Representability theorems for simplicial presheaves

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Introduction

The Brown representability theorem gives a list of conditions for a set-valued contravariant functor defined on the classical pointed homotopy category to be representable. It has had many uses through the years, and has long been part of the canon of Algebraic Topology.

It is entirely reasonable to ask for a more general version of Brown representability, which gives conditions on a closed model category $\mathcal{N}$ and a contravariant set-valued functor G defined on the homotopy category $\text{Ho}(\mathcal{N})$ so that the functor G is representable. One could call this a Brown representability theorem for $\mathcal{N}$, although some might say that it is a "cohomological" Brown representability result [3], [14].

Such a result is proved in this paper, and appears as Theorem 24. The conditions for the Theorem are essentially classical: the model category $\mathcal{N}$ must have a set of compact generators, suitably defined, while the functor G should take coproducts to products and should satisfy a Mayer-Vietoris property. Theorem 24 asserts that G is representable under these circumstances. The proof displayed for this result is the standard argument (see also the proof of Theorem 3.1 in [13], or Heller's purely categorical formulation in [5]), albeit translated into the language of model categories. Theorem 24 and its proof are not new.

Multiple settings in which Theorem 24 applies are displayed in the third section, following the proof. The basic message is that there are classical Brown representability results for all model structures based on pointed simplicial presheaves — these include presheaves of spectra, presheaves of chain complexes, diagrams of spectra, motivic T-spectra and unstable motivic homotopy theories — *so long as* the underlying local model structure on pointed simplicial presheaves is defined on a rather forgiving Grothendieck topology, for which a set of compact generators can be defined in a traditional way.

Many of the standard geometric topologies, such as the étale topology, are not so forgiving, and the classical argument for Brown representability does not work in those cases. The problem is the compact generation requirement, which fails because "small" inductive colimits of fibrant objects may not be fibrant in any reasonable sense. This is overcome by using the observation that inductive colimits of fibrant simplicial presheaves are fibrant provided that the inductive systems are large enough, but at a cost of introducing homotopy coherence issues which cannot be addressed within the traditional framework for Brown representability.

Homotopy coherence problems are often solved in the context of simplicial functors between simplicial model categories, and that is what we do here. The main result of this paper, which is Theorem 16, gives conditions on a pointed simplicial model category $\mathcal{M}$ and a (contravariant) simplicial functor $F : \mathcal{M}^{op} \to s\mathbf{Set}_*$ taking values in pointed simplicial sets, such that F is sectionwise weakly equivalent to a representable functor $\mathbf{hom}(\ ,Y)$ which is defined by an object Y of $\mathcal{M}$ which is fibrant and cofibrant.

The conditions on the simplicial model category $\mathcal{M}$, which are discussed in some detail in the first section of the paper, are abstractions of the behaviour of model categories of pointed simplicial presheaves. These conditions are met by anything that can be built from simplicial presheaves, including presheaves of spectra, presheaves of symmetric spectra and Bousfield localizations of these categories.

The conditions on the functor F are satisfied by representable functors: this functor should preserve weak equivalences between cofibrant objects, and should take homotopy colimits to homotopy inverse limits. The latter condition implies strong forms of both the wedge and Mayer-Veitoris properties that one finds in the conditions for the classical Brown representability theorem, and it gives a way of inductively producing vertices of homotopy inverse limits of big towers. This last device solves the homotopy coherence problem in the formulation and proof of the representability theorem, and the technique appears in the proof of Proposition 9, which is the key step in the derivation of the main result.

Theorem 16 is a very strong representability result and is quite general (compare [2]), but it is perhaps too strong.

Something like Brown representability for simplicial presheaves and presheaves of spectra has been expected since the late 1980s. The original dream was that Brown representability could give a useful descent condition for presheaves of spectra on the étale site.

Theorem 16 does imply a descent criterion. If one can show, for example, that if a presheaf of Ω-spectra E on a big site of schemes represents a functor $\mathbf{hom}(\ ,E)$ which takes étale local weak equivalences between cofibrant objects to weak equivalences, then E satisfies descent for the étale topology.

But there is a problem, in that one doesn't need a representability theorem to derive this condition: it's actually quite easy to prove — see Proposition 17. In the end, this criterion just isolates the interesting part of the descent problem. One needs a descent condition, and perhaps a representability theorem, with weaker conditions. One has to be able to ask for something less than the full class of local weak equivalences to be preserved by the functor F.

1 Simplicial presheaf model categories

Suppose that $\mathcal{M}$ is a closed simplicial model category. Write $\mathbf{hom}(A, B)$ for the function complexes associated with simplicial structure of $\mathcal{M}$, and let $A \otimes \Delta^n_+ \to B$ be the morphisms making up their simplices. As usual, Δ^n_+ is the standard n-simplex Δ^n with a disjoint base point attached.

Suppose that the category $\mathcal{M}$ and its model structure have the following properties:

M1 The category $\mathcal{M}$ has all small limits and colimits, and has an object $*$ which is both initial and terminal.

M2 The model structure on $\mathcal{M}$ is cofibrantly generated, meaning that there is a set I of trivial cofibrations and a set J of cofibrations, with cofibrant source objects in all cases, such that $p : X \to Y$ is a fibration (respectively trivial fibration) if and only if it has the right lifting property with respect to all members of I (respectively J).

Here's a basic fact about pointed simplicial sets:

Lemma 1. *Suppose that $f : X \to Y$ is a map of simplicial sets such that f induces bijections*

$$[K, X] \xrightarrow{\cong} [K, Y]$$

for all finite simplicial sets K. Then f is a weak equivalence.

Proof. It suffices to assume that X and Y are Kan complexes, and that f is a fibration. Then f is a surjective fibration because every simplex $\Delta^n \to Y$ of Y lifts to X up to homotopy, so it lifts.

Suppose given a lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{\alpha} & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \xrightarrow[\beta]{} & Y \end{array} \qquad (1)$$

The map $\beta : \Delta^n \to Y$ lifts to X up to homotopy, so there is a homotopy

$$H : \Delta^n \times \Delta^1 \to Y$$

from β to a composite $p\dot{\omega}$ for some map $\omega : \Delta^n \to X$. The restricted homotopy

$$h = H|_{\partial\Delta^n \times \Delta^1} : \partial\Delta^n \times \Delta^1 \to Y$$

lifts to a homotopy $\zeta : \partial\Delta^n \times \Delta^1 \to X$ which ends at $\omega|_{\partial\Delta^n}$, and the map

$$(\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{1\}) \xrightarrow{(\zeta,\omega)} X$$

extends to a map $H' : \Delta^n \times \Delta^1 \to X$. The map

$$(\Delta^n \times \Lambda^2_2) \cup (\partial\Delta^n \times \Delta^2) \xrightarrow{((f(H'),H,\ ),s_0 h)} Y$$

extends to a map $\theta : \Delta^n \times \Delta^2 \to Y$, and the composite map

$$\Delta^n \times \Delta^1 \xrightarrow{1\times d^2} \Delta^n \times \Delta^2 \xrightarrow{\theta} Y$$

is a pointed homotopy from $\beta : \Delta^n \to Y$ to $f(\beta')$ for some map $\beta' : \Delta^n \to X$, rel boundary, where $\beta'|_{\partial\Delta^n} = \omega|_{\partial\Delta^n}$. It follows that the diagram (1) is homotopic to one for which the lifting problem can be solved, so that the desired lifting exists. □

Corollary 2. *Suppose that $f : X \to Y$ is a map of pointed simplicial sets such that f induces bijections*

$$[L, X] \xrightarrow{\cong} [L, Y]$$

in the pointed homotopy category for all finite pointed simplicial sets L. Then f is a weak equivalence.

