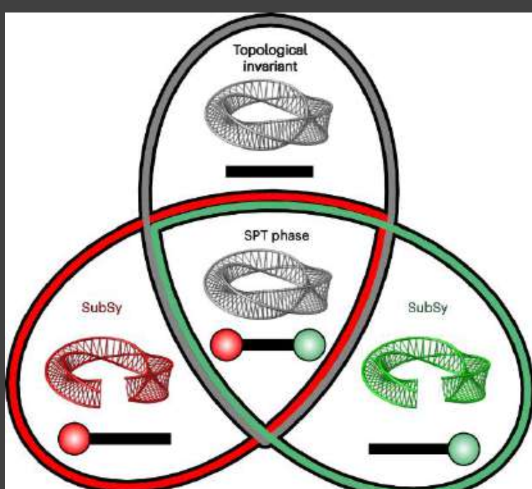
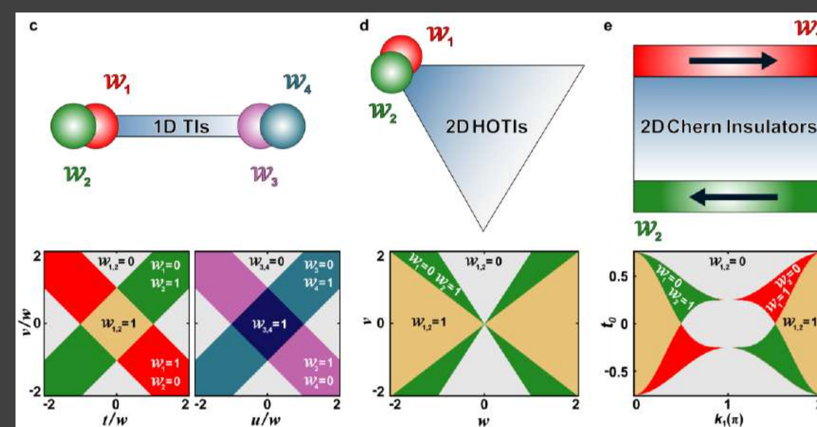
Topology Beyond Conventional Band Theory in Photonic Systems

Dario Jukić

University of Zagreb, Croatia

NYU Abu Dhabi, April 2026

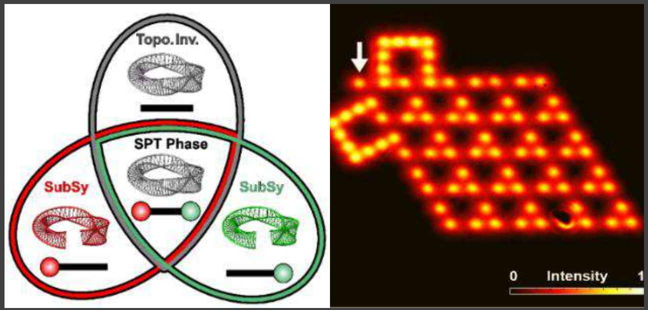




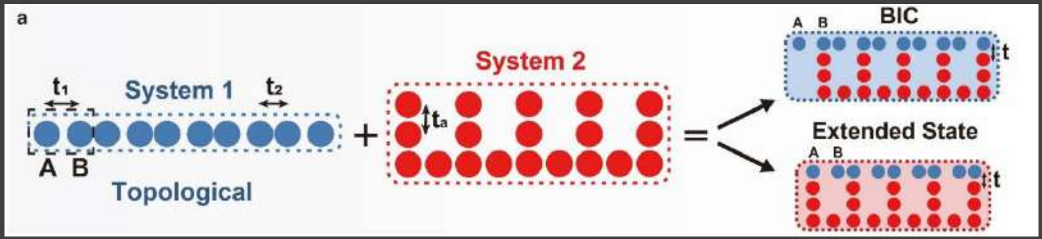
Outline

- Introduction

- **Sub-symmetry-protected topological states,**
  Z. Wang et al. ,
  Nature Phys. 2023

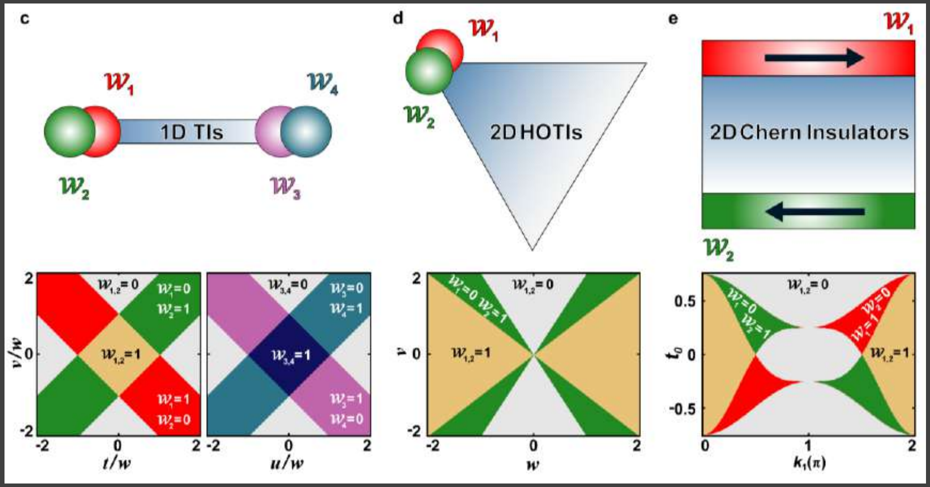


- **Construction of topological bound states in the continuum via sub-symmetry,**
  X. Wang et al.
  ACS Photonics 2024

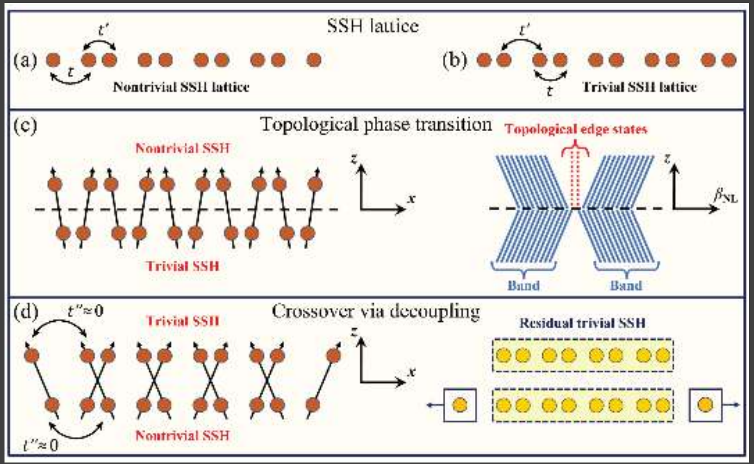


Outline

- **Hidden multi-topological phases mediated by constrained inter-cell coupling,**
  Z. Wang et al. ,
  eLight 2026



- **Nonlinear Topological Photonics: LSA 2020, LSA 2021, PRL 2021, eLight 2023 …**



Collaboration Zagreb-Nankai

Est. 2019.

Hrvoje Buljan

Zhigang Chen

Outline

If time permits:

- **Quantization of charge via classical electrodynamics,**
  Golik, Jukić & Buljan,
  Laser and Photonics Reviews 2024
  https://doi.org/10.1002/lpor.202400217

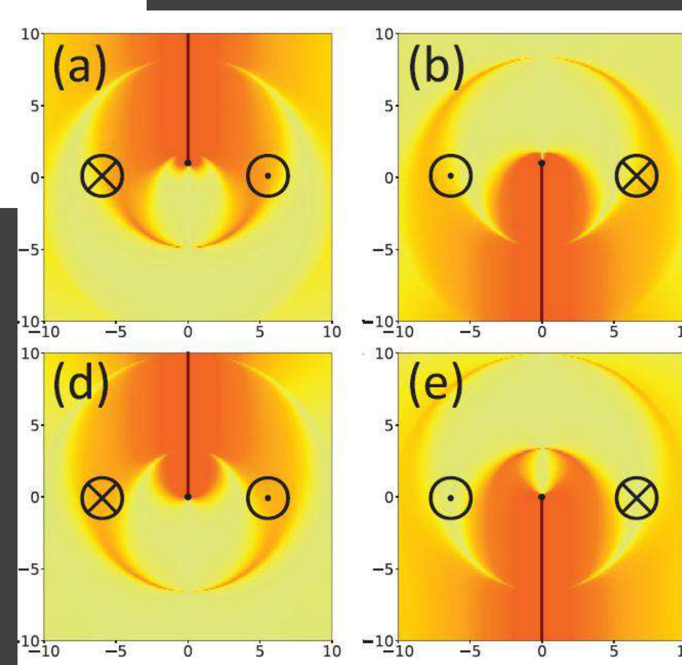
RESEARCH ARTICLE

LASER & PHOTONICS REVIEWS

www.lpr-journal.org

Theory of Classical Electrodynamics with Topologically Quantized Singularities as Electric Charges

*Bruno Golik, Dario Jukić, and Hrvoje Buljan**



Topological photonics

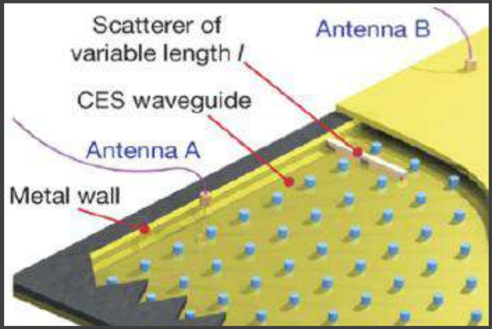
Haldane & Raghu (theory):

- *Analogs of quantum-Hall-effect edge states in photonic crystals* Phys. Rev. A 78, 033834 (2008);
  Phys. Rev. Lett. 100, 013904 (2008).

$$\sigma_{xy} = -\frac{e^2}{2\pi\hbar} C$$

- IQHE concepts ⟶ OPTICS
- Chern number, Berry curvature, …

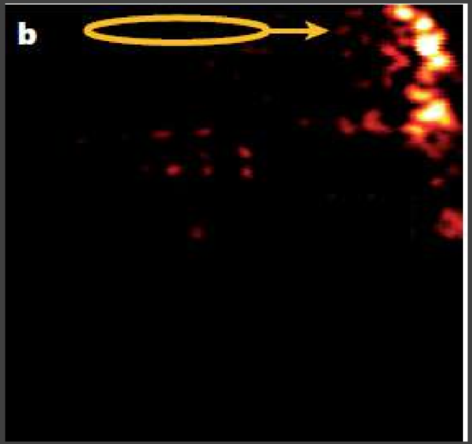
Topological edge states -
- microwaves



Wang et al.
Nature 461, 772 (2009)
Soljačić group, MIT

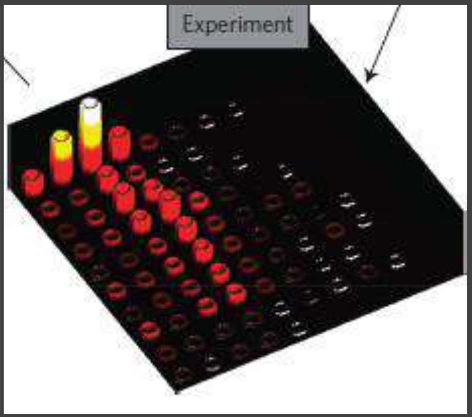
Topological photonics

Floquet topological insulators



Rechtsman *et al.*,
Nature 496, 196 (2013).

Topological edge states



Hafezi et al.
Nat. Phys. 7, 907 (2011)

Reviews:

- T. Ozawa, H.M. Price, A. Amo, N. Goldman, Mohammad Hafezi, Ling Lu, M.C. Rechtsman, D. Schuster, J. Simon, O. Zilberberg, and I. Carusotto, Topological photonics. Rev. Mod. Phys. 91, 015006 (2019).
- L. Lu, J. D. Joannopoulos, M. Soljačić, Nat. Photonics 8, 821 (2014).

Nonlinear and/or NH topological photonics

- Symmetry protected topological phases + nonlinearity
- Nonlinearity breaks the protecting symmetry! What happens to topology?

ARTICLE Open Access

Nontrivial coupling of light into a defect: the interplay of nonlinearity and topology

Shiqi Xia[1], Dario Jukić[2], Nan Wang[1], Daria Smirnova[3], Lev Smirnov[4], Liqin Tang[1,5], Daohong Song[1,5], Alexander Szameit[6], Daniel Leykam[7,8], Jingjun Xu[1,5], Zhigang Chen[1,5,9] and Hrvoje Buljan[1,10]

RESEARCH

TOPOLOGICAL OPTICS

Nonlinear tuning of PT symmetry and non-Hermitian topological states

Shiqi Xia[1*], Dimitrios Kaltsas[2*], Daohong Song[1*], Ioannis Komis[2], Jingjun Xu[1], Alexander Szameit[3], Hrvoje Buljan[1,4†], Konstantinos G. Makris[2,5†], Zhigang Chen[1,6†]

Xia *et al.*, *Science* **372**, 72–76 (2021) 2 April 2021

- Interplay of nonlinearity, topology and non-Hermiticity

ARTICLE Open Access

Nonlinear control of photonic higher-order topological bound states in the continuum

Zhichan Hu[1], Domenico Bongiovanni[1,2], Dario Jukić[3], Ema Jajtić[4], Shiqi Xia[1], Daohong Song[1,5], Jingjun Xu[1,5], Roberto Morandotti[2,6], Hrvoje Buljan[1,4✉] and Zhigang Chen[1,5,7✉]

PHYSICAL REVIEW LETTERS **127**, 184101 (2021)

Dynamically Emerging Topological Phase Transitions in Nonlinear Interacting Soliton Lattices

Domenico Bongiovanni,[1,2] Dario Jukić,[3] Zhichan Hu,[1] Frane Lunić,[4] Yi Hu,[1] Daohong Song,[1] Roberto Morandotti, Zhigang Chen,[1,6,*] and Hrvoje Buljan[1,4,†]

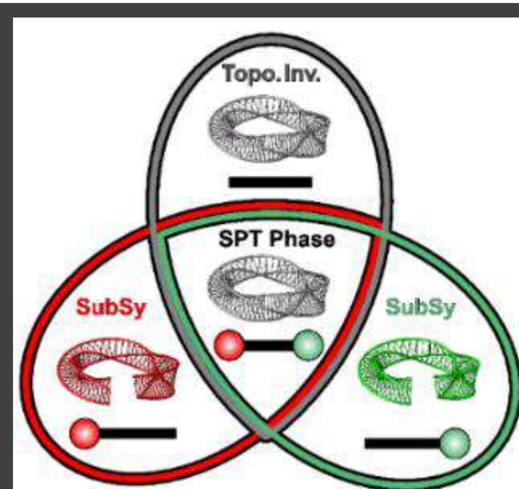
nature physics

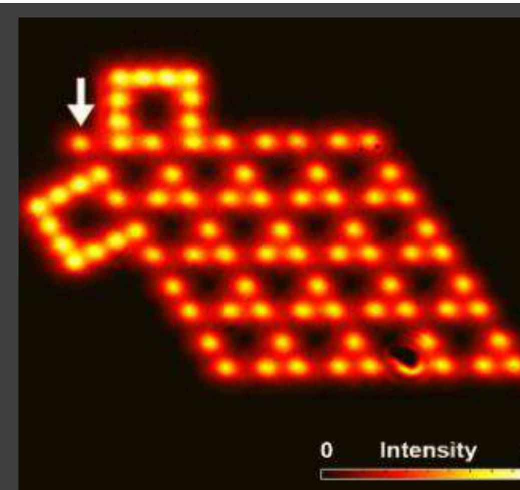
Article

https://doi.org/10.1038/s41567-023-02011-9

Sub-symmetry-protected topological states

Z. Wang et al.,
Nature Physics 19, 992 (2023)



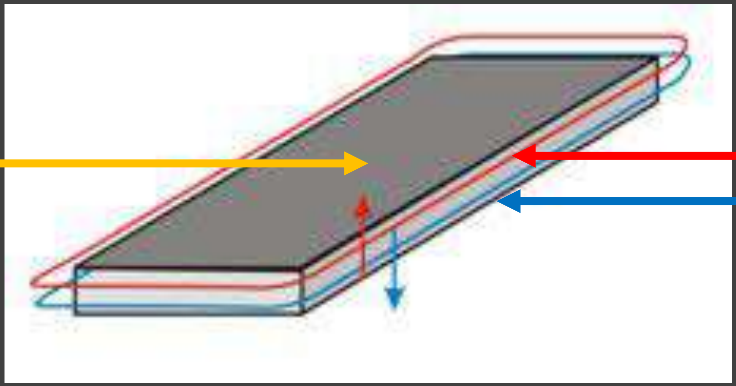


Symmetry protected topological (SPT) phases

- Topological insulators (TIs): protected by time-reversal symmetry (TRS)

  (* any perturbation that respects TRS leaves the boundary states intact *)

Bulk: insulating



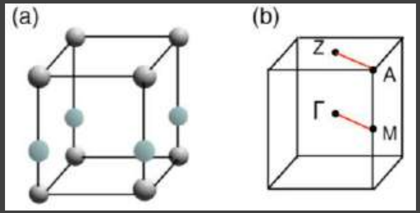
Boundary states (conduction)

- Kane, C. L. & Mele, E. J. PRL **95**, 146802 (2005).
- Xia, Y., Qian, D., Hsieh, D. et al. Nature Phys 5, 398 (2009)
- C-K. Chiu et al., Rev. Mod. Phys. 88, 035005 (2016) (* review *)

Platform: cond-mat

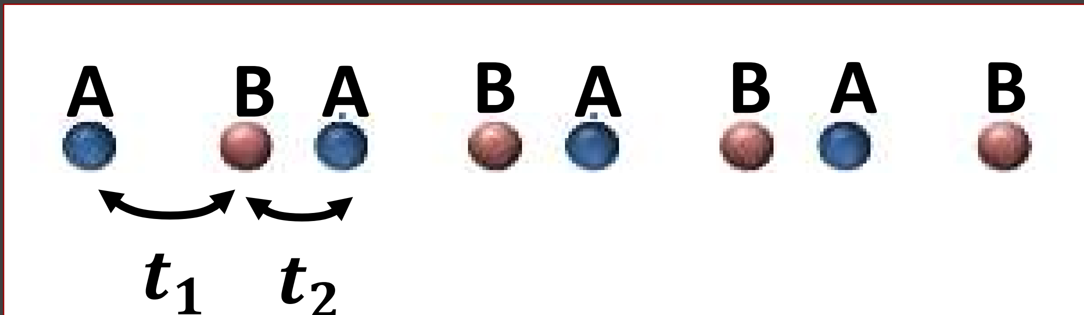
- Topological crystalline insulators (TIs):

  protected by crystalline symmetries



- Fu, L. Topological crystalline insulators. Phys. Rev. Lett. 106, 106802 (2011).

