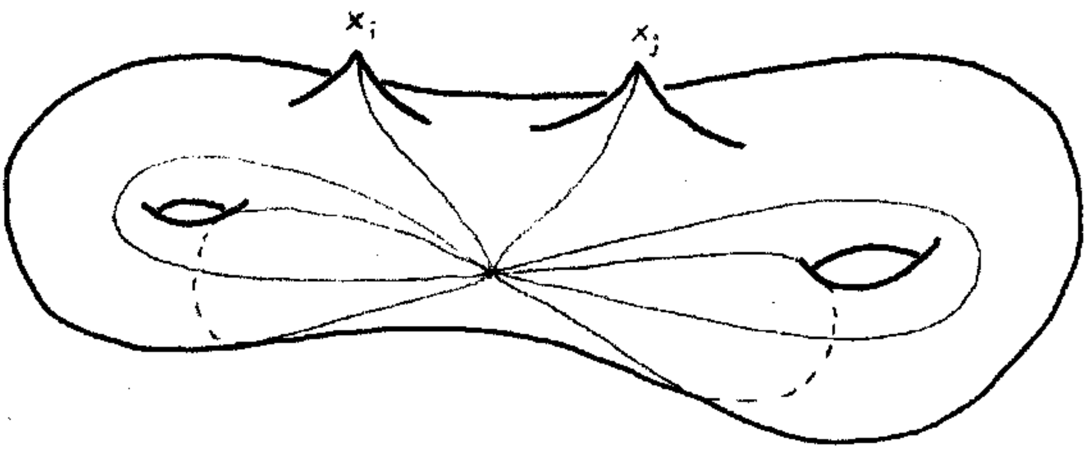
$$q_1 \cdots q_n \prod_{i=1}^{g} [a_i, b_i] = 1\rangle$$

and hence Γ has the desired presentation.

Remark: 1) In the Euclidean and hyperbolic cases Y is contractible so Y/Γ is a $K(\Gamma,1)$.

2) Notice we have proved a slightly stronger statement in that we did not assume $\Gamma \subset \mathrm{Isom}^+(X)$.

Now we assume $\Gamma \subset \mathrm{Isom}^+(X)$. X/Γ is as shown where x_i and x_j are branch points.



If we cut X/Γ along the geodesic paths shown, we get a polygon with $4g + 2n$ sides which is a fundamental domain. The sum of the angles of the polygon is $2\pi + \sum_{i=1}^{n} 2\pi/\alpha_i$. A special case of the Gauss-Bonnet formula states: if P is an m-sided polygon with angles $\theta_1,\ldots,\theta_m$, then $K(\text{area of } P) = \pi(2-m) + \sum_{i=1}^{m} \theta_i$ where $K = -1,-,1$ accordingly as $X = \mathbb{H}^2$, $\mathbb{E}^2$, S^2. In our case

$$K(\text{area of } P) = (2-(4g+2n))\pi + 2\pi(1+\sum_{i=1}^{n} 1/\alpha_i)$$
$$= 2\pi(2-2g-\sum_{i=1}^{n}(\alpha_i-1)/\alpha_i).$$

Therefore X classifies what space Γ acts on in the manner stated in the theorem.

To finish the proof of Theorem 9.2 we must show:

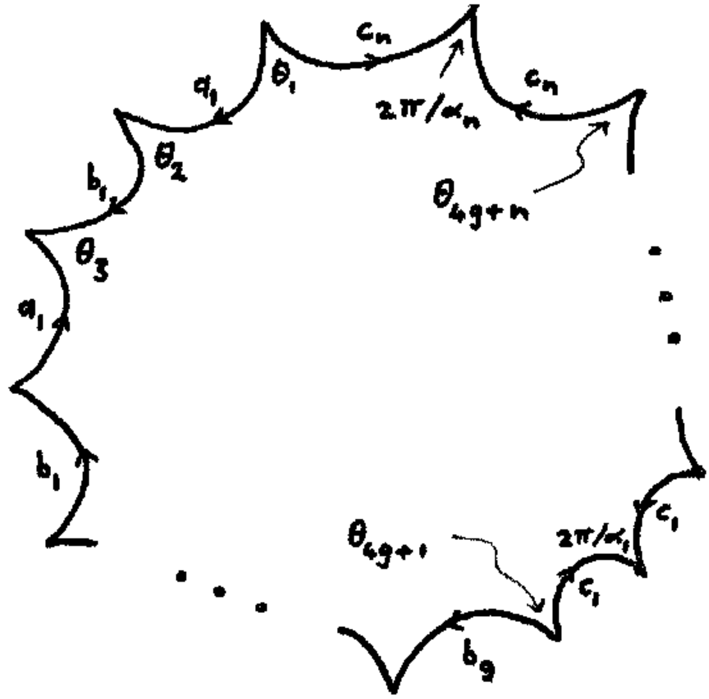
a) Given $\Gamma = \Gamma(g;\alpha_1,\ldots,\alpha_n)$ then Γ can be embedded in $\text{Isom}^+(X)$ as a crystallographic group;

b) The abstract group $\Gamma(g;\alpha_1,\ldots,\alpha_n)$ uniquely determines $g,\alpha_1,\ldots,$

We have already shown by example that if Γ is spherical c Euclidean then $\Gamma \subseteq \text{Isom}^+(X)$. Thus assume $X = \mathbb{H}^2$ and $2-2g - \sum_{i=1}^{n} (\alpha_i-1)/\alpha_i \quad 0$. To show $\Gamma \subseteq \text{Isom}^+(\mathbb{H}^2)$ it suffices to construct a polygon P in $\mathbb{H}^2$ such that P is a fundamental domain for X/Γ. Clearly P must be a polygon of the form:

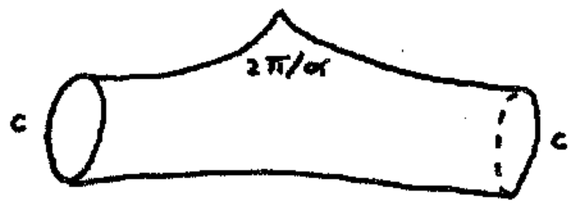
$\sum_{1}^{4g+n} \theta_i = 2\pi$;

similarly labelled sides are of equal length.



We leave the proof of the existence of such a polygon to the reader. However a reference for such a proof is [Gr].

An easier way, due to Scott Wolpert, to show $\Gamma \subsetneq \mathrm{Isom}^+(\mathbb{H}^2)$ is to construct the orbit space directly. This can be done for $g > 0$ by glueing together pieces of the form



c is the length of each S^1 boundary component. This length is fixed but arbitrary. Again the details of this construction are left to the reader.

Finally we must prove b). Note in the Spherical case we have

$$\Gamma(2,2,n) = D_{2n} \text{ of order } 2n$$

$$\Gamma(2,3,3) = A_4 \text{ of order } 12$$

$$\Gamma(2,3,4) = S_4 \text{ of order } 24$$

$$\Gamma(2,3,5) = A_5 \text{ of order } 60.$$

It is a simple exercise to show no two of these groups are isomorphic. Hence this proves b) in the Spherical case.

Now assume $2 - 2g - \sum_{i=2}^{n} (\alpha_i - 1)/\alpha_i \leq 0$ i.e. $\Gamma \subsetneq \mathrm{Isom}^+(X)$ where $X = \mathbb{R}^2$ or $\mathbb{H}^2$. We claim that if $\{1\} \neq F \subseteq \Gamma$ is any finite (cyclic) subgroup then F is conjugate to a subgroup of a unique one of the $\langle q_i \rangle$. To see this note if F is finite then F has a unique fixed point in X. (For $X = \mathbb{R}^2$ see [Ker]; for $X = \mathbb{H}^2$ see [Hel;p. 75]). This fixed point is a branched point for $X \overset{\pi}{\rightarrow} X/\Gamma$ so it lies over one of $x_1,\ldots,x_n$. Say $\pi(y) = x_1$. q_1 is a rotation about some point in $\pi^{-1}(x_1)$ i.e. about some point γy. Then $\gamma F \gamma^{-1}$ fixes y and hence is in $\Gamma_y = \langle q_1 \rangle$ which proves the claim. In particular any maximal finite subgroup is conjugate to a unique one of the $\langle q_i \rangle$. Thus if we consider the set of conjugacy classes of all maximal finite subgroups of Γ, the number of classes is n and the orders of a representative from each conjugacy class will give us the α_i's. Finally if N is the normal subgroup generated by all elements of finite order then Γ/N has presentation

$$\Gamma/N = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle,$$

which allows us to determine g.

<u>Theorem 9.3</u>. Let $\Gamma = \Gamma(g;\alpha_1,\ldots,\alpha_n)$ be a crystallographic group (if $g = 0$ we assume $n \geq 3$) and let X be the corresponding geometry, i.e. X is either S^2, $\mathbb{E}^2$ or $\mathbb{H}^2$. Thus $\Gamma \subsetneq \mathrm{Isom}^+(X) = G$. Then under the above hypotheses we have:

$$G/\Gamma = M(g;(1,2g-2),(\alpha_1,\alpha_1-1),\ldots,(\alpha_n,\alpha_n-1)).$$

<u>Note</u>: $$e(G/\Gamma \to F) = 2 - 2g - \sum \frac{\alpha_i - 1}{\alpha_i}$$
$$= \pm \frac{1}{\pi} \ \mathrm{vol}(X/\Gamma) \quad (\text{if } X \neq \mathbb{E}^2).$$

This number is called $X(\Gamma)$ and was first defined by C.T.C. Wall, see [W]

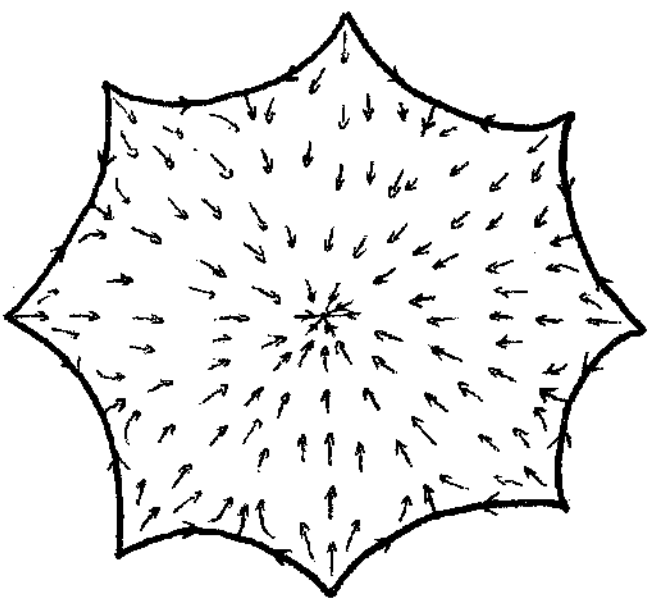
<u>Proof</u> G acts by isometries on X and hence on T^1X the unit tangent bundle of X. This action on T^1X is simply transitive, i.e. given $v_1, v_2 \in T^1X$ there exists a unique $g \in G$ such that $gv_1 = v_2$. Thus if we fix $v_o \in T^1X$ the map $\Phi : G \to T^1X$ given by $\Phi(g) = g \cdot v_o$ is an isomorphism. Here $g \cdot v_o$ is the action of G on T^1X induced by the action of $\Gamma \subset \mathrm{Isom}^+(X)$ on X. If Γ acts on G by left multiplication and on T^1X as just described Φ is Γ-equivariant. Therefore $G/\Gamma \cong T^1X/\Gamma$ and we can describe the Seifert fibered structure as the natural projection $T^1X/\Gamma \overset{\pi}{\to} X/\Gamma$.

If $x \in X/\Gamma$ is the image of $y \in X$, then the fiber over x of the Siefert fibration is:

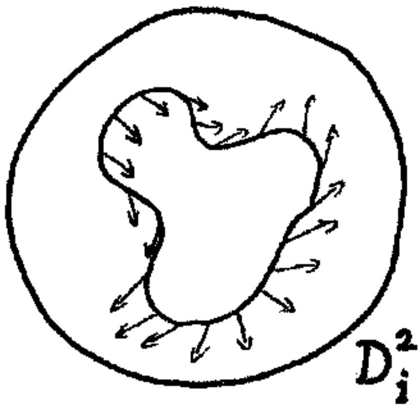
$$\pi^{-1}(x) = T^1_y(X)/\Gamma_y = \begin{cases} T^1_yX/(\mathbb{Z}/\alpha_i) & \text{if } x = x_i \text{ is singular} \\ T^1_yX & \text{otherwise.} \end{cases}$$

Thus $T^1X/\Gamma = M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ for some $\beta_1,\ldots,\beta_n$. Hence we need only determine the β_i. (In fact, our determination of the β_i will also give a second proof that the α_i are correct.)

