Thom Spaces and the Oriented Cobordism Ring

Branko Juran

2020-5-20

Organization of the talk

- Definition of the oriented cobordism groups Ω_n

Organization of the talk

- Definition of the oriented cobordism groups Ω_n
- Isomorphism between Ω_n and a certain homotopy group $\pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Organization of the talk

- Definition of the oriented cobordism groups Ω_n
- Isomorphism between Ω_n and a certain homotopy group $\pi_{n+k}(T(\tilde{\gamma}^k), t_0)$
- Isomorphism $\pi_{n+k}(T(\tilde{\gamma}^k), t_0) \otimes \mathbb{Q} \cong H_n(\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty)) \otimes \mathbb{Q}$.

Oriented Cobordism

Convention

We assume all manifolds to be smooth, compact and oriented.

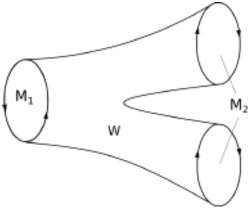
Oriented Cobordism

Convention

We assume all manifolds to be smooth, compact and oriented.

Definition

A *cobordism* between two n-dim. manifolds M_1 and M_2 is an $(n+1)$-dim. manifold with boundary W together with an orientation preserving diffeomorphism $\partial W \cong M_1 \sqcup (-M_2)$.
Two manifolds are said to be *cobordant* if there is a coboridsm between them.



Oriented Cobordism

Lemma

Being cobordant is an equivalence relation on the class of manifolds.

Oriented Cobordism

Lemma

Being cobordant is an equivalence relation on the class of manifolds.

Proof.

□

Lemma

Being cobordant is an equivalence relation on the class of manifolds.

Proof.

- Reflexive: $\partial(M \times [0,1]) \cong M \sqcup (-M)$

$\square$

Oriented Cobordism

Lemma

Being cobordant is an equivalence relation on the class of manifolds.

Proof.

- Reflexive: $\partial(M \times [0,1]) \cong M \sqcup (-M)$
- Symmetric: $\partial(-W) \cong -\partial W \cong (-M_1) \sqcup M_2$

□

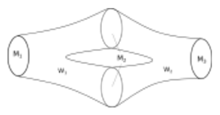
Oriented Cobordism

Lemma

Being cobordant is an equivalence relation on the class of manifolds.

Proof.

- Reflexive: $\partial(M \times [0,1]) \cong M \sqcup (-M)$
- Symmetric: $\partial(-W) \cong -\partial W \cong (-M_1) \sqcup M_2$
- Transitive: For W_1 cobordism between M_1 and M_2, W_2 cobordism between M_2 and M_3 use collar neighborhood theorem for gluing W_1 and W_2 along M_2



□

Oriented Cobordism

Definition

Let Ω_n be the set of cobordism classes of n-dimensional manifolds.

Oriented Cobordism

Definition

Let Ω_n be the set of cobordism classes of n-dimensional manifolds.

Lemma

The disjoint union induces a map $\Omega_n \times \Omega_n \to \Omega_n$ turning Ω_n into an abelian group. This group is called the n-th oriented cobordism group.

Oriented Cobordism

Definition

Let Ω_n be the set of cobordism classes of n-dimensional manifolds.

Lemma

The disjoint union induces a map $\Omega_n \times \Omega_n \to \Omega_n$ turning Ω_n into an abelian group. This group is called the n-th oriented cobordism group.

Proof.

For W cobordism between M_1, M_2 and N another n-dim. manifold, then $W \sqcup N \times [0,1]$ is cobordism between $M_1 \sqcup N$ and $M_2 \sqcup N$. $\square$

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ *turning* Ω_* *into a graded commutative ring. It is called the oriented cobordism ring.*

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ turning Ω_* into a graded commutative ring. It is called the oriented cobordism ring.

Proof.

For W cobordism between M_1 and M_2, $W \times N$ is cobordism between $M_1 \times N$ and $M_2 \times N$ because $\partial(W \times N) \cong (M_1 \times N) \sqcup (-M_2 \times N)$ □

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ turning Ω_* into a graded commutative ring. It is called the oriented cobordism ring.

Proof.

For W cobordism between M_1 and M_2, $W \times N$ is cobordism between $M_1 \times N$ and $M_2 \times N$ because $\partial(W \times N) \cong (M_1 \times N) \sqcup (-M_2 \times N)$ □

Example

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ turning Ω_* into a graded commutative ring. It is called the oriented cobordism ring.