Su-Schrieffer–Heeger (SSH) lattice: SPT



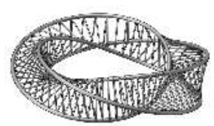
$$H_{SSH} = \sum_n \left(t_1 b_n^+ a_n + t_2 a_{n+1}^+ b_n + H.c. \right)$$

Topologically trivial phase $t_1 > t_2, \;\; W = 0$

Topologically nontrivial phase $t_1 < t_2, \;\; W = 1$

$$W = \frac{1}{2\pi} \int_0^{2\pi/a} dk \frac{d\theta(k)}{dk}$$

Topological invariant

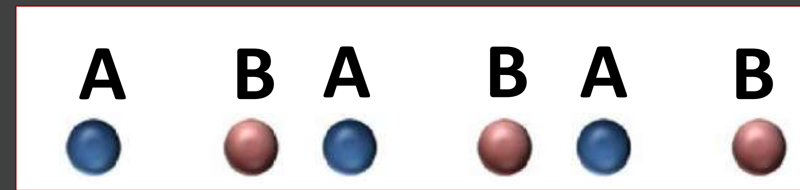


Topologically protected edge states

SPT phase

SSH lattice: Chiral symmetry

$$H_{SSH} = \sum_n \left(t_1 b_n^+ a_n + t_2 a_{n+1}^+ b_n + H.c. \right)$$



Chiral symmetry operator: $\Sigma_z = P_A - P_B$

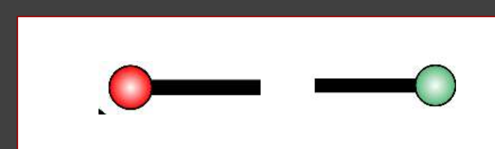
Chiral symmetry: $\Sigma_z H_{SSH} \Sigma_z^{-1} = -H_{SSH}$

Topological protection via symmetry: $H_{SSH} \rightarrow H_{SSH} + H'$

Apply any perturbation H’ respecting chiral symmetry: $\Sigma_z H' \Sigma_z^{-1} = -H'$

- Topological invariant – preserved

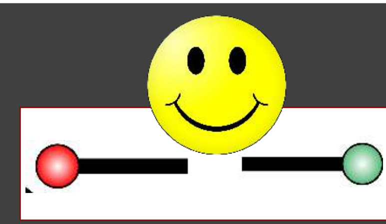
- Topological edge states - preserved



Question (SPT phases):

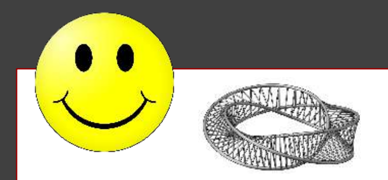
Are there perturbations that

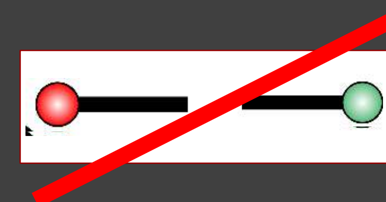
- destroy the topological invariant
  & preserve boundary states?



Poli, C. *et al. 2D Mater.* **4**, 025008 (2017).

- destroy the boundary states
  & preserve the topological invariant?

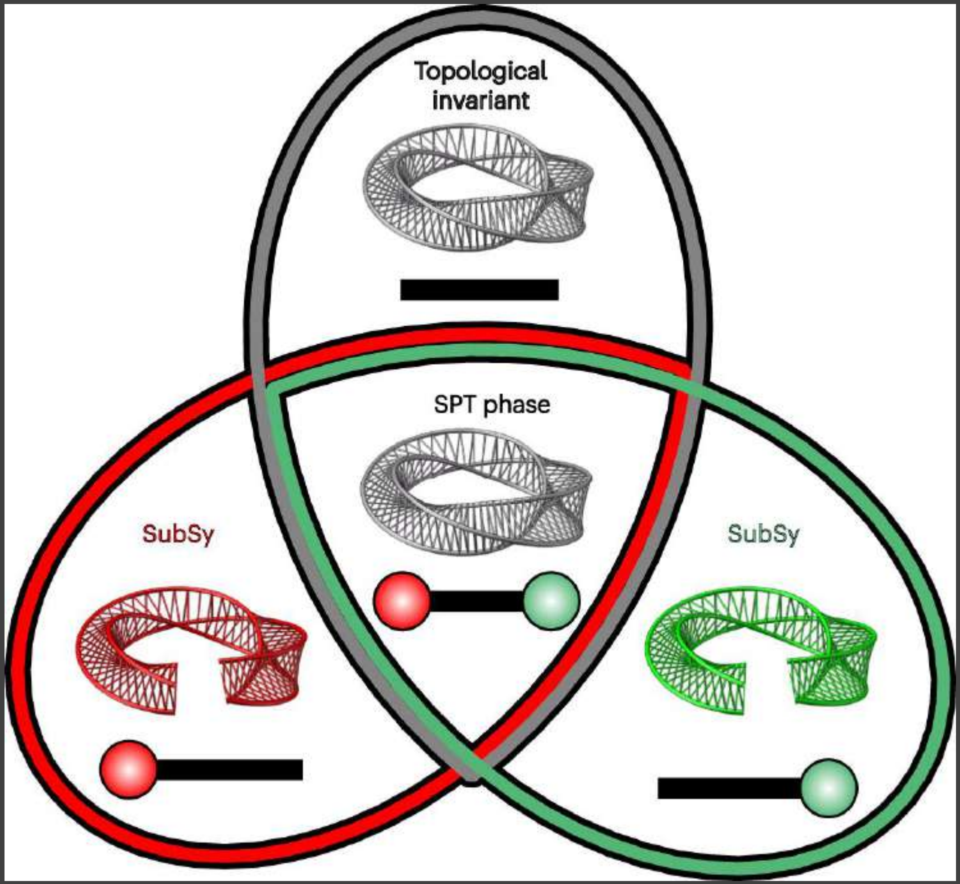




Longhi, S. *Opt Lett* **43**, 4639-4642 (2018).
Jiao, Z. Q. *et al. Phys. Rev. Lett.* **127**, 147401 (2021).

Can we classify such perturbations by using symmetries?

Answer: Yes, by using the SubSy concept

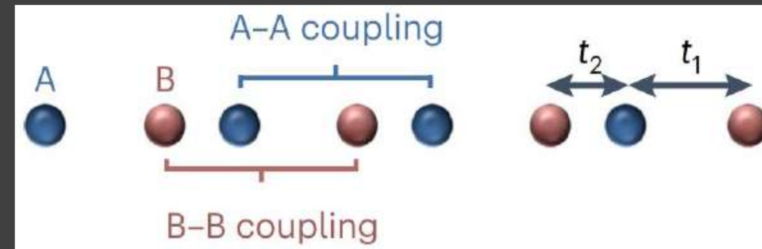


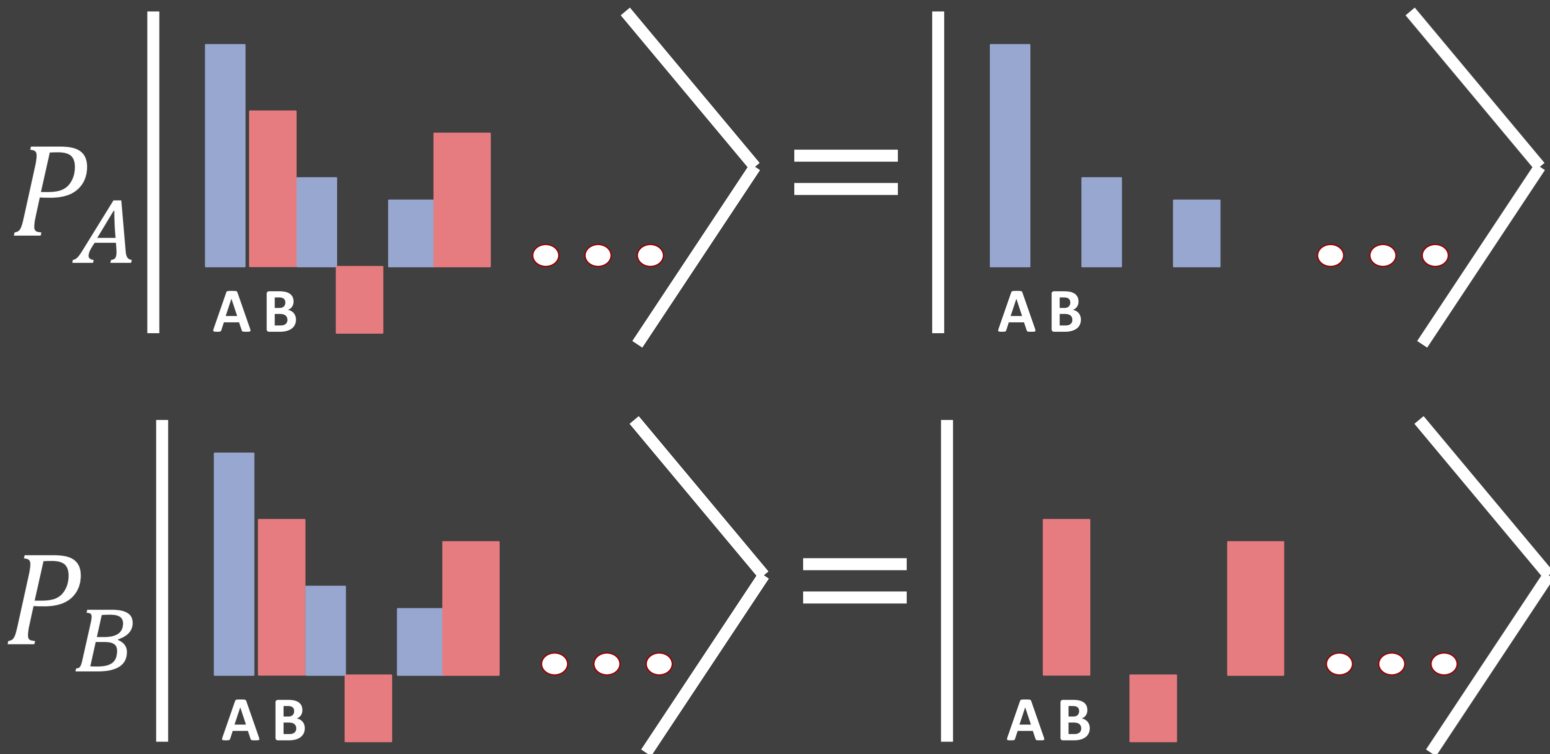
Theory and experiments (photonic platform)
- SSH lattice
- Breathing Kagome lattice (BKL)

Z. Wang et al., Nature Phys. 2023.

Sub-Symetry (SubSy)

Projection operators: $P_A,\ P_B$

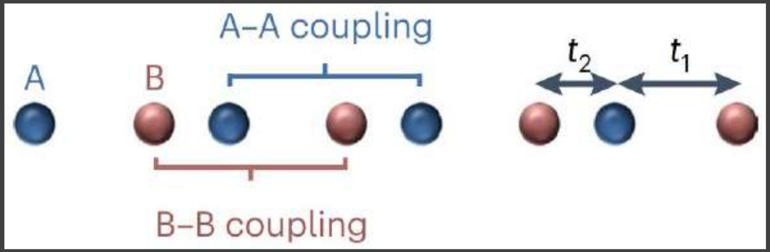




Sub-Symetry (SubSy)

Chiral symmetry operator:

$$\Sigma_z = P_A - P_B$$



Chiral symmetry equation: $\Sigma_z H_{SSH} \Sigma_z^{-1} = -H_{SSH}$ (*)

SubSy: Symmetry equation does not hold on the whole Hilbert space, but only on its subspace!

A-SubSy: $\Sigma_z H_{SSH} \Sigma_z^{-1} P_A = -H_{SSH} P_A$ (*) holds on A sublattice

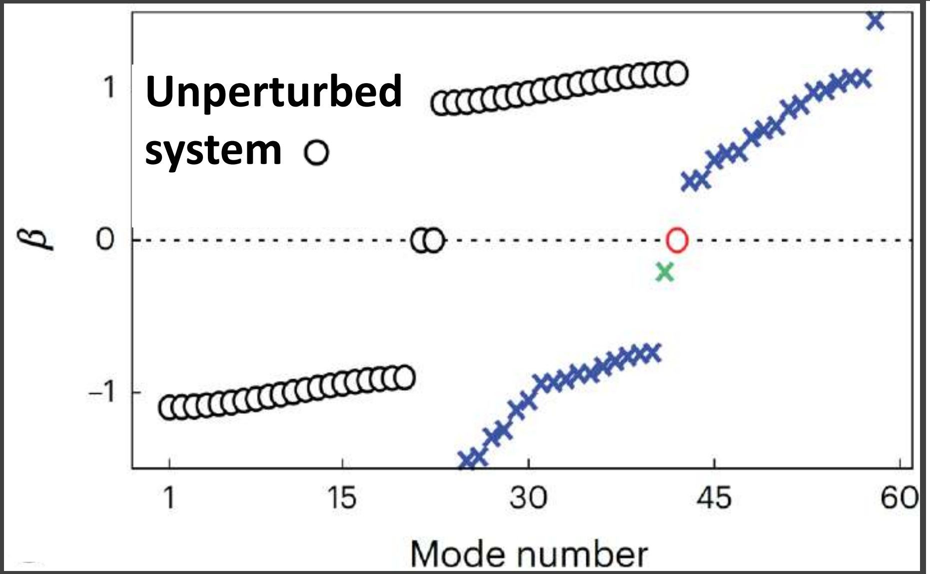
Protects the left edge mode (residing on A sublattice)

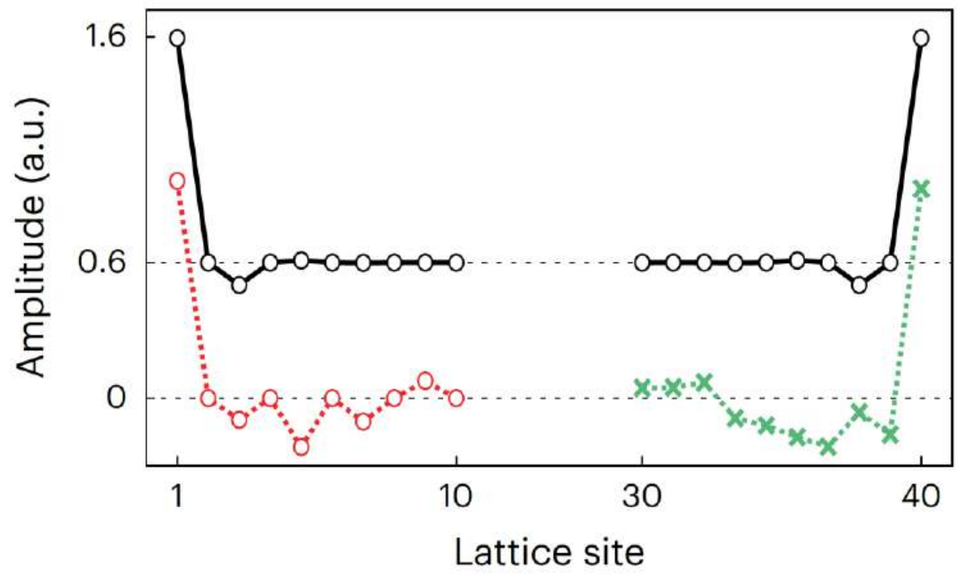
B-SubSy: $\Sigma_z H_{SSH} \Sigma_z^{-1} P_B = -H_{SSH} P_B$ (*) holds on B sublattice

Protects the right edge mode (residing on B sublattice)

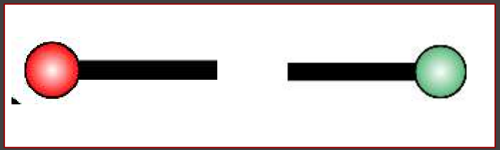
SSH: SubSy-protected edge states

- One A-SubSy-respecting perturbation *H'*: $\Sigma_z H' \Sigma_z^{-1} P_A = -H' P_A$