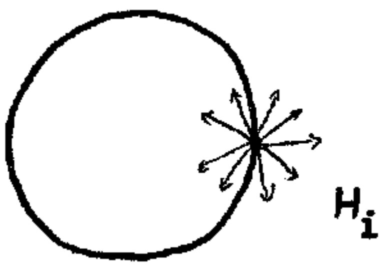
Before continuing the proof, we look at an example. Let Γ be the fundamental group of an orientable surface of genus 2, so $X/\Gamma = F_2$, an orientable surface of genus 2. The Seifert fibration is $T^1F_2 \xrightarrow{\pi} F_2$. It was classically known that $T^1F_2 = M(2;(1,2))$, but let us see how we would prove this from "first principles." To determine the Seifert invariant we must choose a section in $T^1F_2 - \{\bigcup_{i=1}^{n} T_i\}$ where the T_i are disjoint neighborhoods of suitable fibers i.e. disjoint solid tori. This is equivalent to choosing a unit tangent vector field on $F_2 - \{\bigcup_{i=1}^{n} D_i^2\}$ where as the disks D_i^2 we take small disks about the critical points of the vector field. Such a vector field drawn on a fundamental domain looks as follows:



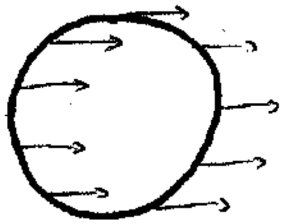
After identifying edges there are six critical points. Each critical point represents a deleted fiber. A solid torus neighborhood T_i of such a fiber is $T_i = T^1D_i^2$, so points of T_i are unit tangent vectors to the disk D_i. Thus the following picture represents a typical closed curve in $T_i = T^1D_i^2$.



Recall in ∂T_i we have the homology relation $\alpha_i Q_i + \beta_i H_i \sim M$. To find (α_i, β_i) we shall represent Q_i, H_i and M in the above form. H_i is a non-exceptional fiber and can be represented as:



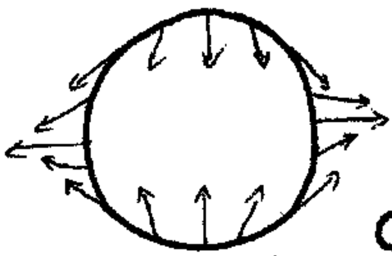
M can be represented as:



M

On F_2 there are four critical points of type:

at such a point Q_i is the curve:

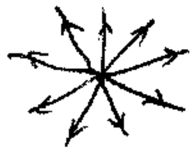


Q_i

Thus $Q_i \sim M_i - H_i$, so $(\alpha_i, \beta_i) = (1,1)$ at such a point. There is one critical point of type:

Here $Q_i \sim M_i + H_i$, so $(\alpha_i, \beta_i) = (1,-1)$.

Finally, there is one critical point of type:

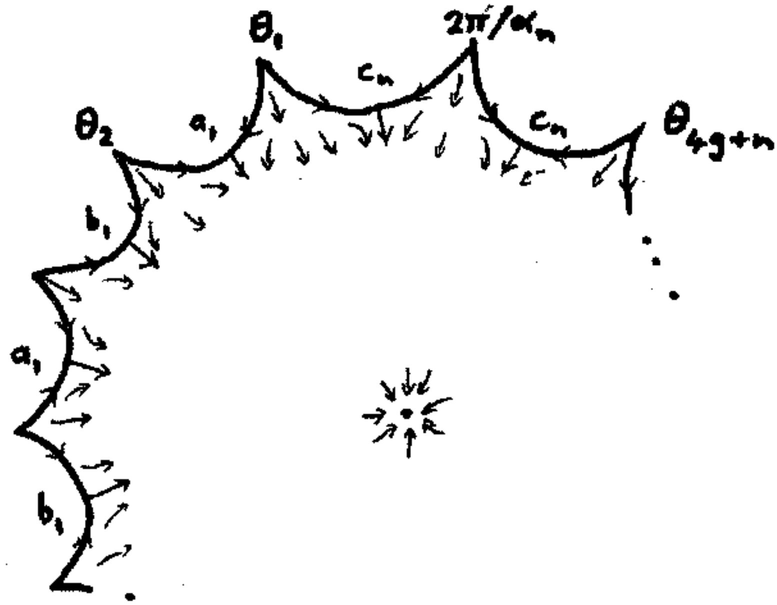


Here $(\alpha,\beta) = (1,-1)$.

Thus $T^1F_2 = M(2;4(1,1),2(1,-1)) \cong M(2;(1,2))$.

The argument we have just given is the one which, in a more general form, was given by Hopf to prove his theorem that the Euler characteristic of any closed manifold is the sum of the indices of the zeroes of any vector field with isolated zeroes on that manifold.

Returning to the proof of our theorem, we can determine the (α_i,β_i) in exactly the same manner. Draw a fundamental domain for X/Γ as on page 60 and a vector field on that domain.



There are $(4g+2n)/2 + 2$ critical points;

$2g + n$ of type  with $(\alpha,\beta) = (1,1)$;

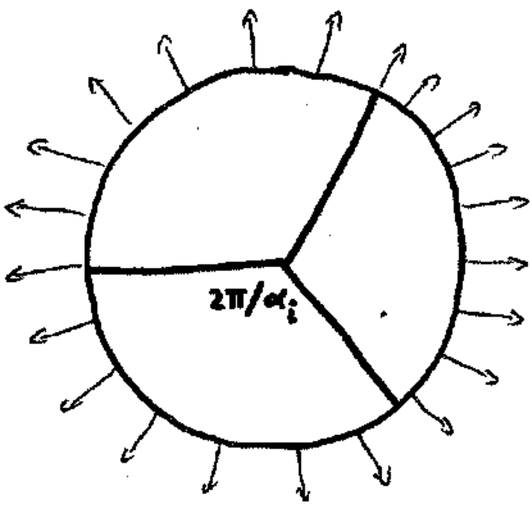
1 of type  with $(\alpha,\beta) = (1,-1)$;

1 of type  with $(\alpha,\beta) = (1,-1)$;

and for each $i = 1,\dots,n$

1 of type  with $(\alpha,\beta) = (\alpha_i,-1)$.

Here the angle is $2\pi/\alpha_i$, and the fact that $(\alpha,\beta) = (\alpha_i,-1)$ follows from the following figure, which shows $\alpha_i Q_i = M_i + H_i$.



$\alpha_i Q_i$

Therefore $G/\Gamma = M(g;(2g+n)(1,1),2(1,-1),(\alpha_1,-1),\dots(\alpha_n,-1))$

$= M(g;(1,2g-2),(\alpha_1,\alpha_1-1),\dots,(\alpha_n,\alpha_n-1))$.

10. Seifert Groups

The aim of this section is to show that the Seifert invariant of $M = M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ is determined by $\pi_1(M)$. Along the way we define the notion of a Seifert group and classify these groups. Again for this section we assume if $g = 0$ then $n \geq 3$. We denote $\pi_1(M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ by $\pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$.

Proposition 10.1. Either $(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)) = (1; \ )$ and $\pi(1; \ ) = \mathbb{Z}^3$, or $\langle h\rangle$ is the complete center $C(\pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ of $\pi = \pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$.

Proof. Recall we have the exact sequence

$$1 \to \langle h\rangle \to \pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)) \to \Gamma(g;\alpha_1,\ldots,\alpha_n) \to 1$$

Claim. $\Gamma = \Gamma(g;\alpha_1,\ldots,\alpha_n)$ has a trivial center unless

$$(g;\alpha_1,\ldots,\alpha_n) = (1; \ ) \qquad (\Gamma = \mathbb{Z}^2)$$
$$= (0;2,2,n) \quad (\Gamma = D_{2n}).$$

Proof of Claim:

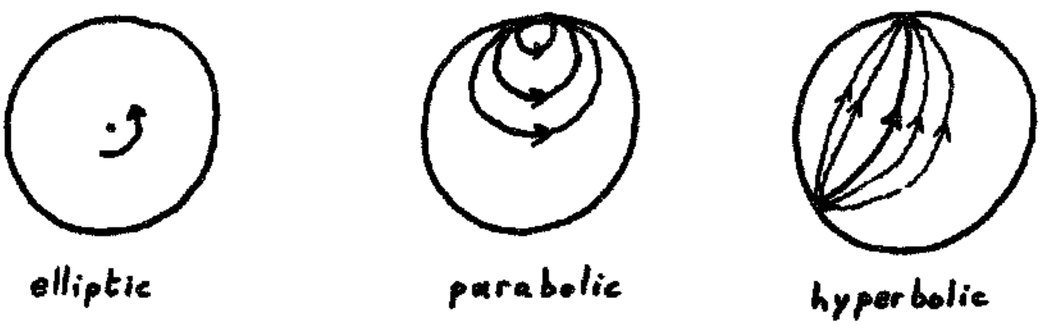
Case 1 Γ is hyperbolic

By the Brouwer fixed point theorem any $g \in \mathrm{Isom}^+(\mathbb{H}^2)$ has a fixed point in the extended plane $\overline{\mathbb{H}^2}$ (the closed disc in the Poincaré model). The elements of $\mathrm{Isom}^+(\mathbb{H}^2) \cong PSL(2,\mathbb{R})$ can be classified into three types according to how many fixed points each $g \in \mathrm{Isom}^+(\mathbb{H}^2)$ has in $\overline{\mathbb{H}^2} \cong D^2$ and where these fixed points are located. This classification can also be given in terms of the absolute value of the trace of g considered as an element of $SL(2,\mathbb{R})$. Let $1 \neq g \in \mathrm{Isom}^+(\mathbb{H}^2)$, then the three types of elements are:

a) g has a fixed point which is in the interior of D^2. In this case g is called an elliptic element and is a rotation about the fixed point with $|\mathrm{tr} g| < 2$.

b) g has just one fixed point and it is on the boundary of D^2. Here g is called a parabolic element and is rotations of horospheres. When g is parabolic $|\mathrm{tr} g| = 2$.

c) g has two fixed points on the boundary of D^2. g is called a hyperbolic element and is a translation along a geodesic and along curves of constant distance from this geodesic. Here $|\mathrm{tr} g| > 2$



Since if $g_1, g_2 \in \mathrm{Isom}^+(X)$ and $g_1 g_2 = g_2 g_1$ then $g_1(X^{g_2}) = X^{g_2}$

(X^{g_2} is the fixed point set of g_2), we can conclude that any abelian subgroup of $Isom^+(\mathbb{H}^2)$ is contained in a subgroup of the following type:

S^1 - rotations about a given point;

$\mathbb{R}$ - 1-parameter family of parabolic elements, with a given fixed point at infinity;

$\mathbb{R}$ - translations along a given geodesic.

Hence in a discrete subgroup of $Isom^+(\mathbb{H}^2)$ any abelian subgroup is cyclic Thus if $c \in C(\Gamma)$ and $c \neq 1$ we can choose g not in the same 1-parameter subgroup. Then $cg \neq gc$ which is a contradiction. Therefore Γ has a trivial center.

<u>Case 2</u>: Γ is Euclidean

The only possible Euclidean subgroups are:

$\Gamma(2,4,4)$

$\Gamma(2,3,6)$

$\Gamma(3,3,3)$

$\Gamma(2,2,2,2)$

$\Gamma(1;)$.

If $\Gamma \neq \Gamma(1;)$, there exists elements of finite order in Γ, namely rotations about a point in $\mathbb{R}^2$. Assume $c \in C(\Gamma)$, then c fixes the fixed point of any such rotation. If x is a fixed point of $\gamma \in \Gamma$ then gx is a fixed point of $g\gamma g^{-1} \in \Gamma$, for any $g \in \Gamma$. Therefore

c fixes infinitely many points, so $c = \text{id}$.

