DISKS, DUALITY AND $\Theta$-CATEGORIES

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The goal of this note is to define a simple concept of infinite dimensional (weak) categories that makes possible the development of a theory [BD, B, HMP, BFSV, T]. Our method is to extend simplicial homotopy theory to an appropriate concept of cellular sets. Our concept of higher categories is imitating the concept of restricted Kan complexes of Boardman and Vogt [BV]. A cellular set is a contravariant functor $X: \Theta \to \text{Sets}$ on a category $\Theta$ of cells. For defining the objects of $\Theta$ we use Batanin's idea of representing higher cells by finite trees [B]. To define the arrows we introduce a dual category $D$ of finite disks. A disk is a combinatorial structure that generalises to higher dimension the idea of an interval (a linearly ordered set with distinct first and last elements). The Euclidean $n$-ball $B^n \subset \mathbb{R}^n$ is an example of $n$-disk. Every finite disk $D$ has a dual $D' = \text{hom}(D, B^\infty)$ which is a polyhedral ball. This defines the geometric realisation of the cells from which we obtain a geometric realisation functor from the category $\hat{\text{Set}}$ of cellular sets to $\text{CW}$-complexes. We define the concepts of horns and of inner horns. A $\theta$-category is then defined to be a cellular set for which inner horns can be filled. The category of finite disks has an increasing filtration $\text{Disk}^1 \subset \text{Disk}^2 \subset \cdots$ defined by disk dimension and there is a corresponding level filtration $\theta^1 \text{Sets} \subset \theta^2 \text{Sets} \subset \cdots$ on the category of cellular sets. A $\theta$-category of level $\leq n$ is called a $\theta^n$-category: it is an infinite dimensional (weak) category for which all cells of dimension $> n$ are invertibles. A $\theta^1$-category is a restricted Kan complex. A concept of (weak) $n$-categories can be obtained by further truncating $\theta^n$-categories to cells of dimension $\leq n + 1$. In a second note we shall prove that $\theta \text{Cat}$, the category of $\theta$-categories, is cartesian closed. We shall define a concept of $h$-isomorphism for arrows in $\theta$-categories and prove that a $\theta$-category is a (cellular) Kan complex iff all its cells are $h$-invertible. We shall define the fundamental $n$-category $\pi_n X$ of a $\theta$-category $X$ and homotopy groups $\pi_n(X, f)$ for cells $f \in X$ and $n > \text{dim} f$. We shall prove a Whitehead theorem.

In what follows we shall define most concepts mentioned above.

§ 1.1 Simplicial sets and intervals

We begin by recalling the connection between simplicial sets and the theory of intervals. It can be phrased in topos theoretic terms as in [MM] but we shall limit the discussion to its combinatorial aspects. The category of finite non-empty ordinals and order preserving maps is denoted $\Delta$. It is standard to denote the ordinal $n + 1 = \{0, \ldots, n\}$ by $[n]$. Recall that a simplicial set is a contravariant functor $X: \Delta \to \text{Sets}$. The opposite category $\Delta^o$ has a direct combinatorial description with the category of finite strict intervals. Let us recall this duality. An interval is a linearly ordered set having a first and a last element respectively denoted 0 and 1, or $\bot$ and $\top$. If $0 \neq 1$ the interval is said to be strict. A morphism is a map $f: I \to J$ preserving the order ($\leq$) and such that $f(0) = 0$ and $f(1) = 1$. We shall denote by $\mathcal{I}$ the category of finite strict intervals. Let us describe the duality

$$(-)^*: \Delta^o \simeq I \quad (-)^*: \mathcal{I}^o \simeq \Delta.$$
By definition \([n]^* = \Delta([n], [1]) = [n + 1] \rightarrow [1]\) where for \(f, g : [n] \rightarrow [1]\) we put \(f \leq g\) iff \(f(i) \leq g(i)\) for every \(i \in [n]\). Similarly, \(I = I(\{2\})\). We shall view the interval \([n]^* = [n + 1]\) as the set of Dedekind cuts on \([n]\). For example, \([3]\) has 5 cuts, hence \([3]^* = [4]\):

\[
| 0132, 0 | 123, 01 | 23, 012 | 30123 |
\]

Similarly, the dual \(I^*\) is the set of inner cuts of \(I\). For example, \([4]\) has four inner cuts:

\[
0 \ast 1234, 01 \ast 234, 012 \ast 340123 \ast 4.
\]

The duality is illustrated by the picture:

\[
| \ast | | \ast | | \ast |
\]

The duals \(f^*\) or \(f\) of a map \(f : [n] \rightarrow [m]\) is obtained by taking inverse images of cuts. Notice that if \(I = [n]^*\) then the set hom\((I, B^1)\) is the geometric simplex of dimension \(n\) where \(B^1\) denotes the real interval \([0, 1]\). It follows from the duality that a simplicial set could be defined as a covariant functor \(X : \mathcal{I} \rightarrow \text{Sets}\).