A edge



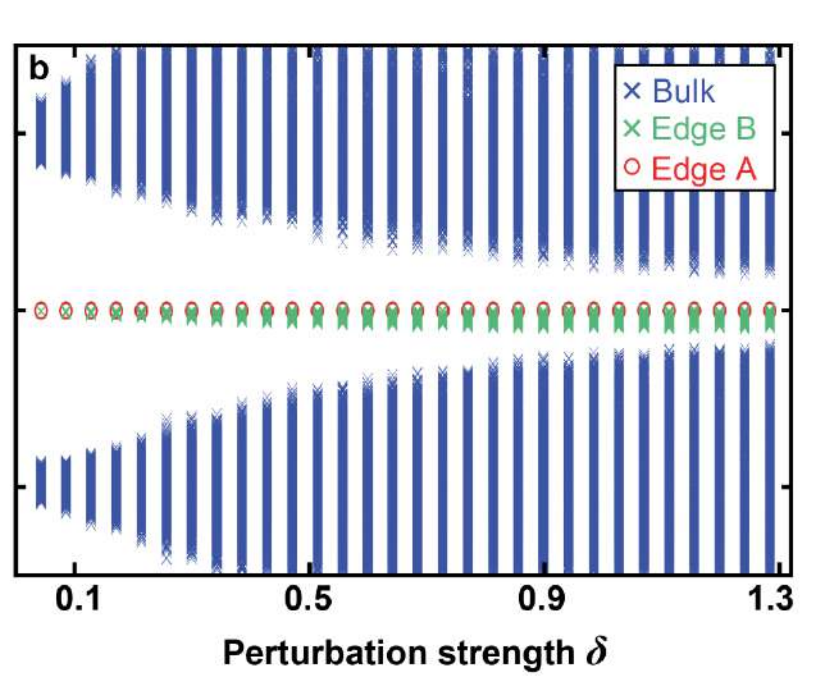
B edge

SSH: SubSy-protected edge states

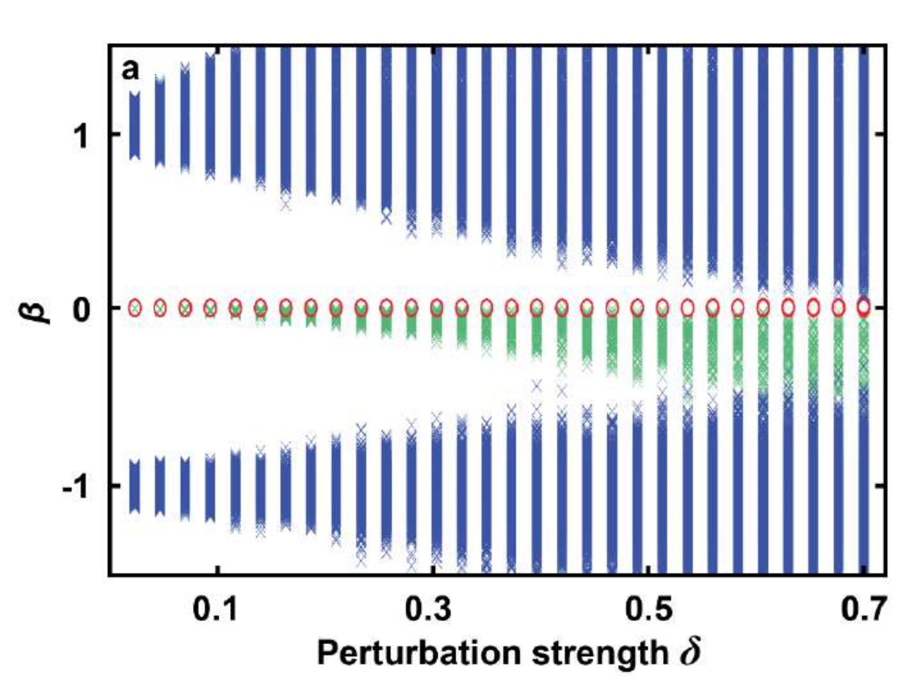
- Many A-SubSy-respecting perturbations *H'*: $\Sigma_z H' \Sigma_z^{-1} P_A = -H' P_A$

70 perturbations for every perturbation strength:

B-B coupling



A-B and B-B coupling

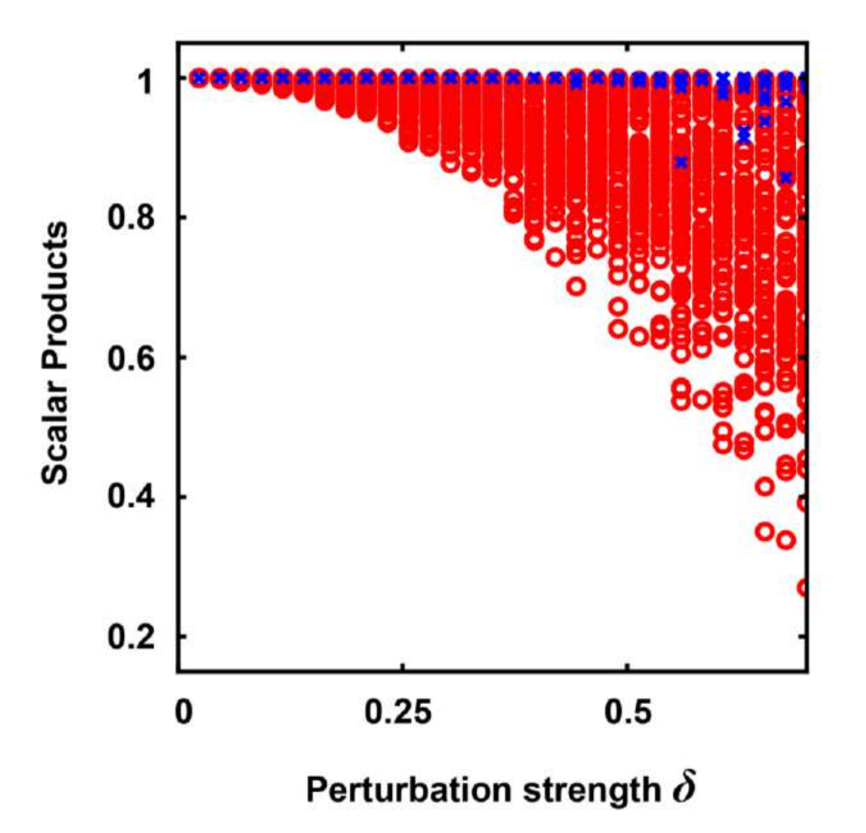


SSH: SubSy-protected edge states

- Many A-SubSy-respecting perturbations *H'*: $\Sigma_z H' \Sigma_z^{-1} P_A = -H' P_A$

(* 70 perturbations for every perturbation strength *)

A-B and B-B coupling

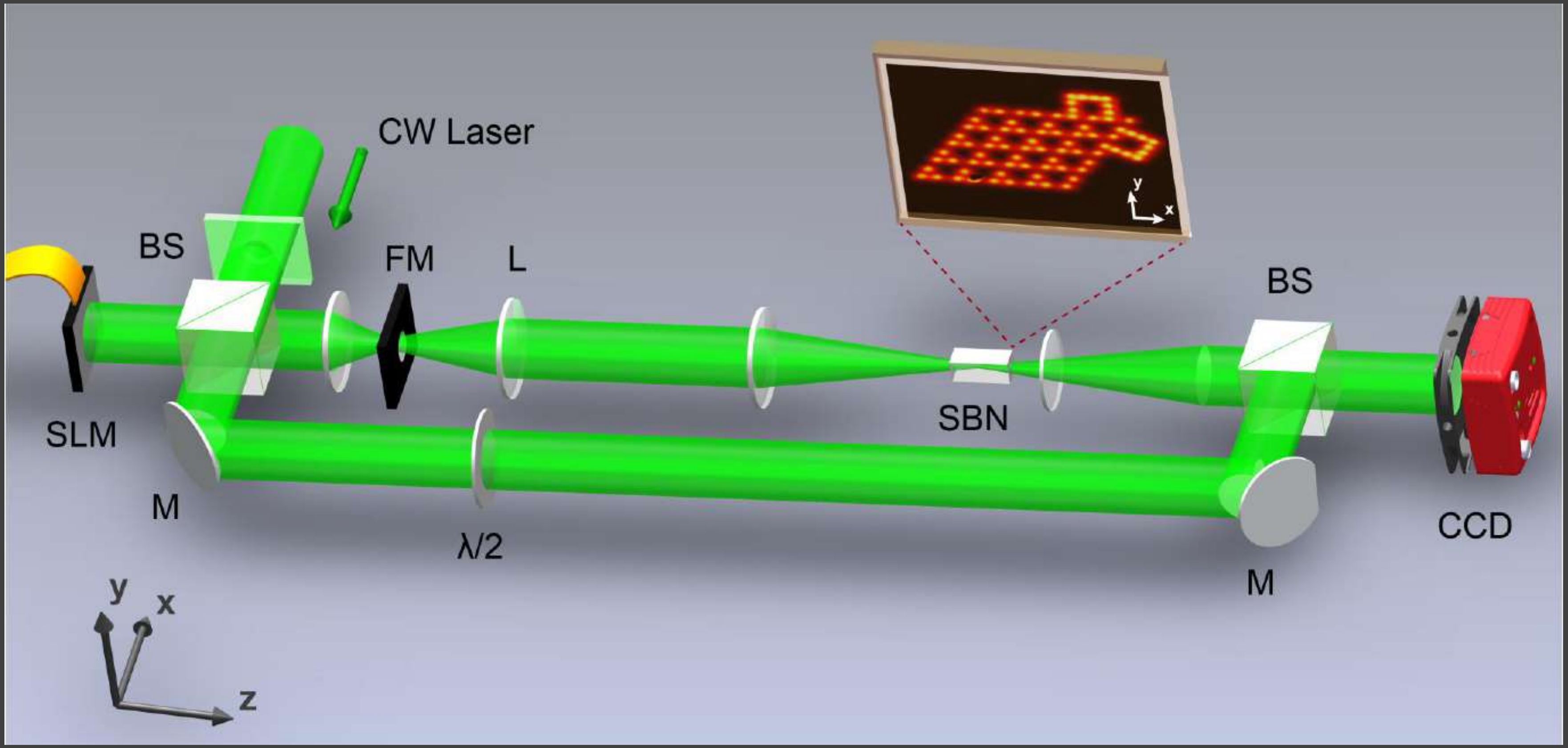


- Scalar product of the perturbed and unperturbed edge state $|\langle A_L' | A_L \rangle|^2$ (red circles)

- Scalar product $|\langle A_L' | P_A A_L' \rangle|^2$ (blue crosses)

Experimental implementation

Optically induced lattice in a photorefractive crystal



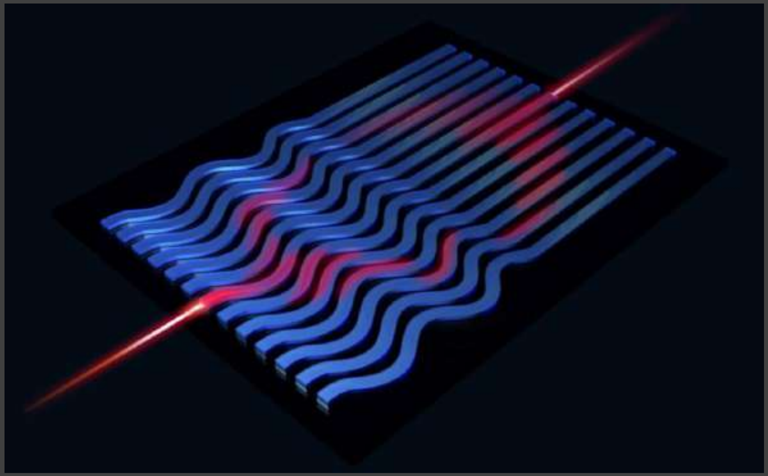
Propagation of light in photonic lattices (waveguide arrays)

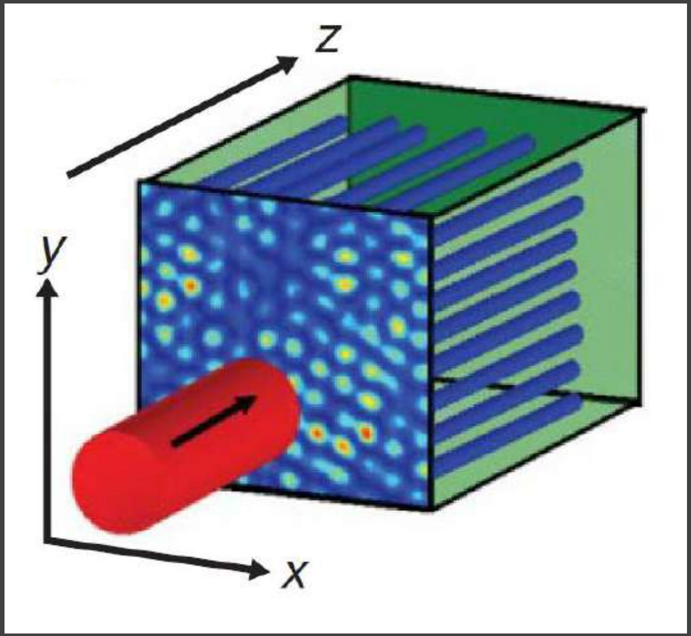
Index of refraction is a function of x and y (independent of z, or slowly varying with z)

Propagation of light along the z-axis

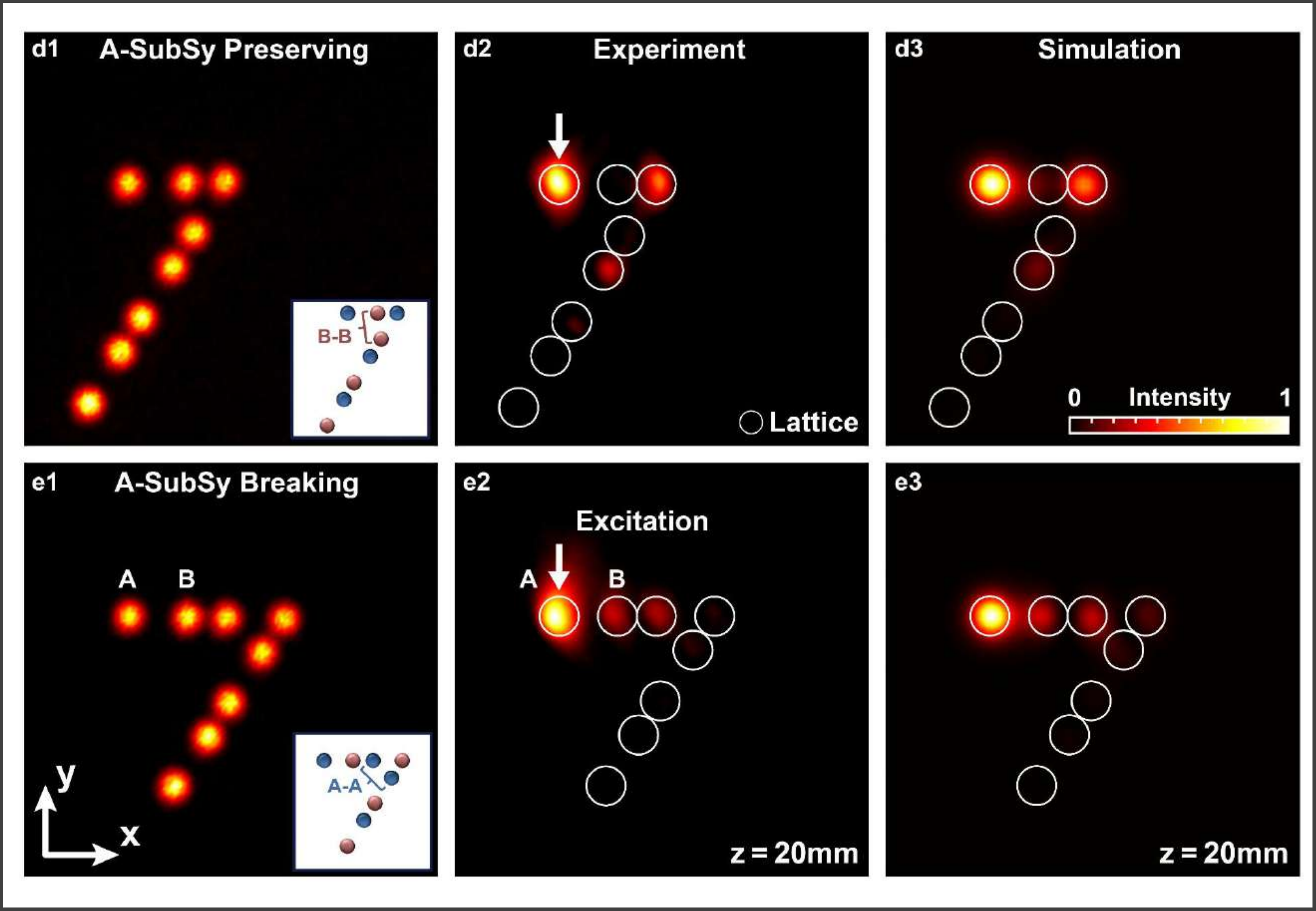
$$i\frac{\partial\psi}{\partial z} = -\frac{1}{2k_0}\nabla^2\psi - \frac{k_0\delta n_L(\mathbf{x})}{n_0}\psi$$

ψ is complex amplitude describing the *electric* field





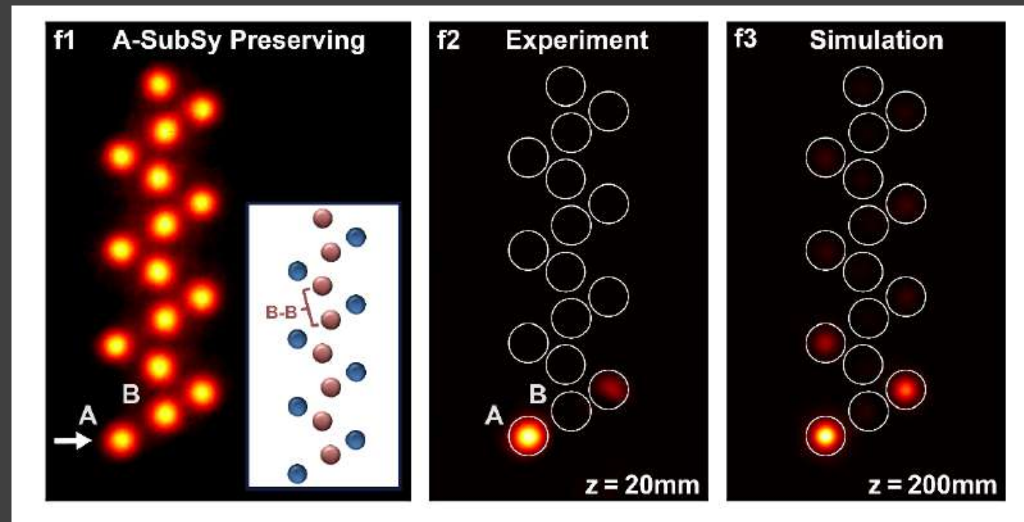
Experiments: Twisted SSH photonic lattice