Proof. If K is a finite simplicial set, then $K_+ = K \sqcup \{*\}$ is a finite pointed simplicial set, and there is a bijection

$$[K_+, X] \cong [K, X]$$

relating morphisms in the pointed and unpointed homotopy categories. □

Corollary 3. *Suppose that $\mathcal{M}$ is a pointed closed simplicial model category which is cofibrantly generated. Suppose that S is the set of all objects $A \otimes L$, where A is either a target or source object of some morphism $A \to B$ appearing in the set of generating cofibrations for $\mathcal{M}$ and L is a finite pointed simplicial set. Then a map $f : X \to Y$ is a weak equivalence of $\mathcal{M}$ if and only if the functions*

$$f_* : [A \otimes L, X] \to [A \otimes L, Y]$$

are bijections for all objects $A \otimes L$ of S.

Proof. We show that f is a weak equivalence if all of the displayed functions are bijections.

It suffices to assume that f is a fibration and that the objects X and Y are fibrant. Suppose that $i : A \to B$ is a generating cofibration for $\mathcal{M}$, and form the diagram

$$\begin{array}{ccc} \mathbf{hom}(B, X) & \xrightarrow{f_*} & \mathbf{hom}(B, Y) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ \mathbf{hom}(A, X) & \xrightarrow[f_*]{} & \mathbf{hom}(A, Y) \end{array}$$

Since there are canonical isomorphisms

$$\mathbf{hom}(L, \mathbf{hom}(A, X)) \cong \mathbf{hom}(A \otimes L, X),$$

the assumptions on the functions f_* and Corollary 2 together guarantee that the horizontal simplicial set maps f_* in the diagram are trivial fibrations. But then this forces the fibration

$$\mathbf{hom}(B, X) \to \mathbf{hom}(A, X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B, Y)$$

to be a trivial fibration, which is therefore surjective. This means that all lifting problems

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

can be solved, so that f is a trivial fibration. □

We have therefore proven that any closed simplicial model category which satisfies the conditions **M1** and **M2** also satisfies the following:

M3 There is a set S of cofibrant objects K such that a map $f : X \to Y$ is a weak equivalence if and only if it induces bijections

$$[K, X] \xrightarrow{\cong} [K, Y]$$

of morphisms in the homotopy category $\mathrm{Ho}(\mathcal{M})$ for all objects K of S.

Here's another condition for the model category $\mathcal{M}$:

M4 There is an infinite cardinal β such that, if $Y : \gamma \to \mathcal{M}$ is an inductive system of fibrant objects Y_s, $s < \gamma$, then the colimit $\varinjlim_{s<\gamma} Y_s$ is fibrant, and the map

$$\varinjlim_{s<\gamma} \mathbf{hom}(K, Y_s) \to \mathbf{hom}(K, \varinjlim_{s<\gamma} Y_s)$$

is a weak equivalence for all $K \in S$.

I say that the closed simplicial model category $\mathcal{M}$ *satisfies conditions* **M*** if it satisfies the conditions **M1**, **M2** and **M4**, and hence **M3**. Suppose that $\mathcal{M}$ satisfies the conditions **M*** henceforth.

The conditions **M*** are abstractions of the behaviour of the injective model structure on the category $s\,\mathrm{Pre}(\mathcal{C})_*$ of pointed simplicial presheaves on a small Grothendieck site $\mathcal{C}$. Recall [6] that this model structure has all monomorphisms for cofibrations, and it weak equivalences are the local (or stalkwise) weak equivalences. The fibrations for this theory, the injective fibrations (sometimes called global fibrations), are the maps which have the right lifting property with respect to all trivial cofibrations. This model structure is proper. It has a simplicial structure, where the n-simplices of the function complex $\mathbf{hom}(A,X)$ are the simplicial presheaf morphisms $A \wedge \Delta^n_+ \to X$. The model structure is cofibrantly generated, where the generating cofibrations the α-bounded cofibrations, and the generating trivial cofibrations are the α-bounded trivial cofibrations. Here, α is an infinite cardinal which is an upper bound for the cardinality of the set $\mathrm{Mor}(\mathcal{C})$ of morphisms of $\mathcal{C}$.

We can, in this case, take S to be the set of all objects $U_+ \wedge L$ where U denotes the representable functor associated to an object U of $\mathcal{C}$ and L is a finite pointed simplicial set. Then a simplicial presheaf map $f : X \to Y$ is a local weak equivalence if and only if all induced functions

$$[U_+ \wedge L, X] \to [U_+ \wedge L, Y]$$

are bijections. To see this, observe that a map $f : X \to Y$ is a local weak equivalence if and only if the induced map $FX \to FY$ of injective fibrant models is a sectionwise equivalence, and this is so if and only if all induced functions

$$[U_+ \wedge L, FX] \to [U_+ \wedge L, FY]$$

are bijections. In effect, there are natural bijections

$$[U_+ \wedge L, Z] \cong [L, Z(U)]$$

for all pointed simplicial sets K and injective fibrant simplicial presheaves Z. We also need to know that a map $L \to L'$ of pointed simplicial sets is a weak equivalence if and only if the function

$$[K, L] \to [K, L']$$

is a bijection for all finite pointed simplicial sets K — this is Corollary 2.

Large inductive diagrams of injective fibrant objects of $s\,\mathrm{Pre}_*(\mathcal{C})$ have injective fibrant colimits. Taking $\beta > 2^\alpha$ does the trick, where α is the upper bound on the cardinality of the set of morphisms of $\mathcal{C}$ which was introduced above. The canonical map

$$\varinjlim_{s<\gamma} \mathbf{hom}(U_+ \wedge K, X_s) \to \mathbf{hom}(U_+ \wedge K, \varinjlim_{s<\gamma} X_s),$$

is an isomorphism for all $\gamma \geq \beta$ since all objects $U_+ \wedge K \in S$ are α-bounded.

Generally speaking, for any closed simplicial model structure on the category of simplicial presheaves with standard function complexes, if that model structure is cofibrantly generated and all cofibrations are monomorphisms, then the condition **M4** holds by choosing suitable upper bound α on the cardinality of all objects appearing in the generating sets of cofibrations and trivial cofibrations as well as the members of the set S, and then one chooses $\beta > 2^\gamma$.

It follows that the projective local model structure on the simplicial presheaf category $s\,\mathrm{Pre}_*(\mathcal{C})$ and all structures intermediate between the projective and injective structures [11] satisfy conditions **M***, as do all Bousfield localizations of the injective structure [4]. This last case includes the motivic model structure for simplicial presheaves on the smooth Nisnevich site $(Sm|_S)_{Nis}$ [12]. The same holds for all categories of presheaves of spectra and symmetric spectra on a small Grothendieck site [8], together with all of their Bousfield localizations [7]. All categories of presheaves of chain complexes (aka, presheaves of simplicial modules), presheaves of unbounded chain complexes, and spectrum and symmetric spectrum objects in presheaves of chain complexes [9] satisfy conditions **M***.

Similar observations apply for the standard (injective) model structure on the category $s\,\mathrm{Shv}(\mathcal{C})_*$ of pointed simplicial sheaves on a small Grothendieck site, and this model category satisfies conditions **M***.

Suppose that I is a small category. Since $\mathcal{M}$ is cofibrantly generated, the category $\mathcal{M}^I$ of I-diagrams, or functors $I \to \mathcal{M}$ and their natural transformations, has a model structure for which a map $X \to Y$ is a fibration (respectively weak equivalence) if all component maps $X_i \to Y_i$ are injective fibrations (respectively weak equivalences) of $\mathcal{M}$. The cofibrations for the theory, which are called *projective cofibrations*, are those maps which have the left lifting property with respect to all trivial fibrations. This is the *projective* model structure for the category $\mathcal{M}^I$ of I-diagrams.

The *injective model structure* on $\mathcal{M}^I$ is defined dually: the cofibrations and fibrations are defined componentwise, and the injective fibrations are those maps which have the right lifting property with respect to all trivial cofibrations. Every projective cofibration is a cofibration for the injective model structure since the generating set for the class of projective cofibrations consists of componentwise cofibrations, but the converse is not true.

