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BRAIDED MONOIDAL CATEGORIES

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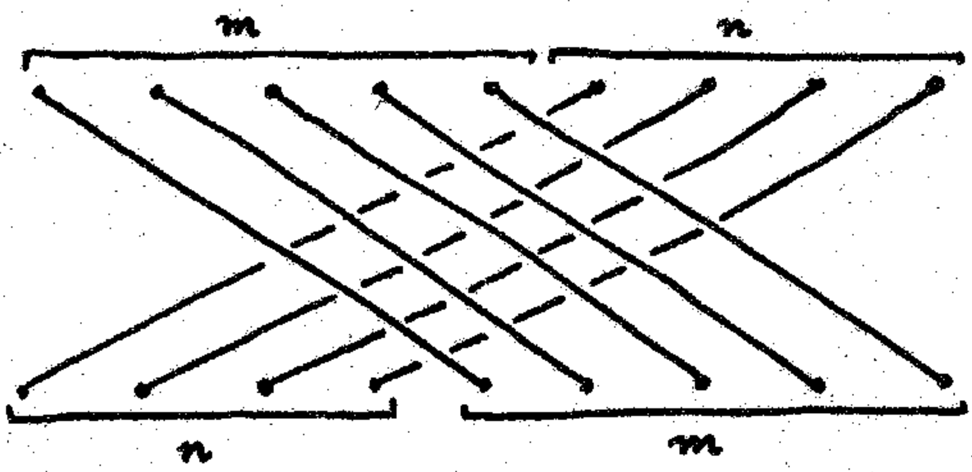

$$c_{m,n} : m+n \longrightarrow n+m$$

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Introduction.

Presumably Mac Lane's coherence theorem [14] and the plethora of examples led Eilenberg-Kelly [5] to define a symmetry for a monoidal category to be an involutary natural isomorphism $c : A \otimes B \longrightarrow B \otimes A$ satisfying B1 below. (Do we detect some regret that a monoidal closed category may admit several distinct symmetries [5 ; p.512] ?) With a symmetry Eilenberg-Kelly could define opposites (duals) and tensor products of enriched categories. Kasangian-Rossi [7] observed that replacement of B1 by the requirement that c should give a monoidal equivalence between the monoidal category and its reverse (this amounts to B3, B4, B5 below) still allowed opposite enriched categories but not tensor products.

Here we examine deletion of the involutary requirement while still asking that the natural isomorphism c and its inverse satisfy B1; this is our notion of braiding for a monoidal category. The relationship between symmetries and the symmetric groups goes back to Mac Lane [14]; braidings bear the same relationship to the braid groups.

This paper was generated by the confluence of a conversation of A. Carboni, F.W. Lawvere and R.F.C. Walters (Sydney, January 1984), and, joint work of M. Tierney and the first author. The conversation concerned the question of tensor products of categories enriched over a bicategory. The results of Street [18] made it clear that to have such tensor products would involve having a global tensor product on the base bicategory itself. Their idea was that when the bicategory had one object (and so amounted to a monoidal category) a generalization of the argument of Eckmann-Hilton [4] (showing that a monoid in the category of monoids is a commutative monoid) would show such a global tensor product to force symmetry. It does not quite.

Joyal-Tierney were concerned with homotopy 3-types: they found that arc-connected, simply connected such spaces could be represented by what we would call braided groups in Cat. They used a 2-groupoidal version of the tensor product of Gray [6]. Recall that Gray used the braid groups to prove his tensor product coherent. The link between homotopy theory and categories with structure is by no means new [17], [16], [9], [1]; indeed, the Eckmann-Hilton argument itself arose out of the proof that the higher homotopy groups are abelian.

Section 1 gives the definition of a braided monoidal category and certain further diagrams are found to be commutative. Examples of braidings which are not symmetries are provided in Section 2; there is also an example which satisfies the further commutative diagrams of Section 1 yet is not a braiding.

The generalized Eckmann-Hilton result appears in Section 3. A monoid in the category of strict monoidal categories and strong monoidal functors is a braided strict monoidal category. General braided monoidal categories are monoidal objects in the 2-category of monoidal categories and strong monoidal functors.

Up to equivalence, the free braided monoidal category on the terminal category is the coproduct of all the braid groups (as one-object categories) with an appropriate braided monoidal structure. This, and its implications for coherence, is the topic of Section 4.

Section 5 points out that categories enriched over braided monoidal categories admit opposites and tensor products. Finally, Section 6 deals with some constructions leading to braidings (notably, convolution).

The second author would like to thank the "Groupe Interuniversitaire en Etudes Catégoriques" directed by Michael Barr for making possible a six-week visit to Montréal during April-May 1985.

§1. Braidings

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I, r, l, a)$ be a monoidal category in the sense of Eilenberg-Kelly [5 ; p.471]. A *braiding* for $\mathcal{V}$ consists of a natural family of isomorphisms

$$c = c_{AB} : A\otimes B \longrightarrow B\otimes A$$

in $\mathcal{V}_0$ such that the following two diagrams commute.

B1.
$$\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, "a"] \arrow[d, "c\otimes 1"'] & A\otimes(B\otimes C) \arrow[r, "c"] & (B\otimes C)\otimes A \arrow[d, "a"] \\
(B\otimes A)\otimes C \arrow[r, "a"'] & B\otimes(A\otimes C) \arrow[r, "1\otimes c"'] & B\otimes(C\otimes A)
\end{tikzcd}$$

B2.
$$\begin{tikzcd}
A\otimes(B\otimes C) \arrow[r, "a^{-1}"] \arrow[d, "1\otimes c"'] & (A\otimes B)\otimes C \arrow[r, "c"] & C\otimes(A\otimes B) \arrow[d, "a^{-1}"] \\
A\otimes(C\otimes B) \arrow[r, "a^{-1}"'] & (A\otimes C)\otimes B \arrow[r, "c\otimes 1"'] & (C\otimes A)\otimes B
\end{tikzcd}$$

A monoidal category together with a braiding is called a *braided monoidal category*. Note that if c is a braiding then so is c^{-1}.

Compare the definition of "braiding" with that of "symmetry" [5 ; p.512]. Axiom B1 is precisely axiom MC7 while axiom B2 is replaced by axiom MC6.

MC6.
$$\begin{tikzcd}
& B\otimes A \arrow[dr, "c"] & \\
A\otimes B \arrow[ur, "c"] \arrow[rr, "1"'] & & A\otimes B
\end{tikzcd}$$

In the presence of MC6, observe that B2 is the inverse of B1; so every symmetry is a braiding. Examples of braidings which are not symmetries will be given in the next section.

<u>Proposition 1.</u> <u>In a braided monoidal category, the solid arrows in the following diagrams</u> B3 – B7 <u>commute</u>. [The symbol ⊗ has been omitted from the objects to save space.]