Experiments: Zig-zag SSH photonic lattice

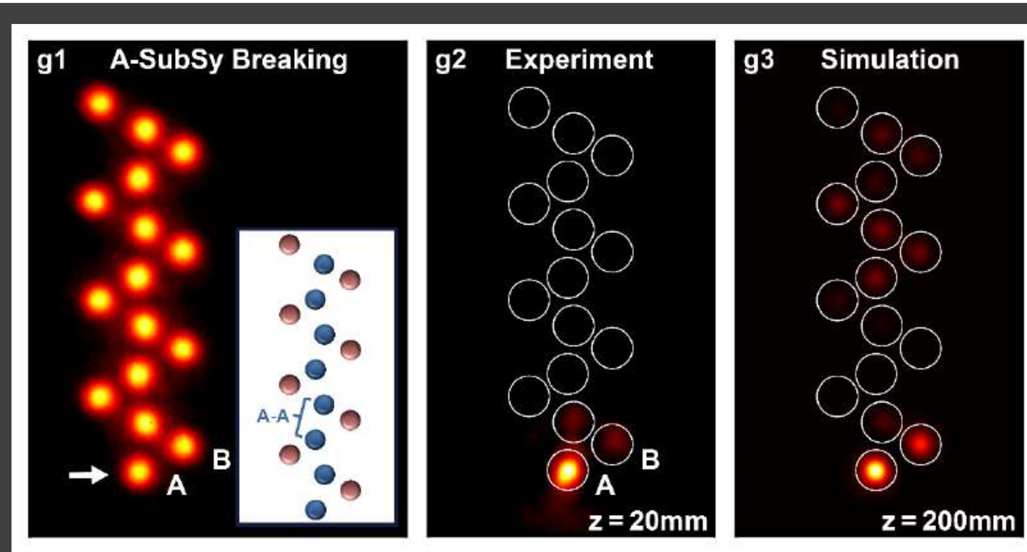
A-SubSy preserving zig-zag lattice

Excitation:
lowest lattice site

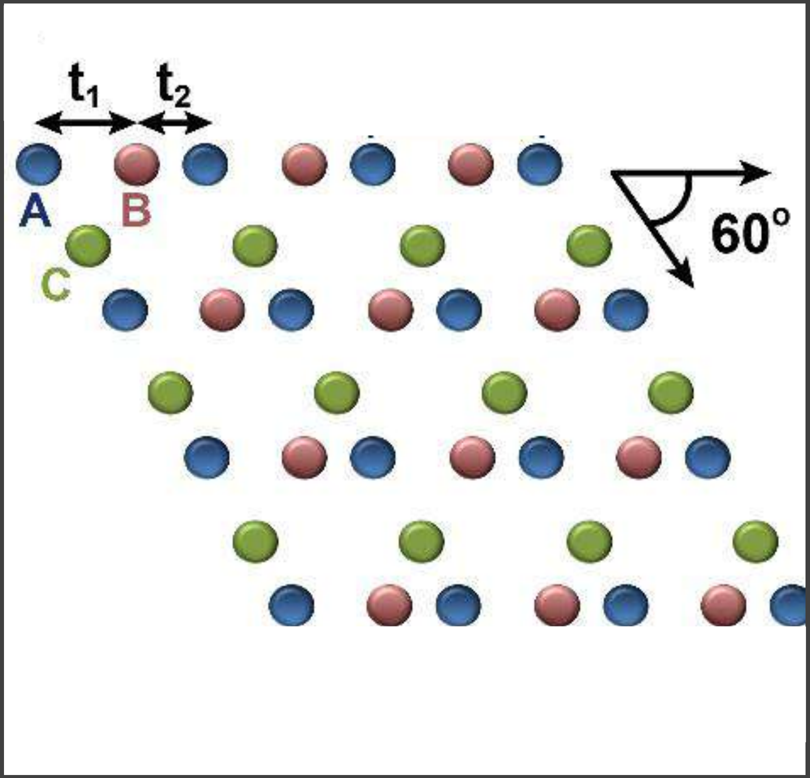


A-SubSy breaking zig-zag lattice

Excitation:
lowest lattice site



Breathing Kagome lattice (BKL) 2D



Parameters:

- intracell hopping amplitude: t_1
- intercell hopping amplitude: t_2

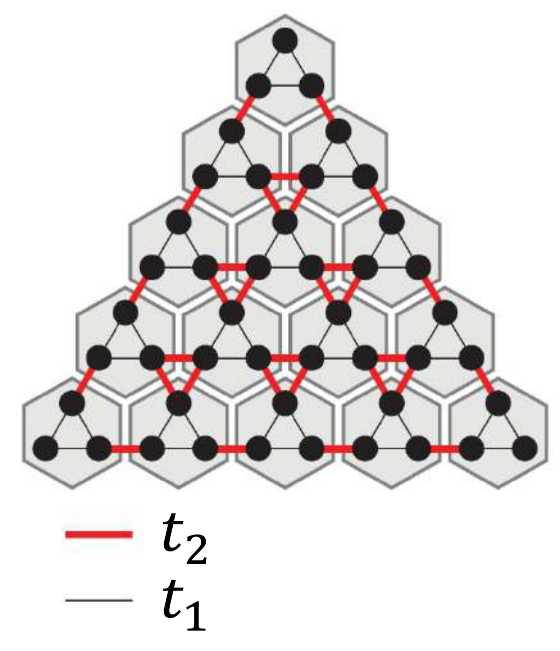
A, B, and C sublattice

BKL Hamiltonian:

$$H_K = \sum_{m,n}\left(t_1 a^{\dagger}_{m,n} b_{m,n} + t_1 a^{\dagger}_{m,n} c_{m,n} + t_1 b^{\dagger}_{m,n} c_{m,n} + H.c.\right) +$$

$$\sum_{m,n}\left(t_2 b^{\dagger}_{m,n} a_{m+1,n} + t_2 c^{\dagger}_{m,n} a_{m,n+1} + t_2 c^{\dagger}_{m,n} b_{m-1,n+1} + H.c.\right).$$

Breathing Kagome lattice (BKL)



HOTI:
Benalcazar, Bernevig, Hughes, *Science* **357**, 61-66 (2017).

BKL:
Ni et al. Nat. Mater. 18, 113 (2019).
Xue et al. Nat. Mater. 18, 108 (2019).
Peterson et al. Science 368, 1114 (2020).
Kirsch et al. Nat. Phys. 17, 995 2021).
etc.

Higher-order topological insulator (HOTI)

Topologically trivial phase $t_1 > t_2$

$$P_x = P_y = 0$$

Corner states: NO

Topologically nontrivial phase $t_1 < t_2$

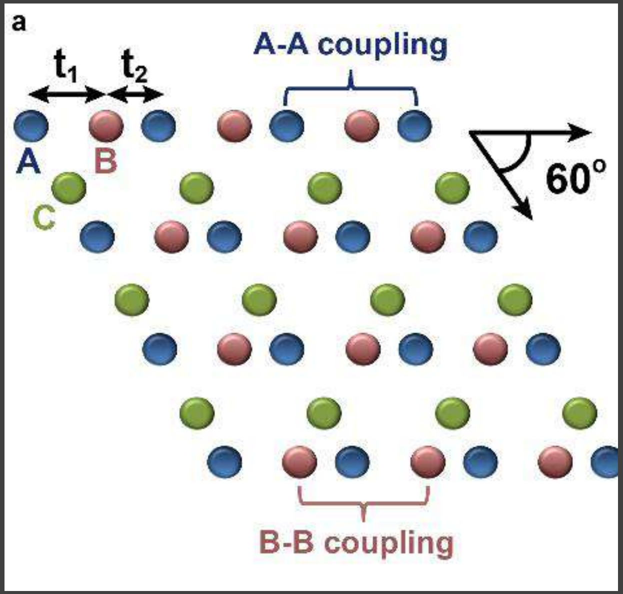
$$P_x = P_y = \frac{1}{3}$$

Corner states: YES

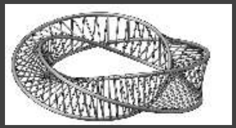
BKL: 2D

Corner states: 0D

BKL: Symmetries & SubSys



A, B, and C sublattice

- C_3 symmetry 
- generalized chiral symmetry:

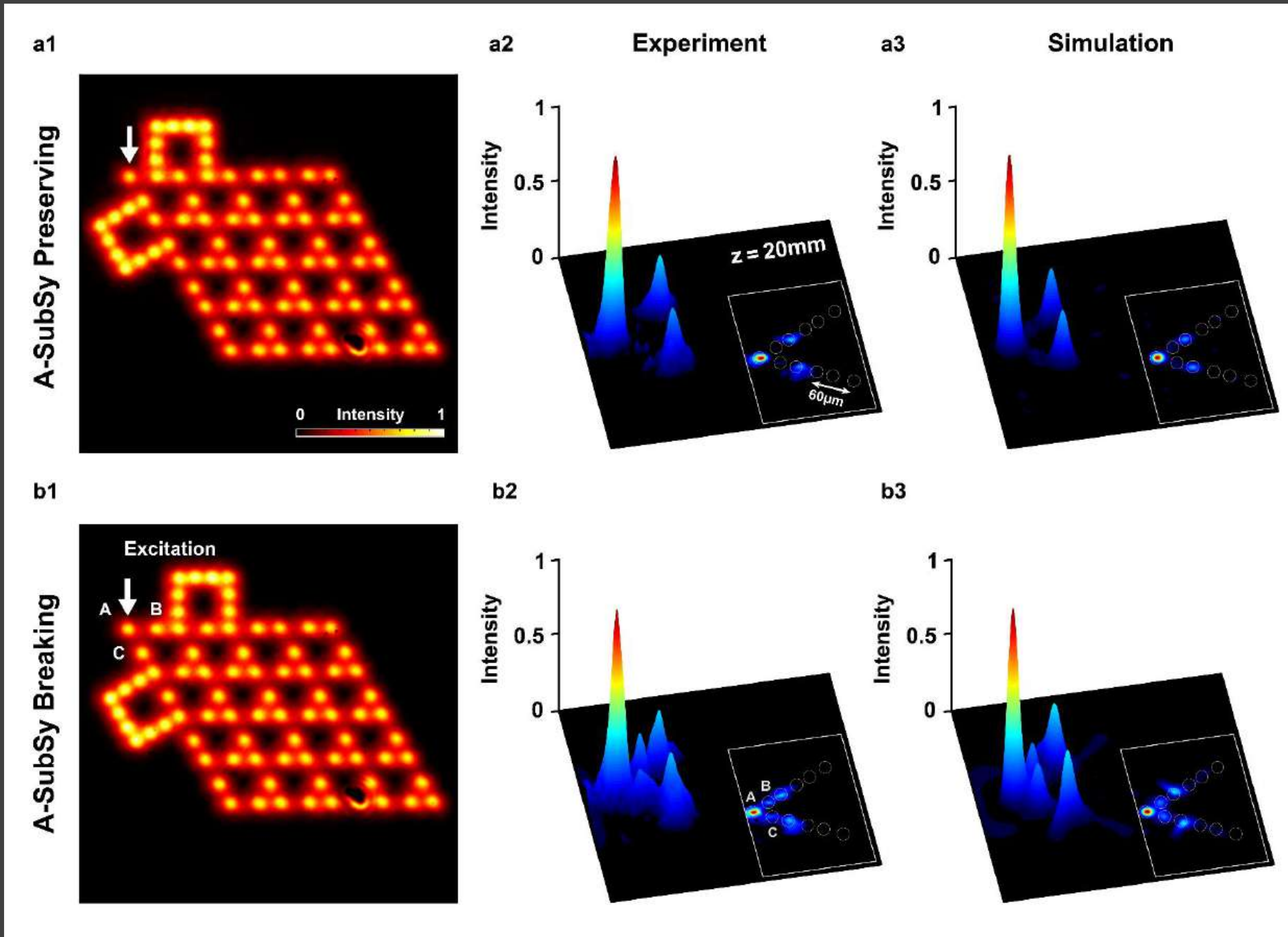
$$\Sigma_3 H_K \Sigma_3^{-1} + \Sigma_3^2 H_K \Sigma_3^{-2} = -H_K$$
$$\Sigma_3 = P_A + e^{i\frac{2\pi}{3}} P_B + e^{-i\frac{2\pi}{3}} P_C$$

Ni et al. Nat. Mater. 18, 113 (2019).

A, B, and C-SubSy

$$\Sigma_3 H_K \Sigma_3^{-1} P_i + \Sigma_3^2 H_K \Sigma_3^{-2} P_i = -H_K P_i$$
$$i \in \{A, B, C\}$$

Experiments: SubSy-protected edge states



ACS Photonics

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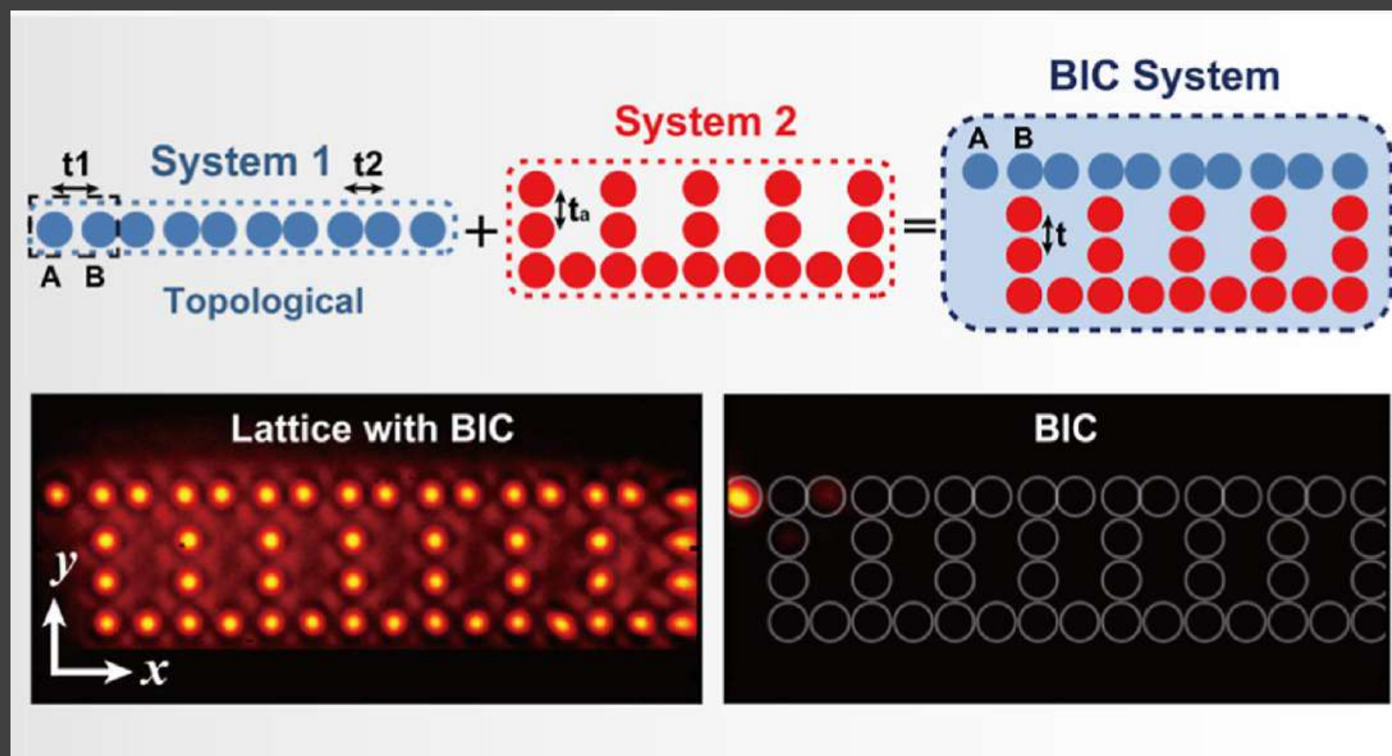
Article