The *homotopy colimit* $\underrightarrow{\mathrm{holim}}_I\, X$ for a diagram $X : I \to \mathcal{M}$ is defined up to weak equivalence by taking a projective cofibrant model $\tilde{X} \to X$ of X (ie. a weak equivalence with $\tilde{X}$ projective cofibrant), and then one sets

$$\underrightarrow{\mathrm{holim}}_I\, X = \varinjlim_I\, \tilde{X}.$$

The *homotopy inverse limit* $\underleftarrow{\mathrm{holim}}_I\, X$ for the diagram X is defined dually: one takes an injective fibrant model $j : X \to FX$ for X (a weak equivalence with FX injective fibrant), and then one sets

$$\underleftarrow{\mathrm{holim}}_I\, X = \varprojlim_I\, FX.$$

Lemma 4. *Suppose that the closed simplicial model category $\mathcal{M}$ satisfies conditions* **M****, and let $X : \gamma \to \mathcal{M}$ be an inductive system defined on a cardinal γ with $\gamma \geq \beta$. Suppose that $K \in S$ is a generator of $\mathcal{M}$. Then the canonical function*

$$\varinjlim_{s<\gamma} [K, X_s] \to [K, \varinjlim_{s<\gamma} X_s]$$

is a bijection.

Proof. There is a trivial projective cofibration $j : X \to Y$ where the diagram Y is projective fibrant, so that all Y_s, $s < \gamma$ are fibrant objects of $\mathcal{M}$. The induced map

$$\varinjlim_{s<\gamma} X_s \to \varinjlim_{s<\gamma} Y_s$$

is a trivial cofibration of $\mathcal{M}$, and the colimit $\varinjlim_{s<\gamma} Y_s$ is fibrant and the simplicial set map

$$\varinjlim_{s<\gamma} \mathbf{hom}(K, Y_s) \to \mathbf{hom}(K, \varinjlim_{s<\gamma} Y_s)$$

is a weak equivalence by **M4**. All generators $K \in S$ are cofibrant, so a comparison of path components shows that the canonical function

$$\varinjlim_{s<\gamma} [K, Y_s] \to [K, \varinjlim_{s<\gamma} Y_s]$$

is a bijection. There is, finally, a commutative diagram

$$\begin{array}{ccc} \varinjlim_{s<\gamma} [K, X_s] & \longrightarrow & [K, \varinjlim_{s<\gamma} X_s] \\ \cong \downarrow & & \downarrow \cong \\ \varinjlim_{s<\gamma} [K, Y_s] & \xrightarrow[\cong]{} & [K, \varinjlim_{s<\gamma} Y_s] \end{array}$$

which shows that the desired function is a bijection. □

If the closed simplicial model category $\mathcal{M}$ satisfies conditions **M***, then both the projective and injective model structures on the diagram category $\mathcal{M}^I$ satisfy conditions **M***.

2 The representability theorem

For this section, suppose that $\mathcal{M}$ is a closed simplicial model category which satisfies the properties **M*** of the first section.

We shall be considering functors

$$F : \mathcal{M}^{op} \to s\mathbf{Sets}_*$$

defined contravariantly on $\mathcal{M}$, and having the following properties:

F1 The space $F(*)$ is contractible.

F2 The functor F takes weak equivalences $f : A \to B$ between cofibrant objects to weak equivalences $f^* : F(B) \to F(A)$.

F3 Suppose that I is a small category, and that $X : I \to \mathcal{M}$ is a projective cofibrant diagram in $\mathcal{M}$. Then the map

$$F(\varinjlim_i X_i) \to \underleftarrow{\mathrm{holim}}_i F(X_i)$$

is a weak equivalence. In other words, F should take homotopy colimits to homotopy inverse limits, up to weak equivalence.

We shall say that a functor F *satisfies the properties* **F*** if it satisfies all three of these properties.

The point of this section is to establish conditions on F which guarantee that F is sectionwise equivalent to a representable functor $\mathbf{hom}(\ ,Y)$ with Y fibrant. A natural transformation $f : F \to G$ of functors $\mathcal{M}^{op} \to s\mathbf{Set}_*$ is said to be a *sectionwise equivalence* if the map $f : F(A) \to G(A)$ is a weak equivalence for each cofibrant object A of $\mathcal{M}$

The three conditions are invariant of sectionwise equivalence: if there is a sectionwise equivalence $f : F \to G$, then F satisfies the properties **F*** if and only if they are satisfied by G. Thus, for example, it is harmless to suppose that F takes values in Kan complexes, since the the natural map

$$j : F \to \mathrm{Ex}^\infty F$$

arising from Kan's Ex^∞ construction is a sectionwise weak equivalence.

The three conditions are satisfies by all representable functors $\mathbf{hom}(\ ,Y)$ with Y fibrant. In particular, if $X : I \to \mathcal{M}$ is a projective cofibrant diagram, then the canonical map

$$\mathbf{hom}(\varinjlim_i X_i, Y) \to \underleftarrow{\mathrm{holim}}_i \mathbf{hom}(X_i, Y)$$

is a weak equivalence, because the I^{op}-diagram defined by $i \mapsto \mathbf{hom}(X_i, Z)$ is injective fibrant (see, for example, [10, p.114-5]), so that the functor $\mathbf{hom}(\ ,Y)$ satisfies **F3**. The conditions **F1** and **F2** are easy to verify in this case. It follows that if a functor F is sectionwise equivalent to a representable functor $\mathbf{hom}(\ ,Y)$ with Y fibrant, then F satisfies conditions **F***.

Example 5. Suppose that $I = \mathrm{Ob}(I)$ is a discrete category on its set of objects, so that it has only identity arrows. A diagram $X : I \to \mathcal{M}$ consists of a set of objects X_i, $i \in I$, and X is projective cofibrant if and only if all of the objects X_i are cofibrant. Then condition **F3** for F in this case asserts that the composite map

$$F(\bigvee_i X_i) \to \prod_i F(X_i)$$

is a weak equivalence. The special case of condition **F3** for discrete diagrams is otherwise known as the *wedge property* for the functor F.

Example 6. Among all diagrams having the shape

$$X_1 \xleftarrow{i_1} X_0 \xrightarrow{i_2} X_2$$

the projective cofibrant ones are the diagrams for which all objects X_i are cofibrant and the two morphisms i_1, i_2 are cofibrations. Then condition **F3** for diagrams of this shape means precisely that the diagram

$$\begin{array}{ccc} F(X_1 \cup_{X_0} X_2) & \longrightarrow & F(X_2) \\ \downarrow & & \downarrow \\ F(X_1) & \longrightarrow & F(X_0) \end{array}$$

is homotopy cartesian. In the presence of condition **F3**, and because all of the objects X_i are cofibrant, this diagram will be homotopy cartesian if just one of the maps i_1, i_2 is a cofibration. This is a strong form of the *Mayer-Veitoris property* for F — see the description of the Mayer-Veitoris property in Section 3.

Example 7. Suppose that γ is an infinite cardinal number, and that $A : \gamma \to \mathcal{M}$ is a directed system indexed by γ. Suppose that the system is projective cofibrant — this means that A_0 is cofibrant, all maps $A_s \to A_{s+1}$ are cofibrations, and that the maps $\varinjlim_{s<t} \to A_t$ are cofibrations for all limit ordinals $t < \gamma$. Observe that the restriction of the diagram A to any limit ordinal $\beta < \gamma$ is a projective cofibrant β-diagram.

Applying the functor F to all A_s, $s < \gamma$ gives a "tower" $F(A) : \gamma^{op} \to s\mathbf{Set}_*$. Suppose that $j : F(A) \to Z$ is an injective fibrant model in the category of γ^{op}-diagrams. Then Z_0 is fibrant, the maps $Z_{s+1} \to Z_s$ are fibrations for all $s < \gamma$, and the maps $Z_t \to \varprojlim_{s<t} Z_s$ are fibrations for all limit ordinals $t < \gamma$.

The assumption that the functor F satisfies condition **F3** means that the map

$$F(\varinjlim_{s<t} A_s) \to \varprojlim_{s<t} Z_s$$

is a weak equivalence for all limit ordinals $t \leq \gamma$.