B3.
$$\begin{tikzcd} AI \arrow[rr, "c"] \arrow[dr, "r"'] & & IA \arrow[dl, "l"] \\ & A & \end{tikzcd}$$
,

B4.
$$\begin{tikzcd} IC \arrow[rr, "c"] \arrow[dr, "l"'] & & CI \arrow[dl, "r"] \\ & C & \end{tikzcd}$$
,

B5.
$$\begin{tikzcd} & (AB)C \arrow[rr, "c\otimes 1"] \arrow[dl, "a"'] & & (BA)C \arrow[d, "c"] \\ A(BC) \arrow[d, "1\otimes c"'] & & & C(BA) \\ A(CB) \arrow[rr, "c"'] & & (CB)A \arrow[ur, "a"'] & \end{tikzcd}$$
,

B6.
$$\begin{tikzcd} (AB)(CD) \arrow[r, "c\otimes 1"] \arrow[d, "1\otimes c"'] \arrow[dr, phantom, "\otimes\ \text{functor}"] & (BA)(CD) \arrow[r, "a"] \arrow[d, dashed, "1\otimes c"] \arrow[dr, phantom, "a\ \text{natural}"] & B(A(CD)) \arrow[rr, "1\otimes a^{-1}"] \arrow[d, dashed, "1\otimes(1\otimes c)"] & & B((AC)D) \arrow[d, "1\otimes c"] \\ (AB)(DC) \arrow[r, dashed, "c\otimes 1"'] \arrow[d, "a^{-1}"'] \arrow[dr, phantom, "a\ \text{natural}"] & (BA)(DC) \arrow[r, dashed, "a"] \arrow[d, dashed, "a^{-1}"] \arrow[dr, phantom, "\text{MC3}"] & B(A(DC)) \arrow[dr, dashed, "1\otimes a^{-1}"] & \text{B2} & B(D(AC)) \arrow[dd, "1\otimes a^{-1}"] \\ ((AB)D)C \arrow[r, dashed, "(c\otimes 1)\otimes 1"] \arrow[d, "a\otimes 1"'] \arrow[dr, phantom, "\text{B1}"] & ((BA)D)C \arrow[r, dashed, "a\otimes 1"] & (B(AD))C \arrow[r, dashed, "a"] \arrow[d, dashed, "(1\otimes c)\otimes 1"] \arrow[dr, phantom, "a\ \text{natural}"] & B((AD)C) \arrow[dr, dashed, "1\otimes(c\otimes 1)"] & \\ (A(BD))C \arrow[r, "c\otimes 1"'] & ((BD)A)C \arrow[r, "a\otimes 1"'] & (B(DA))C \arrow[rr, "a"'] & & B((DA)C) \end{tikzcd}$$
,

B7.

$$\begin{tikzcd}[column sep=small]
 & & A(CB) \arrow[r, "a^{-1}"] \arrow[rrrddd, dashed, "c"] & (AC)B \arrow[r, "c\otimes 1"] & (CA)B \arrow[dr, "a"] \arrow[d, phantom, "\underline{B1}"] & & \\
 & A(BC) \arrow[ur, "1\otimes c"] \arrow[rrrddd, dashed, "c"'] & & \arrow[d, phantom, "\underline{c\ \mathrm{natural}}"] & & C(AB) \arrow[dr, "1\otimes c"] & \\
(AB)C \arrow[ur, "a"] \arrow[dr, "c\otimes 1"'] & & & & & & C(BA) \\
 & (BA)C \arrow[dr, "a"'] \arrow[r, phantom, "\underline{B1}"] & & & & (CB)A \arrow[ur, "a^{-1}"'] & \\
 & & B(AC) \arrow[r, "1\otimes c"'] & B(CA) \arrow[r, "a^{-1}"'] & (BC)A \arrow[ur, "c\otimes 1"'] & &
\end{tikzcd}$$

Proof. B3. Take $B = C = I$ in B1, use the coherence of a, r, l, and the invertibility of $c_{A,I}$.

B4. Take $A = B = I$ in B2, use the coherence of a, r, l, and the invertibility of $c_{I,C}$.

B5 becomes B7 on replacing the bottom c using B1 and the right-hand c using B2.

B6 and B7 are proved by using the dotted arrows in the diagrams. □

§2. Examples.

1. Graded modules.

For a commutative ring K, let $\mathcal{V}_0$ be the category GMod_K of graded K-modules with tensor product given by

$$(A\otimes B)_n = \sum_{p+q=n} A_p \otimes_K B_q .$$

The associativity $a : (A\otimes B)\otimes C \longrightarrow A\otimes(B\otimes C)$ is given by $a((x\otimes y)\otimes z) = x\otimes(y\otimes z)$.

Braidings $c : A\otimes B \to B\otimes A$ for this monoidal structure on $G\mathrm{Mod}_K$ are in bijection with invertible elements $k \in K$ via the formula

$$c(x\otimes y) = k^{pq}(y\otimes x) \quad \text{where } x\in A_p,\ y\in B_q .$$

The proof can be extracted from [5; pp. 558-559] where it is shown that symmetries are in bijection with elements $k\in K$ satisfying $k^2 = 1$.

Note that, by taking k non-invertible and defining c as above, we still obtain a natural c satisfying B1, B2. Hence the requirement that a braiding be an isomorphism is independent of the other requirements.

2. Homotopy structure of the 2-sphere.

Let R be any (semi-)ring, although the case which relates to the 2-sphere is $R = \mathbb{Z}$. Objects of $\mathcal{V}_0$ are the elements of R and the homsets are given by

$$\mathcal{V}_0(x,y) = \begin{cases} R & \text{when } x = y, \\ \emptyset & \text{when } x \neq y. \end{cases}$$

Composition in $\mathcal{V}_0$ is addition in R. The tensor product is given by

$$(x \xrightarrow{u} x)\otimes(y \xrightarrow{v} y) = (x+y \xrightarrow{u+v} x+y).$$

This defines a strict monoidal category $\mathcal{V}$ with identity object $I = 0$;

A braiding for $\mathcal{V}$ is given by

$$c = (x+y \xrightarrow{xy} y+x).$$

Notice that B1 amounts to the distributive law $x(y+z) = xy + xz$ and B2 to $(x+y)z = xz + yz$. Diagram B7 amounts to the equation $xy + xz + yz = yz + xz + xy$.

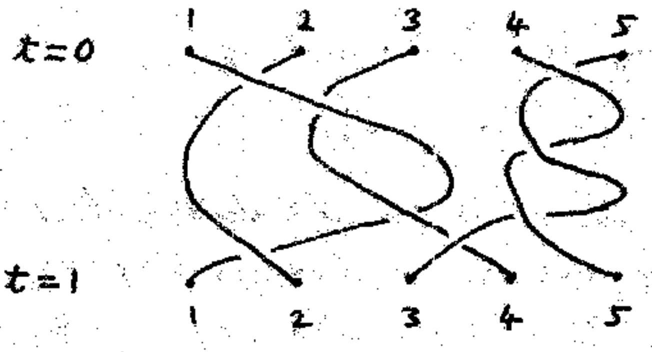
Since there are structures R satisfying all the ring axioms except for one or both of the distributive laws, it follows that:

B1 does not imply B2; and,
B7 implies neither B1 nor B2.