Construction of Topological Bound States in the Continuum Via Subsymmetry

X. Wang et al., ACS Photonics 11, 3213 (2024)



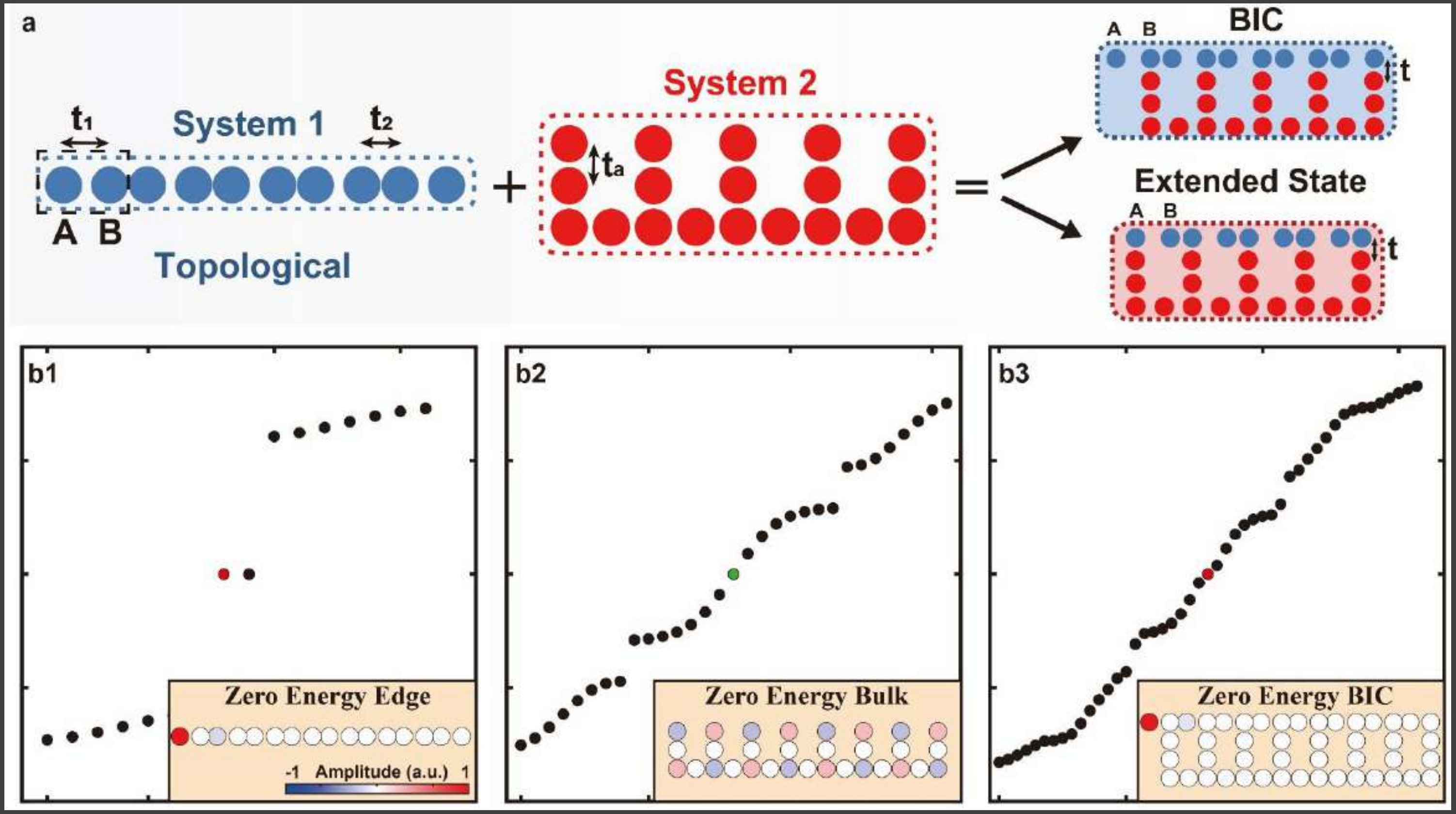
SYSTEM 1: In-gap topological states ψ_0

$$H = \begin{pmatrix} H_1 & K \\ K^\dagger & H_2 \end{pmatrix}$$

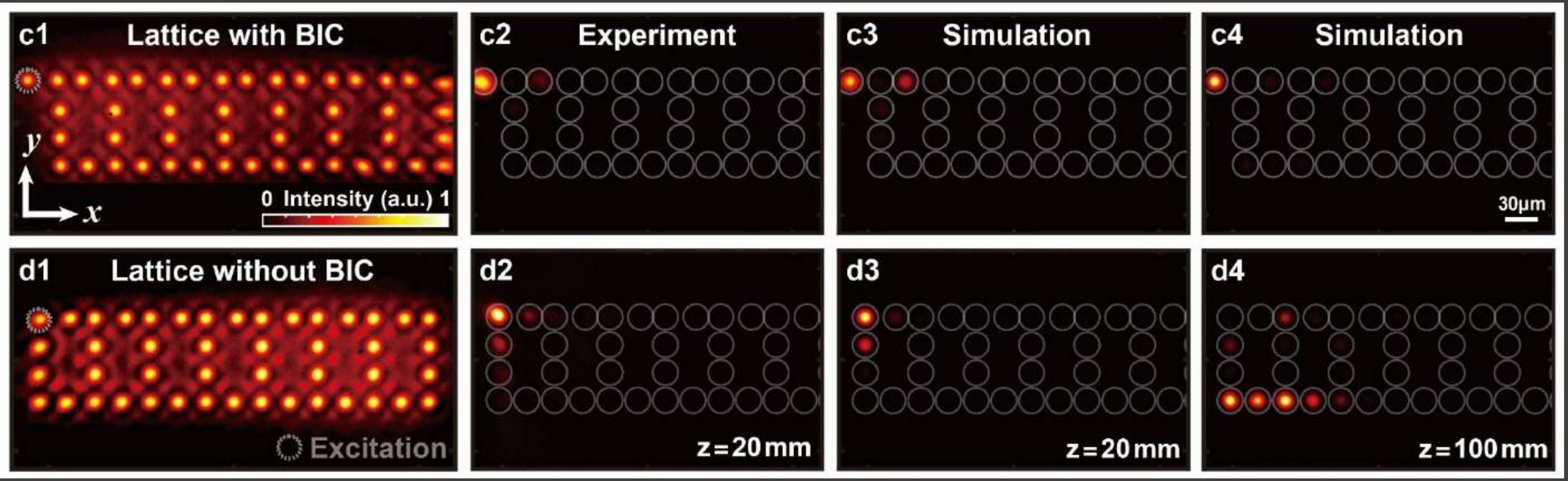
SYSTEM 2: Continuous spectrum

Conditions for the existence of BICs:

- Condition 1: $K^\dagger \psi_0 = 0$
- Condition 2: $(H_1 - K^\dagger H_2^{-1} K)\psi'(\mathbf{k}) = 0$

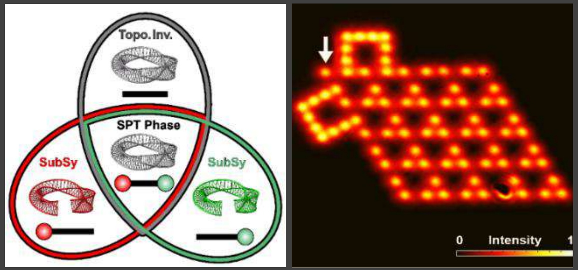


Experiments: topological BICs via sub-symmetry

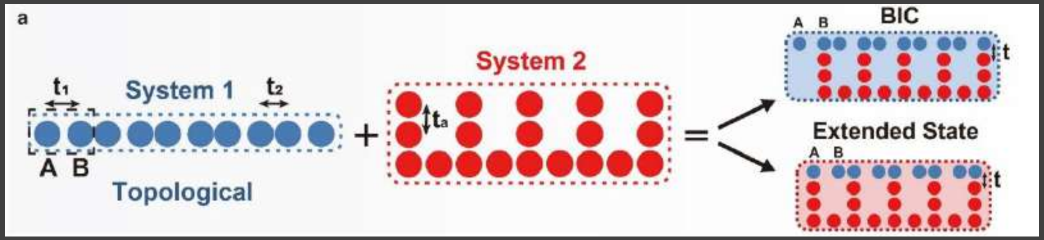


Summary (so far)

- **Sub-symmetry-protected topological states,**
  Z. Wang et al.,
  Nature Phys. 2023



- **Construction of topological bound states in the continuum via sub-symmetry,**
  X. Wang et al.,
  ACS Photonics 2024



Hidden Multi-Topological Phases Mediated by Constrained Inter-Cell Coupling

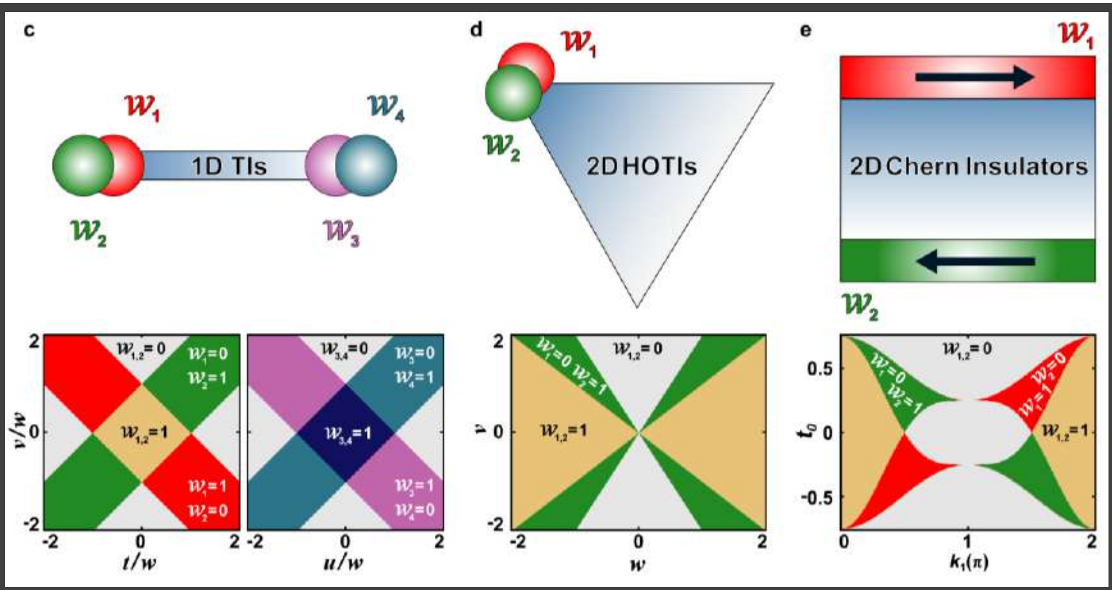
Wang *et al. eLight* (2026) 6:2
https://doi.org/10.1186/s43593-025-00118-5
elight.springeropen.com

RESEARCH ARTICLE **Open Access**

Check for updates

Hidden multi-topological phases mediated by constrained inter-cell coupling

Z. Wang et al., eLight 2026



Hidden Multi-Topological Phases Mediated by Constrained Inter-Cell Coupling

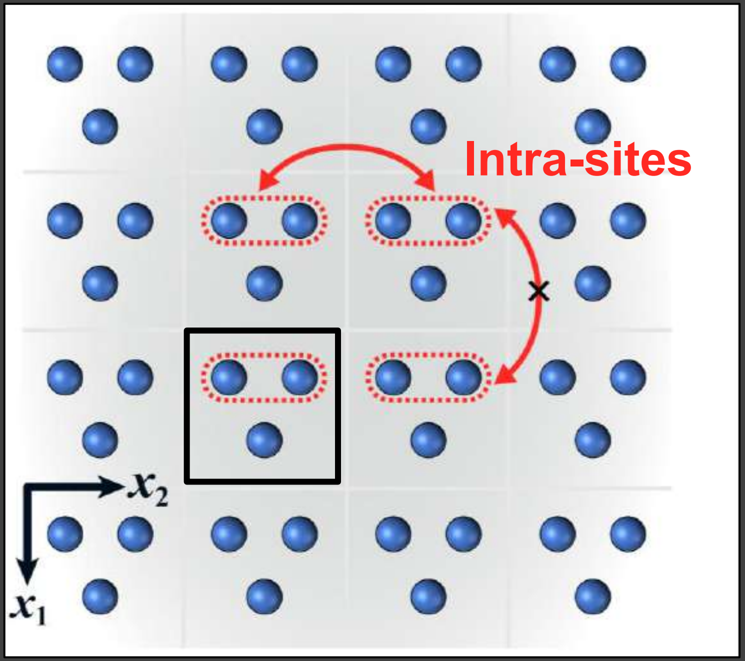
Lattice restrictions:

- $N + J$ sites in a unit cell
- $N + J$ sublattices
- N intra sites – lack coupling to neighboring unit cells along some dimensions
- J sites – no restrictions

$$H(\boldsymbol{k}) = \begin{pmatrix} h_{\text{intra}}(\boldsymbol{k}_{\text{intra}}) & f(\boldsymbol{k}) \\ f^{\dagger}(\boldsymbol{k}) & \delta(\boldsymbol{k}) \end{pmatrix}$$

- $h_{\text{intra}}(\boldsymbol{k}_{\text{intra}})$ is a $N \times N$ matrix
- $\delta(\boldsymbol{k})$ is a $J \times J$ matrix
- $H(\boldsymbol{k})$ can be trivial in conventional band topology

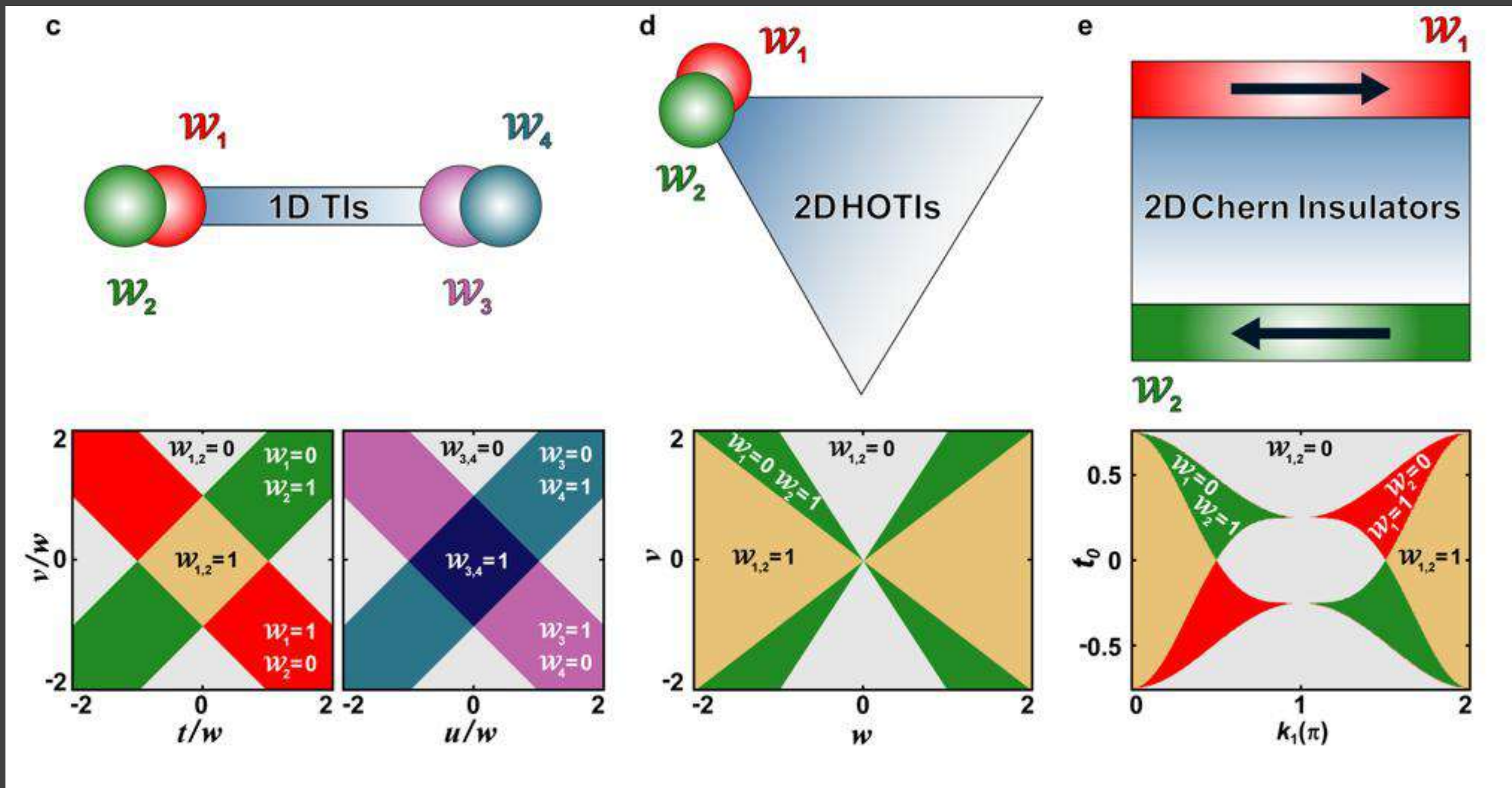
Example:



- $N = 2$, lack coupling along x_1
- J=1
- $N + J$=3 sublattices

Z. Wang et al., eLight 2026

Multi-Topological Phases Mediated by Constrained Inter-Cell Coupling



Z. Wang et al., eLight 2026

Original Hamiltonian w/ restricted coupling

WELL-DEFINED PROCEDURE →

Auxiliary Hamiltonians

H1, H2, H3, …

→

Topological invariants

W1…, W2…, W3…, …

→

Boundary states

Set1, Set2, Set3, …

Auxiliary Hamiltonians, chiral structure, topology

Original Hamiltonian:

$$H(\boldsymbol{k}) = \begin{pmatrix} h_{\text{intra}}(\boldsymbol{k}_{\text{intra}}) & f(\boldsymbol{k}) \\ f^{\dagger}(\boldsymbol{k}) & \delta(\boldsymbol{k}) \end{pmatrix}$$

WELL-DEFINED PROCEDURE

Auxiliary Hamiltonians ($i = 1..N$):

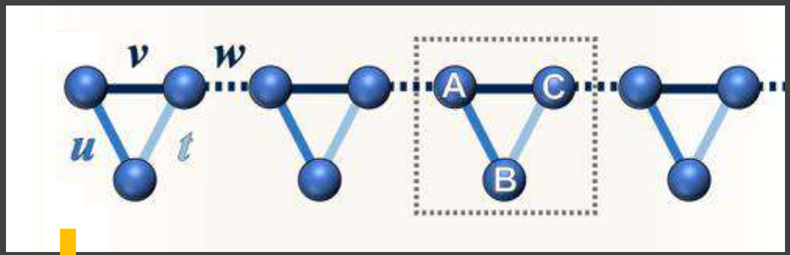
$$H_c^i(\boldsymbol{k}) = \begin{pmatrix} 0 & F_i(\boldsymbol{k}) \\ F_i^{\dagger}(\boldsymbol{k}) & 0_{J\times J} \end{pmatrix}$$

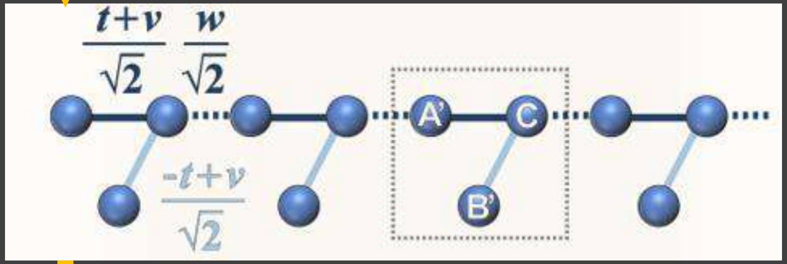
Chiral symmetry: $\Sigma_z H_c^i(\boldsymbol{k}) \Sigma_z^{-1} = -H_c^i(\boldsymbol{k})$

Topological invariant:
winding number

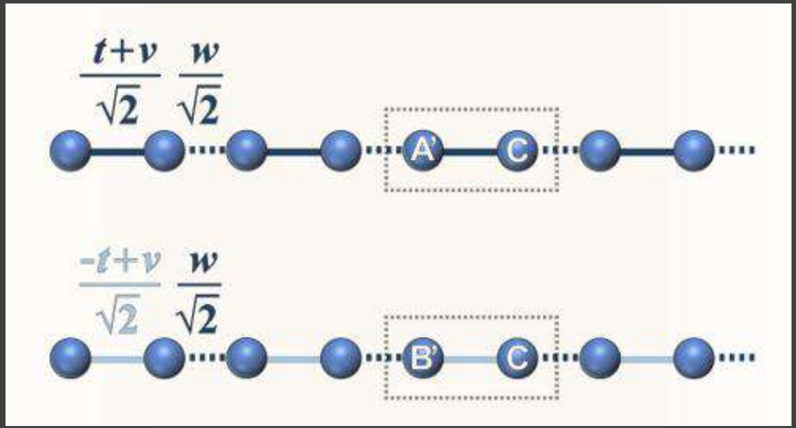
Topological boundary states:
Exactly coincide with the boundary states of the original system

Example 1:



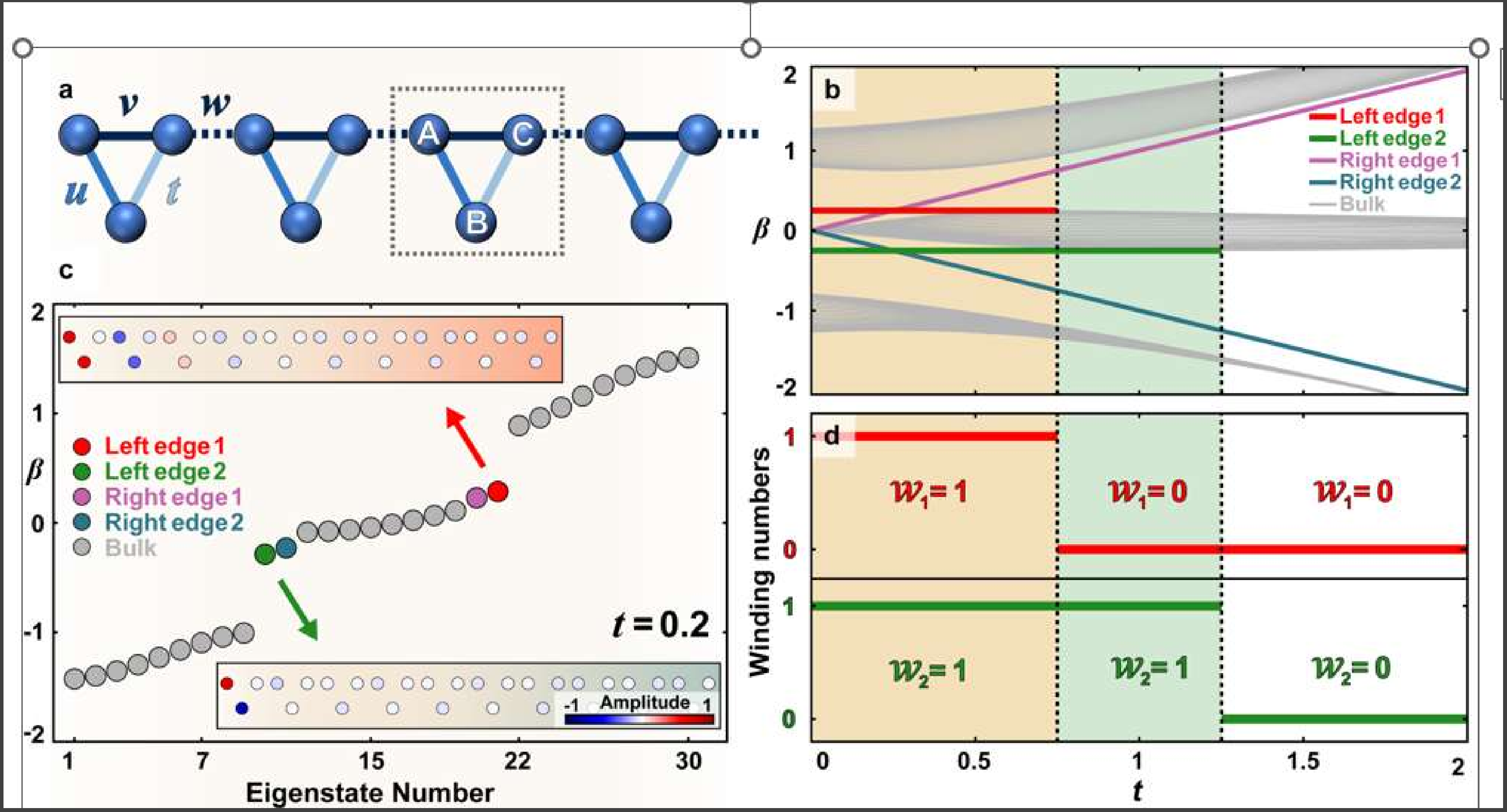


WELL-DEFINED PROCEDURE

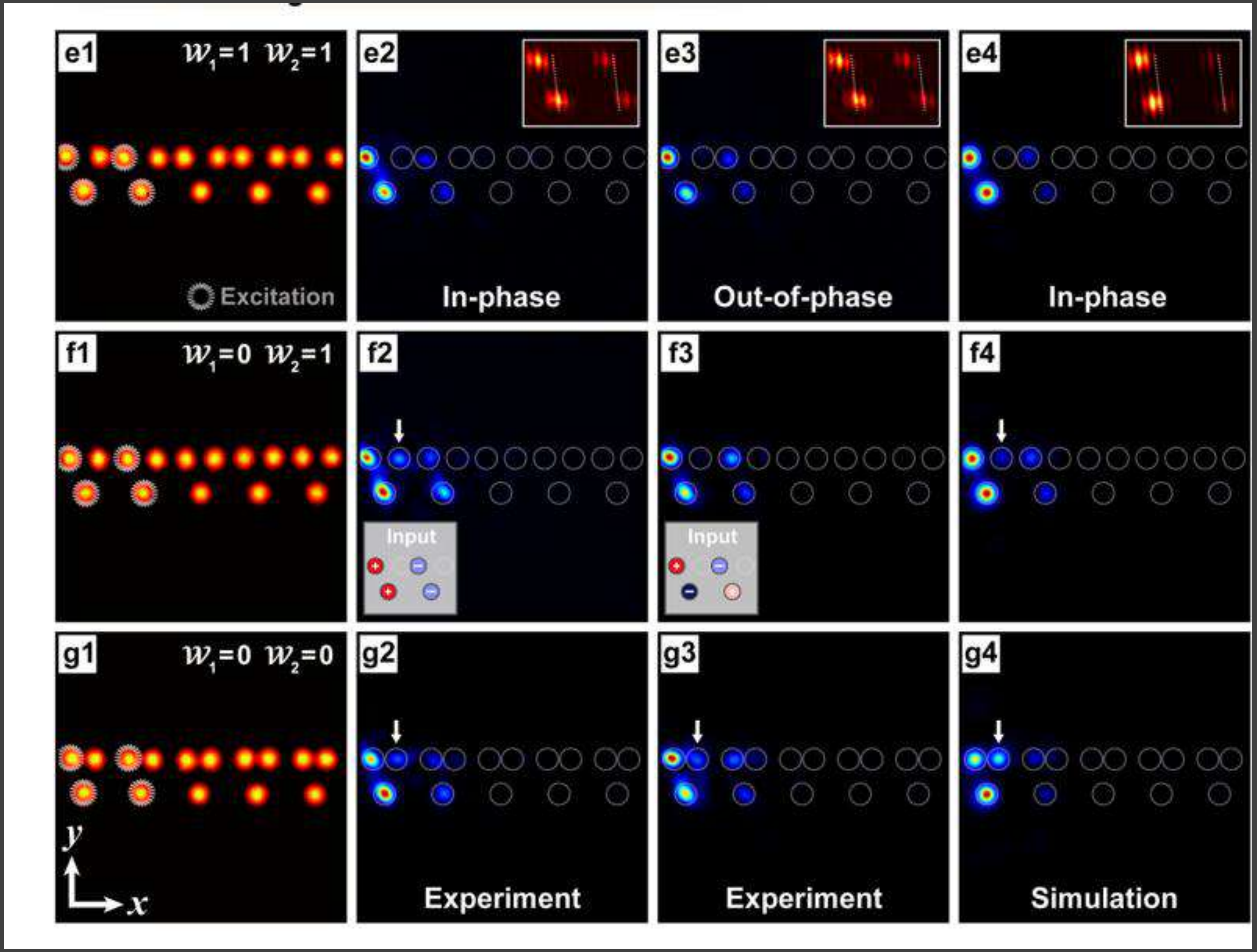


Topological boundary states of auxiliary Hamiltonians
=
Boundary states of the original system

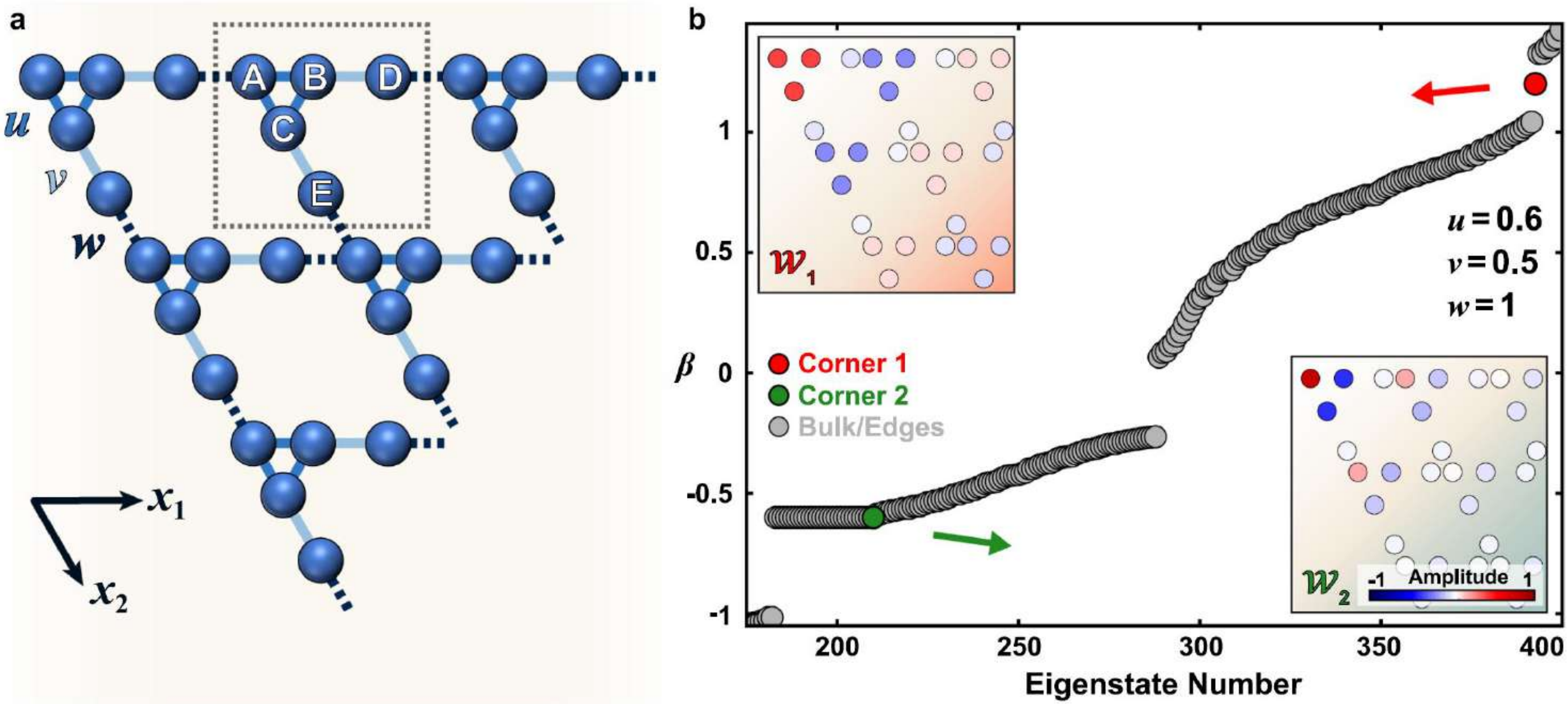
Example 1:



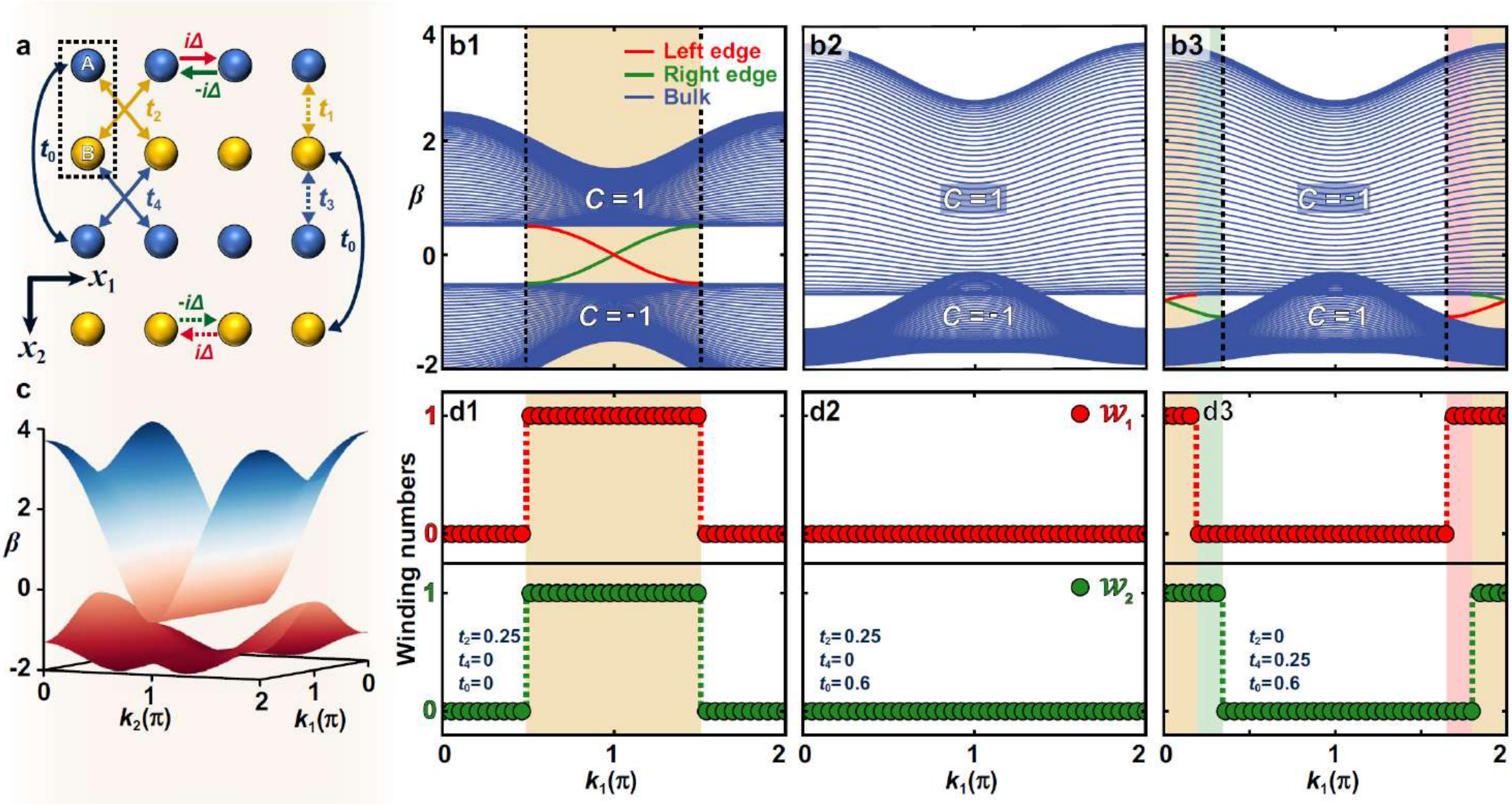
Topological boundary states of the original system experiments



MTPs in 2D HOTI



MTPs in 2D Chern insulators



Nonlinear topological photonics

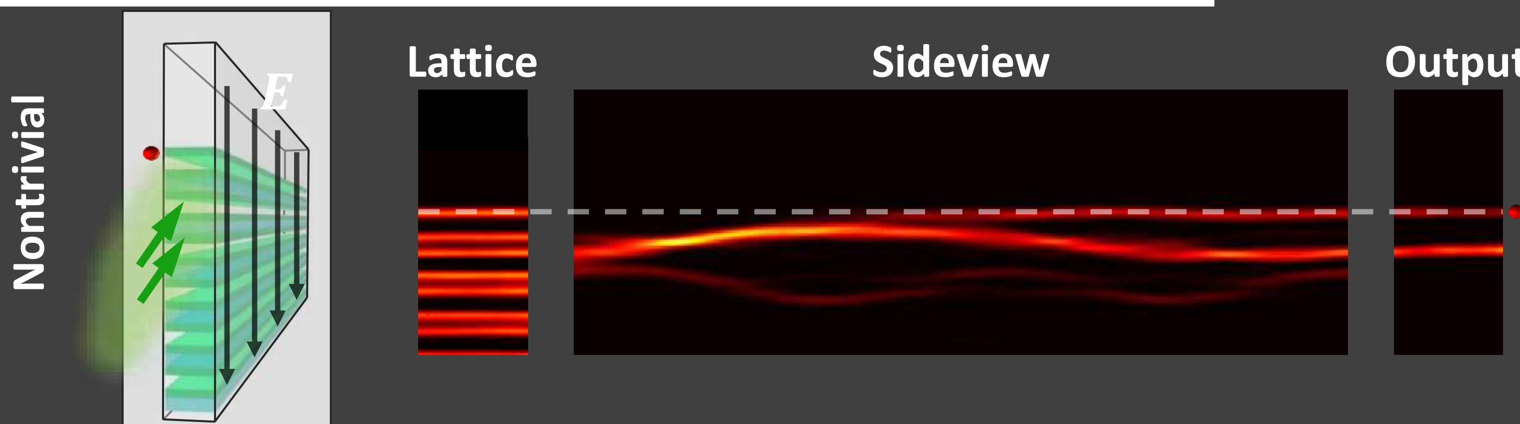
(* Some of our contributions, in collaboration with
Z. Chen, A. Szameit *)


Xia et al. *Light: Science & Applications* (2020)9:147
https://doi.org/10.1038/s41377-020-00371-y
Official journal of the CIOMP 2047-7538
www.nature.com/lsa

ARTICLE Open Access

Nontrivial coupling of light into a defect: the interplay of nonlinearity and topology

Shiqi Xia[1], Dario Jukić[2], Nan Wang[1], Daria Smirnova[3], Lev Smirnov[4], Liqin Tang[1,5], Daohong Song[1,5], Alexander Szameit[6], Daniel Leykam[7,8], Jingjun Xu[1,5], Zhigang Chen[1,5,9] and Hrvoje Buljan[1,10]



- SSH - Symmetry protected topological phase
- Nonlinearity breaks the protecting symmetry! What happens to topology?
- INHERITED TOPOLOGICAL PROPERTIES (from the linear system, quantified)!