Case 3: Γ is spherical

The proof is just a case by case verification which we omit.

This completes the proof of the claim. The claim implies that $C(\pi) = \langle h \rangle$ except for possibly $\pi(1;\ )$ or $\pi(0;(2,1)(2,1),(n,\beta))$. These two cases can be checked individually (exercise), completing the proof of Proposition 10.1

Theorem 10.2. (spherical case) If $(\alpha_1,\alpha_2,\alpha_3)$ is one of $(2,2,n)$, $(2,3,3),(2,3,4)$ or $(2,3,5)$ then $\pi = \pi(0;(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3))$ determines the Seifert invariant (up to sign).

Proof. $\pi/C(\pi) = \Gamma(\alpha_1,\alpha_2,\alpha_3)$ which by Theorem 9.2 allows us to recover $\alpha_1,\alpha_2,\alpha_3$. Since $|\pi/[\pi,\pi]| = \alpha_1\alpha_2\alpha_3|e(M\to F)|$ we can recover $|e(M\to F)|$. It is sufficient to show $\alpha_1,\alpha_2,\alpha_3,e$ completely determine the Seifert invariant. That is, we must compute each β_i (mod α_i) in terms of $\alpha_1,\alpha_2,\alpha_3$, and e.

Case 1: $(\alpha_1,\alpha_2,\alpha_3) = (2,2,n)$. Recalling that the α_i and β_i are relatively prime, the β_i (mod α_i) are given by:

$$\beta_1 \equiv 1 (\text{mod } 2), \quad \beta_2 \equiv 1(\text{mod } 2), \quad \beta_3 \equiv -ne(\text{mod } n).$$

Case 2: $(\alpha_1,\alpha_2,\alpha_3) = (2,3,3)$. Up to exchanging β_2 and β_3 there are three possible cases:

a) $\beta_1 \equiv 1 \pmod 2$, $\beta_2 \equiv 1 \pmod 3$, $\beta_3 \equiv 1 \pmod 3$, $e \equiv \frac{5}{6} \pmod 1$

b) $\beta_1 \equiv 1 \pmod 2$, $\beta_2 \equiv 1 \pmod 3$, $\beta_3 \equiv 2 \pmod 3$, $e \equiv \frac{1}{2} \pmod 1$

c) $\beta_1 \equiv 1 \pmod 2$, $\beta_2 \equiv 2 \pmod 3$, $\beta_3 \equiv 2 \pmod 3$, $e \equiv \frac{1}{6} \pmod 1$.

These three possibilities are thus distinguished by the value of $e \pmod 1$

The final two cases are completely analogous.

<u>Case 3</u>: $(\alpha_1, \alpha_2, \alpha_3) = (2,3,4)$

$\beta_1 \pmod 2$	$\beta_2 \pmod 3$	$\beta_3 \pmod 4$	$e \pmod 1$
1	1	1	$\frac{11}{12}$
1	1	3	$\frac{5}{12}$
1	2	1	$\frac{7}{12}$
1	2	3	$\frac{1}{12}$

Case 4: $(\alpha_1,\alpha_2,\alpha_3) = (2,3,5)$

$\beta_1 \pmod 2$	$\beta_2 \pmod 3$	$\beta_3 \pmod 5$	$e \pmod 1$
1	1	1	$\frac{29}{30}$
1	1	2	$\frac{23}{30}$
1	1	3	$\frac{17}{30}$
1	1	4	$\frac{11}{30}$
1	2	1	$\frac{19}{30}$
1	2	2	$\frac{13}{30}$
1	2	3	$\frac{7}{30}$
1	2	4	$\frac{1}{30}$

Definition 10.3. A Seifert group $\pi = \pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ is a group with presentation:

$$\langle a_1,b_1,\ldots,a_g,b_g,q_1,\ldots,q_n,h \,|\, [a_i,h] = [b_i,h] = [q_j,h] = 1$$
$$i = 1,\ldots,g, j = 1,\ldots,n, q_j^{\alpha_j} h_j^{\beta_j} = 1, \prod_{j=1}^{n} q_j \prod_{i=1}^{g} [a_i,b_i] = 1\rangle.$$

Here we do not assume the (α_i,β_i) are relatively prime, but we do assume $\chi = 2 - 2g + \Sigma(\alpha_i - 1)/\alpha_i$ is not positive.

Remark: As before, and by a similar argument we can show if π is a Seifert group and $\pi \neq \pi(1;\ ) = \mathbb{Z}^3$ then the center of π is $C(\pi) = \langle h\rangle$. Thus

$$1 \to \mathbb{Z} \to \pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)) \to \Gamma(g;\alpha_1,\ldots,\alpha_n) \to 1$$

is exact. By the theory of group extensions (see for example [Mac]), this extension is classified by an element $a \in H^2(\Gamma;\mathbb{Z})$. In our situation we can say more.

Theorem 10.4. If π,Γ and a are as above then $H^2(\Gamma;\mathbb{Z}) = \mathrm{Ab}\langle X_0,X_1,\ldots,X_n \mid \alpha_i X_i = X_0, i=1,\ldots,n\rangle$ and $a = \beta_1 X_1 + \cdots + \beta_n X_n$. Also any automorphism f of Γ induces $f^*\colon H^2(\Gamma;\mathbb{Z}) \to H^2(\Gamma;\mathbb{Z})$ of the following type: either $f^*(X_i) = X_{i'}$, for each i, where $i \mapsto i'$ is a permutation with $\alpha_i = \alpha_{i'}$, or $f^*(X_i) = -X_{i'}$, for each i, with $i \mapsto i'$ as before. We say $f\colon \Gamma \to \Gamma$ is "orientation preserving" or "orientation reversing" correspondingly.

Remark: In view of this theorem, the classifying element $a \in H^2(\Gamma;\mathbb{Z})$ is equivalent to the Seifert invariant, in that the one determines and is determined by the other (up to sign), so the Seifert invariant of a Seifert group is an abstract invariant of the group.

Observe also that if we tensor with $\mathbb{R}$ we have $H^2(\Gamma;\mathbb{R}) = H^2(\Gamma;\mathbb{Z}) \otimes \mathbb{R} \cong \mathbb{R}$ by $X_i \mapsto 1/\alpha_i \in \mathbb{R}$, and under this identification of $H^2(\Gamma;\mathbb{R})$ with $\mathbb{R}$, the classifying element a becomes the Euler number. This Euler number is still an invariant for Seifert groups, not just for Seifert manifolds.

Before proving Theorem 10.4, we need a brief summary of the theory of central extensions.

Fix an abstract group Γ, and an abelian group A. We define an equivalence relation on the set of short exact sequences of the form $1 \to A \to \pi \to \Gamma \to 1$ where $A \subseteq C(\pi)$ as follows: Given two short exact sequences

$$E_1 : 1 \to A \to \pi_1 \to \Gamma \to 1, \quad E_2 : 1 \to A \to \pi_2 \to \Gamma \to 1$$

we say $E_1 \sim E_2$ if there exists a group isomorphism $\varphi : \pi_1 \to \pi_2$ such that

$$\begin{array}{ccccccccc} 1 & \to & A & \to & \pi_1 & \to & \Gamma & \to & 1 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 1 & \to & A & \to & \pi_2 & \to & \Gamma & \to & 1 \end{array}$$

commutes. We define $\mathrm{Ext}(\Gamma,A) = \{1 \to A \to \pi \to \Gamma \to 1 \mid A \subseteq \mathcal{C}(\pi)\}/\sim$, where $\sim$ is the equivalence relation just defined.

Given a homomorphism $f : \Gamma' \to \Gamma$ we define $f^* : \mathrm{Ext}(\Gamma,A) \to \mathrm{Ext}(\Gamma',A)$ by means of the following pullback diagram:

$$\begin{array}{ccccccccc} 1 & \to & A & \to & f^*\pi & \xrightarrow{\psi} & \Gamma' & \to & 1 \in \mathrm{Ext}(\Gamma',A) \\ & & \| & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle f}\downarrow & & \\ 1 & \to & A & \longrightarrow & \pi & \xrightarrow{p} & \Gamma & \to & 1 \in \mathrm{Ext}(\Gamma,A) \end{array}$$

where $f^*\pi = \{(\pi,\gamma') \in \pi \times \Gamma' \mid p(\pi) = f(\gamma')\}$ and the maps $\varphi : f^*\pi \to \pi$ and $\psi : f^*\pi \to \Gamma'$ are the obvious projections onto the first and second coordinates respectively. Given a homomorphism $g : A \to A'$ we define $g_* : \mathrm{Ext}(\Gamma,A) \to \mathrm{Ext}(\Gamma,A')$ by means of the following pushout diagram:

$$\begin{array}{ccccccccc} 1 \longrightarrow & A & \longrightarrow & \pi & \longrightarrow & \Gamma \longrightarrow 1 & \in \mathrm{Ext}(\Gamma,A) \\ & g\downarrow & & \varphi\downarrow & & \| & \\ 1 \longrightarrow & A' & \longrightarrow & g_*\pi & \xrightarrow{\psi} & \Gamma \longrightarrow 1 & \in \mathrm{Ext}(\Gamma,A') \end{array}$$

where $g_*\pi = (\pi \times A')/\{(a,g(a)) \mid a \in A\}$ and φ,ψ are the obvious maps. Therefore what we have shown is that $\mathrm{Ext}(\Gamma,A)$ is a functor which is covariant in A and contravariant in Γ.

Let $E,E' \in \mathrm{Ext}(\Gamma,A)$ be represented by $1 \to A \to \pi \to \Gamma \to 1$ and $1 \to A \to \pi' \to \Gamma \to 1$ respectively. We define $E \times E' \in \mathrm{Ext}(\Gamma\times\Gamma, A\times A)$ to be the equivalence class represented by

$$1 \to A \times A \to \pi \times \pi' \to \Gamma \times \Gamma \to 1.$$

Define $\Delta\colon \Gamma \to \Gamma \times \Gamma$ by $\gamma \mapsto (\gamma,\gamma)$ and $\nabla\colon A \times A \to A$ by $(a,b) \mapsto a+b$. Then we let $E \oplus E' = \nabla_*\Delta^*(E\times E') = \Delta^*\nabla_*(E\times E')$. We leave it as an exercise to show that these two definitions of $E \oplus E'$ are the same and that this Baer sum turns $\mathrm{Ext}(\Gamma,A)$ into a group. It is a classical result that $\mathrm{Ext}(\Gamma,A) = H^2(\Gamma;A)$, see Maclane [Mac, p. 137] for a historical discussion. For our purposes, with $A = \mathbb{Z}$ we can take it as a definition of $H^2(\Gamma;\mathbb{Z})$.