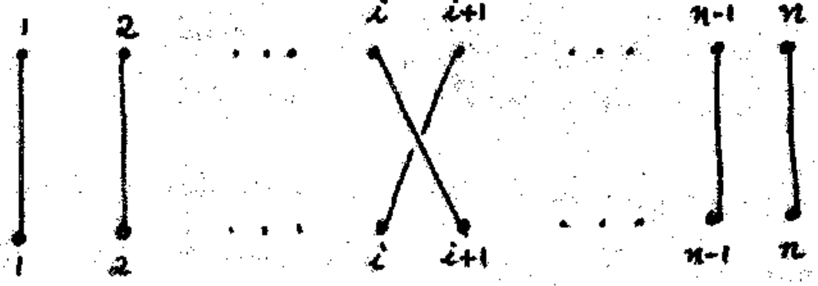
Finally notice that the braiding is a symmetry precisely when the ring satisfies the axiom $xy + yx = 0$ (which is false for $R = \mathbb{Z}$).

3. <u>The braid category</u> $\mathbb{B}$.

Let P denote a Euclidean plane with n distinct collinear points distinguished and labelled $1, 2, \ldots, n$. Let $\binom{P}{n}$ denote the space of subsets of P of cardinality n. The braid group $\mathbb{B}_n$ on n strings is the fundamental group of $\binom{P}{n}$. A loop $\omega : [0,1] \to \binom{P}{n}$ at the point $\{1, 2, \ldots, n\}$ of this space can be depicted by a diagram in Euclidean space of the form



where a horizontal cross-section by P at level $t \in [0,1]$ intersects the curves in the subset $\omega(t)$ of P. Let τ_i be the homotopy class of the loop depicted by the following diagram



for $i=1,\dots,n-1$. A presentation of $\mathbb{B}_n$ is given by the generators $\tau_1, \tau_2, \dots, \tau_{n-1}$ and relations

BG1. $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ for $i=1,\dots,n-2$,

BG2. $\tau_i \tau_j = \tau_j \tau_i$ for $|i-j|>1$, $i,j=1,\dots,n-1$.

For details see [A],[B],[C].

There are relations between the various $\mathbb{B}_n$. There are group homomorphisms

$$h: \mathbb{B}_m \longrightarrow \mathbb{B}_{m+n}, \quad k: \mathbb{B}_n \longrightarrow \mathbb{B}_{m+n}$$

given by: $h(\tau_i) = \tau_i$ for $i=1,\dots,m-1$,

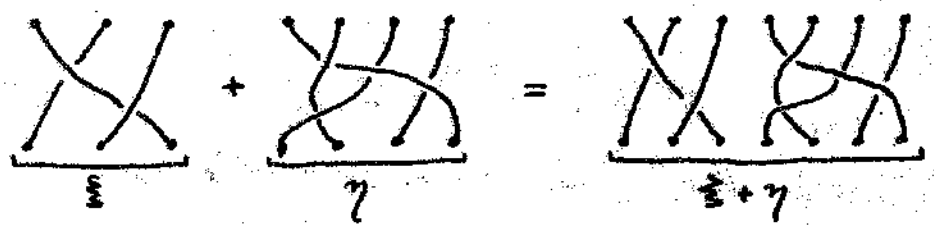
$k(\tau_i) = \tau_{m+i}$ for $i=1,\dots,n-1$.

Note that τ_m is not in the image of h or k.

By BG2, elements in the image of h commute with elements in the image of k; hence the function

$$+ \ : \ \mathbb{B}_m \times \mathbb{B}_n \longrightarrow \mathbb{B}_{m+n}$$

given by $\xi + \eta = h(\xi)\,k(\eta)$ is a group homomorphism called <u>addition of braids</u>. Pictorially addition of braids amounts to juxtaposition of diagrams.



<u>The braid category</u> $\mathbb{B}$ is the coproduct of the $\mathbb{B}_n$ as one-object categories. More explicitly, the objects of $\mathbb{B}$ are the natural numbers $0, 1, 2, \dots$, the homsets are given by

$$\mathbb{B}(m,n) = \begin{cases} \mathbb{B}_n & \text{when } m = n, \\ \emptyset & \text{when } m \neq n, \end{cases}$$

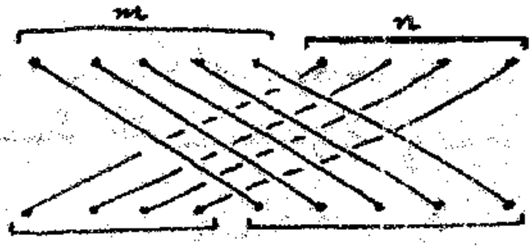
and, the composition is multiplication in the braid groups.

The tensor product $+ : \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$ is given on objects by addition of natural numbers and on arrows by addition of braids. This defines a strict monoidal structure on $\mathbb{B}$ with identity object $I = 0$.

Now we come to the definition of a braiding

$$c = c_{m,n} : m + n \longrightarrow n + m$$

for the strict monoidal category $\mathbb{B}$. The idea is illustrated by the following diagram for $c_{5,4} \in \mathbb{B}_9$.



To describe this algebraically, put

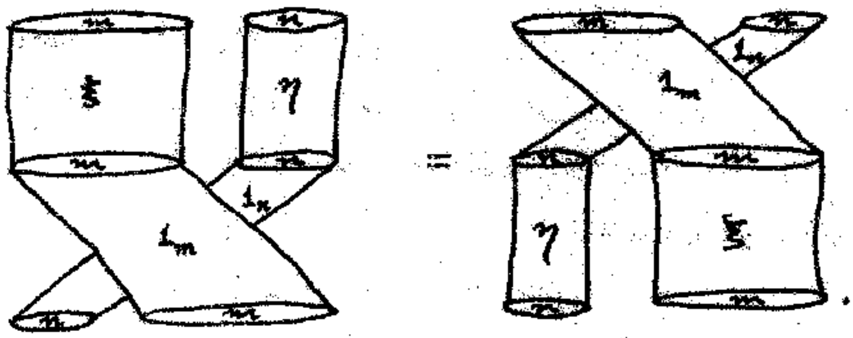
$$\gamma = \tau_n \tau_{n-1} \dots \tau_2 \tau_1 \in \mathbb{B}_{n+1} \text{ and}$$

$$\gamma^{(p)} = 1_p + \gamma + 1_{m-p-1} \in \mathbb{B}_{m+n} \text{ for } p = 0, 1, \dots, m-1.$$