In the special case that the map $\varinjlim_{s<t} A_s \to A_t$ is an isomorphism for all limit ordinals $t < \gamma$, we could, in the construction of Z, set

$$Z_t = \varinjlim_{s<t} Z_s$$

for all limit ordinals t.

Suppose that X is cofibrant and that $u \in F(Y)_0$ with Y cofibrant. If Y is also fibrant then an evaluation map

$$u^* : [X, Y] \to \pi_0 F(X)$$

can be defined by the assignment $[\alpha] \mapsto [\alpha^*(u)]$. If Y is not fibrant, let $j : Y \to LY$ be a fibrant model (with j a trivial cofibration). Then there is a unique element $[v] \in \pi_0 F(LY)$ such that $[v] \mapsto [u]$ under the isomorphisms $\pi_0 F(LY) \to \pi_0 F(Y)$, and then the *evaluation map* u^* is defined to be the composite

$$[X, Y] \xrightarrow{\cong} [X, LY] \xrightarrow{v^*} \pi_0 F(X).$$

The definition of u^* is independent of the choice of fibrant model LY.

Suppose that Y is a cofibrant object of $\mathcal{M}$. An element u of $F(Y)_0$ is said to be *universal* if the evaluation map

$$u^* : [K, Y] \to \pi_0 F(K)$$

is an isomorphism for all generators K.

The near-term goal of the following is to show that every functor F satisfying conditions **F*** has a universal element. Suppose that F satisfies conditions **F*** for the rest of this section.

Lemma 8. *Suppose that A is a cofibrant object of $\mathcal{M}$ and that $u \in F(A)_0$. Then there is a cofibration $i : A \to B$ and an element $v \in F(B)_0$ with $i^*([v]) = [u]$ in $\pi_0 F(A)_0$, and such that in the diagram*

$$\begin{array}{ccc} [K, A] & \xrightarrow{i_*} & [K, B] \\ & {\scriptstyle u^*}\searrow & \downarrow{\scriptstyle v^*} \\ & & \pi_0 F(K) \end{array}$$

the following hold:

1) *the map v^* is surjective,*

2) *if $u^*(\alpha) = u^*(\beta)$ then $i_*(\alpha) = i_*(\beta)$.*

Proof. We can suppose that A is fibrant.

Form the coproduct

$$A \vee (\bigvee_{\lambda} K)$$

over all

$$\lambda \in \pi_0 F(K),\ K \in S.$$

Take the list of all pairs of elements $[\alpha], [\beta] \in [K, A]$ such that $u^*[\alpha] = u^*[\beta]$ in $\pi_0 F(K)$, and choose representatives $\alpha, \beta : K \to A$ for all such pairs of elements. Form the pushout diagram

$$\begin{array}{ccccc} \bigvee_{(\alpha,\beta)} (K \vee K) & \longrightarrow & A & \longrightarrow & A \vee (\bigvee_{\lambda} K) \\ \downarrow & & & & \downarrow \\ \bigvee_{(\alpha,\beta)} (K \wedge \Delta^1_+) & & \longrightarrow & & B \end{array}$$

and observe that all objects in the diagram are cofibrant.

Write j for the composite cofibration

$$A \to A \vee (\bigvee_{\lambda} K) \to B.$$

There is an element $[w] \in \pi_0 F(A \vee (\bigvee_\lambda K))$ which restricts to $[u] \in \pi_0 F(A)$ and all $\lambda \in \pi_0 F(K)$, by the wedge property. There is an element $[v] \in \pi_0 F(B)$ which restricts simultaneously to $[w]$ and the sequence $(u^*[\alpha] = u^*[\beta])$, since the diagram

$$\begin{array}{ccc} F(B) & \longrightarrow & F(\bigvee_{(\alpha,\beta)}(K \wedge \Delta^1_+)) \\ \downarrow & & \downarrow \\ F(A \vee (\bigvee_\lambda K)) & \longrightarrow & F(\bigvee_{(\alpha,\beta)}(K \vee K)) \end{array}$$

is homotopy cartesian (Mayer-Veitoris property).

The map $v^* : [K, B] \to \pi_0 F(K)$ is surjective for all K by construction. All pairs of elements $[\alpha], [\beta] \in [K, A]$ such that $u^*[\alpha] = u^*[\beta]$ in $\pi_0 F(K)$ also have the same image in $[K, B]$. □

Proposition 9. *Suppose that A is cofibrant and that u is a vertex of $F(A)$. Then there is a cofibration $i : A \to Y$ with a universal element $v \in F(Y)_0$ such that $i^*([v]) = [u] \in \pi_0 F(A)$.*

Proof. We construct a projective cofibrant inductive diagram $A : \beta \to \mathcal{M}$ together with an inductive fibrant model $j : F(A) \to Z$, by induction on $s < \beta$.

Set $A_0 = A$ and $u_0 = u$. Suppose that $t < \beta$ and that A_s and the maps $j : F(A_s) \to Z_s$ have been defined for $s < t$. Suppose further that vertices $v_s \in Z_s$ have been chosen which are compatible in the sense that if $s' \leq s < t$ then $v_s \mapsto v_{s'}$ under the fibration $p : Z_s \to Z_{s'}$.

It $t = s + 1$ then the map $F(A_s) \to Z_s$ is a weak equivalence, so there is a vertex $u_s \in F(A_s)$ such that $j_*[u_s] = [z_s] \in \pi_0 Z_s$. Choose a cofibration $i : A_s \to A_{s+1}$ with $u_{s+1} \in F(A_{s+1})$ according to the construction of Lemma 8. Form a diagram

$$\begin{array}{ccc} F(A_{s+1}) & \xrightarrow{j} & Z_{s+1} \\ \downarrow & & \downarrow p \\ F(A_s) & \xrightarrow[j]{} & Z_s \end{array}$$

such that j is a trivial cofibration and p is a fibration. Then $[p(j(u_{s+1}))] = [z_s] \in \pi_0 Z_s$ and p is a fibration so there is a vertex $z_{s+1} \in Z_{s+1}$ such that $p(z_{s+1}) = z_s$ and $[z_{s+1}] = [j(u_{s+1})] \in \pi_0 Z_{s+1}$.

It t is a limit ordinal, set $A_t = \varinjlim_{s<t} A_s$, set $Z_t = \varprojlim_{s<t} F(A_s)$, and let $j : F(A_t) \to Z_t$ be the canonical map. Then $j : F(A_t) \to Z_t$ is a weak equivalence since F takes homotopy colimits to homotopy inverse limits. Let the vertex z_t be the map $* \to \varinjlim_{s<t} Z_s$ which is determined by all z_s, $s < t$.

Suppose that $Y = \varinjlim_{s<\beta} A_s$, and set $Z_\beta = \varprojlim_{s<\beta} Z_s$. Let z_β be the vertex of Z_β which is defined by all the z_s, $s < \beta$. The natural transformation $j : F(A) \to Z$ induces a weak equivalence

$$F(Y) \xrightarrow{j_*} Z_\beta$$

again by condition **F3**. It follows that there is a vertex $v \in F(Y)$ such that

$$[j_*(v)] = [z_\beta] \in \pi_0 Z_\beta.$$

Then there are commutative diagrams

$$\begin{array}{ccccc} [K, A_s] & \longrightarrow & [K, A_{s+1}] & \longrightarrow & [K, Y] \\ & {\scriptstyle u_s^*} \searrow & & {\scriptstyle u_{s+1}^*} \searrow & \downarrow {\scriptstyle v^*} \\ & & & & \pi_0 F(K) \end{array}$$

and the map v^* is an isomorphism for all $K \in S$. In effect, the function

$$\varinjlim_{s<\gamma} [K, A_s] \to [K, Y]$$

is a bijection for all $K \in S$, since β is sufficiently large. If $i : A \to Y$ is the canonical map, then $i^*([v]) = [u] \in \pi_0 F(A)$. □

Lemma 10. *Suppose that $f : Y \to Y'$ is a morphism of cofibrant objects of $\mathcal{M}$. Suppose that $u \in F(Y)_0$ and $u' \in F(Y')_0$ are universal, and that $f^*([u']) = [u] \in \pi_0 F(Y)$. Then f is a weak equivalence.*

Proof. This is a consequence of the commutativity of the diagrams

$$\begin{array}{ccc} [K, Y] & \xrightarrow{f_*} & [K, Y'] \\ & {\scriptstyle u^*} \searrow \cong & \cong \downarrow {\scriptstyle (u')^*} \\ & & \pi_0 F(K) \end{array}$$

which are associated to all generators K. □

Lemma 11. *Suppose given maps*

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i} \downarrow & & \\ B & & \end{array}$$

of $\mathcal{M}$ such that i is a cofibration and A and Y are cofibrant. Suppose that $u \in F(Y)_0$ is universal and that $x \in F(B)$ satisfies $i^*([x]) = f^*([u]) \in \pi_0 F(A)$. Then f extends to a map $g : B \to Y$ in the homotopy category such that $g^*([u]) = [x] \in \pi_0 F(B)$.