<u>Proof of Theorem 10.4</u>. Recall $\pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ and $\Gamma(g;\alpha_1,\ldots,\alpha_n)$ have presentations

$$\pi(g;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n)) = \langle a_1,b_1,\dots,a_g,b_g,q_1,\dots,q_n,h \mid$$

$$[a_i,h] = [b_i,h] = [q_j,h] = 1, \quad q_j^{\alpha_j}h^{\beta_j} = 1,$$

$$i = 1,\dots,g, j = 1,\dots,n, \prod_{j=1}^{n} q_j \prod_{i=1}^{g} [a_i,b_i] = 1\rangle$$

$$\Gamma(g;\alpha_1,\dots,\alpha_n) = \langle a_1,b_1,\dots,a_g,b_g,q_1,\dots,q_n \mid q_j^{\alpha_j} = 1, j = 1,\dots,n$$

$$\prod_{j=1}^{n} q_j \prod_{i=1}^{g} [a_i,b_i] = 1\rangle.$$

Let

$$1 \to \mathbb{Z} \to \pi \to \Gamma \to 1$$

be a representative of an element of $\mathrm{Ext}(\Gamma,\mathbb{Z})$. If we choose lifts $\overline{a}_i$, $\overline{b}_i$, $\overline{q}_j$ of a_i, b_i, q_j $i = 1,\dots,g$, $j = 1,\dots,n$ respectively, it is easily seen that π has presentation

$$\langle \overline{a}_1,\overline{b}_1,\dots,\overline{a}_g,\overline{b}_g,\overline{q}_1,\dots,\overline{q}_n,h \mid [\overline{a}_i,h] = [\overline{b}_i,h] = [\overline{q}_j,h] = 1,$$

$$\overline{q}_j^{\alpha_j} = h^{s_j}, i = 1,\dots,g, j = 1,\dots,n, \prod_{j=1}^{n} \overline{q}_j \prod_{i=1}^{g} [\overline{a}_i,\overline{b}_i] = h^{s_0}\rangle$$

for some integers $s_0,\dots,s_n$. We denote this presentation by $\pi(s_0,\dots,s_n)$. We denote the element of $\mathrm{Ext}(\Gamma,\mathbb{Z})$ represented by

$$1 \to \mathbb{Z} \to \pi(s_0,\dots,s_n) \to \Gamma \to 1$$

by $E(s_0,\dots,s_n)$. Notice that we made a choice for the lifts of a_i, b_i, q_j. We could have chosen different lifts $\overline{\overline{a}}_i = \overline{a}_i h^{\ell_i}$,

$\bar{\bar{b}}_i = \bar{b}_i h^{m_i}$, $\bar{\bar{q}}_j = \bar{q}_j h^{k_j}$ in which case π would have presentation

$$<\bar{\bar{a}}_1,\bar{\bar{b}}_1,\ldots,\bar{\bar{a}}_g,\bar{\bar{b}}_g,\bar{\bar{q}}_1,\ldots,\bar{\bar{q}}_n,h \,|\, [\bar{\bar{a}}_i,h] = [\bar{\bar{b}}_i,h] = [\bar{\bar{q}}_j,h] = 1,$$

$$\bar{\bar{q}}_j^{\alpha_j} = h^{s_j+\alpha_j k_j}, i = 1,\ldots,g, j = 1,\ldots,n,$$

$$\prod_{j=1}^{n} \bar{\bar{q}}_j \prod_{i=1}^{g} [\bar{\bar{a}}_i,\bar{\bar{b}}_i] = h^{s_0+k_1+\cdots+k_n}>$$

Thus any element $E = E(s_0,\ldots,s_n) \in \mathrm{Ext}(\Gamma,\mathbb{Z})$ corresponds to an element $[s_0,\ldots,s_n] \in \mathbb{Z}^{n+1}/I$ where $I = \{(k_1+\cdots+k_n,\alpha_1 k_1, \cdots, \alpha_n k_n) | k_i \in \mathbb{Z}\}$ span $\{(1,\alpha_1,0,\ldots,0),(1,0,\alpha_2,0,\ldots,0),\ldots,(1,0,\ldots,0,\alpha_n)\}$. Therefore we have an injective map $F: \mathrm{Ext}(\Gamma,\mathbb{Z}) \to \mathbb{Z}^{n+1}/I$. We leave as an exercise the proof that $E(s_0,\ldots,s_n) \oplus E(s_0',\ldots,s_n') = E(s_0+s_0',\ldots,s_n+s_n')$ and hence F is a group homomorphism.

$\mathbb{Z}^{n+1}/I = <X_0,X_1,\ldots,X_n | \alpha_i X_i = X_0, i = 1,\ldots,n>$. (This can be seen by representing $X_0 = (1,0,\ldots,0)$, $X_1 = (0,-1,0,\ldots,0),\ldots, X_n = (0,\ldots,0,-1)$). Therefore to see $H^2(\Gamma;\mathbb{Z})$ has the desired form we must show F is onto.

To this end it suffices to show that given $[s_0,\ldots,s_n] \in \mathbb{Z}^{n+1}/I$ we get $E(s_0,\ldots,s_n) \in \mathrm{Ext}(\Gamma,\mathbb{Z})$. This requires showing that given $[s_0,\ldots,s_n]$ the kernel of the map $p: \pi(s_0,\ldots,s_n) \to \Gamma$ is $\mathbb{Z}$ and not a non-trivial quotient of $\mathbb{Z}$. We define a function $\nu: \mathbb{Z}^{n+1}/I \to S(\mathbb{Z}) = \{\text{subgroups of } \mathbb{Z}\}$ by $[s_0,\ldots,s_n] \mapsto \ker(\mathbb{Z} \to \pi(s_0,\ldots,s_n))$. Thus

(**) $$1 \to \mathbb{Z}/\nu([s_0,\ldots,s_n]) \to \pi(s_0,\ldots,s_n) \to \Gamma \to 1$$

is exact. Note ν has the following properties:

1) $\nu(k[s_0,\ldots,s_n]) = k\nu([s_0,\ldots,s_n])$ for all $k \in \mathbb{Z}$

2) $\nu([s_0,\ldots,s_n] + [s_0',\ldots,s_n']) \subseteq \nu([s_0,\ldots,s_n]) + \nu([s_0',\ldots,s_n'])$

for any $[s_0,\ldots,s_n],[s_0',\ldots,s_n'] \in \mathbb{Z}^{n+1}/I$.

Since (**) is exact, if we can show the image of ν is trivial, we will have shown $H^2(\Gamma;\mathbb{Z}) = \langle X_0,X_1,\ldots,X_n | \alpha_i X_i = X_0, i = 1,\ldots,n\rangle$. Note that for every $[s_0,\ldots,s_n] \in \mathbb{Z}^{n+1}/I$ there exists $k,m \in \mathbb{Z}$ such that $k[s_0,\ldots,s_n] = mX_0$. Therefore by property 1 it is sufficient to show $\nu(mX_0) = \langle 0\rangle$. In fact it is sufficient to show $\nu(\beta_1X_1 + \cdots + \beta_nX_n) = \langle 0\rangle$ for some $\beta_1X_1 + \cdots + \beta_nX_n$ with $\sum_{i=1}^{n} \beta_i/\alpha_i \neq 0$ since $\alpha_1\ldots\alpha_n(\beta_1X_1+\ldots+\beta_nX_n) = mX_0$ with $m \neq 0$. In the hyperbolic case we saw $(2g-2)X_0 + \sum_{i=1}^{n} ((\alpha_i - 1)/\alpha_i)X_i$ classifies $(T^1\mathbb{H})/\Gamma$ and

$$\begin{array}{ccccccccc}
1 \to & \pi_1(T^1\mathbb{H}) & \to & \pi_1(T^1\mathbb{H}/\Gamma) & \to & \Gamma & \to 1 \\
 & \| & & \| & & \| & \\
 & \mathbb{Z} & \longrightarrow & \pi & \longrightarrow & \Gamma &
\end{array}$$

is exact. Hence $\nu((2g-2)X_0 + \sum_{i=1}^{n} ((\alpha_i-1)/\alpha_i)X_i) = \langle 0\rangle$.

For the Euclidean case we recall that an earlier computation showed the universal abelian covers of the Seifert manifolds

$M(0;(2,\beta_1),(3,\beta_2),(6,\beta_3))$

$M(0;(2,\beta_1),(4,\beta_2),(4,\beta_3))$

$M(0;(3,\beta_1),(3,\beta_2),(3,\beta_3))$

$M(0;(2,\beta_1),(2,\beta_2),(2,\beta_3),(2,\beta_4))$

are

$M(1;(1,1))$

$M(1;(1,2))$

$M(1;(1,3))$

$M(1;(1,4))$

respectively, provided we assume $e(M \to F) < 0$ in each case. That is the universal abelian cover is the genuine S^1-bundle over T^2 of Euler number $-1,-2,-3$ or -4. Thus h has infinite order in the fundamental group of the cover and consequently in the fundamental group of the Seifert manifold in question.

It remains to show that automorphisms γ of Γ induce isomorphisms γ^* of $H^2(\Gamma;\mathbb{Z})$ of the desired type. As we saw in the proof of Theorem 9.2, γ must map $\langle q_j \rangle$ to a conjugate of some $\langle q_{j'} \rangle$ with $\alpha_j = \alpha_{j'}$. For simplicity of notation we assume $j = j'$ and hence $\gamma(q_j) = g_j^{-1}(q_j^{m_j})g_j$ with $m_j \in \mathbb{Z}$ g.c.d. $(m_j,\alpha_j) = 1$ and $g_j \in \Gamma$. Consider the action induced by γ on $\pi(s_0,\ldots,s_n)$. $\gamma(q_j^{\alpha_j}) = g_j^{-1}h^{-s_j m_j}g_j = h^{s_j m_j}$. Consequently for the element X_j of $H^2(\Gamma;\mathbb{Z})$ we have $\gamma^*(X_j) = m_j X_j$. γ^* is an isomorphism hence we must have $m_j = \pm 1$.

Remark: Much of the above proof can be put in a more general setting. If we let $\Gamma = \langle X_1,\ldots,X_g | W_1 = 1,\ldots,W_r = 1\rangle$ and A be an abelian group, then by a completely analagous construction as in the proof of Theorem 10.4 we obtain a map $F\colon \mathrm{Ext}(\Gamma,A) \to A^r/I$ where $I = \mathrm{Im}(W\colon A^g \to A^r)$ and $W = (W_1^{ab},\ldots,W_r^{ab})$ (W_i^{ab} is the abelianization of W_i). More precisely: given

$$1 \to A \to E \to \Gamma \to 1$$

a representative of an element of $\mathrm{Ext}(\Gamma,A)$, E has presentation:

$$(*) \qquad \langle \overline{X}_1,\ldots,\overline{X}_n,\text{"}A\text{"} | \overline{W}_1 = a_1,\ldots,\overline{W}_r = a_r, \text{"relations in } A\text{"}, \text{"}A \text{ central"}\rangle$$

(Here by "A" we mean a set of generators of A, by "relations in A" we mean a set of defining relations of A and by "A central" we mean A is central in E.) As before $\overline{X}_i$, $\overline{W}_i$ are lifts of X_i and W_i respectively. If we had chosen different $\overline{X}_i$'s say $\overline{\overline{X}}_i = \overline{X}_i a_i$ we would have $\overline{W}_i(\overline{X}_1 a_1,\ldots,\overline{X}_n a_n) = \overline{W}_i(\overline{X}_1,\ldots,\overline{X}_n)W_i^{ab}(a_1,\ldots,a_n)$. Thus by letting $F(E) = (a_1,\ldots,a_r)$ we get a well-defined map $F\colon \mathrm{Ext}(\Gamma,A) \to A^r/\mathrm{Im}(W_1^{ab},\ldots,W_r^{ab})$.

We define a map $\nu\colon A^r/I \to S(A) = \{\text{subgroups of } A\}$ by $[a_1,\ldots,a_r] \mapsto \ker(A \to \pi(a_1,\ldots,a_r))$ where $\pi(a_1,\ldots,a_r)$ is the group presented as in (*) above.

Hence $1 \to A/\nu([a_1,\ldots,a_r]) \to \pi(a_1,\ldots,a_r) \to \Gamma \to 1$ is exact. ν as defined in this more general way still satisfies properties 1) and 2) as listed in the proof of Theorem 10.4. Therefore in general, given a presentation of Γ and an abelian group A we get a subgroup I of A^r a map $\nu: A^r/I \to S(A)$ such that $\mathrm{Ext}(\Gamma,A) = \nu^{-1}(0)$ and $\mathrm{Ext}(\Gamma,A/nA) = \nu^{-1}(nA)/n(A^r/I)$.

References

[Gr] L. Greenberg, Finiteness theorems for Fuchsian and Kleinian groups, in Discrete Groups and Automorphic Functions, ed. W.J. Harvey, (Academic Press, 1977) 199-257.

[Hel] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, (Academic Press, 1978).

[Ker] B. Von Kerékjártó, Über die periodischen Transformationen der Kreisscheibe und der Kugelflache, Math. Ann. 80 (1921) 36-38.

[Mac] S. MacLane, Homology, (Springer Verlag, 1963).

[N-R] W. Neumann and F. Raymond, Seifert manifolds, plumbing, μ-invariant and orientation reversing maps, in Proceedings Sympos., University of California, Santa Barbara, CA 1977, Lecture Notes in Math., 664 (Springer Verlag, 1978) 163-196.

[O-R] P. Orlik and F. Raymond, Actions of SO(2) on 3 manifolds, in Proceedings of the Conference on Transformation Groups, (Springer Verlag, 1968).