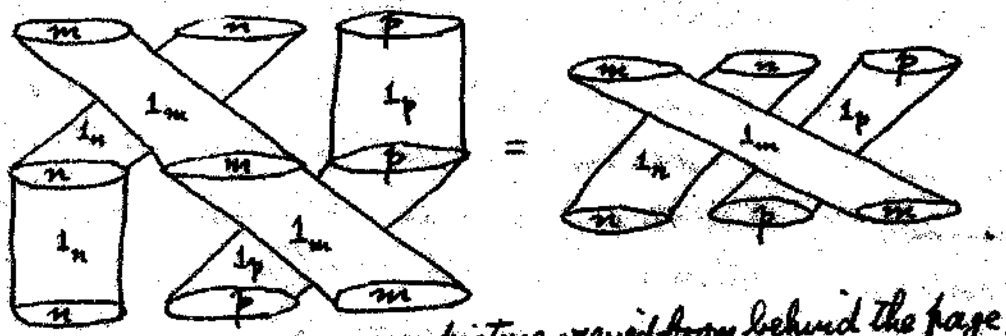
Then

$$c_{m,n} = \gamma^{(0)} \gamma^{(1)} \dots \gamma^{(m-1)} \in \mathbb{B}_{m+n}.$$

Naturality of $c_{m,n}$ is proved pictorially by:



Axiom B1 is proved pictorially by:



Axiom B2 is the same picture viewed from behind the page. Algebraic proofs seem to add nothing to our discussion and so will not be included. This braiding is not a symmetry since $c_{1,1} = \tau_1 \in \mathbb{B}_2$ and $\tau_1 \tau_1 \neq 1_2$.

4. Ribbons and braids.

After a lecture on the material of this paper [19], Fred Linton suggested the following modification of the braid example. The monoidal category $\mathbb{L}$ is defined similarly to $\mathbb{B}$ except that the arrows are braids on ribbons (instead of on strings) where twisting of the ribbons is allowed. Each ribbon has two edges which act as strings so there is a faithful strict monoidal functor $\mathbb{L} \longrightarrow \mathbb{B}$ taking n to $2n$. Let $c_{m,n} : m+n \longrightarrow n+m$ be defined for $\mathbb{L}$ as for $\mathbb{B}$ except that the ribbons are also given one twist each through 2π. Then $c_{m,n}$ <u>is a natural isomorphism which satisfies</u> B3-B7 <u>of Proposition 1 but does not satisfy</u> B1 <u>or</u> B2. For, B1, B2 involve an odd number of c's whereas all the others involve an even number.

§3. Multiplications on monoidal categories.

For monoidal categories $\mathcal{V}, \mathcal{V}'$, a *monoidal functor* $\Phi = (\varphi, \tilde{\varphi}, \varphi^0) : \mathcal{V} \to \mathcal{V}'$ consists of a functor $\varphi : \mathcal{V}_0 \to \mathcal{V}'_0$, a natural transformation $\tilde{\varphi} = \tilde{\varphi}_{AB} : \varphi A \otimes \varphi B \to \varphi(A \otimes B)$, and, an arrow $\varphi^0 : I \to \varphi I$, satisfying axioms MF1, MF2, MF3 [5; p.473]. Call Φ *strong* when $\tilde{\varphi}, \varphi^0$ are invertible. Call Φ *strict* when $\tilde{\varphi}, \varphi^0$ are identities.

Let MC_L denote the 2-category of (small) monoidal categories, monoidal functors, and, monoidal natural transformations. The 2-category of more interest here is the sub-2-category MC of MC_L with the same objects, with strong monoidal functors as arrows, and, with monoidal natural transformations as 2-cells. There is also the 2-category MC_S of strict monoidal categories, strict monoidal functors, and, monoidal natural transformations; so MC_S is the 2-category of monoids in the 2-category Cat of categories.

The 2-category MC_L admits products and the projections are strict monoidal functors; so the sub-2-categories MC, MC_S are closed under formation of products. The product of $\mathcal{V}, \mathcal{V}'$ in MC_L has $(\mathcal{V} \times \mathcal{V}')_0 = \mathcal{V}_0 \times \mathcal{V}'_0$, $(A, A') \otimes (B, B') = (A \otimes B, A' \otimes B')$, $I = (I, I)$. The terminal object MC_L is the category $\mathbb{1}$ with one object and one arrow enriched with its unique monoidal structure.

A binary operation on an object $\mathcal{V}$ of MC is a strong monoidal functor $\Phi : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$. For reference we give axiom MF3 in this case.

MF3.
$$\begin{tikzcd}
(\varphi(A,A')\varphi(B,B'))\varphi(C,C') \arrow[r, "a"] \arrow[d, "\tilde{\varphi}\otimes 1"'] & \varphi(A,A')(\varphi(B,B')\varphi(C,C')) \arrow[d, "1\otimes\tilde{\varphi}"] \\
\varphi(AB,A'B')\varphi(C,C') \arrow[d, "\tilde{\varphi}"'] & \varphi(A,A')\varphi(BC,B'C') \arrow[d, "\tilde{\varphi}"] \\
\varphi((AB)C,(A'B')C') \arrow[r, "\varphi(a,a)"'] & \varphi(A(BC),A'(B'C'))
\end{tikzcd}$$

A *pseudo-identity* for this binary operation in MC (as in any 2-category) is a nullary operation $\mathbb{1} \to \mathcal{V}$ which acts as an identity for Φ up to an invertible 2-cell. Each $\mathbb{1} \to \mathcal{V}$ in MC is isomorphic to the particular strong monoidal functor $\mathbb{1} \to \mathcal{V}$ whose underlying functor has value I at the one object of $\mathbb{1}$. Hence Φ admits a pseudo-identity if and only if there exist isomorphisms $\lambda_A : A \cong \varphi(I,A)$, $\rho_A : A \cong \varphi(A,I)$ such that $\lambda_I = \varphi^0 = \rho_I$ and the following diagrams commute.