Proof.

For W cobordism between M_1 and M_2, $W \times N$ is cobordism between $M_1 \times N$ and $M_2 \times N$ because $\partial(W \times N) \cong (M_1 \times N) \sqcup (-M_2 \times N)$ □

Example

- $\Omega_0 \cong \mathbb{Z}$. Spanned by point with positive orientation.

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ turning Ω_* into a graded commutative ring. It is called the oriented cobordism ring.

Proof.

For W cobordism between M_1 and M_2, $W \times N$ is cobordism between $M_1 \times N$ and $M_2 \times N$ because $\partial(W \times N) \cong (M_1 \times N) \sqcup (-M_2 \times N)$ □

Example

- $\Omega_0 \cong \mathbb{Z}$. Spanned by point with positive orientation.
- $\Omega_1 \cong 0$. S^1 is boundary of D^2

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ turning Ω_* into a graded commutative ring. It is called the oriented cobordism ring.

Proof.

For W cobordism between M_1 and M_2, $W \times N$ is cobordism between $M_1 \times N$ and $M_2 \times N$ because $\partial(W \times N) \cong (M_1 \times N) \sqcup (-M_2 \times N)$ □

Example

- $\Omega_0 \cong \mathbb{Z}$. Spanned by point with positive orientation.
- $\Omega_1 \cong 0$. S^1 is boundary of D^2
- $\Omega_2 \cong 0$. S^2 and genus g surfaces bound

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ turning Ω_* into a graded commutative ring. It is called the oriented cobordism ring.

Proof.

For W cobordism between M_1 and M_2, $W \times N$ is cobordism between $M_1 \times N$ and $M_2 \times N$ because $\partial(W \times N) \cong (M_1 \times N) \sqcup (-M_2 \times N)$ □

Example

- $\Omega_0 \cong \mathbb{Z}$. Spanned by point with positive orientation.
- $\Omega_1 \cong 0$. S^1 is boundary of D^2
- $\Omega_2 \cong 0$. S^2 and genus g surfaces bound
- $\Omega_3 \cong 0$. (Rohlin, 1951)

Oriented Cobordism

Lemma

The product induces a map $\Omega_m \times \Omega_n \to \Omega_{m+n}$ turning Ω_* into a graded commutative ring. It is called the oriented cobordism ring.

Proof.

For W cobordism between M_1 and M_2, $W \times N$ is cobordism between $M_1 \times N$ and $M_2 \times N$ because $\partial(W \times N) \cong (M_1 \times N) \sqcup (-M_2 \times N)$ □

Example

- $\Omega_0 \cong \mathbb{Z}$. Spanned by point with positive orientation.
- $\Omega_1 \cong 0$. S^1 is boundary of D^2
- $\Omega_2 \cong 0$. S^2 and genus g surfaces bound
- $\Omega_3 \cong 0$. (Rohlin, 1951)
- $\Omega_4 \cong \mathbb{Z}$. Spanned by $\mathbb{C}P^2$

Cobordism classes of complex projective spaces

Theorem (Pontryagin)

As $(i_1, \ldots, i_k)$ ranges over all partitions of r, the manifolds

$$\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_k}$$

represent linearly independent elements of Ω_{4r}.

Cobordism classes of complex projective spaces

Theorem (Pontryagin)

As $(i_1, \ldots, i_k)$ ranges over all partitions of r, the manifolds

$$\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_k}$$

represent linearly independent elements of Ω_{4r}.

Proof.

Follows immediatly from the facts that

□

Cobordism classes of complex projective spaces

Theorem (Pontryagin)

As $(i_1, \ldots, i_k)$ ranges over all partitions of r, the manifolds

$$\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_k}$$

represent linearly independent elements of Ω_{4r}.

Proof.

Follows immediatly from the facts that

- Pontryagin numbers are additive, i.e. $p_I(M) + p_I(N) = p_I(M \sqcup N)$.

□

Cobordism classes of complex projective spaces

Theorem (Pontryagin)

As $(i_1, \dots, i_k)$ ranges over all partitions of r, the manifolds

$$\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_k}$$

represent linearly independent elements of Ω_{4r}.

Proof.