[O-V-Z] P. Orlik, E. Vogt, and H. Zieschang. Zur Topologie gefaserter dreidimensionaler Mannigfaltigkeiten, Topology 6 (1967) 49-64.

[Wa 1] F. Waldhausen, Eine Klasse von 3-dimensionalen Mannigfaltigkeiten I, II, Invent. Math. 3 (1967) 308-333, ibid 4 (1967) 87-117.

[Wa 2] F. Waldhausen, Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten, Topology 6 (1967) 505-517.

[W] C.T.C. Wall, Rational Euler characteristics, Proc. Cambridge Philos. Soc. 57 (1961) 182-184.

GEOMETRY OF QUASIHOMOGENEOUS SURFACE SINGULARITIES

Walter D. Neumann*
University of Maryland
College Park, MD 20742

This is a slightly expanded version of my talk, the first part of which was purely expository, describing the analytic classification of normal quasihomogeneous surface singularities (from now on QH-singularity for short, our base field is $\mathbb{C}$). This classification is well known; the earliest version seems to be Conner and Raymond [CR, especially §13], and the most explicit probably Pinkham [P1, Theorem 2.1], who bases his version on Orlik and Wagreich's in [OW] (see also [W2]). Our approach is different from either, and seemed worth reproducing here. It is in terms of "Seifert line bundles" on complex curves a concept easily generalizable to higher dimensional base spaces and maybe of use in other contexts.

The second part of the talk, and the part to which the title refers, is new material. The main result is a natural "geometric structure" on the link M of any QH-singularity (V,p), such that the geometric structure on M determines the analytic structure on (V,p) and vice versa, in a one-one fashion.

---

*Partially supported by the National Science Foundation.

Reprinted from Proceedings of Symposia in Pure Mathematics, vol. 40, by permission of the American Mathematical Society.

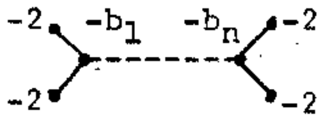
The archetypal example of this is due to Klein [Kl 1]. Let $I \subset SO(3)$ be the icosahedral group, that is the group of orientation preserving symmetries of an icosahedron. The universal cover of $SO(3)$ is the group $SU(2)$, also describable as the group S^3 of unit quaternions, and if we lift I to S^3 we get the binary icosahedral group $I' \subset S^3$ of order 120. Klein gave a homeomorphism

$$I'\backslash S^3 \cong \Sigma(2,3,5)$$

where $\Sigma(2,3,5) = V(2,3,5) \cap S^5$ with $V(2,3,5) = \{z \in \mathbb{C}^3 | z_1^2+z_2^3+z_3^5 = 0\}$. That is, the link $\Sigma(2,3,5)$ of the QH-singularity $(V(2,3,5),0)$ has the geometric structure $I'\backslash S^3$. Klein gave other examples, including a treatment for $V(2,3,7)$ in [Kl 2] and [KF], see also Brieskorn [B]. These results were generalized by Milnor [M] to arbitrary $V(a_1,a_2,a$ and by Dolgachev [D] and the author [N1] to complete intersections of $n-2$ copies of the n-dimensional Brieskorn variety $V(a_1,\dots,a_n)$ in general position. The present result generalizes all of these, without however giving as explicit a description of the geometric structure as was possible in these special examples. In the general case of rational base curve these more explicit results still hold however, see Section 7 and [N4].

Returning to $V(2,3,5)$, we should say that Klein's formulation is an analytic isomorphism between $I'\backslash\mathbb{C}^2$ and $V(2,3,5)$. Our general correspondence is analogous. Namely the relevant geometries are certain simply connected

3-dimensional homogeneous spaces $\mathbb{X} = G/K$. To each of them we give a G-invariant complex structure on $\mathbb{X} \times \mathbb{R}_+$. To a geometric structure $M = \Pi\backslash\mathbb{X}$ on a singularity link M is thus associated a complex structure $\Pi\backslash\mathbb{X} \times \mathbb{R}_+$ on $M \times \mathbb{R}_+$, and this is then $V - \{p\}$.

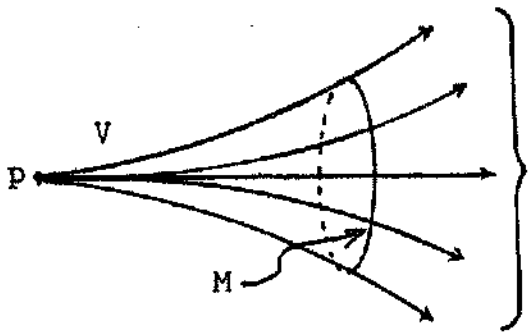
The result is slightly more general than stated in this introduction. Also we give an analogous result for cusp singularities and singularities with dual resolution graphs of the form 

. Since in these cases both the geometric structure on the link, and the singularity itself are rigid, this is of more philosophical interest. No other singularity links M^3 admit geometric structures.

The proofs of the geometric results will appear in detail in [N3]. Here we just sketch them in a typical case (section 6), using the analytic classification of the first part of this paper

1. Topology of a QH-singularity

By a QH-singularity we mean a normal surface singularity (V,p) with a good $\mathbb{C}^*$-action. "Good" means p is in the closure of every $\mathbb{C}^*$-orbit. Denote $V_0 = V - \{p\}$. $V_0/\mathbb{C}^*$ is a complex curve, which we denote X. Consider $\mathbb{R}_+ \subset \mathbb{C}^*$ acting on V. After renormalizing by the automorphism $t \mapsto t^{-1}$ of $\mathbb{C}^*$ if necessary, the $\mathbb{R}_+$-orbits are rays emanating from p, as in the schematic picture.



We can thus identify $V_0/\mathbb{R}_+$ with the link M of the singularity. Since $\mathbb{C}^* = \mathbb{R}_+ \times S^1$, we have $M/S^1 = V_0/\mathbb{C}^* = X$, and the map $M \to X$ is a Seifert fibration of M whose fibers are the S^1-orbits of M.

The topology of V_0, and hence also (V,p), is determined by M, since V_0 is $\mathbb{C}^*$-equivariantly homeomorphic to $M \times \mathbb{R}_+$. M itself is determined (up to S^1-equivariant orientation preserving diffeomorphism) by its Seifert invariant

$$\{g;b;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)\}.$$

Here $g = \text{genus}(X)$, and each pair (α_i,β_i) satisfies $0 < \beta_i < \alpha_i$ and $\gcd(\alpha_i,\beta_i) = 1$, and it codes the topology near a singular orbit of the S^1-action on M. The invariant

$$e(M\to X) = -b - \sum \beta_i/\alpha_i$$

generalizes the usual euler number of a non-singular S^1-bundle.

<u>Proposition</u> ([NR,§5], [P1,§2]). M <u>occurs as the link of a QH-singularity if and only if</u> $e(M\to X) < 0$.

A sharper result is in [N2, Corollary 6]. This proposition completes the classification of topological types of QH-singularities.

We are using the orientation conventions for Seifert invariants most prevalent in the literature. A reversal of orientation replaces β_i by $\alpha_i - \beta_i$, b by $-b - r$, and these values are sometimes used instead, particularly in the context of resolution of singularities, where they occur naturally. This is true for instance in [P1], quoted above.

2. Analytic Classification

With notation as in §1, let x_i denote the point in X over which the (α_i,β_i) orbit lies.

<u>Theorem</u> ([CR], [OW], [P1]). <u>The set of isomorphism types of</u> QH-<u>singularities</u> (V,p) <u>with fixed Seifert invariant and fixed analytic type of</u> $(X,x_1,\dots,x_n)$ <u>is isomorphic to</u> Jac(X) <u>modulo an action of</u> $\mathrm{Aut}(X,x_1,\dots,x_n)$.

We just sketch our argument; the details are not hard to fill in. Let $\bar{V} = V_0 \times_{\mathbb{C}^*} \mathbb{C}$, that is, $V_0 \times \mathbb{C}$ factored by the $\mathbb{C}^*$-action $t(v,z) = (tv,t^{-1}z)$. Writing $\bar{V} = (V_0 \times_{\mathbb{C}^*} \mathbb{C}^*) \cup (V_0 \times_{\mathbb{C}^*} \{0\}) = V_0 \cup X$, we see that $\bar{V}$ is obtained from V_0 by adding a 0 to each $\mathbb{C}^*$ orbit. We call $X \subset \bar{V}$ the <u>zero section</u>. The projection $V_0 \times \mathbb{C} \to V_0$ induces $\bar{V} \to V_0/\mathbb{C}^* = X$ which is a "Seifert line bundle", that is, a complex line bundle except that for each i the fiber over x_i is a singular fiber of the form $\mathbb{C}/\mu_{\alpha_i}$, where μ_{α_i} is the group of α_i-th roots of unity.

We shall prove the theorem by classifying Seifert line bundles, so we digress to explain how the Seifert line bundle $E = (\bar{V}\to X)$ determines (V,p). We can form $\bar{V} = V_0 \times_{\mathbb{C}^*} \mathbb{C}$ for any Seifert $\mathbb{C}^*$-bundle $V_0 \to X$, not just complements of QH-singular points, so Seifert $\mathbb{C}^*$-bundles and Seifert line bundles are equivalent concepts. We define the Seifert invariant of such a bundle to be the Seifert-invariant of the associated Seifert fibered $M^3 = V_0/\mathbb{R}_+$, and the euler number $e(V_0\to X) = e(\bar{V}\to X)$ to be $e(M\to X)$. The latter equals the self-intersection number $X\cdot X$ of the zero section $X \subset \bar{V}$ ($\bar{V}$ is a $\mathbb{Q}$-homology manifold and so intersection numbers can be defined $\mathbb{Q}$-Poincaré dual to cup product). By Grauert's criterion [G], which can be generalized to this situation, we can blow down $X \subset \overline{V}$ to get a normal singularity (V,p) if and only if $X \cdot X < 0$. $\overline{V}$ is in fact the first stage in a resolution of (V,p). It has only cyclic quotient singularities, at the points $x_i \in X \subset \overline{V}$, and these resolve by linear configurations of exceptional curves to give the familiar star shaped resolution configuration for (V,p).

Let $\mathrm{Pic}(X,(x_1,\alpha_1),\ldots,(x_n,\alpha_n))$ be the set of isomorphism classes of Seifert line bundles $E = (\overline{V} \to X)$ which have singular fibers only over $\{x_1,\ldots,x_n\}$ and which are locally isomorphic, near x_i, to the following:

$$(D \times \mathbb{C})/\mu_{\alpha_i}^{\beta_i} \to D/\mu_{\alpha_i} \cong D,$$

for some β_i with $0 \le \beta_i < \alpha_i$. Here D denotes $\{z \in \mathbb{C} \mid |z| < 1\}$ and μ_α^β denotes the α-th roots of unity μ_α acting on $D \times \mathbb{C}$ by $t(z_1,z_2) = (tz_1,t^{-\beta}z_2)$. We do not require $\gcd(\alpha_i,\beta_i) = 1$. Thus the Seifert invariant will have the form $(g;b,(\alpha_1',\beta_1'),\ldots,(\alpha_n',\beta_n'))$, where $\alpha_i' = \alpha_i/\gcd(\alpha_i,\beta_i)$, $\beta_i' = \beta_i/\gcd(\alpha_i,\beta_i)$, and pairs $(\alpha_i',\beta_i') = (1,0)$ refer to non-singular fibers and may be deleted.