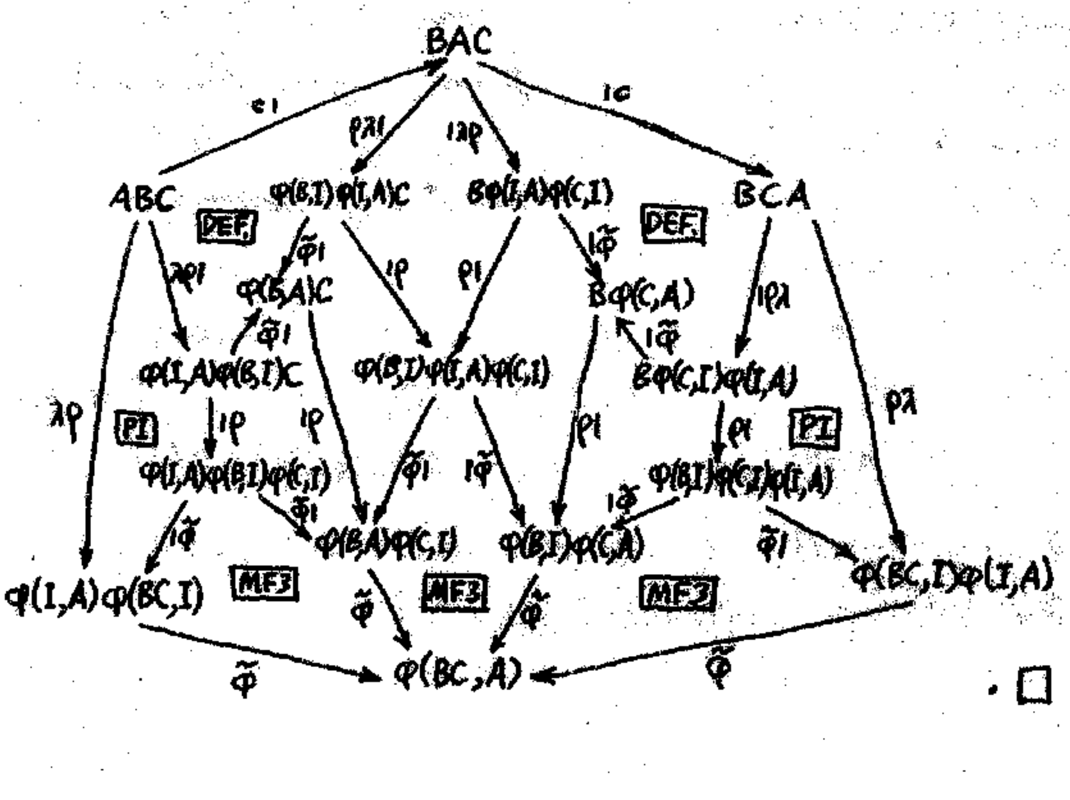
PI.
$$\begin{tikzcd}
AB \arrow[r, "\lambda"] \arrow[d, "\lambda\otimes\lambda"'] & \varphi(I,AB) \\
\varphi(I,A)\varphi(I,B) \arrow[r, "\tilde{\varphi}"'] & \varphi(II,AB) \arrow[u, "\varphi(\ell_I,1)"']
\end{tikzcd}, \quad
\begin{tikzcd}
AB \arrow[r, "\rho"] \arrow[d, "\rho\otimes\rho"'] & \varphi(AB,I) \\
\varphi(A,I)\varphi(B,I) \arrow[r, "\tilde{\varphi}"'] & \varphi(AB,II) \arrow[u, "\varphi(1,r_I)"']
\end{tikzcd}.$$

A binary operation Φ together with λ, ρ as above will be called a *multiplication on* $\mathcal{V}$.

<u>Proposition 2</u>. <u>*For any multiplication* (Φ, λ, ρ) *on a monoidal category* $\mathcal{V}$, *the following diagram defines a braiding* c *for* $\mathcal{V}$.</u>

$$\begin{tikzcd}
A\otimes B \arrow[rr, "c_{A,B}"] \arrow[d, "\lambda\otimes\rho"'] & & B\otimes A \arrow[d, "\rho\otimes\lambda"] \\
\varphi(I,A)\otimes\varphi(B,I) \arrow[d, "\tilde{\varphi}"'] & & \varphi(B,I)\otimes\varphi(I,A) \arrow[d, "\tilde{\varphi}"] \\
\varphi(I\otimes B, A\otimes I) \arrow[dr, "\varphi(\ell,r)"'] & & \varphi(B\otimes I, I\otimes A) \arrow[dl, "\varphi(r,\ell)"] \\
& \varphi(B,A) &
\end{tikzcd}$$

<u>Proof</u>. *Since each arrow in the definition of* c *is a natural isomorphism, it remains to prove that* B1, B2 *commute. The following diagram proves* B1 *for* $\mathcal{V}$ *strict monoidal* $(a, r, l$ *identities). We leave it to the reader to modify the diagram in the general case.*



. □

Proposition 3. *If c is a braiding for a monoidal category $\mathcal{V}$ then a multiplication (Φ, λ, ρ) on $\mathcal{V}$ is defined by:*

$$\varphi = - \otimes - : \mathcal{V}_0 \times \mathcal{V}_0 \longrightarrow \mathcal{V}_0 ;$$

$$\begin{tikzcd}
(A\otimes A')\otimes(B\otimes B') \arrow[r, "\tilde{\varphi} = m"] \arrow[d, "(1\otimes a^{-1})a"'] & (A\otimes B)\otimes(A'\otimes B') \arrow[d, "(1\otimes a^{-1})a"] \\
A\otimes((A'\otimes B)\otimes B') \arrow[r, "1\otimes(c_{A',B}\otimes 1)"'] & A\otimes((B\otimes A')\otimes B');
\end{tikzcd}$$

$$\lambda_A = l_A^{-1} : A \longrightarrow I\otimes A ; \text{ and, } \quad \rho_A = r_A^{-1} : A \longrightarrow A\otimes I.$$

The braiding c is recovered from this multiplication by the construction of Proposition 2. If (Φ', λ', ρ') is a multiplication on $\mathcal{V}$ which leads to the braiding c using Proposition 2 then (Φ', λ', ρ') is isomorphic (in the obvious sense) to the multiplication (Φ, λ, ρ) defined above.

Proof. Conditions MF1, MF2, PI follow from B3, B4 of Proposition 1 whereas MF3 comes from B6. The last two sentences of the Proposition are straightforward. $\square$

For braided monoidal categories $\mathcal{V}, \mathcal{V}'$ a monoidal functor $\Phi : \mathcal{V} \longrightarrow \mathcal{V}'$ is said to be *braided* when the following diagram commutes:

$$\begin{tikzcd}
\varphi A \otimes \varphi B \arrow[r, "c"] \arrow[d, "\tilde{\varphi}"'] & \varphi B \otimes \varphi A \arrow[d, "\tilde{\varphi}"] \\
\varphi(A\otimes B) \arrow[r, "\varphi c"'] & \varphi(B\otimes A).
\end{tikzcd}$$

Let BMC_L denote the 2-category of braided monoidal categories, braided monoidal functors, and, monoidal natural transformations. Restricting to braided strong monoidal functors, we have the 2-category BMC. Further restricting to braided strict monoidal categories and braided strict monoidal functors, we have the 2-category BMC_s.

The results of this section can be summarized as an equivalence between the 2-category BMC and the obvious 2-category Mult(MC) of monoidal categories with multiplication.

§4. Coherence for braidings.