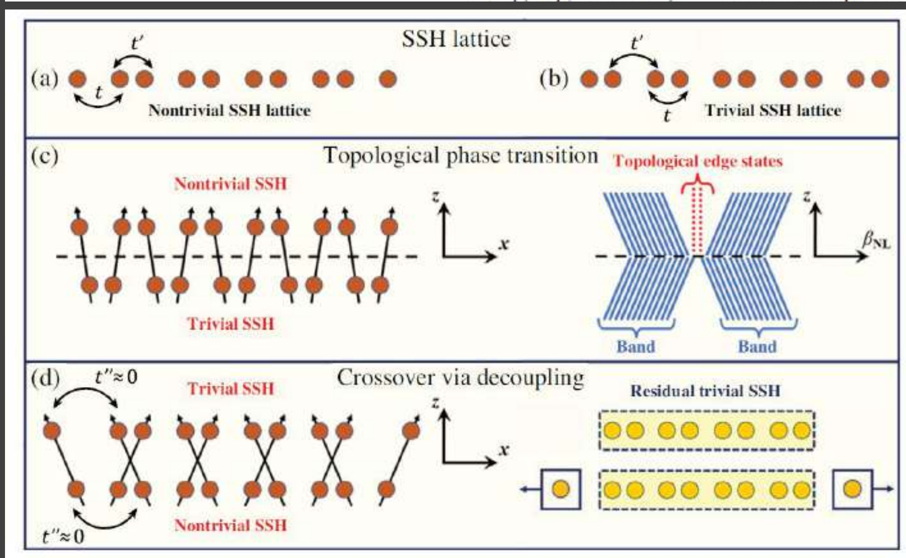
Nonlinear topological photonics

(* Some of our contributions, in collaboration with
Z. Chen, R. Morandotti *)

PHYSICAL REVIEW LETTERS **127**, 184101 (2021)

Dynamically Emerging Topological Phase Transitions in Nonlinear Interacting Soliton Lattices

Domenico Bongiovanni,[1,2] Dario Jukić,[3] Zhichan Hu,[1] Frane Lunić,[4] Yi Hu,[1] Daohong Song,[1] Roberto Morandotti
Zhigang Chen,[1,6,*] and Hrvoje Buljan[1,4,†]



- EMERGENT TOPOLOGICAL PROPERTIES!
- Without nonlinearity there is no topology

Nonlinear topological photonics

(* Some of our contributions, in collaboration with Z. Chen, R. Morandotti *)

PHYSICAL REVIEW LETTERS **127**, 184101 (2021)

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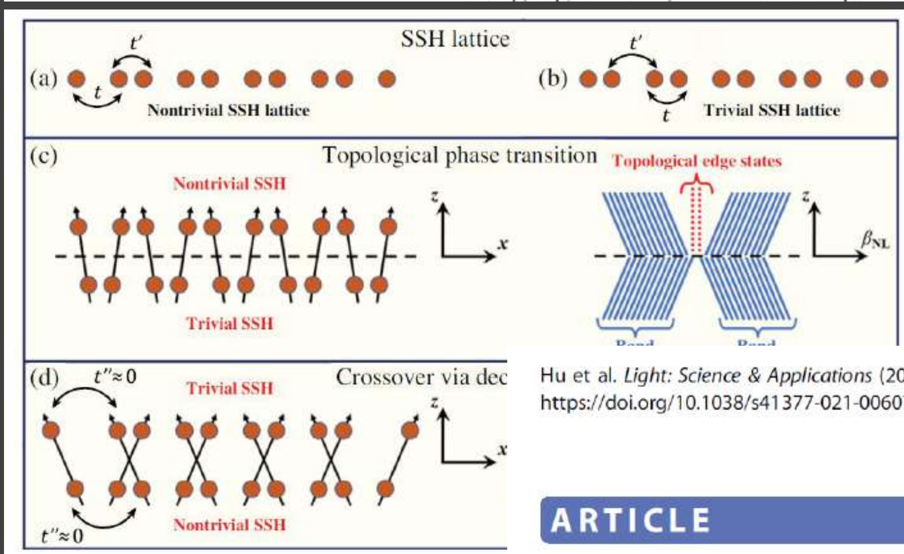

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Hu et al. *Light: Science & Applications* (2021)10:164
https://doi.org/10.1038/s41377-021-00607-5
Official journal of the CIOMP 2047-7538
www.nature.com/lsa

ARTICLE Open Access

Nonlinear control of photonic higher-order topological bound states in the continuum

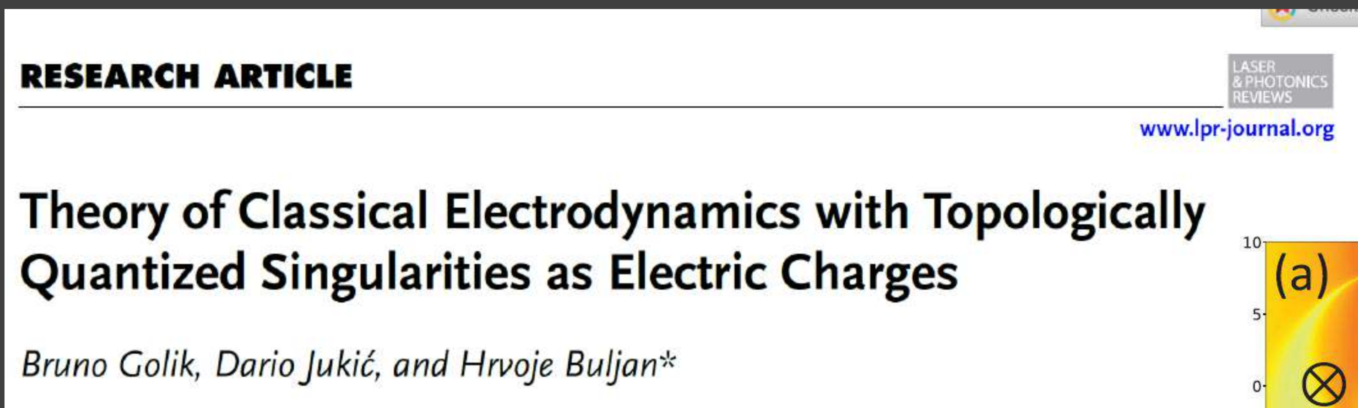
Zhichan Hu[1], Domenico Bongiovanni[1,2], Dario Jukić[3], Ema Jajtić[4], Shiqi Xia[1], Daohong Song[1,5], Jingjun Xu[1,5], Roberto Morandotti[2,6], Hrvoje Buljan[1,4✉] and Zhigang Chen[1,5,7✉]

Quantization of charge via classical electrodynamics

Bruno Golik†, Dario Jukić†, Hrvoje Buljan

University of Zagreb, Zagreb, Croatia

†These authors contributed equally to this work



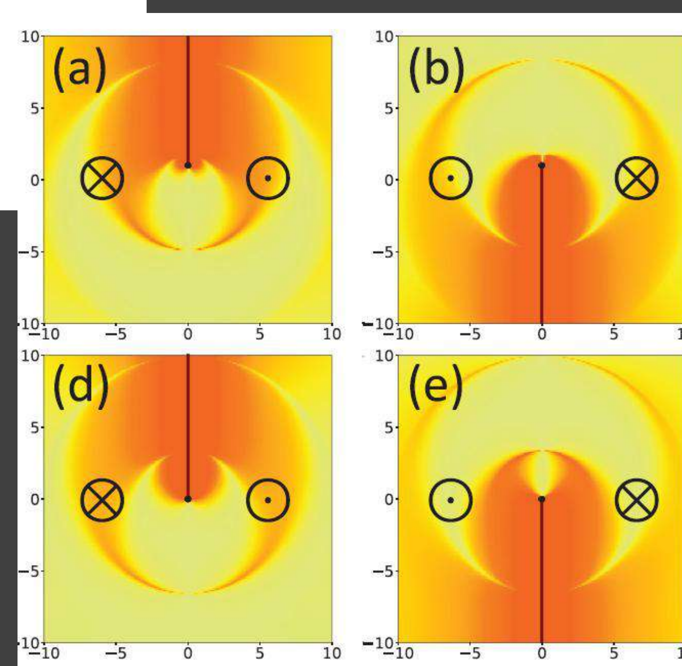
RESEARCH ARTICLE

LASER & PHOTONICS REVIEWS

www.lpr-journal.org

Theory of Classical Electrodynamics with Topologically Quantized Singularities as Electric Charges

*Bruno Golik, Dario Jukić, and Hrvoje Buljan**



- **Quantization of charge via classical electrodynamics, Golik, Jukić & Buljan, Laser and Photonics Reviews 2024 https://doi.org/10.1002/lpor.202400217**

Classical electrodynamics (Maxwell's eqs.)

(i) $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho,$

(ii) $\nabla \cdot \mathbf{B} = 0,$

(iii) $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0,$

(iv) $\nabla \times \mathbf{B} - \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J},$

Charges: $\rho(\mathbf{r}, t)$

Currents: $\mathbf{J}(\mathbf{r}, t)$

(* sources *)

Electromagnetic fields: $\mathbf{E}(\mathbf{r}, t)$ $\mathbf{B}(\mathbf{r}, t)$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}),$$

Lorentz's force

Dirac’s magnetic monopoles (1931)

Suppose that magnetic charges exist, what are the implications?

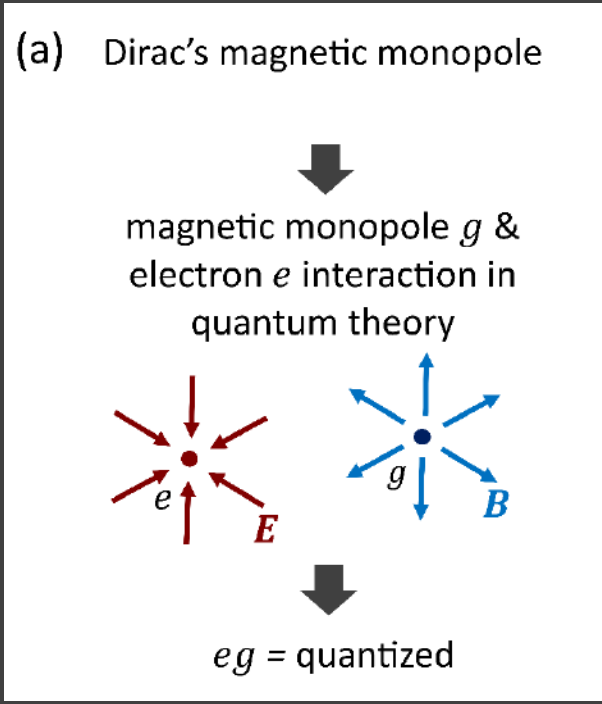
$$\text{(i)}\quad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho_e,$$

$$\text{(ii)}\quad \nabla \cdot \mathbf{B} = \mu_0 \rho_m,$$

$$\text{(iii)}\quad \nabla \times \mathbf{E} = -\mu_0 \mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t},$$

$$\text{(iv)}\quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_e + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Maxwell’s eqs. acquire symmetric form with magnetic charges and currents



Charge is quantized!

Experiments so far: no evidence for magnetic monopoles!

Can we account for charge quantization in classical electrodynamics?

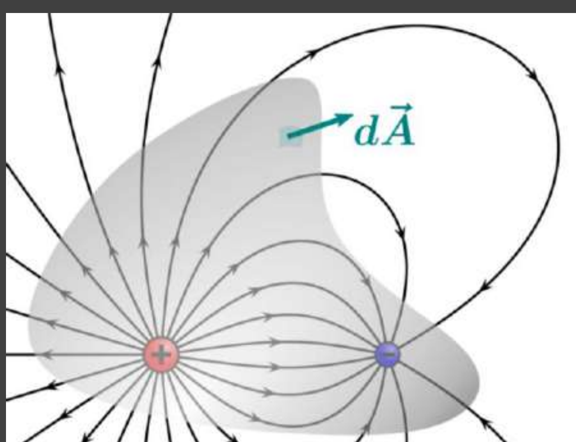
Conventional Gauss law
(Maxwell's theory)

Differential form:

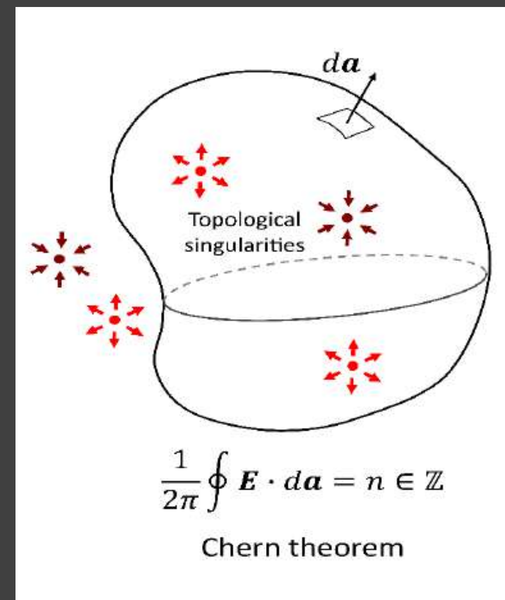
$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho_e,$$

Integral form:

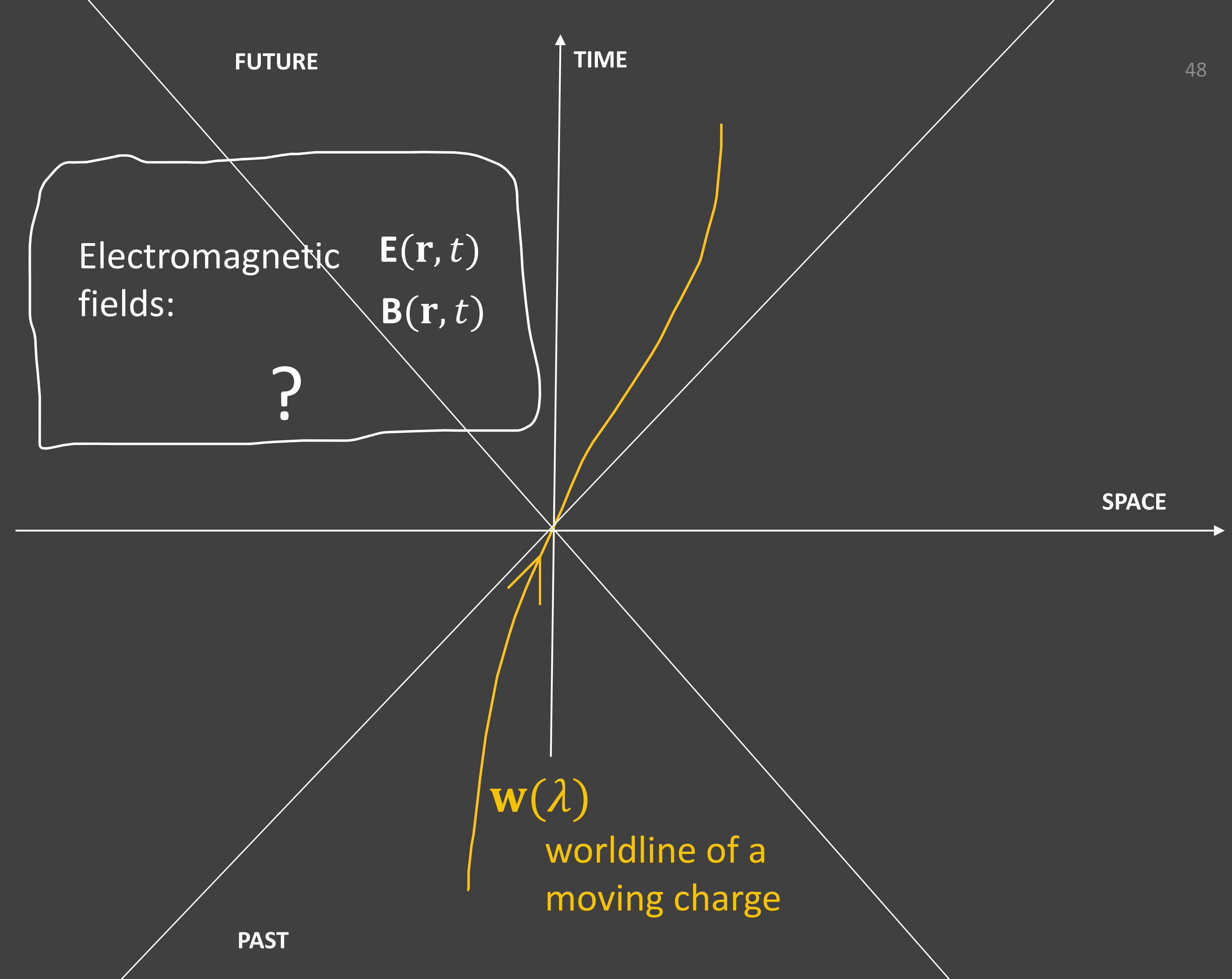
$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}},$$



- In our (topological) formulation of classical electrodynamics,
  Gauss law becomes the Chern theorem



- the only admissible electric charges are topological singularities in the EM field (+ & -)
- The sign of the charge corresponds to the chirality of the topological singularity
- Dirac magnetic monopoles are not needed


FUTURE
TIME
Electromagnetic fields:
$\mathbf{E}(\mathbf{r}, t)$
$\mathbf{B}(\mathbf{r}, t)$
?
SPACE
$\mathbf{w}(\lambda)$
worldline of a moving charge
PAST

Classical electrodynamics, alternative formulation

Weyl equation in frequency-wavevector space (R=right):

$$c\frac{\partial}{\partial\omega}\tilde{\psi}_R(k^\mu) = -(\sigma_x\frac{\partial}{\partial k_x} + \sigma_y\frac{\partial}{\partial k_y} + \sigma_z\frac{\partial}{\partial k_z})\tilde{\psi}_R(k^\mu),$$

Look for eigenvalues and eigenstates (2-component spinors):

$$\tilde{\psi}_R(k^\mu) = \psi_R(x^\mu)\exp(i\mathrm{k}\cdot\boldsymbol{\rho} - i\frac{\omega}{c}\rho^0),$$

(* auxiliary field *)

$$\rho^\mu = \Lambda^\mu_\nu(x^\nu - w^\nu).$$

(* displacement 4-vector *)