Tensor product $\otimes$ is defined for Seifert line bundles (do it locally in the cyclic cover and then factor by μ_{α_i}, or globally: use Seifert $\mathbb{C}^*$-bundles and form $V_0 \oplus V_0'$ by a pullback, factor $V_0 \oplus V_0'$ by the C^*-action $t(v,v') = (tv,t^{-1}v')$, and then normalize to get $V_0 \otimes V_0'$). This gives a semigroup structure on $\mathrm{Pic}(X,\{(x_i,\alpha_i)\})$ such that

$\beta_i : \mathrm{Pic}(X,\{(x_i,\alpha_i)\}) \to \mathbb{Z}/\alpha_i$ is a homomorphism. The kernel of $\beta = \beta_i \times \cdots \times \beta_n : \mathrm{Pic}(X,\{(x_i,\alpha_i)\}) \to \prod \mathbb{Z}/\alpha_i$ consists of line bundles with no singular fibers, that is, $\mathrm{Ker}(\beta) = \mathrm{Pic}(X)$, which is a group. Hence $\mathrm{Pic}(X,\{(x_i,\alpha_i)\})$ is a group and moreover its identity component will be $\mathrm{Pic}_0(X) = \mathrm{Jac}(X)$. Denote $\mathrm{Pic}(X,\{(x_i,\alpha_i)\})/\mathrm{Jac}(X)$ by H. We obtain

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Jac}(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{d} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & \mathrm{Jac}(X) & \longrightarrow & \mathrm{Pic}(X,\{(x_i,\alpha_i)\}) & \xrightarrow{c} & H & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \prod \mathbb{Z}/\alpha_i & = & \prod \mathbb{Z}/\alpha_i & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns, where the top row is classical, with d equal to degree or chern class, and the right hand column is induced by the rest of the diagram. The projection map c should be thought of as a <u>chern class for Seifert bundles</u> By the following lemma it is an abstract version of the Seifert invariant.

<u>Lemma</u>. (i) $H = \langle g_0, g_1, \ldots, g_n \mid \alpha_i g_i = g_0 \rangle$ <u>with</u> $i : \mathbb{Z} \to H$ <u>given by</u> $i(1) = g_0$. <u>In particular any</u> $g \in H$ <u>can be uniquely written in the form</u> $g = bg_0 + \sum \beta_i g_i$ <u>with</u> $0 \leq \beta_i < \alpha_i$.
(ii) <u>If</u> $E \in \mathrm{Pic}(X,\{(x_i,\alpha_i)\})$ <u>has</u> $c(E) = -(bg_0 + \sum \beta_i g_i)$ <u>with</u>

$0 \le \beta_i < \alpha_i$ <u>then its Seifert invariant is</u> $(g;b;(\alpha_1',\beta_1'),\dots,(\alpha_n',\beta_n'))$ <u>with</u> $\alpha_i' = \alpha_i/\gcd(\alpha_i,\beta_i)$, $\beta_i' = \beta_i/\gcd(\alpha_i,\beta_i)$.

The content of this lemma is a formula for the Seifert invariant for $\overline{V} \otimes \overline{V}'$ in terms of the Seifert invariants of $\overline{V}$ and $\overline{V}'$. It follows easily from the definition of the Seifert invariant of $\overline{V}$ in terms of a nonzero continuous section to $\overline{V}$ over $X - \{x_i\}$ (see [NR]) and the observation that such sections in $\overline{V}$ and $\overline{V}'$ give one in $\overline{V} \otimes \overline{V}'$. The lemma can also be proved using only naturality properties of the euler number for nonsingular bundles to give an independent introduction to the Seifert invariant (exercise).

The exact sequence

$$0 \longrightarrow \mathrm{Jac}(X) \longrightarrow \mathrm{Pic}(X,\{(x_i,\alpha_i)\}) \longrightarrow H \longrightarrow 0$$

is the desired classification of Seifert bundles. This is also the form in which it was proved, by very different methods, in [CR]. In [CR] H arises as the second cohomology group $H^2(\Gamma)$ of a certain Fuchsian or euclidean group Γ (unless (V,p) is a quotient singularity). The connection will become clear in Section 6.

3. Geometry of 3-manifolds

A <u>geometric structure</u> on a manifold M shall mean a complete locally homogeneous riemannian metric of finite volume. Locally homogeneous means any two points have isometric neighborhoods. On the universal cover $\mathbb{X}$ of M such local isometries extend to global ones, so $\mathbb{X}$ is a homogeneous space.

We can thus write

$$\mathbb{X} = G/K, \qquad M = \Pi\backslash\mathbb{X} = \Pi\backslash G/K,$$

where $G = G(\mathbb{X})$ is the isometry group of $\mathbb{X}$, $K = G_x$ the isotropy group of some point $x \in \mathbb{X}$ (so K is well defined up to inner automorphisms of G), $\Pi \subset G$ is a discrete subgroup which acts freely on $\mathbb{X} = G/K$ with finite volume quotient Π is of course isomorphic to $\pi_1(M)$. $G^+ = G^+(\mathbb{X})$ will denote orientation preserving isometries.

To avoid trivial distinctions, we consider two metrics on $\mathbb{X}$ with the same isometry group G to be equivalent, and we only allow maximally symmetric metrics, that is, we assume no metric on $\mathbb{X}$, has strictly larger isometry group than G. The <u>geometry</u> $(\mathbb{X},G)$ will be relevant to 3-manifold theory if $\dim \mathbb{X} = 3$ and $\mathbb{X}$ admits finite volume quotients $\Pi\backslash\mathbb{X}$ with $\Pi \subset G$ discrete. Thurston [T] has pointed out that there are exactly 8 such geometries. They are most easily computed in terms of K_0, the identity component of the group $K = G_x$ of isometries fixing a point $x \in \mathbb{X}$.

$K_0 =$	$\mathbb{X} =$			$K =$
$SO(3)$	$\mathbb{S}^3$	$\mathbb{E}^3$	$\mathbb{H}^3$	$O(3)$
$SO(2)$	$\mathbb{S}^2 \times \mathbb{E}^1$		$\mathbb{H}^2 \times \mathbb{E}^1$	$O(2) \times O(1)$
		N	$\widetilde{PSL}$	$O(2)$
$\{1\}$		S		D_8

Here $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$ are spherical, euclidean, and hyperbolic geometry. N, $\widetilde{PSL}$, and S are certain lie groups with left invariant metrics. Namely $\widetilde{PSL}$ is the universal cover of $PSL(2,\mathbb{R}) = G^+(\mathbb{H}^2)$; this geometry can also be described as the universal cover of the unit tangent bundle $T^1\mathbb{H}^2$ of $\mathbb{H}^2$, with natural metric. N is the group of real 3×3 upper triangular unipotent matrices. It is more usefully described for us as the group structure:

$$(a,b;c)(a',b';c') = (a+a',b+b';c+c'+ab'-a'b) \quad \text{on} \quad \mathbb{R}^2 \times \mathbb{R}.$$

Its center is $\{0\} \times \mathbb{R}$, giving a central extension $0 \to \mathbb{R} \to N \to \mathbb{R}^2 \to 1$. Finally S is a split extension $1 \to \mathbb{R}^2 \to S \to \mathbb{R} \to 1$, and can be described explicitely as the group structure

$$(a,b;c)(a',b';c') = (a+e^c a',b+e^{-c}b';c+c') \quad \text{on} \quad \mathbb{R}^2 \times \mathbb{R}.$$

The isometry group for each of $\mathbb{X} = \widetilde{PSL}$, N, and S is a semidirect product $G(\mathbb{X}) = T\cdot K$, where T is the group $\widetilde{PSL}$, N, or S respectively, acting by left translations, and K is given in the table and acts on T as follows. For $\mathbb{X} = \widetilde{PSL}$, $K = O(2) \subset G(\mathbb{H}^2)$ acts by conjugation on $G^+(\mathbb{H}^2) = PSL(2,\mathbb{R})$, so lift this action to an action on $T = \widetilde{PSL}$. For N, with coordinates $\mathbb{R}^2 \times \mathbb{R}$ as above, $K = O(2)$ acts standardly on $\mathbb{R}^2$ and by determinant on $\mathbb{R}$. For S, K is dihedral of order 8 generated by $\tau : (a,b;c) \mapsto (b,a;-c)$ and

$\sigma : (a,b;c) \mapsto (-a,b;c)$.

If $M^3 \to F$ is a Seifert fibration of a closed connected oriented M^3 over a possibly non-orientable surface F, the Seifert invariant $(g;b;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ is still defined but we use negative g to indicate non-orientable F. Thus the euler number

$$e(M \to F) = -b - \sum_{i=1}^{n} \beta_i/\alpha_i$$

is still defined. We also define, with $\chi(F)$ equal to euler characteristic:

$$\chi(M \to F) = \chi(F) - \sum_{i=1}^{n} (\alpha_i - 1)/\alpha_i.$$

<u>Theorem</u>. <u>Let</u> M <u>be a closed connected oriented</u> 3-<u>manifold which admits a geometric structure</u>. <u>Then</u>:

(i) <u>The geometry</u> $\mathbb{X}$ <u>in question is uniquely determined by</u> M.

(ii) <u>If</u> M <u>admits an</u> $\mathbb{H}^3$ <u>structure we won't discuss it</u>.

(iii) M <u>admits an</u> S-<u>structure if and only if either</u>:

a) M <u>can be fibered over</u> S^1 <u>with fiber</u> $S^1 \times S^1$ <u>such that the monodromy</u> $h \in SL(2,\mathbb{Z})$ <u>has</u> $|\text{trace}(h)| \geq 3$, <u>or</u>

b) M <u>is a twisted double of the orientation</u> [0,1]-<u>bundle over the Klein bottle, but cannot be Seifert fibered</u>.

(iv) <u>M admits an</u> $\mathbb{X}$<u>-structure</u>, $\mathbb{X} \neq \mathbb{H}^3, S$, <u>if and only if</u> M <u>admits a Seifert fibering</u> $M \to F$. <u>The relevant geometry is as follows</u>:

e \ χ	>0	=0	<0
=0	$\mathbb{S}^2 \times \mathbb{E}^1$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{E}^1$
≠0	$\mathbb{S}^3$	N	$\widetilde{PSL}$

$\chi = \chi(M \to F)$

$e = e(M \to F)$

We call the six geometries of part (iv) the <u>Seifert geometries</u>.

<u>Remarks</u>. 1) For $\mathbb{X} = N$ or $\widetilde{PSL}$, $G(\mathbb{X}) = G^+(\mathbb{X})$, so all manifolds with an $\mathbb{X}$-structure are orientable. For $\mathbb{X} = S^3$ the latter is still true but less obvious.

2) The theorem is valid with minor changes also for non-orientable and/or noncompact M, and with slightly more change even for arbitrary lattices $\Pi \subset G(\mathbb{X})$, $\mathbb{X} \neq \mathbb{H}^3$, see [N3].

3) The invariants χ and e arise naturally for QH-singularities. For instance, Dolgachev has shown, and it also follows easily from Pinkham [P1, Theorem 5.1], that the Poincaré series for the graded affine ring of (V,p) is a rational function of the form

$$p(t) = -et/(1-t)^2 + \chi/2(1-t) + P(t)/Q(t),$$

where $Q(t)$ is cyclotomic and not divisible by $(t-1)$. See also [W2, §2] and [N4, §4].

4) With a natural normalization of the metrics on the

geometries, the volume of M is $4\pi^2\chi^2/|e|$ in the Seifert case when $e\chi \neq 0$, and is indeterminate, depending on the geometric structure, in all other cases with $\mathbb{X} \neq \mathbb{H}^3$. For $\mathbb{X} = N$ it is $\ell^2|e|$, where ℓ = (length of a fiber of M).

If $\mathbb{X}$ is a Seifert geometry other than $\mathbb{S}^3$ or $\mathbb{E}^3$, then the set of isometries which fix a point $x \in \mathbb{X}$ fixes a tangent direction at that point up to sign, so $\mathbb{X}$ has a $G(\mathbb{X})$-invariant tangent line field. This line field gives a foliation of $\mathbb{X}$, which in fact fibers $\mathbb{X}$ over $\mathbb{S}^2$, $\mathbb{E}^2$, or $\mathbb{H}^2$ in a way which is obvious for $\mathbb{X} = \mathbb{S}^2 \times \mathbb{E}$ and $\mathbb{H}^2 \times \mathbb{E}$, and which is visible from our description of $\mathbb{X}$ for $\mathbb{X} = N$ and $\mathbb{X} = \widetilde{PSL}$. It is this <u>vertical fibration</u> of $\mathbb{X}$ which induces the Seifert fibration of $M = \Pi\backslash\mathbb{X}$ (except for some $\mathbb{S}^2 \times \mathbb{E}^1$-structures on $S^2 \times S^1$). Each of the geometries $\mathbb{S}^3$ and $\mathbb{E}^3$ also fibers geometrically over $\mathbb{S}^2$ and $\mathbb{E}^2$ respectively ($\mathbb{S}^3 \to \mathbb{S}^2$ is Hopf fibration), but this fibration is only well defined up to isometries, so the subgroup $G^{fib} \subset G$ which preserves this fibration is only determined up to conjugation. The different conjugates of G^{fib} correspond to the different "<u>vertical fibrations</u>" of $\mathbb{X}$. If $M = \Pi\backslash\mathbb{X}$ is a geometric 3-manifold with $\mathbb{X} = \mathbb{S}^3$ or $\mathbb{E}^3$, then Π is in some conjugate of G^{fib} and a Seifert fibration of M can again be induced from the corresponding vertical fibration of $\mathbb{X}$. If $M = T^3 = S^1 \times S^1 \times S^1$ one must choose the conjugate of G^{fib} correctly—otherwise one just gets a foliation of M.