One form of the coherence theorem for monoidal categories is that every monoidal category $\mathcal{V}$ is equivalent in MC to a strict monoidal category (= monoid in Cat) [14],[1],[15]. However, it is not true that every symmetric (*a fortiori*, braided) monoidal category is equivalent in BMC to a commutative monoid in Cat. The reason for this is that, in general, $1 \neq c : A \otimes A \to A \otimes A$ and this distinction is preserved by equivalence.

Theorem 4. *For each braided monoidal category* $\mathcal{V}$, *evaluation at* $1 \in \mathbb{B}$ *is an equivalence of categories*

$$\mathrm{BMC}(\mathbb{B}, \mathcal{V}) \simeq \mathcal{V}_0 .$$

If $\mathcal{V}$ *is strict monoidal, this restricts to an isomorphism of categories*

$$\mathrm{BMC}_s(\mathbb{B}, \mathcal{V}) \cong \mathcal{V}_0 .$$

Proof. Let $\mathcal{M}$ be a strict monoidal category with $\mathcal{V} \simeq \mathcal{M}$ in MC. Clearly the braiding on $\mathcal{V}$ transports to a braiding on $\mathcal{M}$ such that $\mathcal{V} \simeq \mathcal{M}$ lifts to BMC. There is a commutative diagram of functors:

$$\begin{tikzcd}
\mathrm{BMC}(\mathbb{B}, \mathcal{V}) \arrow[r, "ev_1"] \arrow[d, "\simeq"'] & \mathcal{V}_0 \arrow[d, "\simeq"] \\
\mathrm{BMC}(\mathbb{B}, \mathcal{M}) \arrow[r, "ev_1"'] & \mathcal{M}_0 .
\end{tikzcd}$$

So it suffices to prove the Theorem for $\mathcal{V}$ strict monoidal.

For each object A of $\mathcal{V}_0$, we shall describe the unique braided strict monoidal functor $\Phi : \mathbb{B} \to \mathcal{V}$ with $\varphi(1) = A$. To preserve tensor product, we are forced to put $\varphi(n) = A^n$ (where again we put $A \otimes B = AB$). To give φ on arrows we must define a monoid homomorphism $\varphi : \mathbb{B}_n \longrightarrow \mathcal{V}_0(A^n, A^n)$ for each n. Since Φ is to be braided and $c_{1,1} = \tau_1 : 2 \to 2$ in $\mathbb{B}$, we are forced to have $\varphi(\tau_1) = c_{A,A} : A^2 \longrightarrow A^2$. But then the equality $\tau_i = 1_{i-1} + \tau_1 + 1_{n-i-1} : (i-1)+2+(n-i-1) \longrightarrow (i-1)+2+(n-i-1)$ in $\mathbb{B}$ forces the definition

$$\varphi(\tau_i) = 1_{A^{i-1}}\, c_{A,A}\, 1_{A^{n-i-1}} : A^{i-1}AAA^{n-i-1} \longrightarrow A^{i-1}AAA^{n-i-1}.$$

To see that this gives the desired monoid homomorphism we must see that the relations of the braid group are preserved: BG1 follows from B7 of Proposition 1 and BG2 from functoriality of $\otimes$ in $\mathcal{V}$. Naturality of the equality $\varphi(m)\varphi(n) = \varphi(m+n)$ in $m, n \in \mathbb{B}$ follows from the definition of addition of braids (look at the images of h, k separately). Hence we have a strict monoidal functor $\Phi : \mathbb{B} \to \mathcal{V}$. Properties B1, B2 in $\mathbb{B}$ and the fact that $n = 1 + \ldots + 1$ show that each $c_{m,n}$ is built up from $c_{1,1} = \tau_1 : 2 \to 2$ using the monoidal structure; since $c_{A,A} = \varphi(\tau_1)$, we have that Φ is braided.

Now we show that evaluation at 1 is fully faithful. Take braided strong (not necessarily strict) monoidal functors $\Phi, \Psi : \mathbb{B} \to \mathcal{V}$ and an arrow $f : \varphi(1) \to \psi(1)$ in $\mathcal{V}_0$. Let $\tilde{\varphi}^n : \varphi(1)^n \longrightarrow \varphi(n)$ be defined inductively by: $\tilde{\varphi}^0 = \varphi^0 : I \to \varphi(0)$ and

$$\tilde{\varphi}^{n+1} = \left(\varphi(1)^{n+1} = \varphi(1)^n \varphi(1) \xrightarrow{\tilde{\varphi}^n 1} \varphi(n)\varphi(1) \xrightarrow{\tilde{\varphi}} \varphi(n+1) \right).$$

In order to have a monoidal natural transformation $\alpha : \Phi \to \Psi$ with $\alpha_1 = f$, we are forced to define α_n by the commutative diagram

$$\begin{tikzcd} \varphi(1)^n \arrow[r, "f^n"] \arrow[d, "\tilde{\varphi}^n"', "\cong"] & \psi(1)^n \arrow[d, "\tilde{\psi}^n", "\cong"'] \\ \varphi(n) \arrow[r, "\alpha_n"'] & \psi(n). \end{tikzcd}$$

The naturality of α follows from the naturality of $\tilde{\varphi}, \tilde{\psi}$ and the braidedness of Φ, Ψ (we only need prove $\psi(\tau_i)\,\alpha_n = \alpha_n\,\varphi(t_i)$). That α is monoidal follows from commutativity (see [13] for coherence of monoidal functors) of the following diagram and the similar one for Ψ:

or prove directly by induction

$$\begin{tikzcd} \varphi(1)^m \varphi(1)^n \arrow[r, "\tilde{\varphi}^m \tilde{\varphi}^n"] \arrow[d, equal] & \varphi(m)\varphi(n) \arrow[d, "\tilde{\varphi}"] \\ \varphi(1)^{m+n} \arrow[r, "\tilde{\varphi}^{m+n}"'] & \varphi(m+n). \end{tikzcd} \quad \square$$

(slightly modified)