Λ = Lorentz transformation containing information of moving charges (velocity, acceleration):
Λ = Rotation*Boost

w = world line of a moving charge

C.E., alternative formulation, retarded time

Eigenvalue equation (space and time parameters):

$$H\psi_R(x^\mu) = \boldsymbol{\sigma} \cdot \boldsymbol{\rho}\psi_R(x^\mu) = \rho^0\psi_R(x^\mu),$$

Eigenvalues:

$\rho^0 = \pm|\boldsymbol{\rho}|$ equivalently $\rho^\mu\rho_\mu = 0$

$\rho^\mu = \Lambda^\mu_\nu(x^\nu - w^\nu)$ implies $(x^\nu - w^\nu)(x_\nu - w_\nu) = 0$

Positive eigenvalue: $|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r)$

Associated eigenstate: $|\psi_{R,p}\rangle$

} Retarded time

~~Negative eigenvalue: $|\mathbf{r} - \mathbf{w}(t_a)| = -c(t - t_a)$~~

~~Associated eigenstate: $|\psi_{R,n}\rangle$~~

} Advanced time

Weyl semimetals analogy:

$$\boldsymbol{\sigma} \cdot \mathbf{k}\,\psi = E\psi$$

$$E = \pm|\mathbf{k}|$$

Berry connections ➡ Berry curvatures ➡ Electromagnetic fields

$$\rho^\mu = \Lambda^\mu_\nu (x^\nu - w^\nu).$$

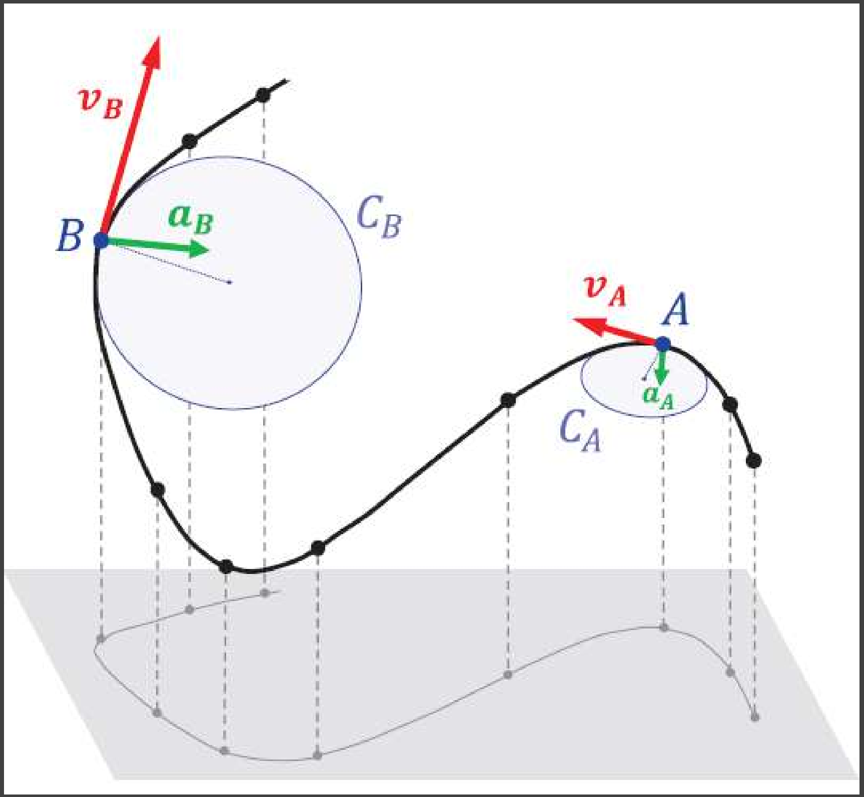
Λ = Lorentz transformation containing information of moving charges (velocity, acceleration):

$$\Lambda = R(\theta(t_r)) * B(\mathbf{v}(t_r))$$

$$\theta(t_r) = \hat{\mathbf{n}} \cdot \int^{t_r} (\boldsymbol{\omega}_{Th}(t') \cdot \hat{\mathbf{n}}) dt',$$

$$\boldsymbol{\omega}_{Th}(t') = \frac{1}{c^2} \frac{\gamma^2}{\gamma + 1} (\mathbf{a}(t') \times \mathbf{v}(t'))$$

$\hat{n}$ is a unit vector parallel to $\mathbf{a} \times \mathbf{v}$



Berry connections ➡ Berry curvatures ➡ Electromagnetic fields

Eigenstates $|\psi_{R,p}\rangle$ and $|\psi_{L,p}\rangle$:
provide electric and magnetic fields:

$$\mathbf{A} = i\langle\psi|\nabla|\psi\rangle, \quad V = -\frac{1}{c^2} i\langle\psi|\frac{\partial}{\partial t}|\psi\rangle.$$

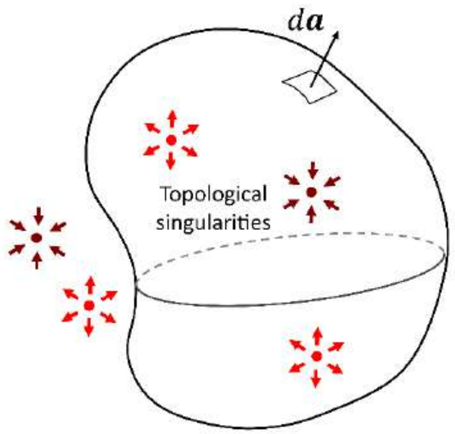
$$\mathbf{E} = \nabla \times \mathbf{A}, \quad \mathbf{B} = \nabla V + \frac{1}{c^2}\frac{\partial \mathbf{A}}{\partial t}.$$

Mathematically IDENTICAL to solutions of the Maxwell's equations!!!

$$\mathbf{E}_M = \frac{q}{2\pi\epsilon_0}\mathbf{E}, \quad \mathbf{B}_M = \frac{q}{2\pi\epsilon_0}\mathbf{B},$$

Topology guarantees charge quantization:

$$\oint_S \mathbf{E}\cdot d\mathbf{a} = 2\pi \times \text{integer},$$



$$\frac{1}{2\pi}\oint \boldsymbol{E}\cdot d\boldsymbol{a} = n \in \mathbb{Z}$$

Chern theorem

Worldline 1: Motion of a charge along a straight line (z-axis; e.g., Brehmstrallung)

Weyl "Hamiltonian" with space and time as "parameters"; w=world line of a charge

$$H = x\sigma_x + y\sigma_y + \gamma(t_r)(z - w_z(t_r) - v_z(t_r)(t - t_r))\sigma_z.$$

Eigenvalues: Provide retarded position of a charge:

$$|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r),$$

Eigenstates: Provide electric and magnetic fields:

$$\mathbf{A} = i\langle\psi|\nabla|\psi\rangle, \quad V = -\frac{1}{c^2} i\langle\psi|\frac{\partial}{\partial t}|\psi\rangle.$$

$$\mathbf{E} = \nabla \times \mathbf{A}, \quad \mathbf{B} = \nabla V + \frac{1}{c^2}\frac{\partial \mathbf{A}}{\partial t}.$$

Worldline 1: Motion of a charge along a straight line (e.g., Brehmstrallung)

Worldline 2: Motion of a charge in a plane (e.g., Synchrotron radiation)

Worldline 3: General motion in 3D

$$\mathrm{E} = \pm\frac{1}{2}\frac{s}{(\boldsymbol{s}\cdot\boldsymbol{u})^3}\left[(c^2 - v^2)\boldsymbol{u} + \boldsymbol{s}\times(\boldsymbol{u}\times\boldsymbol{a})\right],\ \mathrm{B} = \frac{1}{c}\hat{\boldsymbol{s}}\times\mathrm{E}.$$

Connection with Maxwell:

$$\mathrm{E}_M = \frac{q}{2\pi\epsilon_0}\mathrm{E},\ \mathrm{B}_M = \frac{q}{2\pi\epsilon_0}\mathrm{B},$$

$$\mathbf{F} = \frac{q^2}{2\pi\varepsilon_0}(\mathbf{E} + \mathbf{v}\times\mathbf{B}),$$

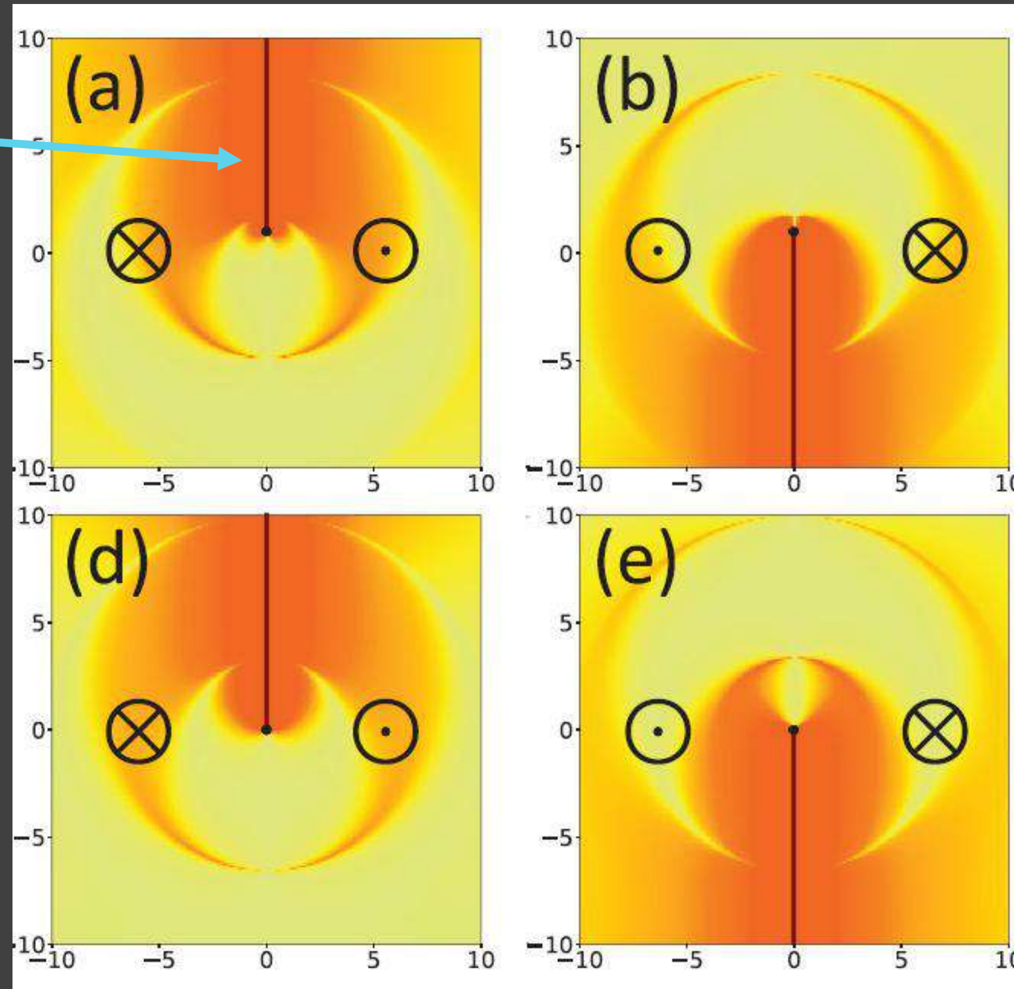
Maxwell's equations allow continuous distributions of charges and currents!
This theory allows only point charges with two opposite (positive and negative) values of the charge!!! Here, chirality of topological singularities is charge.

Dirac string:

Berry connections

for an oscillating

topological singularity

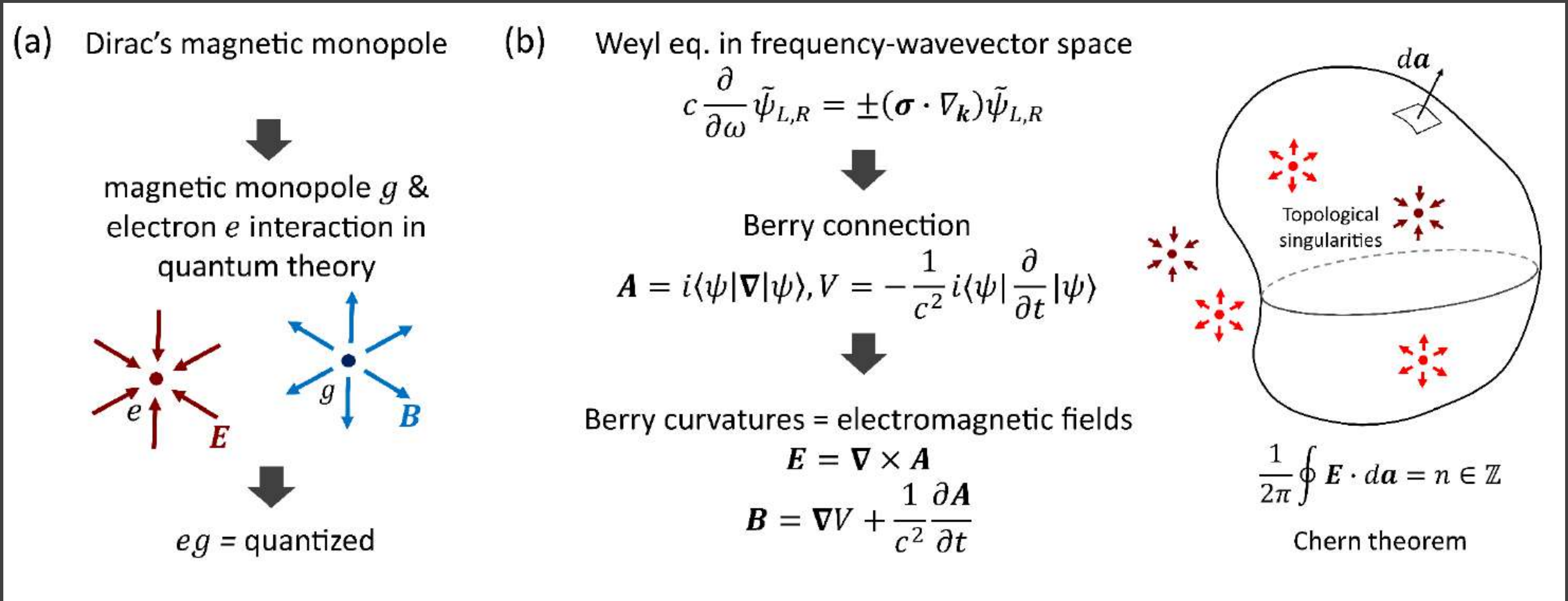


$$\mathbf{E} = \nabla \times \mathbf{A}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = \nabla \cdot \mathbf{E} = \mathbf{0}$$ NOT AT THE SINGULARITY

https://doi.org/10.1002/lpor.202400217
Golik, Jukić, Buljan: Laser and Photonics Reviews 2024

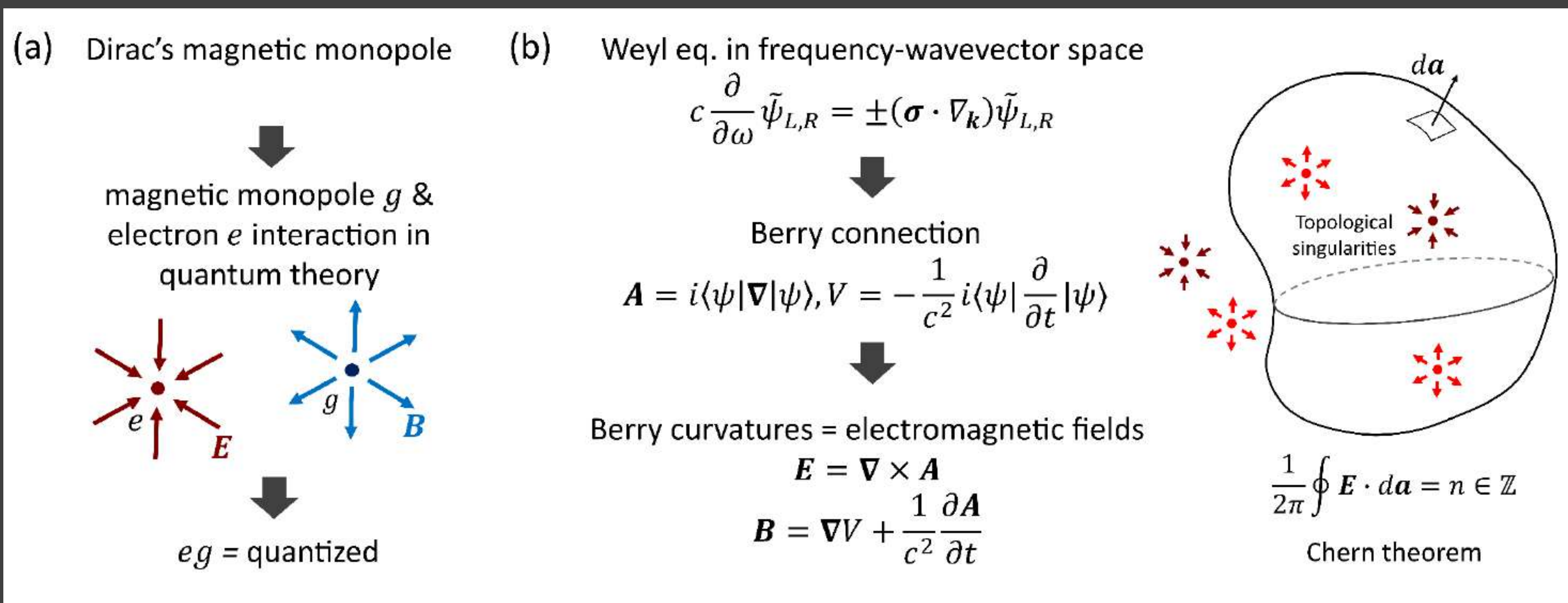
- We have formulated a theory of classical electrodynamics where the only admissible electric charges are topological singularities in the EM field
- Gauss law = Chern theorem
- The sign of the charge corresponds to the chirality of the topological singularity
- Given the trajectory w(t) of the singularity, one can calculate electric and magnetic fields identical to those produced by Maxwell's equations for a moving point charge



- We have formulated a theory of classical electrodynamics where the only admissible electric charges are topological singularities in the EM field
- Gauss law = Chern theorem
- Th
- Gi … identical to

Is there deeper meaning of the field ψ?

At this point we treat ψ as an auxiliary mathematical field!



Thank you for attention!