Follows immediatly from the facts that

- Pontryagin numbers are additive, i.e. $p_I(M) + p_I(N) = p_I(M \sqcup N)$.
- All Pontryagin numbers of the boundary of a $(4r+1)$-dimensional manifold are 0.

□

Cobordism classes of complex projective spaces

Theorem (Pontryagin)

As $(i_1, \dots, i_k)$ ranges over all partitions of r, the manifolds

$$\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_k}$$

represent linearly independent elements of Ω_{4r}.

Proof.

Follows immediatly from the facts that

- Pontryagin numbers are additive, i.e. $p_I(M) + p_I(N) = p_I(M \sqcup N)$.
- All Pontryagin numbers of the boundary of a $(4r+1)$-dimensional manifold are 0.
- Pontryagin numbers define a group homomorphism $\Omega_{4r} \to \mathbb{Z}^{p(r)}$

$\square$

Cobordism classes of complex projective spaces

Theorem (Pontryagin)

As $(i_1, \ldots, i_k)$ ranges over all partitions of r, the manifolds

$$\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_k}$$

represent linearly independent elements of Ω_{4r}.

Proof.

Follows immediatly from the facts that

- Pontryagin numbers are additive, i.e. $p_I(M) + p_I(N) = p_I(M \sqcup N)$.
- All Pontryagin numbers of the boundary of a $(4r+1)$-dimensional manifold are 0.
- Pontryagin numbers define a group homomorphism $\Omega_{4r} \to \mathbb{Z}^{p(r)}$
- The above manifolds have linearly independent Pontryagin numbers

□

The Thom Space of a Euclidean Vector Bundle

Definition

Let ξ be a k-dim. Euclidean vector bundle. Let $A \subset E(\xi)$ be the subset of all vectors v with $|v| \geq 1$. The *Thom space* $T(\xi)$ of ξ is defined as $E(\xi)/A$. Let t_0 denote the canonical base point.

The Thom Space of a Euclidean Vector Bundle

Definition

Let ξ be a k-dim. Euclidean vector bundle. Let $A \subset E(\xi)$ be the subset of all vectors v with $|v| \geq 1$. The *Thom space* $T(\xi)$ of ξ is defined as $E(\xi)/A$. Let t_0 denote the canonical base point.

Remark

If ξ has a compact base space, then $T(\xi)$ is homeomorphic to the one-point-compactification of $E(\xi)$.

The Thom Space of a Euclidean Vector Bundle

Definition

Let ξ be a k-dim. Euclidean vector bundle. Let $A \subset E(\xi)$ be the subset of all vectors v with $|v| \geq 1$. The *Thom space* $T(\xi)$ of ξ is defined as $E(\xi)/A$. Let t_0 denote the canonical base point.

Remark

If ξ has a compact base space, then $T(\xi)$ is homeomorphic to the one-point-compactification of $E(\xi)$.

Proof.

Extend the $E(\xi) - A \to E(\xi), v \mapsto v/(1 - |v|)$ to a map $T(\xi) \to E(\xi) \cup \{\infty\}$. □

Theorem of Thom

Definition

Let $\tilde{\gamma}^k$ denote the universal oriented k-bundle over $\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty)$.

Theorem of Thom

Definition

Let $\tilde{\gamma}^k$ denote the universal oriented k-bundle over $\tilde{\text{Gr}}_k(\mathbb{R}^\infty)$.

Theorem (Thom, 1954)

There is an isomorphism $\pi_{n+k}(T(\tilde{\gamma}^k), t_0) \cong \Omega_n$ *for* $k \geq n + 2$.

The Thom-Pontryagin Construction: $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

- For $[M] \in \Omega_n$, choose an embedding $M \hookrightarrow \mathbb{R}^{n+k}$.