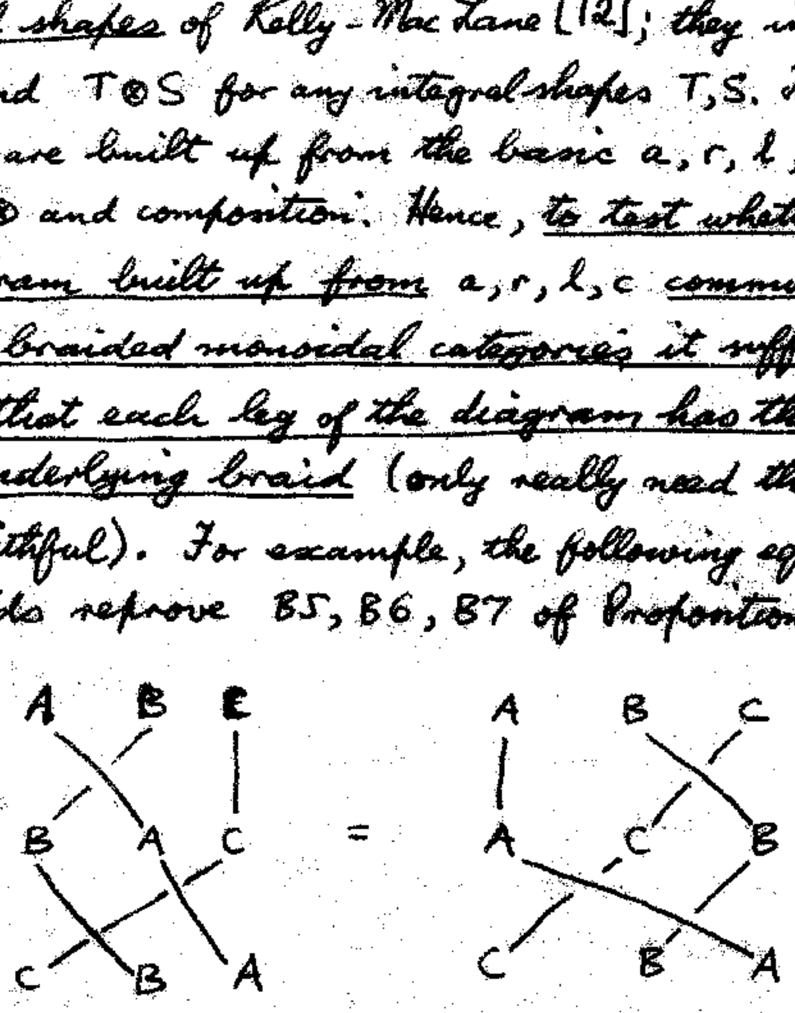
The work of Kelly [8] on "clubs" gives the 2-monadicity of BMC_s over Cat and that the 2-monad is determined by the free object on $\mathbb{1}$. The second sentence of Theorem 4 gives precisely that $\mathbb{B}$ is the free braided strict monoidal category on $\mathbb{1}$.

[8], [10], [11]

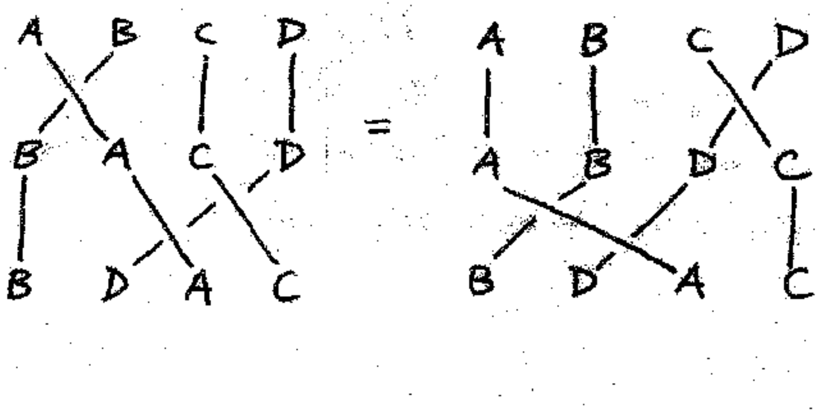
Again from Kelly ~~[illegible]~~ we know that the 2-category of braided monoidal categories and braided strict monoidal functors is 2-monadic over Cat. In this sense, the free braided monoidal category $\mathcal{B}$ on $\mathbf{1}$ is such that the category of braided strict monoidal functors $\mathcal{B} \longrightarrow \mathcal{V}$ is isomorphic to $\mathcal{V}_0$ 2-naturally in braided monoidal $\mathcal{V}$. Let $\Gamma : \mathcal{B} \to \mathbb{B}$ correspond to $1 \in \mathbb{B}$ under this isomorphism.

Theorem 4 implies that $\Gamma : \mathcal{B} \longrightarrow \mathbb{B}$ *is an equivalence in* BMC. The objects of $\mathcal{B}$ are the *integral shapes* of Kelly-Mac Lane [12]; they include I, 1 and $T \otimes S$ for any integral shapes T, S. The arrows are built up from the basic a, r, l, c using $\otimes$ and composition. Hence, *to test whether a diagram built up from* a, r, l, c *commutes in all braided monoidal categories it suffices to see that each leg of the diagram has the same underlying braid* (only really need that Γ is faithful). For example, the following equalities of braids reprove B5, B6, B7 of Proposition 1.

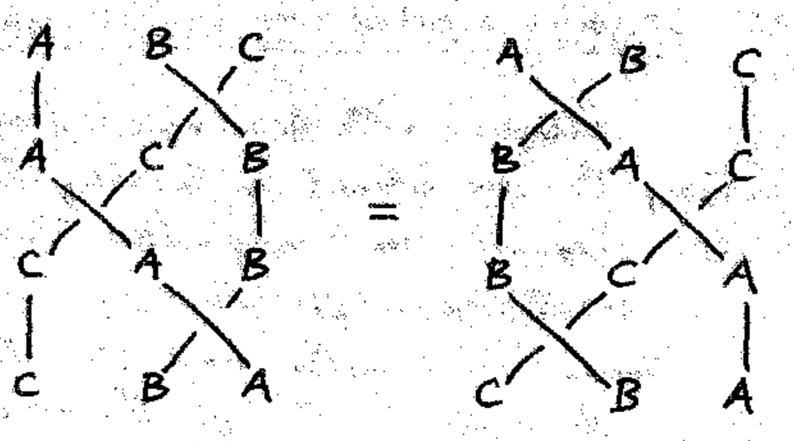
B5.



B6.



B7.



§5. Categories enriched over braided monoidal categories.

Categories with homs enriched in a monoidal category $\mathcal{V}$ were defined by Eilenberg-Kelly [5; pp.495-496]; they are more briefly called $\mathcal{V}$-categories. There is a 2-category (= hypercategory) $\mathcal{V}$-Cat of (small) $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations. We write $\mathcal{V}\text{-Cat}^{co}$ for the 2-category obtained from $\mathcal{V}$-Cat by reversing 2-cells (but not 1-cells).

Proposition 5. *Suppose $\mathcal{V}$ is a braided monoidal category. For each $\mathcal{V}$-category $\mathcal{A}$ there is a $\mathcal{V}$-category $\mathcal{A}^{op}$ defined by the following data:*

(i) *objects of $\mathcal{A}^{op}$ are those of $\mathcal{A}$;*

(ii) $\mathcal{A}^{op}(A,B) = \mathcal{A}(B,A)$;

(iii) $j : I \longrightarrow \mathcal{A}^{op}(A,A)$ is $j : I \longrightarrow \mathcal{A}(A,A)$; and,

(iv)
$$\begin{tikzcd}
\mathcal{A}^{op}(B,C)\otimes\mathcal{A}^{op}(A,B) \arrow[rr, "M"] \arrow[d, equal] & & \mathcal{A}^{op}(A,C) \arrow[d, equal] \\
\mathcal{A}(C,B)\otimes\mathcal{A}(B,A) \arrow[dr, "c"'] & & \mathcal{A}(C,A) \\
& \mathcal{A}(B,A)\otimes\mathcal{A}(C,B) \arrow[ur, "M"'] &
\end{tikzcd}$$
.