Proof. Form the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i} \downarrow & & \downarrow {\scriptstyle i_*} \\ B & \longrightarrow & B \cup_A Y \end{array} \qquad (2)$$

Then there is an element $[w] \in \pi_0 F(B \cup_A Y)$ which restricts to $[x] \in \pi_0 F(B)$ and $[u] \in \pi_0 F(Y)$ by the Mayer-Vietoris property. There is a cofibration $j : B \cup_A Y \to Y'$ such that Y' has a universal element $v \in F(Y')_0$ such that $j^*([v]) = [w] \in \pi_0 F(B \cup_A Y)$, by Lemma 8. The composite map

$$Y \xrightarrow{i_*} B \cup_A Y \to Y'$$

is a weak equivalence by Lemma 10, and is therefore an isomorphism in the homotopy category. □

Proposition 12. *Suppose that $F : \mathcal{M}^{op} \to s\mathbf{Set}_*$ is a functor which satisfies conditions $\mathbf{F*}$. Then there is a cofibrant object Y of $\mathcal{M}$ and a vertex $u \in F(Y)_0$ such that the evaluation map*

$$u^* : [X, Y] \xrightarrow{\cong} \pi_0 F(X)$$

is a bijection for all cofibrant objects X of $\mathcal{M}$.

Proof. By Proposition 9 applied to some $v \in F(*)_0$ (note that $F(*)$ is contractible hence non-empty) there is a cofibrant object Y of $\mathcal{M}$ with a universal element $u \in F(Y)$. We show that the induced map

$$u^* : [X, Y] \to \pi_0 F(X)$$

is a bijection for all cofibrant X.

Suppose that $v \in F(X)_0$. Then applying Lemma 11 to the diagram

$$\begin{array}{ccc} * & \longrightarrow & Y \\ \downarrow & & \\ X & & \end{array}$$

gives a map $g : X \to Y$ in the homotopy category $\mathrm{Ho}(\mathcal{M})$ such that $g^*([u]) = [v]$, so that $u^*[g] = [v]$. It follows that the function u^* is surjective.

To prove the injectivity of u^*, we can suppose that Y is fibrant. Suppose that $u^*[g_0] = u^*[g_1] = [v] \in \pi_0 F(X)$ for maps $g_0, g_1 : X \to Y$. Consider the maps

$$\begin{array}{ccc} X \vee X & \xrightarrow{(g_0, g_1)} & Y \\ {\scriptstyle (d^0, d^1)} \downarrow & & \\ X \otimes \Delta^1 & & \end{array}$$

and choose an element $[w] \in \pi_0 F(X \otimes \Delta^1)$ which restricts to $[v]$ along the functions induced by the maps $F(d^0) = F(d^1) : F(X \otimes \Delta^1) \to F(X)$. Then Lemma 11 says that (g_0, g_1) extends to a map $h : X \otimes \Delta^1 \to Y$ in the homotopy category $\mathrm{Ho}(\mathcal{M})$ such that $h^*([u]) = [w]$. But then g_0 and g_1 represent the same map in $\mathrm{Ho}(\mathcal{M})$. □

Suppose now that

$$F : \mathcal{M}^{op} \to s\mathbf{Set}_*$$

is a *simplicial functor*, and that it satisfies conditions **F***.

The simplicial functor F associates a simplicial set map $\alpha^* : F(B) \wedge \Delta^n_+ \to F(A)$ to every morphism $\alpha : A \otimes \Delta^n_+ \to B$. In particular there is a map $f_n : F(A \otimes \Delta^n_+) \wedge \Delta^n_+ \to F(A)$ to the identity map $A \otimes \Delta^n_+ \to A \otimes \Delta^n_+$. Taking adjoints of the f_n gives maps

$$f_{n*} : F(A \otimes \Delta^n_+) \to \mathbf{hom}(\Delta^n_+, F(A)),$$

Every map $\theta : \Delta^m \to \Delta^n$ induces a commutative diagram

$$\begin{array}{ccc} F(A \otimes \Delta^n_+) & \xrightarrow{f_{n*}} & \mathbf{hom}(\Delta^n_+, F(A)) \\ {\scriptstyle (1\wedge\theta)^*}\downarrow & & \downarrow{\scriptstyle \theta^*} \\ F(A \otimes \Delta^m_+) & \xrightarrow[f_{m*}]{} & \mathbf{hom}(\Delta^m_+, F(A)) \end{array}$$

It follows that there is a map

$$f : F(A \otimes L) \to \mathbf{hom}(L, F(A))$$

which is natural in pointed simplicial presheaves A and pointed simplicial sets L. The map

$$f : \hom(A \otimes L, Z) \to \mathbf{hom}(L, \mathbf{hom}(A, Z))$$

is a canonical isomorphism.

In simplicial degree 0, maps $f : A \to B$ in $\mathcal{M}$ are identified with maps $\tilde{f} : A \otimes \Delta^0_+ \to B$ via the diagrams

$$\begin{array}{ccc} A \otimes \Delta^0_+ & & \\ {\scriptstyle \cong}\downarrow & \searrow^{\tilde{f}} & \\ A & \xrightarrow[f]{} & B \end{array}$$

and there are commutative diagrams

$$\begin{array}{ccc} F(B) \wedge \Delta^0_+ & \xrightarrow{\tilde{f}^*} & F(A) \\ {\scriptstyle \cong}\downarrow & \nearrow_{f^*} & \\ F(B) & & \end{array}$$

It follows that the map $f_0 : F(A \otimes \Delta^0_+) \wedge \Delta^0_+ \to F(A)$ is the composite of canonical isomorphisms

$$F(A \otimes \Delta^0_+) \wedge \Delta^0_+ \xrightarrow{\cong} F(A \otimes \Delta^0_+) \xrightarrow{\cong} F(A).$$

It also follows that the adjoint map

$$f_{0*} : F(A \otimes \Delta^0_+) \to \mathbf{hom}(\Delta^0_+, F(A))$$

is an isomorphism.

Lemma 13. *Suppose that the simplicial functor F satisfies conditions* **F***, *and that F takes values in Kan complexes. Then the natural map*

$$f : F(A \otimes L) \to \mathbf{hom}(L, F(A))$$

is a weak equivalence for all pointed simplicial sets L and all cofibrant objects A of $\mathcal{M}$.

Proof. The map

$$f : F(A \otimes \Delta^0_+) \to \mathbf{hom}(\Delta^0_+, F(A))$$

is an isomorphism, and both functors involved in the natural map f preserve weak equivalences, so that all maps

$$f : F(A \otimes \Delta^n_+) \to \mathbf{hom}(\Delta^n_+, F(A))$$

are weak equivalences.

Now proceed by induction on n for the skeleta $\mathrm{sk}_n\, L$ of L. The pushout squares

$$\begin{array}{ccc} \bigvee_{\sigma \in NL_n} \partial\Delta^n_+ & \longrightarrow & \mathrm{sk}_{n-1}(L) \\ \downarrow & & \downarrow \\ \bigvee_{\sigma \in NL_n} \Delta^n_+ & \longrightarrow & \mathrm{sk}_n(L) \end{array}$$

are mapped to homotopy cartesian diagrams by both functors (recall that F takes homotopy colimits to homotopy inverse limits), and so the maps

$$f : F(A \otimes \mathrm{sk}_n(L)) \to \mathbf{hom}(\mathrm{sk}_n\, L, F(A))$$

are weak equivalences for all pointed simplicial sets L, and for all $n \geq 0$.