We call the Seifert fibration of $M = \Pi\backslash\mathbb{X}$, induced as above from a vertical fibration of $\mathbb{X}$, a <u>geometric Seifert fibration</u>. It is unique for $\mathbb{X} \neq \mathbb{S}^3, \mathbb{E}^3$, but may depend on a choice of conjugate of G^{fib} containing Π for $\mathbb{X} = \mathbb{S}^3$ or $\mathbb{E}^3$.

If $\mathbb{X}$ is a Seifert geometry, let $G_c = G_c(\mathbb{X}) \subset G^+(\mathbb{X})$ be the subgroup preserving a vertical fibration on $\mathbb{X}$ as an oriented fibration. Then $G_c = (G^{fib})_o$ if $\mathbb{X} = \mathbb{S}^3$ or $\mathbb{E}^3$ and $G_c = G_o$ otherwise. The centre C of G_c is S^1 for $\mathbb{X} = \mathbb{S}^3$ and is $\mathbb{R}$ otherwise.

<u>Proposition</u>. <u>Let</u> $M = \Pi \backslash \mathbb{X}$ <u>with</u> $\mathbb{X}$ <u>a Seifert geometry</u>. <u>Then</u> $\Pi \subset G_c$ (<u>respectively</u> Π <u>is in some conjugate of</u> G_c <u>if</u> $\mathbb{X} = \mathbb{S}^3$ <u>or</u> $\mathbb{E}^3$) <u>if and only if</u> M <u>can be Seifert fibered with orientable base</u>. <u>If</u> $M \neq S^2 \times S^1$, <u>then</u> $C/C \cap \Pi \cong S^1$ <u>and it acts on</u> M <u>inducing the geometric Seifert fibration</u> $M \to M/S^1$ (<u>for</u> $M = T^3$ <u>this holds only for suitable conjugates of</u> G_c <u>containing</u> Π).

Not every geometric structure on $S^2 \times S^1$ admits a geometric Seifert fibration, but any Seifert fibration, of any M, is geometric for some geometric structure on M.

4. <u>Geometry of holomorphic Seifert $\mathbb{C}^2$-bundles</u>

For each Seifert geometry $\mathbb{X}$ let $G_{\mathbb{C}}(\mathbb{X}) = G_c(\mathbb{X}) \times \mathbb{R}_+$, acting transitively on $\mathbb{X} \times \mathbb{R}_+$ in the obvious way. We shall describe a complex analytic structure on $\mathbb{X} \times \mathbb{R}_+$ such that $G_{\mathbb{C}} = G_{\mathbb{C}}(\mathbb{X})$ acts by complex analytic maps.

On $\mathbb{E}^1 \times \mathbb{R}_+$ take the complex structure

$$\mathbb{E}^1 \times \mathbb{R}_+ \to \mathbb{C}, \qquad (\theta,r) \mapsto \theta - i\cdot \ell n(r).$$

The notation is chosen to suggest that $\mathbb{E}^1 \times \mathbb{R}_+$ is polar coordinates in $\tilde{\mathbb{C}}^*$, the universal cover of $\mathbb{C}^*$. Now the complex structures on $\mathbb{S}^2 \times \mathbb{E}^1 \times \mathbb{R}_+$, $\mathbb{E}^2 \times \mathbb{E}^1 \times \mathbb{R}_+$, and $\mathbb{H}^2 \times \mathbb{E}^1 \times \mathbb{R}_+$ are the obvious ones: $\mathbb{C}P^1 \times \mathbb{C}$, $\mathbb{C} \times \mathbb{C}$, $\mathbb{H} \times \mathbb{C}$ (we use $\mathbb{H}$, as opposed to $\mathbb{H}^2$, to denote the upper half plane with complex structure, instead of hyperbolic metric).

On $\mathbb{S}^3 \times \mathbb{R}_+$ take the obvious structure as $\mathbb{C}^2-\{0\}$, by considering $\mathbb{S}^3 \times \mathbb{R}_+$ as polar coordinates, with G_c acting as $U(2)$.

On $N \times \mathbb{R}_+$, coordinatize N as in §3 and take the complex structure

$$N \times \mathbb{R}_+ \to \mathbb{C} \times \mathbb{C}, \qquad ((a,b;c),r) \mapsto (a+ib, c-\tfrac{i}{2}(a^2+b^2-2\ \ell n(r))).$$

For $\widetilde{PSL} \times \mathbb{R}_+$ recall first that we can identify $PSL = \mathbb{Z}\backslash\widetilde{PSL}$ with the unit tangent bundle $T^1\mathbb{H}^2$ of $\mathbb{H}^2$. Thus $PSL \times \mathbb{R}_+$ can be taken as polar coordinates in $T_0\mathbb{H}$ (the bundle of non-zero tangent vectors), so $\widetilde{PSL} \times \mathbb{R}_+$ is identified with $(T_0\mathbb{H})^\sim$. Since $T_0\mathbb{H} \cong \mathbb{H} \times \mathbb{C}^*$, we have $\widetilde{PSL} \times \mathbb{R}_+ \cong (\mathbb{H} \times \mathbb{C}^*)^\sim \cong \mathbb{H} \times \mathbb{C}$.

In each case denote $\mathbb{X}_{\mathbb{C}} = \mathbb{X} \times \mathbb{R}_+$ with the above complex structure. The center $C \times \mathbb{R}_+$ of $G_{\mathbb{C}}$ can be identified as $\mathbb{C}^*$ acting by multiplication on $\mathbb{X}_{\mathbb{C}} = \mathbb{C}^2-\{0\}$ when $\mathbb{X}^3 = \mathbb{S}^3$, and as $\mathbb{C}$, acting by translations in the second factor of

$\mathbb{X}_{\mathbb{C}} = (-) \times \mathbb{C}$ in each of the other cases.

Let $M = \Pi\backslash\mathbb{X}$ be a geometric Seifert manifold with geometric Seifert fibration, as in the last proposition of §3. Thus $\Pi \subset G_c$ and $C/C \cap \Pi = S^1$. Then

$$M \times \mathbb{R}_+ = \Pi\backslash\mathbb{X} \times \mathbb{R}_+ = \Pi\backslash\mathbb{X}_{\mathbb{C}}$$

gives a complex structure on $M \times \mathbb{R}_+$ and $S^1 \times \mathbb{R}_+$ acts as $(C/C \cap \Pi) \times \mathbb{R}_+ \cong \mathbb{C}^*$, acting holomorphically. That is, $M \times \mathbb{R}_+$ receives the structure of a holomorphic Seifert $\mathbb{C}^*$-bundle.

Theorem. Normalize by fixing vol(M) in the $\chi = 0$ cases and fixing the length of a fiber of M in the $e = 0$ cases (see Remark 4 on p. 13). Then if double brackets represent "set of equivalence classes of ..." with an appropriate isomorphism concept, the above construction defines a bijection

1) $\left\{\left\{\text{Geometric structures on } M \text{ with geometric Seifert fibration.}\right\}\right\} \leftrightarrow \left\{\left\{\text{Holomorphic structures on } M \times \mathbb{R}_+ \text{ as a Seifert } \mathbb{C}^*\text{-bundle.}\right\}\right\}$.

In particular:

2) $\left\{\left\{\text{Geometric structures on } M \text{ with geometric Seifert fibration with negative } e.\right\}\right\} \leftrightarrow \left\{\left\{\text{QH-singularities } (V,p) \text{ with link homeomorphic to } M.\right\}\right\}$

Unless (V,p) is a cyclic quotient singularity, so M is a lens space, the singularity (V,p) has a unique good $\mathbb{C}^*$-action and M has a unique geometric Seifert fibration. So excluding lens spaces, and assuming M admits a Seifert fibration with negative e, we get:

3) {{Geometric structures on M}} $\leftrightarrow$ {Normal singularities (V,p) with link homeomorphic to M which admit a good $\mathbb{C}^*$-action.}

For a cyclic quotient singularity (V,p) the geometric structure on its link M is unique and geometric Seifert fibrations of M correspond one-one with $\mathbb{C}^*$-actions on (V,p).

5. Cusps and the geometry S

There is a correspondence, analogous to 3) above, connected with the geometry S. Coordinatize S as $\{(a,b;c) \mid a,b,c \in \mathbb{R}\}$ as in §3. Let $G_c \subset G(S)$ be the subgroup of index 4, generated by $G_0 = S$ (acting on itself by left translations) and the element $\tau : (a,b;c) \to (b,a;-c)$ mentioned in §3. Then G_c acts on $\mathbb{H} \times \mathbb{H}$ as follows:

$$(a,b;c)(z_1,z_2) = (e^c z_1 + a, e^{-c} z_2 + b)$$

$$\tau(z_1,z_2) = (z_2,z_1).$$

The map $\mathbb{H} \times \mathbb{H} \to S \times \mathbb{R}_+$, $(z_1,z_2) \mapsto ((\mathrm{Re}\, z_1, \mathrm{Re}\, z_2; \ln \mathrm{Im}\, z_1), \mathrm{Im}\, z_1 \cdot \mathrm{Im}\, z_2)$ is a G_c-equivariant homeomorphism. Thus an S-structure $M \cong \Pi\backslash S$, with $\Pi \subset G_c$, on a manifold M leads to a complex structure $M \times \mathbb{R}_+ \cong \Pi\backslash S \times \mathbb{R}_+ \cong \Pi\backslash(\mathbb{H} \times \mathbb{H})$ on $M \times \mathbb{R}_+$.

Theorem. 1) A manifold M which admits an S-structure admits a unique one of any chosen volume.

2) The following are equivalent:

   a) $M \cong \Pi\backslash S$ with $\Pi \subset S$;

   b) M fibers over S^1 with fiber T^2 and monodromy of trace ≥ 3;

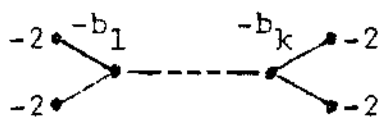
   c) M is homeomorphic (preserving orientation) to the link of a cusp singularity (that is a singularity (V,p) with cyclic resolution graph);

   d) Statement c) is true and $V-\{p\} = \Pi\backslash(\mathbb{H} \times \mathbb{H})$.

3) The following are equivalent:

   a) $M \cong \Pi\backslash S$ with $\Pi \subset G_c$, $\Pi \not\subset S$

   b) M is homeomorphic (respecting orientation) to the link of a singularity (V,p) with resolution graph of the form

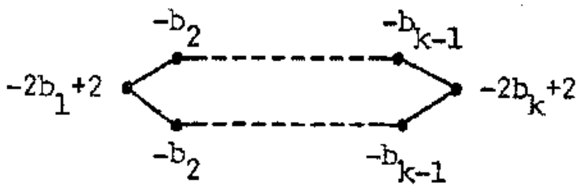


$b_i \geq 2$ for all i,
$b_i \geq 3$ for some i.

   c) Statement b) is true and $V-\{p\} = \Pi\backslash(\mathbb{H} \times \mathbb{H})$.

Remarks. 1. The analytic structure on these singularities is unique, by Karras [K]. Their description as "cusps" of discrete quotients of $\mathbb{H} \times \mathbb{H}$ is how they originally arose, in the work Hirzebruch et. al. on Hilbert modular surfaces.