The assignment $\mathcal{A} \longmapsto \mathcal{A}^{op}$ is the object function of an isomorphism of 2-categories
$$(\ )^{op} : \mathcal{V}\text{-Cat}^{co} \longrightarrow \mathcal{V}\text{-Cat}.$$

Proof. Compare [5; p.514]. The only difference is that we cannot appeal to Mac Lane's coherence to prove commutativity of the top hexagon in the diagram on p.515 of [5]. However the hexagon does commute by Proposition 1, B5. $\square$

A word of warning: $(\mathcal{A}^{op})^{op} \neq \mathcal{A}$. The isomorphism $(\ )^{op}$ is not involutory unless the braiding is a symmetry. However, since $(\ )^{op}$ is an isomorphism the principle of duality can be used in the general braided case.

Proposition 6. Suppose $\mathcal{V}$ is a braided monoidal category and $\mathcal{A}$, $\mathcal{B}$ are $\mathcal{V}$-categories. The following data define a $\mathcal{V}$-category $\mathcal{C}$ denoted by $\mathcal{A} \otimes \mathcal{B}$:

(i) objects of $\mathcal{C}$ are ordered pairs (A, B) of objects A, B of $\mathcal{A}$, $\mathcal{B}$, respectively;

(ii) $\mathcal{C}((A,B),(A',B')) = \mathcal{A}(A,A') \otimes \mathcal{B}(B,B')$;

(iii)
$$\begin{tikzcd} I \arrow[r, "j"] \arrow[d, "l^{-1}"'] & \mathcal{C}((A,B),(A,B)) \arrow[d, equal] \\ I \otimes I \arrow[r, "j \otimes j"'] & \mathcal{A}(A,A) \otimes \mathcal{B}(B,B) \end{tikzcd}$$
; and,

(iv)
$$\begin{tikzcd} \mathcal{C}((A',B'),(A'',B'')) \otimes \mathcal{C}((A,B),(A',B')) \arrow[rr, "M"] \arrow[d, equal] & & \mathcal{C}((A,B),(A'',B'')) \arrow[d, equal] \\ (\mathcal{A}(A',A'') \otimes \mathcal{B}(B',B'')) \otimes (\mathcal{A}(A,A') \otimes \mathcal{B}(B,B')) \arrow[d, "m"'] & & \mathcal{A}(A,A'') \otimes \mathcal{B}(B,B'') \\ (\mathcal{A}(A',A'') \otimes \mathcal{A}(A,A')) \otimes (\mathcal{B}(B',B'') \otimes \mathcal{B}(B,B')) \arrow[urr, "M \otimes M"'] & & \end{tikzcd}$$

(where m is the "middle-four interchange" appearing in Proposition 3). With the definitions just as given by Eilenberg-Kelly [5; p.519], $\mathcal{V}$-Cat becomes a braided monoidal 2-category.

Proof. The only difference here from [5] is that we cannot appeal to Mac Lane's coherence to prove commutativity of the top hexagon in the bottom diagram of p. 518. The hexagon does commute in our case too by Proposition 1, B6. $\square$

Note that __monoids in__ $\mathcal{V}$ amount to one-object $\mathcal{V}$-categories. So, in a braided monoidal category, we can speak of opposite monoids and tensor products of monoids. A monoid A is __commutative__ when the diagram below commutes.

$$\begin{tikzcd} A\otimes A \arrow[rr, "c_{A,A}"] \arrow[dr, "M"'] & & A\otimes A \arrow[dl, "M"] \\ & A & \end{tikzcd}$$

Proposition 6 also allows us to define __monoidal__ $\mathcal{V}$-__categories__ and __braidings__ thereon.

§6. Convolution and bilinearity.

Convolution of monoidal structures was discussed by Day [2]; we centre attention on pages 17–29 of that paper.

Let $\mathcal{V}$ be a braided monoidal category. Using $(\;)^{\varphi}$ and $\otimes$ as in Section 5 above, we can carry over the definition of promonoidal $\mathcal{V}$-category $\mathcal{P}$ as it appears in [2; pp. 17–18] (although "premonoidal" was used there). Also Day's definition of "symmetry" (p. 23) gives the notion of braiding for a promonoidal $\mathcal{V}$-category when PC3 is deleted and PC4 (being the analogue of B1) is augmented by the obvious analogue of B2.

Proposition 7. Suppose $\mathcal{V}$ is a cocomplete braided monoidal closed category and $\mathcal{P}$ is a small braided monoidal $\mathcal{V}$-category. Then the $\mathcal{V}$-functor $\mathcal{V}$-category $[\mathcal{P}, \mathcal{V}]$ with the convolution structure is a cocomplete braided monoidal closed category.

Proof. All the necessary diagrams appear in Day [ ]. □

In particular, each braided monoidal $\mathcal{V}$-category gives rise to a braided promonoidal $\mathcal{V}$-category [2; p. 26]. So the examples of Section 2 can be convoluted to give more examples. The particular case $[\mathbb{B}, \mathrm{Set}]$ seems worthy of detailed study (elsewhere).

The example of graded modules in Section 2 can be viewed as a convolution where the base $\mathcal{V}$ is the category AbGp of abelian groups. For a commutative ring K, let $\mathcal{P}$ be the additive category whose objects are integers and whose homs are given by

$$\mathcal{P}(p,q) = \begin{cases} K \text{ as an additive group for } p=q, \\ \{0\} \text{ for } p \neq q; \end{cases}$$

composition in $\mathcal{P}$ is multiplication in K. Now $\mathcal{P}$ becomes strict monoidal by defining

$$(p \xrightarrow{u} p') \otimes (q \xrightarrow{v} q') = (p+q \xrightarrow{uv} p'+q').$$

Each ~~unit~~ invertible element k of K gives a braiding

$$c_{p,q} = k^{pq} : p+q \longrightarrow q+p .$$

The additive functor category $[\mathcal{P}, \mathrm{AbGp}]$ is equivalent to GMod_K and the convolution structure transports to the braided monoidal structure in Section 2 Example 1.

For braided monoidal $\mathcal{V}$-categories $\mathcal{A}, \mathcal{B}$, there is an obvious braiding on the monoidal $\mathcal{V}$-category $\mathcal{A} \otimes \mathcal{B}$; but there are also other braidings which involve a "twist" by means of a "bilinear map". The braidings on $\mathcal{P}$ (in the last paragraph) and in the 2-sphere example (Section 2 Example 2) are both examples of this.

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