The Thom-Pontryagin Construction: $\alpha \colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

- For $[M] \in \Omega_n$, choose an embedding $M \hookrightarrow \mathbb{R}^{n+k}$.
- Choose tubular neighborhood M in $\mathbb{R}^{n+k}$, i.e. extend embedding to embedding $i \colon E(\nu_M) \hookrightarrow \mathbb{R}^{n+k}$

The Thom-Pontryagin Construction: $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

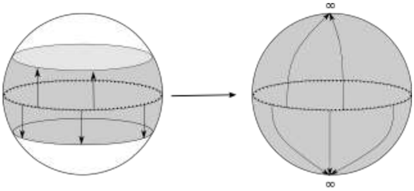
- For $[M] \in \Omega_n$, choose an embedding $M \hookrightarrow \mathbb{R}^{n+k}$.
- Choose tubular neighborhood M in $\mathbb{R}^{n+k}$, i.e. extend embedding to embedding $i\colon E(\nu_M) \hookrightarrow \mathbb{R}^{n+k}$
- Get Thom-Pontryagin collapse map $S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to T(\nu_M)$

The Thom-Pontryagin Construction: $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

- For $[M] \in \Omega_n$, choose an embedding $M \hookrightarrow \mathbb{R}^{n+k}$.
- Choose tubular neighborhood M in $\mathbb{R}^{n+k}$, i.e. extend embedding to embedding $i\colon E(\nu_M) \hookrightarrow \mathbb{R}^{n+k}$
- Get Thom-Pontryagin collapse map $S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to T(\nu_M)$
  - $i^{-1}\colon i(E(\nu_M)) \to E(\nu_M) \subset T(\nu_M)$ on $i(E(\nu_M))$

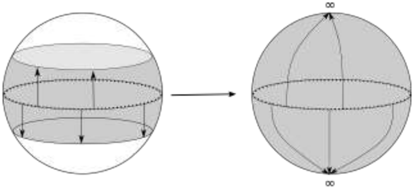
The Thom-Pontryagin Construction: $\alpha \colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

- For $[M] \in \Omega_n$, choose an embedding $M \hookrightarrow \mathbb{R}^{n+k}$.
- Choose tubular neighborhood M in $\mathbb{R}^{n+k}$, i.e. extend embedding to embedding $i \colon E(\nu_M) \hookrightarrow \mathbb{R}^{n+k}$
- Get Thom-Pontryagin collapse map $S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to T(\nu_M)$
  - $i^{-1} \colon i(E(\nu_M)) \to E(\nu_M) \subset T(\nu_M)$ on $i(E(\nu_M))$
  - ∞ outside tubular neighborhood



The Thom-Pontryagin Construction: $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

- For $[M] \in \Omega_n$, choose an embedding $M \hookrightarrow \mathbb{R}^{n+k}$.
- Choose tubular neighborhood M in $\mathbb{R}^{n+k}$, i.e. extend embedding to embedding $i\colon E(\nu_M) \hookrightarrow \mathbb{R}^{n+k}$
- Get Thom-Pontryagin collapse map $S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to T(\nu_M)$
  - $i^{-1}\colon i(E(\nu_M)) \to E(\nu_M) \subset T(\nu_M)$ on $i(E(\nu_M))$
  - ∞ outside tubular neighborhood



- Define $\alpha([M]) = [f]$ where $f\colon S^{n+k} \to T(\nu_M) \xrightarrow{\text{Gauss}} T(\tilde{\gamma}^k)$

Well-definiteness of $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Independence of choice of tubular neighborhood: Any two tubular neighborhoods are isotopic.

Well-definiteness of $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Independence of choice of tubular neighborhood: Any two tubular neighborhoods are isotopic.
Independence of choice of embedding: Two embeddings $M \to \mathbb{R}^{n+k}$ are isotopic if $k \geq n+2$.

Well-definiteness of $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Independence of choice of tubular neighborhood: Any two tubular neighborhoods are isotopic.
Independence of choice of embedding: Two embeddings $M \to \mathbb{R}^{n+k}$ are isotopic if $k \geq n+2$.
Independence of choice of M.

- α additive

Well-definiteness of $\alpha \colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Independence of choice of tubular neighborhood: Any two tubular neighborhoods are isotopic.
Independence of choice of embedding: Two embeddings $M \to \mathbb{R}^{n+k}$ are isotopic if $k \geq n+2$.
Independence of choice of M.

- α additive
- Let W be $(n+1)$-dim. manifold with boundary. Want to show: Thom-Pontryagin collapse map of ∂W null-homotopic

Well-definiteness of $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Independence of choice of tubular neighborhood: Any two tubular neighborhoods are isotopic.

Independence of choice of embedding: Two embeddings $M \to \mathbb{R}^{n+k}$ are isotopic if $k \geq n+2$.

Independence of choice of M.