### § 1.2 Disks

We now describe a combinatorial concept of \(n\)-disks, an interval being a 1-disk. The euclidian ball \(B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}\) is the main geometric example of \(n\)-disk. Observe that the projection \(p_n : B^n \rightarrow [-1, 1]\) is surjective and that its fibers \(p^{-1}(x)\) are \(n - 1\)-disks except for \(x = \pm 1\) where they degenerate to a point. There is a complementary view with the projection \(q : B^n \rightarrow B^{n-1}\). The fiber \(q^{-1}(x)\) is a 1-disk except if \(x \in \partial B^{n-1}\) where it degenerates to a point. We shall say that \(q\) is a bundle of intervals. If we order the coordinates we obtain a sequence of bundles of intervals:

\[
1 \leftarrow B^1 \leftarrow B^2 \leftarrow \cdots B^{n-1} \leftarrow B^n
\]

Each projection \(p : B^{n+1} \rightarrow B^n\) has two canonical sections \(d_0, d_1 : B^n \rightarrow B^{n+1}\) that correspond to the bottom and top hemispheres. We have \(d_0(x) = d_1(x)\) iff \(x \in \partial B^n\). Observe also that \(\partial B^{n+1} = d_0(B^n) \cup d_1(B^n)\).

We now define the combinatorial concept of bundles of intervals in the category \(\text{Sets}\). Shortly, a bundle of intervals over \(B \in \text{Sets}\) is an interval object in the category \(\text{Sets}/B\). More explicitly, it is a map \(p : X \rightarrow B\) whose fibers have an interval structure. It has then a top and a bottom section \(d_0, d_1 : B \rightarrow X\) and we have \(x \in \{d_0(px), d_1(px)\}\) for every \(x \in X\). We shall say that the equaliser of \(d_0\) and \(d_1\) is the singular set of the bundle.

**Definition 1:** A disk \(D\) of dimension \(\leq N\) is a sequence of length \(N\) of bundles of intervals:

\[
1 = D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow \cdots D_{N-1} \leftarrow D_N
\]

such that the singular set of \(p : D_{n+1} \rightarrow D_n\) is equal to \(d_0(D_{n-1}) \cup d_1(D_{n-1})\) for every \(0 \leq n < N\). If \(n = 0\) this condition means that \(D_1\) is a strict interval.

Disks of dimension \(\leq \infty\) are defined by using infinite sequences. It follows from the definition that each map \(d_0, d_1 : D_{i-1} \rightarrow D_i\) equalizes the pair \(d_0, d_1 : D_i \rightarrow D_{i+1}\), hence \(d_0d_0 = d_1d_0\) and \(d_0d_1 = d_1d_1\). We define the boundary \(\partial D_n\) to be \(d_0(D_{n-1}) \cup d_1(D_{n-1})\) and the interior \(i(D_n)\) to be \(D_n \setminus \partial D_n\). By convention \(\partial D_0 = \emptyset\).
Lemma 1. We have \( p(x(D_n)) \subseteq t(D_{n-1}) \) for every \( n \).

Proof: If \( p(x) \in \partial D_{n-1} \) then \( d_0(px) = d_1(px) \) by the condition in the definition. But \( x \in [d_0(px), d_1(px)] \) and it follows that \( x = d_0(px) \in \partial D_n \). QED

By restricting the projections to the interiors we obtain a sequence of maps

\[
1 \leftarrow t(D_0) \leftarrow t(D_1) \leftarrow t(D_2) \leftarrow t(D_3) \leftarrow \cdots
\]

This is a (generalised) planar tree since the fibers of each projection \( t(D_i) \to t(D_{i-1}) \) are linearly ordered. We shall say that it is the interior of \( D \) and denote it by \( i(D) \). We define the level of \( D \) to be the height of \( i(D) \). We define the volume \( |D| \) of \( D \) to be the cardinality of \( i(D) \) minus 1 (we do not include \( D_0 = 1 \) in counting the volume). If \( |D| < \infty \) we shall say that \( D \) is finite.