Finally, the space $F(A \otimes L)$ is naturally equivalent to the homotopy inverse limit of the spaces $F(A \otimes \mathrm{sk}_n\, L)$, and the space $\mathbf{hom}(L, F(A))$ is naturally equivalent to the homotopy inverse limit of the spaces $\mathbf{hom}(\mathrm{sk}_n\, L, F(A))$. The result follows. □

A morphism

$$u : \mathbf{hom}(\ , Y) \to F$$

of simplicial functors is completely determined by the element $u = u(1_Y) \in F(Y)_0$ such that $u \mapsto *$ under the map $F(Y) \to F(*)$ which is induced by the morphism $* \to Y$. There is a homotopy

$$\omega : \mathbf{hom}(\ , Y) \wedge \Delta^1_+ \to F$$

between such functors u, u' if and only if there is a path $\omega : u \to u'$ in the fibre $\tilde{F}(Y)$ over the base point of the map $F(Y) \to F(*)$. It also follows there is a natural isomorphism

$$\mathbf{Nat}(\mathbf{hom}(\ ,Y), F) \cong \tilde{F}(Y)$$

where $\mathbf{Nat}(G, F)$ denotes the function space of natural transformations between simplicial functors G and F.

As noted before, we are entitled to vary the simplicial functor F up to sectionwise equivalence. In particular Kan's Ex^∞-construction $j : X \to \mathrm{Ex}^\infty X$ determines a simplicial functor $\mathrm{Ex}^\infty(F)$ with $\mathrm{Ex}^\infty(F)(A) = \mathrm{Ex}^\infty(F(A))$, and the weak equivalences $j : F(A) \to \mathrm{Ex}^\infty(F(A))$ define a natural morphism

$$j : F \to \mathrm{Ex}^\infty F$$

of simplicial functors. This map j is a sectionwise weak equivalence.

Remark 14. Suppose that G is a simplicial functor $\mathcal{M}^{op} \to s\mathbf{Set}_*$. Suppose in addition that the pointed simplicial set $G(*)$ is contractible. The canonical maps $t : A \to *$ in $\mathcal{M}$ induce cofibrations $t^* : G(*) \to G(A)$, which together determine a map $t^* : G(*) \to G$ of simplicial functors, where $G(*)$ is the constant simplicial diagram on $\mathcal{M}$ associated to the pointed simplicial set $G(*)$. Form the quotient $G/G(*))$ and consider the canonical transformation $p : G \to G/G(*)$. The map p is a sectionwise equivalence since $G(*)$ is contractible, and there is an isomorphism

$$(G/G(*))(*) = G(*)/G(*) \cong *.$$

Kan's Ex^∞ functor preserves points, so the composite

$$G \xrightarrow{p} G/G(*) \xrightarrow{j} \mathrm{Ex}^\infty(G/G(*)) =: \tilde{G}$$

gives a sectionwise equivalence $G \to \tilde{G}$ such that $\tilde{G}$ takes values in Kan complexes and $\tilde{G}(*) = *$.

Lemma 15. *Suppose that $u \in F(Y)_0$ is a universal element which determines a map*

$$u : \mathbf{hom}(\ ,Y) \to F$$

of simplicial functors, where Y is cofibrant and fibrant and F takes values in Kan complexes and satisfies condition **F***. *Then the map u is a sectionwise weak equivalence.*

Proof. We show that the induced map

$$[L, \mathbf{hom}(A, Y)] \xrightarrow{u_*} [L, F(A)]$$

is a bijection for each pointed simplicial set L and cofibrant object A.

In the diagram

$$\begin{array}{ccc} \mathbf{hom}(A \otimes L, Y) & \xrightarrow{u_*} & F(A \otimes L) \\ \downarrow f & & \downarrow f \\ \mathbf{hom}(L, \mathbf{hom}(A, Y)) & \xrightarrow[u_*]{} & \mathbf{hom}(L, F(A)) \end{array}$$

the vertical maps f are weak equivalences by Lemma 13, and the map

$$u : \mathbf{hom}(A \otimes L, Y) \to F(A \otimes L)$$

is an isomorphism in path components by Proposition 12. But then the map

$$u_* : \mathbf{hom}(L, \mathbf{hom}(A, Y)) \to \mathbf{hom}(L, F(A))$$

induces an isomorphism in path components. □

The following is the main result of this section. It is the *representability theorem* of the section title.

Theorem 16. *Suppose that $\mathcal{M}$ is a closed simplicial model category which satisfies conditions* **M***. *Suppose that $F : \mathcal{M}^{op} \to s\mathbf{Sets}_*$ is a simplicial functor which satisfies conditions* **F***. *Then there are sectionwise equivalences*

$$F \xrightarrow{\simeq} \tilde{F} \xleftarrow{\simeq} \mathbf{hom}(\ ,Y),$$

where Y is some fibrant object of $\mathcal{M}$.

Proof. The construction of Remark 14 gives a sectionwise weak equivalence $F \to \tilde{F}$ such that $\tilde{F}$ takes values in Kan complexes and $\tilde{F}(*) = *$. The simplicial functor satisfies the conditions **F***, and therefore has a universal element $u \in \tilde{F}(Y)$ for some cofibrant (and fibrant) object Y of $\mathcal{M}$ by Proposition 9. This element u defines a morphism of simplicial functors

$$u : \mathbf{hom}(\ ,Y) \to \tilde{F}$$

by the construction of the simplicial functor $\tilde{F}$, and the morphism u is a sectionwise weak equivalence by Lemma 15. □

Suppose that Z is a pointed simplicial presheaf on a site $\mathcal{C}$, and suppose that Z is fibrant for the injective model structure on $\mathcal{C}^{op}$-diagrams. The cofibrations of the injective model structure for the pointed simplicial presheaf category $s\,\mathrm{Pre}_*(\mathcal{C})$ are the monomorphisms, and hence coincide for all possible topologies on $\mathcal{C}$. It follows that the projective cofibrant diagrams in the diagram category $s\,\mathrm{Pre}_*(\mathcal{C})^I$ in the I-diagram category also coincide for all topologies on $\mathcal{C}$ and for all small categories I. It also follows that the functor $\mathbf{hom}(\ ,Z)$ satisfies condition **F3** (as well as **F1**) for the injective model structure on $s\,\mathrm{Pre}_*(\mathcal{C})$, for all topologies on $\mathcal{C}$.

Now fix a topology $\mathcal{T}$ on the category $\mathcal{C}$. If the functor $\mathbf{hom}(\ ,Z)$ satisfies condition **F2**, then since $\mathbf{hom}(*, Z) = *$ the methods of this section (specifically, Proposition 12 and Lemma 15) imply that there is a pointed simplicial presheaf Y on $\mathcal{C}$ which is injective fibrant for the topology $\mathcal{T}$, and a sectionwise weak equivalence

$$u : \mathbf{hom}(\ ,Y) \to \mathbf{hom}(\ ,Z)$$

of simplicial functors. This map is induced by a pointed simplicial presheaf map $u : Y \to Z$, and evaluating the simplicial functor u at all (cofibrant) pointed simplicial presheaves V_+ corresponding to objects V of $\mathcal{C}$ shows that all of the maps maps

$$u : Y(V) \to Z(V)$$

are weak equivalences of pointed simplicial sets. It follows that Z satisfies descent with respect to the chosen topology on $\mathcal{C}$. In effect, if $j : Z \to FZ$ is an injective fibrant model for that topology, then the composite

$$Y \xrightarrow{u} Z \xrightarrow{j} FZ$$

is a local weak equivalence of injective fibrant objects, and is therefore a weak equivalence in all sections. It follows that all maps $j : Z(V) \to FZ(V)$ are weak equivalences, so that Z satisfies descent.