- α additive
- Let W be $(n+1)$-dim. manifold with boundary. Want to show: Thom-Pontryagin collapse map of ∂W null-homotopic
- Embedding $\partial W \to S^{n+k}$ extends to neat embedding $W \hookrightarrow D^{n+k+1}$

Well-definiteness of $\alpha \colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Independence of choice of tubular neighborhood: Any two tubular neighborhoods are isotopic.
Independence of choice of embedding: Two embeddings $M \to \mathbb{R}^{n+k}$ are isotopic if $k \geq n+2$.
Independence of choice of M.

- α additive
- Let W be $(n+1)$-dim. manifold with boundary. Want to show: Thom-Pontryagin collapse map of ∂W null-homotopic
- Embedding $\partial W \to S^{n+k}$ extends to neat embedding $W \hookrightarrow D^{n+k+1}$
- Intersection of S^{n+k} and a tubular neighborhood of W in D^{n+k+1} is a tubular neighborhood of ∂W in S^{n+k}

Well-definiteness of $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$

Independence of choice of tubular neighborhood: Any two tubular neighborhoods are isotopic.
Independence of choice of embedding: Two embeddings $M \to \mathbb{R}^{n+k}$ are isotopic if $k \geq n+2$.
Independence of choice of M.

- α additive
- Let W be $(n+1)$-dim. manifold with boundary. Want to show: Thom-Pontryagin collapse map of ∂W null-homotopic
- Embedding $\partial W \to S^{n+k}$ extends to neat embedding $W \hookrightarrow D^{n+k+1}$
- Intersection of S^{n+k} and a tubular neighborhood of W in D^{n+k+1} is a tubular neighborhood of ∂W in S^{n+k}
- Use Thom-Pontryagin construction for W:

$$\begin{array}{ccccc} S^{n+k} & \longrightarrow & T(\nu_{\partial W}) & & \\ \big\downarrow & & \big\downarrow & \searrow & \\ & & & & T(\tilde{\gamma}^k) \\ & & & \nearrow & \\ D^{n+k+1} & \longrightarrow & T(\nu_W) & & \end{array}$$

The inverse map $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n$

- How do we get back M from the map $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ representing $\alpha([M])$? Solution: $M = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ (inverse of the zero-section).

The inverse map $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n$

- How do we get back M from the map $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ representing $\alpha([M])$? Solution: $M = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ (inverse of the zero-section).
- Idea: For $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$: $\beta([f]) = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$.

The inverse map $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n$

- How do we get back M from the map $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ representing $\alpha([M])$? Solution: $M = f^{-1}(\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty))$ (inverse of the zero-section).
- Idea: For $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$: $\beta([f]) = f^{-1}(\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty))$.
- Problem: $f^{-1}(\mathrm{Gr}_k(\mathbb{R}^\infty))$ does not need to be a manifold (even if f is smooth!)

The inverse map $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n$

- How do we get back M from the map $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ representing $\alpha([M])$? Solution: $M = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ (inverse of the zero-section).
- Idea: For $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$: $\beta([f]) = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$.
- Problem: $f^{-1}(\mathrm{Gr}_k(\mathbb{R}^\infty))$ does not need to be a manifold (even if f is smooth!)
- Need *transversality*.

Sard's Theorem

Definition

Let $f\colon M \to N$ be a smooth map. A point $y \in N$ is a *regular value* of f if for all $x \in f^{-1}(y)$, the map $T_x f\colon T_x M \to T_y N$ is surjective.

Sard's Theorem

Definition

Let $f\colon M \to N$ be a smooth map. A point $y \in N$ is a *regular value* of f if for all $x \in f^{-1}(y)$, the map $T_x f\colon T_x M \to T_y N$ is surjective.

Theorem (Sard)

Let $f\colon M \to N$ be a smooth map. The set of regular values of f is dense in N.

Transversality

Definition

Let M, N be manifolds, X a subset of M and Y a submanifold of N. A smooth function $f\colon M \to N$ is *transverse* to Y throughout X if $T_x M \xrightarrow{T_x f} T_{f(x)} N \to T_{f(x)} N / T_{f(x)} Y$ is surjective for all $x \in f^{-1}(Y) \cap X$.