Proposition 1. A disk is determined by its interior (up to unique isomorphism). Any tree is the interior of a disk.

Proof: It follows from definition 1 that the square

\[
\begin{array}{ccc}
\partial D_{n-1} & \xrightarrow{d_1} & d_1(D_{n-1}) \\
d_0 \downarrow & & \downarrow \\
d_0(D_{n-1}) & \longrightarrow & \partial D_n
\end{array}
\]

is cartesian. It is thus a pushout square and we have \( \partial D_n \simeq D_{n-1} \cup_{\partial D_{n-1}} D_{n-1} \). The result then follows from the formula \( D_n = i(D_n) \cup \partial D_n \) by induction on \( n \).

A morphism \( S \to T \) of planar trees is a sequence of maps \( f_n : S_n \to T_n \) commuting with the projections

\[
1 \leftarrow S_1 \leftarrow S_2 \leftarrow S_3 \leftarrow \cdots
\]

\[
\begin{array}{ccc}
1 & \xrightarrow{f_1} & T_1 \\
f_3 & & \downarrow \\
T_3 & \leftarrow & T_2 \leftarrow T_3 \leftarrow \cdots
\end{array}
\]

and respecting the order of the fibers. A morphism \( D \to D' \) of disks is a morphism of planar trees that respects the endpoints of each fiber. For any planar tree \( S \) we shall denote by \( \overline{S} \) the unique disk such that \( S = i(\overline{S}) \). This is a functorial construction.

Proposition 2. The functor \( S \mapsto \overline{S} \) is left adjoint to the forgetful functor from disks to planar trees.

Remark: The construction \( S \mapsto \overline{S} \) is a monad on the category of planar trees. The category of algebras for this monad is the category of disks. It follows from proposition 1 that every disk is free on its interior. Hence the category of disks is also a Kleisly category.
A map \( f : D \to D' \) between disks is open if \( f(\iota(D)) \subseteq \iota(D') \). The category of planar trees is equivalent to the category of disks with open maps.

The suspension \( \sigma(T) \) of a planar tree is defined by shifting: \( \sigma(T)_i = T_{i-1} \) for \( i > 0 \) where \( T_0 = 1 \). There is also the operation of concatenation for trees: \( (S \star T)_i = S_i \cup T_i \) for \( i > 0 \) where \( \cup \) is disjoint union for \( i > 1 \) and the ordinal sum for \( i = 1 \). The suspension \( S(D) \) of a disk is the disk

\[
1 \leftarrow [2] \leftarrow D_1 \cup [1] \leftarrow D_2 \cup [1] \leftarrow \cdots
\]

where the map \( D_1 \cup [1] \to [2] \) sends \( D_1 \) to the mid point of \( [2] \) and is equal to \( d_1 : [1] \to [2] \) on \( [1] \). All the other maps are the obvious ones. The suspension of disks is defined so that \( \overline{S(T)} = \overline{\sigma(T)} \). These operations are functorial on disks and trees. There is also an operation of composition \( \circ \) for disks so that \( \overline{S \circ T} = \overline{(S \star T)} \) but it has a limited functoriality (it is of course functorial for open maps).

§ 1.3 Duality and Cellular sets

For the next result we need the infinite dimensional euclidian ball \( B^\infty \):

\[
1 \leftarrow B^1 \leftarrow B^2 \leftarrow \cdots
\]

**Proposition 3.** Let \( D \) be a finite disk. Then the set \( \text{hom}(D, B^\infty) \) has the structure of an euclidian ball of dimension \( |D| \).