Recall that a simplicial presheaf X *satisfies descent* for the topology $\mathcal{T}$ if there is a local weak equivalence $f : X \to X'$ with X' injective fibrant such that the maps

$$f : X(V) \to X'(V)$$

are weak equivalences for all objects V of $\mathcal{C}$. The choice of weak quivalence does not matter: in the presence of the map f, if $g : X \to X''$ is another local weak equivalence with X'' injective fibrant, then all maps $X(V) \to X''(V)$ are weak equivalences of pointed simplicial sets.

We have effectively proved the following:

Proposition 17. *Suppose that a pointed simplicial presheaf X has an injective fibrant model $X \to Z$ in $\mathcal{C}^{op}$-diagrams such that the representable functor* $\mathbf{hom}(\ ,Z)$ *satisfies condition* **F2** *for the topology $\mathcal{T}$, then X satisfies descent for $\mathcal{T}$.*

Proposition 17 is a descent criterion for pointed simplicial presheaves, and there are obvious analogues of this result in related categories, such as presheaves of spectra. The hope for representability techniques such as those displayed in this section has been that they would give such a descent criterion. This is certainly true, but there is a far more direct proof:

Proof. Suppose that Z is an injective fibrant $\mathcal{C}^{op}$ diagram which satisfies **F2** as above, and let $j : Z \to FZ$ be an injective fibrant model for the topology on $\mathcal{C}$. All pointed simplicial presheaves are cofibrant, so that the local weak equivalence $j : Z \to FZ$ induces a weak equivalence

$$j^* : \mathbf{hom}(FZ, Z) \xrightarrow{\simeq} \mathbf{hom}(Z, Z).$$

In particular, there is a map $g : FZ \to Z$ such that the composite $g \cdot j$ is pointed homotopic to the identity 1_Z on Z. Then $j \cdot g \cdot j = j$ up to simplicial homotopy. But precomposition with j defines a weak equivalence

$$j^* : \mathbf{hom}(FZ, FZ) \to \mathbf{hom}(Z, FZ)$$

so that $j \cdot g$ coincides with the identity 1_{FZ} up to simplicial homotopy. It follows that the map $j : Z \to FZ$ is a simplicial homotopy equivalence, and therefore consists of weak equivalences $Z(V) \to FZ(V)$ for all objects V of $\mathcal{C}$.

Finally, the composite $X \to Z \to FZ$ is an injective fibrant model for X, and this composite consists of weak equivalences

$$X(V) \to Z(V) \to FZ(V)$$

for all objects V of $\mathcal{C}$, so that X satisfies descent. □

3 Classical Brown representability

Here are the properties that we shall require for the closed model category $\mathcal{N}$ in this section:

N1 The category $\mathcal{N}$ has all small colimits. The initial object $*$ of $\mathcal{N}$ is also terminal, so that $\mathcal{N}$ is a pointed model category.

N2 There is a set S of compact cofibrant objects $\{K\}$ such that a map $f : X \to Y$ is a weak equivalence if and only if it induces a bijection

$$[K, X] \xrightarrow{\cong} [K, Y]$$

of morphisms in the homotopy category $\mathrm{Ho}(\mathcal{N})$, for all objects K in S.

An object K of $\mathcal{N}$ is said to be *compact* if the function

$$\varinjlim_i [K, Y_i] \to [K, \varinjlim Y_i]$$

is a bijection *for all* inductive systems

$$Y_0 \to Y_1 \to Y_2 \to \dots$$

I shall say that a model category $\mathcal{N}$ satisfies the conditions **N*** if it satisfies properties **N1** and **N2**. It is typical to say, under such circumstances, that the model category $\mathcal{N}$ is *compactly generated*, and that the elements K of S are *compact generators* for $\mathcal{N}$.

Remark 18. If $\mathcal{N}$ satisfies the conditions **N***, then inductive colimits preserve weak equivalences.

More specifically, suppose that the map $f : X \to Y$ is a comparison of inductive systems $X, Y : \alpha \to \mathcal{N}$ (where α is any cardinal) such that all maps $f_s : X_s \to Y_s$, $s < \alpha$, are weak equivalences. Then the induced map

$$\varinjlim X_s \to \varinjlim Y_s$$

is a weak equivalence. In effect, there are bijections

$$[K, \varinjlim X_s] \cong \varinjlim [K, X_s] \cong \varinjlim [K, Y_s] \cong [K, \varinjlim Y_s].$$

for all objects K of S.

We shall now consider functors

$$G : \mathcal{N}^{op} \to \mathbf{Sets}_*$$

which take values in pointed sets, and have the following properties:

G1 G takes weak equivalences to bijections.

G2 The set $G(*)$ is the one-point set.

G3 (wedge property) For any coproduct $\bigvee_i X_i$ of a set of cofibrant objects $\{X_i\}$ of $\mathcal{N}$, the canonical induced map

$$G(\bigvee_i X_i) \to \prod_i G(X_i)$$

is a bijection.

G4 (Mayer-Vietoris property) Suppose that the diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \longrightarrow & B \cup_A X \end{array}$$

is a pushout, where i is a cofibration and all objects are cofibrant. Then the induced function

$$G(B \cup_A X) \to G(B) \times_{G(A)} G(X)$$

is surjective.

I say that a functor $G : \mathcal{N}^{op} \to \mathbf{Sets}_*$ *satisfies the conditions* **G*** if it has the properties **G1** – **G4**.

Example 19. Suppose that Z is an object of $\mathcal{N}$. Then the functor

$$G(X) = [X, Z]$$

defined by morphisms in the homotopy category is pointed by the composite $X \to * \to Z$, and satisfies the conditions **G***.

Lemma 20. *Suppose that*

$$Y_0 \xrightarrow{\alpha} Y_1 \xrightarrow{\alpha} \cdots \tag{3}$$

is a countable sequence of maps of $\mathcal{N}$ where all objects Y_i are cofibrant, and that the functor f satisfies the conditions **G***. *Then the canonical function*

$$G(\varinjlim_i Y_i) \to \varprojlim_i G(Y_i)$$

is surjective.

Proof. Take a sequence of cylinder objects

$$\begin{array}{ccc} Y_i \vee Y_i & & \\ {\scriptstyle (d^0,d^1)}\downarrow & \searrow^{\nabla} & \\ Y_i \otimes I & \xrightarrow[s]{\simeq} & Y \end{array}$$

in $\mathcal{N}$, and form the pushout diagram

$$\begin{array}{ccc} \bigvee_i (Y_i \vee Y_i) & \xrightarrow{(1,\alpha)} & \bigvee_i Y_i \\ {\scriptstyle (d^0,d^1)}\downarrow & & \downarrow \\ \bigvee_i (Y_i \otimes I) & \longrightarrow & L \end{array}$$

Then the object L is the telescope construction for the diagram (3), and it is canonically weakly equivalent to the colimit $\varinjlim_i Y_i$.

There is a pullback diagram

$$\begin{array}{ccc} \varprojlim_i G(Y_i) & \longrightarrow & \prod_i G(Y_i) \\ \downarrow & & \downarrow {\scriptstyle (1,\alpha^*)} \\ \prod_i G(Y_i) & \xrightarrow[\Delta]{} & \prod_i (G(Y_i) \times G(Y_i)) \end{array}$$

and there are isomorphisms $G(Y_i) \cong G(Y_i \otimes I)$, so that the map $G(L) \to \varprojlim G(Y_i)$ is surjective by the Mayer-Vietoris property. □

Suppose that Y is an object of $\mathcal{N}$. An element u of $G(Y)$ is said to be *universal* if the evaluation map

$$u^* : [K, Y] \to G(K)$$

defined by $\alpha \mapsto \alpha^*(u)$ is an isomorphism for all compact generators K.