Transversality

Definition

Let M, N be manifolds, X a subset of M and Y a submanifold of N. A smooth function $f : M \to N$ is *transverse* to Y throughout X if $T_xM \xrightarrow{T_xf} T_{f(x)}N \to T_{f(x)}N/T_{f(x)}Y$ is surjective for all $x \in f^{-1}(Y) \cap X$.

Lemma

If $f : M \to N$ is transverse to $Y \subset N$, then $f^{-1}(Y)$ is a smooth manifold. The normal bundle of Y in N pulls back to the normal bundle of $f^{-1}(Y)$ in M. In particular, $f^{-1}(Y)$ inherits an orientation from an orientation on M and an orientation of the normal bundle of Y in N.

Transversality

Definition

Let M, N be manifolds, X a subset of M and Y a submanifold of N. A smooth function $f : M \to N$ is *transverse* to Y throughout X if $T_x M \xrightarrow{T_x f} T_{f(x)} N \to T_{f(x)} N / T_{f(x)} Y$ is surjective for all $x \in f^{-1}(Y) \cap X$.

Lemma

If $f : M \to N$ is transverse to $Y \subset N$, then $f^{-1}(Y)$ is a smooth manifold. The normal bundle of Y in N pulls back to the normal bundle of $f^{-1}(Y)$ in M. In particular, $f^{-1}(Y)$ inherits an orientation from an orientation on M and an orientation of the normal bundle of Y in N.

Proof.

If φ is a local defining function for Y in N, then $\varphi \circ f$ is one for $f^{-1}(Y)$ in M. □

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f : W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g : W \to \mathbb{R}^k$ such that

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f\colon W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g\colon W \to \mathbb{R}^k$ such that

- *$f = g$ outside compact subset*

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f\colon W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g\colon W \to \mathbb{R}^k$ such that

- $f = g$ *outside compact subset*
- *Origin regular value throughout* $X \cup K$

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f\colon W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g\colon W \to \mathbb{R}^k$ such that

- *$f = g$ outside compact subset*
- *Origin regular value throughout $X \cup K$*
- $|f(x) - g(x)| \leq \epsilon$

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ *open subset,* $f\colon W \to \mathbb{R}^k$ *smooth, origin regular value throughout closed subset* $X \subset W$*,* K *a compact subset of* W *and* $\varepsilon > 0$*. There exists smooth* $g\colon W \to \mathbb{R}^k$ *such that*

- $f = g$ *outside compact subset*
- *Origin regular value throughout* $X \cup K$
- $|f(x) - g(x)| \leq \epsilon$

Proof.

□

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f\colon W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g\colon W \to \mathbb{R}^k$ such that

- *$f = g$ outside compact subset*
- *Origin regular value throughout $X \cup K$*
- $|f(x) - g(x)| \leq \epsilon$

Proof.

- Construct map $\lambda\colon W \to [0,1]$ such that $\lambda(x) = 1$ in a neighborhood of K and λ vanishes outside compact set.

□

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f\colon W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g\colon W \to \mathbb{R}^k$ such that

- *$f = g$ outside compact subset*
- *Origin regular value throughout $X \cup K$*
- $|f(x) - g(x)| \leq \epsilon$

Proof.

- Construct map $\lambda\colon W \to [0, 1]$ such that $\lambda(x) = 1$ in a neighborhood of K and λ vanishes outside compact set.
- Set $g(x) = f(x) - \lambda(x)y$ for y regular value of f and $|y| < \varepsilon$

□

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f\colon W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g\colon W \to \mathbb{R}^k$ such that

- *$f = g$ outside compact subset*
- *Origin regular value throughout $X \cup K$*
- $|f(x) - g(x)| \leq \epsilon$

Proof.

- Construct map $\lambda\colon W \to [0,1]$ such that $\lambda(x) = 1$ in a neighborhood of K and λ vanishes outside compact set.
- Set $g(x) = f(x) - \lambda(x)y$ for y regular value of f and $|y| < \varepsilon$
- If $g(x) = 0$ for $x \in K$, then $f(x) = y \implies T_x f = T_x g$ full rank at x

□

Thom's Transversality Theorem

Lemma

Let $W \subset \mathbb{R}^m$ open subset, $f\colon W \to \mathbb{R}^k$ smooth, origin regular value throughout closed subset $X \subset W$, K a compact subset of W and $\varepsilon > 0$. There exists smooth $g\colon W \to \mathbb{R}^k$ such that

- *$f = g$ outside compact subset*
- *Origin regular value throughout $X \cup K$*
- $|f(x) - g(x)| \leq \epsilon$

Proof.