**Proof:** We only sketch the construction. Let us denote respectively by \( \text{hom}_d(D, D') \) and \( \text{hom}_t(T, T') \) the maps in the categories of disks and trees. If \( T = \iota(D) \) then we have \( \text{hom}_d(D, B^\infty) = \text{hom}_t(T, B^\infty) \) by adjointness. We shall describe the polyhedron \( R(T) = \text{hom}_t(T, B^\infty) \) by structural induction on trees. If \( T = 0 \) then \( R(T) = 1 \). If \( T = \sigma(A) \) then \( R(T) \) is obtained as a pushout square

\[
\begin{array}{ccc}
\{-1,1\} \times R(A) & \longrightarrow & [-1,1] \times R(A) \\
\downarrow & & \downarrow \\
\{-1,1\} & \longrightarrow & R(T)
\end{array}
\]

and this shows that \( R(T) \) is the suspension \( SR(A) \) of \( R(A) \). Observe now that there is a structure map \( q : SR(A) \to [-1,1] \). If \( T = \sigma A_1 \star \cdots \star \sigma A_n \) then there is a cartesian square

\[
\begin{array}{ccc}
R(T) & \longrightarrow & \prod_{i=1}^n SR(A_i) \\
\downarrow & & \downarrow q^{\times n} \\
\Delta^n & \longrightarrow & [-1,1]^n
\end{array}
\]

where \( (x_1, \ldots, x_n) \in \Delta^n \) iff \( x_1 \leq \cdots \leq x_n \).
We shall denote by Disks (resp. Trees) the category of finite disks (resp. of finite trees). We define the category $\Theta$ of Batanin cells to be the opposite of Disks. We shall write $C = D^\circ$ and $D = C^\circ$ for a disk and its opposite cell. For any $C \in \Theta$ let us put $R(C) = \text{hom}(C^\circ, B^\infty)$. This defines the geometric realisation functor

$$R : \Theta \to CW-\text{complexes}.$$

The dimension of $C = D^\circ$ is defined to be $|D|$. If $t$ is a finite tree we shall denote the corresponding disk by $\overline{t}$ and the dual cell by $[t] = (\overline{t})^\circ$. The dimension of $[t]$ is thus the cardinality of $t$ (not counting the root). We shall denote by $0$ the null tree (with only the root). Recall the set of planar (rooted) trees can be constructed inductively from 0 by using the operation of (non-empty) concatenation $\ast$ and the operation of succession $\sigma$. Let us put $1 = \sigma(0)$ and $n + 1 = n \ast 1$. The disk $\overline{n}$ is the interval $\{0, 1, 2, \ldots, n, n + 1\}$ and its dual is $[n]$. The cells $[n]$ for $n \in \mathbb{N}$ are the classical simplicial cells. The tree $\sigma^n(0)$ is a chain of $n$ nodes. The corresponding cell has dimension $n$ and we shall denote it $\delta_n$. From the inclusion $\sigma^n-1(0) \subset \sigma^n(0)$ we obtain an arrow $\delta_n \rightarrow \delta_{n-1}$.

We define a cellular set to be a functor $X : \text{Disks} \to \text{Sets}$, that is, a functor $X : \Theta^\circ \to \text{Sets}$. We shall denote by $\Theta\text{Sets}$ the category of cellular sets. We shall use the Yoneda embedding $\Theta \subset \Theta\text{Sets}$ to identify $\Theta$ with a full subcategory of $\Theta\text{Sets}$. By Kan extension we obtain a geometric realisation functor $R : \Theta\text{Sets} \to CW-\text{complexes}$.

Theorem 1. In the category $\Theta\text{Sets}$ the sequence $\overline{\delta} = (\delta_n)$

$$1 = \delta_0 \leftarrow \delta_1 \leftarrow \delta_2 \leftarrow \cdots$$

has the structure of a disk. It is the universal disk in a topos theoretic sense.