Lemma 21. *Suppose that X is a cofibrant object of $\mathcal{N}$ and that $v \in G(X)$. Suppose that the functor f satisfies the conditions* **G***. *Then there is a cofibration $i : X \to Y$ such that there is a universal element $u \in G(Y)$ with $i^*(u) = v$.*

Proof. Suppose that Z is a cofibrant object of $\mathcal{N}$ and that $z \in G(Z)$. By using the methods of proof of Lemma 8 one can show that there is a cofibration $j : Z \to Y$ with $w \in G(Y)$ such that $j^*(w) = z$, and such that in the diagram

$$\begin{array}{ccc} [K, Z] & \xrightarrow{j_*} & [K, Y] \\ & {\scriptstyle z^*}\searrow & \downarrow {\scriptstyle w^*} \\ & & G(K) \end{array}$$

w^* is surjective, and if $z^*[\alpha] = z^*[\beta]$ then $j_*[\alpha] = j_*[\beta] \in [K, Y]$ for all compact generators K.

Set $Y_0 = X$ and $u_0 = v$. Then there is a countable sequence of cofibrations $j : Y_n \to Y_{n+1}$ with elements $u_n \in G(Y_n)$ such that $j^*(u_{n+1}) = u_n \in G(Y_n)$, and in all diagrams

$$\begin{array}{ccc} [K, Y_n] & \xrightarrow{j_*} & [K, Y_{n+1}] \\ & \searrow^{u_n^*} & \downarrow{\scriptstyle u_{n+1}^*} \\ & & G(K) \end{array} \tag{4}$$

the map u_{n+1}^* is surjective, and if $u_n^*[\alpha] = u_n^*[\beta]$ then $j_*[\alpha] = j_*[\beta] \in [K, Y_{n+1}]$.

The function

$$G(Y) = G(\varinjlim Y_n) \to \varprojlim G(Y_n)$$

is surjective by Lemma 20, so that one can pick $u \in G(Y)$ such that u restricts to all u_n. Then there are commutative triangles

$$\begin{array}{ccc} [K, Y_n] & \longrightarrow & [K, Y] \\ & \searrow^{u_n^*} & \downarrow{\scriptstyle u^*} \\ & & G(K) \end{array}$$

and $[K, Y] \cong \varinjlim [K, Y_n]$ since all K are compact. The map u^* is a bijection by the construction of the cofibrations j_* in the diagram (4). □

We now have the following analogues of Lemma 10, Lemma 11 and Proposition 12, respectively, with the same proofs.

Lemma 22. *Suppose that G satisfies the conditions* **G***. *Suppose that $\alpha : Y \to Y'$ is a morphism of cofibrant objects of $\mathcal{N}$. Suppose that $u \in G(Y)$ and $u' \in G(Y')$ are universal, and that $\alpha^*(u') = u$. Then α is a weak equivalence.*

Lemma 23. *Suppose that the functor G satisfies the conditions* **G***. *Suppose given maps*

$$\begin{array}{ccc} A & \xrightarrow{\beta} & Y \\ {\scriptstyle i}\downarrow & & \\ B & & \end{array}$$

of $\mathcal{N}$ such that i is a cofibration and all objects are cofibrant. Suppose that $u \in G(Y)$ is universal and that $x \in G(B)$ satisfies $i^*(x) = \beta^*(u) \in G(A)$. Then β extends to a map $\gamma : B \to Y$ in the homotopy category $\mathrm{Ho}(\mathcal{N})$ *such that $\gamma^*(u) = x$.*

Theorem 24. *Suppose that $\mathcal{N}$ is a closed model category which satisfies the conditions* **N***. *Suppose that the functor $G : \mathcal{N}^{op} \to \mathbf{Set}_*$ satisfies the conditions* **G***. *Then there is an object Y of $\mathcal{N}$ and a natural bijection*

$$[X, Y] \xrightarrow{\cong} G(X)$$

for all objects X of $\mathcal{N}$.

Theorem 24 is the analogue of the classical Brown representability theorem for compactly generated pointed closed model categories $\mathcal{N}$, such as the category $s\mathbf{Set}_*$ of pointed simplicial sets. The finite pointed simplicial sets form a set of compact generators for $s\mathbf{Set}_*$, by Lemma 1. Theorem 24 applies to the ordinary categories of spectra and symmetric spectra — the shifted suspension objects $\Sigma^\infty K[n]$ associated to finite pointed simplicial sets K gives the required set of compact generators in each case.

Theorem 24 applies to a small list of well-behaved pointed simplicial presheaf categories and their associated categories of spectra and symmetric spectra. These include the categories of simplicial presheaves on the scheme category $Sch|_S$ where the local weak equivalences are determined by either the Zariski or Nisnevich topologies. In both cases, the collections of objects $K \wedge U_+$ arising from finite pointed simplicial sets K and S-schemes U give a set of compact generators, by the Brown-Gersten and Nisnevich descent theorems, respectively (see [1], [12]). In effect, in both contexts, if $s \mapsto Y_s$, $s < \alpha$, is an inductive system of injective fibrant objects, then the respective descent theorems imply that the colimit $\varinjlim_{s<\alpha} Y_s$ satisfies descent, and so the objects $K \wedge U_+$ are compact.

Testing the applicability of Theorem 24 for simplicial presheaves and related categories amounts, in all cases, to displaying a compact set of generators. Theorem 24 specializes to a Brown representability result for the motivic model structure on pointed simplicial presheaves for the smooth Nisnevich site $Sm|_S$ on a scheme S: the objects $K \wedge V_+$ with K a finite pointed simplicial set and V a smooth S-scheme form a compact set of generators because colimits of inductive systems of motivic fibrant simplicial presheaves satisfy motivic descent. This set of objects can be parlayed into a family of compact objects $\Sigma^\infty_T(K \wedge V_+)[n]$ for both the categories of motivic T-spectra and motivic symmetric T-spectra, by applying the suspension spectrum construction and all shifts [7], for all of the standard suspension objects T.

In many other cases of interest, however, such as pointed simplicial presheaves or presheaves of spectra on an étale site, we only have Theorem 16, which has stronger conditions and a stronger conclusion.

References

[1] Kenneth S. Brown and Stephen M. Gersten. Algebraic K-theory as generalized sheaf cohomology. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 266–292. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[2] B. Chorny. Brown representability for space-valued functors. Preprint, 2007, arXiv: math.AT/0707.0904v2.

[3] J. Daniel Christensen, Bernhard Keller, and Amnon Neeman. Failure of Brown representability in derived categories. *Topology*, 40(6):1339–1361, 2001.

[4] P. G. Goerss and J. F. Jardine. Localization theories for simplicial presheaves. *Canad. J. Math.*, 50(5):1048–1089, 1998.

[5] Alex Heller. On the representability of homotopy functors. *J. London Math. Soc. (2)*, 23(3):551–562, 1981.

[6] J. F. Jardine. Simplicial presheaves. *J. Pure Appl. Algebra*, 47(1):35–87, 1987.

[7] J. F. Jardine. Motivic symmetric spectra. *Doc. Math.*, 5:445–553 (electronic), 2000.

[8] J. F. Jardine. Presheaves of symmetric spectra. *J. Pure Appl. Algebra*, 150(2):137–154, 2000.

[9] J. F. Jardine. Presheaves of chain complexes. *K-Theory*, 30(4):365–420, 2003. Special issue in honor of Hyman Bass on his seventieth birthday. Part IV.

[10] J. F. Jardine. Categorical homotopy theory. *Homology, Homotopy Appl.*, 8(1):71–144 (electronic), 2006.

[11] J. F. Jardine. Intermediate model structures for simplicial presheaves. *Canad. Math. Bull.*, 49(3):407–413, 2006.

[12] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, 90:45–143 (2001), 1999.

[13] Amnon Neeman. The Grothendieck duality theorem via Bousfield's techniques and Brown representability. *J. Amer. Math. Soc.*, 9(1):205–236, 1996.

[14] Jiří Rosický. Generalized Brown representability in homotopy categories. *Theory Appl. Categ.*, 14:no. 19, 451–479 (electronic), 2005.

[15] Jiří Rosický. Erratum: "Generalized Brown representability in homotopy categories" [Theory Appl. Categ. **14** (2005), no. 19, 451–479]. *Theory Appl. Categ.*, 20:No. 2, 18–24, 2008.