2. The double cover of the singularity of 3b) above, determined by $\Pi \cap S \subset \Pi$, must be a cusp. It is the cusp with resolution graph



3. In sections 4 and 5 we have dealt with every singularity link which admits a geometric structure. Indeed, a singularity link M cannot admit an $\mathbb{H}^3$-structure, since no plumbed manifold admits an $\mathbb{H}^3$-structure. We have dealt with all Seifert manifolds which are singularity links by [N2]. Thus the only case remaining was S-structures. One can either compute all orientable S-manifolds and compare with [N2]. This is not hard, they turn out to be all 3-manifolds which can be plumbed according to a cyclic plumbing graph or a graph as in 3b) above but without the restrictions on b_1 and b_k. It is easier to observe that an S-manifold has solvable fundamental group. We have all singularity links with solvable fundamental group by Wagreich's list [W1].

6. Ideas of proof

We illustrate the proof of the theorem of Section 3 by sketching why a manifold M^3, Seifert fibered as $M \to X$ with orientable base, has a $\widetilde{PSL}$ structure when $\chi < 0$ and $e \neq 0$. Let Π be $\pi_1(M)$. The center of Π is $\mathbb{Z}$, generated by the class of a non-singular fiber, and we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Pi \longrightarrow \Gamma \longrightarrow 1,$$

where $\Gamma = \Gamma(g;\alpha_1,\ldots,\alpha_n)$ is a Fuchsian group with signature $(g;\alpha_1,\ldots,\alpha_n)$ (if χ had been $=0$ or >0 then Γ would be euclidean or spherical respectively, instead of Fuchsian).

Now the group H of Section 2 is in fact $H^2(\Gamma;\mathbb{Z})$ and the "chern class" $c \in H$ described there is just the classifying element for this exact sequence. Note that $H^2(\Gamma;\mathbb{R}) = H \otimes \mathbb{R} \cong \mathbb{R}$ by the map $g_0 \mapsto 1$, and if $i:\mathbb{Z} \to \mathbb{R}$ is the inclusion, then $i_*c \in H^2(\Gamma;\mathbb{R}) = \mathbb{R}$ is just $e(M \to X)$, by the lemma in §2.

We need to embed Π in $G_0 = G_0(PSL)$ in the following way

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Pi & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \downarrow ? & & \downarrow ? & & \downarrow \alpha & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & G_0 & \longrightarrow & (G^+(\mathbb{H}^2) = PSL(2,\mathbb{R})) & \longrightarrow & 1 \,.
\end{array}$$

It is not hard to see that this is just what is necessary to give $\Pi\backslash\widetilde{PSL}$ the desired Seifert fibered structure. The map α is given to us by choosing a complex structure (or a hyperbolic orbifold structure) on $(X,x_1,\ldots,x_n)$. Let us form the following pushout and pullback extensions $(*)$ and $(**)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Pi & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \downarrow i & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & i_*\Pi & \longrightarrow & \Gamma & \longrightarrow & 1
\end{array} \qquad (*)$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \alpha^*\Gamma & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow \alpha & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & G_0 & \longrightarrow & PSL(2,\mathbb{R}) & \longrightarrow & 1
\end{array} \qquad (**)$$

It suffices to show that (*) and (**) are isomorphic. But we have already observed that (*) is classified by $i_*c = e = e(M\to X) \in \mathbb{R} = H^2(\Gamma,\mathbb{R})$, and it is not hard to show that (**) is classified by $\chi \in \mathbb{R} = H^2(\Gamma,\mathbb{R})$ (this follows, for instance, from the computation of Seifert invariants of $\Gamma\backslash PSL(2,\mathbb{R})$, see [EHN] or [RV]). Thus, letting $\beta:\mathbb{R}\to\mathbb{R}$ be multiplication by χ/e, the following diagram can be completed, and we are done.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & i_*\Pi & \longrightarrow & \Gamma & \longrightarrow & 1 \qquad (*) \\ & & \downarrow \beta & & \downarrow \cong & & \| & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \alpha^*\Gamma & \longrightarrow & \Gamma & \longrightarrow & 1 \qquad (**) \end{array}$$

The various different $\widetilde{PSL}$-structures on M lying over a fixed structure on X are classified by the different homomorphisms $\gamma:\Pi\to G_0$ for which

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Pi & \xrightarrow{\pi} & \Gamma & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \gamma & & \downarrow \alpha & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & G_0 & \longrightarrow & PSL(2,\mathbb{R}) & \longrightarrow & 1 \end{array}$$

commutes, up to automorphisms of Π fixing Γ. Given one γ, any other γ' is given by $\gamma'(g) = \gamma(g)\cdot\psi\pi(g)$ for some $\psi \in Hom(\Gamma,\mathbb{R})$, with $\psi \in Hom(\Gamma;\mathbb{Z})$ if γ' is related to γ by an automorphism of Π. Thus $Hom(\Gamma,\mathbb{R})/Hom(\Gamma,\mathbb{Z})$ classifies the $\widetilde{PSL}$-structures on M. The correspondence of §4 sets up a map $Hom(\Gamma,\mathbb{R})/Hom(\Gamma,\mathbb{Z}) \to Jac(X)$ and to prove the theorem of §4 for this case, we must show this is bijective. But it is a map of real tori of equal dimension, so it is enough to show injectivity. A computation shows that this is equivalent to the image of the period mapping $H^0(X,\Omega^1) \to \mathrm{Hom}(\Gamma,\mathbb{C}) = H^1(X,\mathbb{C})$ being transverse to $\mathrm{Hom}(\Gamma,\mathbb{R}) = H^1(X,\mathbb{R})$, which is true, by classical complex curve theory.

The proofs of the results of §3 and §4 in the other cases are similar. Details will be given in [N3].

7. An application

One interpretation of the results of §4 is that the cocycles which are used in other approaches to classify (Seifert) $\mathbb{C}^*$-bundles over a curve X can be put in a very special form. This implies that one can interpret the affine coordinate ring of a QH-singularity (V,p) as a suitable ring of automorphic forms which transform by characters, rather by some general cocycle "multiplier system." But this is really a non-application, in that it would be of interest only if the type of ring of X-automorphic forms which arises were a type which has independent interest, say to number theorists, which it isn't. Nevertheless, as an application of this train of thought we get:

__Theorem__. __Let__ (V,p) __be a__ QH-__singularity with__ $V_0/\mathbb{C}^* = (X,x_1,\ldots,x_n)$ __and Seifert invariant__ $(g;b;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$. __Suppose__ X __is rational, that is,__ $g = 0$. __Let__ $n \geq 3$ (__i.e.__, (V,p) __is not a cyclic quotient singularity__). __Then the universal__ *abelian cover* $(\tilde{V}^{ab},p)$ *of* (V,p), *branched only at* p, *is isomorphic to* $(V_A(\alpha_1,\ldots,\alpha_n),0)$, *where* $V_A(\alpha_1,\ldots,\alpha_n)$ *is the Brieskorn complete intersection:*

$$V_A(\alpha_1,\ldots,\alpha_n) = \{z \in \mathbb{C}^n \mid a_{i1}z_1^{\alpha_1}+\cdots+a_{in}z_n^{\alpha_n} = 0, \quad i=1,\ldots,n-2\}$$

for suitable coefficient matrix $A = (a_{ij})$. *In fact, if one determines* $\lambda_1,\ldots,\lambda_{n-3} \in \mathbb{C}$ *by the unique analytic isomorphism* $(X,x_1,\ldots,x_n) \cong (\mathbb{C}\cup\infty,\lambda_1,\ldots,\lambda_{n-3},1,0,\infty)$, *then one can take*

$$A = \begin{pmatrix} 1 & & & & 1 & \lambda_1 \\ & \ddots & & 0 & \vdots & \vdots \\ & & 1 & & 1 & \lambda_{n-3} \\ & 0 & & 1 & 1 & 1 \end{pmatrix}$$

Proof. We just sketch the proof, for a reason given below Assume $\chi < 0$. We first note a general fact that if, for $i = 1,2$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Pi_i & \longrightarrow & \Gamma & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \gamma_i|\mathbb{Z}} & & \downarrow{\scriptstyle \gamma_i} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

is a map of central extensions, then $\gamma_1([\Pi_1,\Pi_1]) = \gamma_2([\Pi_2,\Pi_2])$ Now our $V_0 = V-p$ is classified by an embedding $\Pi \to G_0 = G_0(\widetilde{PSL})$ which fits in a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Pi & \longrightarrow & \Gamma & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & G_0 & \longrightarrow & PSL(2,\mathbb{R}) & \longrightarrow & 1 . \end{array}$$

Also $\gamma|[\Pi,\Pi]$ corresponds to the universal abelian cover of V_0, so our initial comment shows that if we prove the theorem for just one (V,p) with given α, then it follows for all such (V,p). But such a proof was given (in fact for $V_0 = \alpha(\Gamma)\backslash T_0\mathbb{H})$ in [N1], see also [D].

So this proves the case $\chi < 0$. The analogous proof applies for $\chi > 0$, and would apply too for $\chi = 0$ except that in this case the necessary examples had only been analyzed by ad hoc means in [M] and [N1], and this analysis does not give the information we need. This gap could be filled but this is not necessary, since a much more elementary proof of the theorem will appear elsewhere in these Proceedings [N4]. The above is however essentially how I found the result, with the observation about commutator subgroups replaced by the corresponding observation about rings of χ-automorphic forms.

REFERENCES

[B] E. Brieskorn, The unfolding of exceptional singularities, (to appear in Nova Acta Leopoldina, 1981).

[CR] P. E. Conner, F. Raymond, Holomorphic Seifert fibrations, Proc. Conf. on Transformation Groups, Amhearst 1973, Springer Lecture Notes No. 299 (1974), 124-204.

[D] I. V. Dolgachev, In preparation, see also, Automorphic forms and quasihomogeneous singularities, Funkcional Anal. i Priložen 9 (2) (1975), 67-68.

[EHN] D. Eisenbud, U. Hirsch, W. D. Neumann, Transverse foliations of Seifert bundles and self homeomorphisms of the circle, Comm. Math. Helv. (to appear).

[G] H. Grauert, Über Modifikation und exceptionelle analytische Mengen, Math. Ann. 146 (1962), 331-368.

[K] U. Karras, Klassifikation 2-dimensionaler Singularitäten mit auflösbaren lokalen Fundamentalgruppen, Math. Ann. 213 (1975), 231-255.

[Kl 1] F. Klein, Lectures on the icosahedron and the solution of equations of the fifth degree (1888), reprinted Dover Publications Inc. 1956.

[Kl 2] _____, Über die Transformation siebenter Ordnung der elliptischen Funktionen, Math Ann. 14 (1878), 428-471.

[KF] F. Klein, R. Fricke, Vorlessungen über die Theorie der elliptischen Modulfunktionen, (Teubner, Leipzig 1890).

[M] J. Milnor, On the 3-dimensional Brieskorn manifold M(p,q,r), papers dedicated to the memory of R. H. Fox, Ann. of Math. Studies No. 84, Princeton Univ. Press (1975), 175-225.

[N1] W. D. Neumann, Brieskorn complete intersections and automorphic forms, Inv. Math. 42 (1977), 285-293.

[N2] _____, A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves, to appear in Transactions A.M.S.

[N3] _____, Geometry of 3-manifolds and normal surfaces singularities (tentative title), in preparation.

[N4] _____, Abelian covers of quasihomogeneous surface singularities, these Proceedings.

[NR] W. D. Neumann, F. Raymond, Seifert manifolds, plumbing, μ-invariant, and orientation reversing maps, Proc. Alg. and Geom. Top (Santa Barbara 1977), Springer Lecture Notes 644).

[OW] P. Orlik, P. Wagreich, Algebraic surfaces with k*-action, Acta Math. 138 (1977), 43-81.

[P1] H. Pinkham, Normal surface singularities with ℂ*-action Math. Ann 227 (1977), 183-193.

[P2] _____, Deformations of normal surface singularities with ℂ*-action, Math. Ann. 232 (1978), 65-84.

[RV] F. Raymond, A. T. Vasquez, 3-manifolds whose universal coverings are Lie groups, Topology and its applications 12 (1981), 161-179.

[T] W. P. Thurston, Three dimensional manifolds, Kleinian groups and hyperbolic geometry (to appear).

[W1] P. Wagreich, Singularities of complex surfaces with solvable local fundamental group, Topology 11 (1971), 51-72.

[W2] _____, The structure of quasi-homogeneous singularities these Proceedings.