For the purpose of the discussion we shall say that a functor between two Grothendieck toposes is a realisation functor if it preserves colimits and finite limits (that is, it is the inverse image part of a geometric morphism). The universal property in theorem 1 means that for any topos $E$ and for any disk $D \in E$ there is a unique realisation functor $r : \Theta\text{Sets} \to E$ such that $r(\overline{\delta}) = D$. For example the topos $SSets$ of simplicial sets classifies strict intervals [MM]. There is a realisation functor $SSets \to \Theta\text{Sets}$ that transforms the classical simplicial interval $\Delta[1]$ into $\delta_1$. The category $\Theta\text{Sets}$ is equipped with many operations that can be defined from its universal property. As an example, the truncated sequence $\delta_1 \leftarrow \delta_2 \leftarrow \cdots$ is almost a disk in the category $\Theta\text{Sets}/\delta_1$. It is not only because the bundle of intervals $\delta_2 \leftarrow \delta_1$ degenerates on $\partial\delta_1 \neq \emptyset$. This is easy to cure with the subtopos $\Theta\text{Sets}/\delta(\delta_1)$. The objects of this category are the maps $X \to \delta_1$ that are isomorphism (if pulled back) over $\partial\delta_1 \subset \delta_1$. The operation of suspension $S : \Theta\text{Sets} \to \Theta\text{Sets}/\delta(\delta_1)$ is the realisation functor that send $\delta$ to this truncated sequence. We obtain a suspension functor $S : \Theta\text{Sets} \to \Theta\text{Sets}$ by further composing with the forgetful functor $\Theta\text{Sets}/\delta(\delta_1) \to \Theta\text{Sets}$. We then have $S^n(1) = \delta_n$ for every $n \geq 0$ and this indicate that the topos $\Theta\text{Sets}$ is essentially generated by an abstract operation of suspension. It might be interesting to investigate the properties of such an operation. There are many realisation functors from $\Theta\text{Sets}$ to $SSets$. Let us define the suspension of a simplicial set $X \in SSets$ by putting $S(X) = \Delta[1] \times X/\sim$ where the equivalence relation $\sim$ is collapsing $\{0\} \times X$ to $\{0\}$ and $\{1\} \times X$ to $\{1\}$. We obtain an infinite disk $S^0(pt) \leftarrow S^1(pt) \leftarrow \cdots$ in $SSets$. We could also choose the suspension defined by the formula $S(X) = X \ast 1/X$ where $\ast$ is the simplicial join.
To identify the horns in \( \partial \text{Sets} \) we need to understand the (codimension 1) faces of a cell \( C \in \Theta \). A (codimension 1) face of \( C = [t] \) is a quotient of \( \tilde{t} \) that identifies exactly two points. The tree \( t \) is a disjoint union of intervals \( J(x) = p^{-1}(x) \). Let us say that a pair \( \{a, b\} \) of vertices of \( t \) is \textit{contractible} if \( a \) and \( b \) are adjacent to each other in the same interval of \( t \). In this case we obtain a quotient tree \( t/\{a, b\} \) by first identifying \( a \) with \( b \) and then \textit{schuffling} the intervals \( J(a) \) and \( J(b) \). Let us called such a quotient an \textit{inner contraction}.

Let us say that a vertex \( \alpha \in t \) is \textit{bounding} if it is a leaf and an extremal point of its interval. In this case we shall say that a vertex \( \alpha' \in \partial \Theta \) adjacent to \( \alpha \) in \( \tilde{t} \) is a \textit{compagnon} of \( \alpha \). We can then form a quotient disk \( \tilde{t}/\{a, a'\} \). Its interior is the tree \( s = t \setminus \{a\} \) obtained by deleting \( a \) since \( \alpha \) is now identified to a boundary point. Let us call such a quotient \textit{boundary contraction}. Notice that the map \( \tilde{t} \rightarrow \tilde{s} \) depends on the choice of \( \alpha' \).

**Proposition 4.** Let \( t \) be a finite planar tree. Then the faces of codimension 1 of \([t]\) are of the following two kinds:

(i) the faces \([s] \subset [t]\) where \( \tilde{t} \rightarrow \tilde{s} \) is an inner contraction \( t/\{a, b\} \);

(ii) the faces \([s] \subset [t]\) where \( t \rightarrow \tilde{s} \) is a boundary contraction \( t \setminus \{a, a'\} \).

We call a face of type (i) \textit{inner} and of type (ii) \textit{outer}. A codimension 1 face \([s] \subset [t]\) is inner iff the dual map \( \tilde{s} \rightarrow \tilde{t} \) is open. The \textit{boundary} \( \partial[t] \) of the cell \([t]\) is the union of its faces of codimension 1. If \([s]\) is a face of codimension 1 the \textit{horn} \( \Lambda^s[t] \) is the union of the faces of codimension 1 other than \([s]\). If \( X \) is a cellular set a map \( \xi : \Lambda^s[t] \rightarrow X \) is a \textit{horn} in \( X \). A \textit{filler} for \( \xi \) is an extension \( \xi' : [t] \rightarrow X \).

We say that a horn \( \Lambda^s[t] \subset [t] \) is \textit{inner} if \([s]\) is an inner face of \([t]\).

**Definition 2.** A \( \Theta \)-category (resp. a \textit{cellular} Kan complex) is a cellular set \( X \) in which every inner horn (resp. every horn) \( \xi : \Lambda^s[t] \rightarrow X \) has a filler.

**References**


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