- Construct map $\lambda\colon W \to [0,1]$ such that $\lambda(x) = 1$ in a neighborhood of K and λ vanishes outside compact set.
- Set $g(x) = f(x) - \lambda(x)y$ for y regular value of f and $|y| < \varepsilon$
- If $g(x) = 0$ for $x \in K$, then $f(x) = y \implies T_x f = T_x g$ full rank at x
- We can choose partial derivatives of f and g uniformly close to each other $\implies$ origin regular value throughout X □

Thom's Transversality Theorem

Theorem

Every map $S^m \to T(\xi)$ is homotopic to a map $\hat{f}$ which is smooth throughout $\hat{f}^{-1}(T(\xi) - t_0)$ and transverse to the zero-section. The map $\pi_{n+k}(T(\xi), t_0) \to \Omega_n$, $f \mapsto [\hat{f}^{-1}(B(\xi))]$ is well-defined.

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \ldots, W_k$, images contained in local trivialization of vector bundle ξ

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \ldots, W_k$, images contained in local trivialization of vector bundle ξ
- Cover $f_0^{-1}(B)$ by compacts $K_i \subset W_i$ $(1 \leq i \leq k)$ such that $f_0^{-1}(B) \subset \operatorname{int}(K_1 \cup \cdots \cup K_k)$. $\implies$ $|f_0(x)| \geq c > 0$ outside $K_1 \cup \cdots \cup K_k$

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \dots, W_k$, images contained in local trivialization of vector bundle ξ
- Cover $f_0^{-1}(B)$ by compacts $K_i \subset W_i$ $(1 \leq i \leq k)$ such that $f_0^{-1}(B) \subset \operatorname{int}(K_1 \cup \cdots \cup K_k)$. $\implies |f_0(x)| \geq c > 0$ outside $K_1 \cup \cdots \cup K_k$
- Obtain $\hat{f} = f_k$ by inductively define maps $f_i \colon S^m \to T(\xi)$ smooth throughout $f_i^{-1}(T - t_0) = f_0^{-1}(T - t_0)$ such that

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \ldots, W_k$, images contained in local trivialization of vector bundle ξ
- Cover $f_0^{-1}(B)$ by compacts $K_i \subset W_i$ $(1 \leq i \leq k)$ such that $f_0^{-1}(B) \subset \operatorname{int}(K_1 \cup \cdots \cup K_k)$. $\implies |f_0(x)| \geq c > 0$ outside $K_1 \cup \cdots \cup K_k$
- Obtain $\hat{f} = f_k$ by inductively define maps $f_i \colon S^m \to T(\xi)$ smooth throughout $f_i^{-1}(T - t_0) = f_0^{-1}(T - t_0)$ such that
  - $f_i = f_{i-1}$ outside compact subset of W_i

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \dots, W_k$, images contained in local trivialization of vector bundle ξ
- Cover $f_0^{-1}(B)$ by compacts $K_i \subset W_i$ $(1 \leq i \leq k)$ such that $f_0^{-1}(B) \subset \operatorname{int}(K_1 \cup \dots \cup K_k)$. $\implies |f_0(x)| \geq c > 0$ outside $K_1 \cup \dots \cup K_k$
- Obtain $\hat{f} = f_k$ by inductively define maps $f_i \colon S^m \to T(\xi)$ smooth throughout $f_i^{-1}(T - t_0) = f_0^{-1}(T - t_0)$ such that
  - $f_i = f_{i-1}$ outside compact subset of W_i
  - f_i transverse to B throughout $K_1 \cup \dots \cup K_i$

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \ldots, W_k$, images contained in local trivialization of vector bundle ξ
- Cover $f_0^{-1}(B)$ by compacts $K_i \subset W_i$ $(1 \leq i \leq k)$ such that $f_0^{-1}(B) \subset \operatorname{int}(K_1 \cup \cdots \cup K_k)$. $\implies |f_0(x)| \geq c > 0$ outside $K_1 \cup \cdots \cup K_k$
- Obtain $\hat{f} = f_k$ by inductively define maps $f_i \colon S^m \to T(\xi)$ smooth throughout $f_i^{-1}(T - t_0) = f_0^{-1}(T - t_0)$ such that
  - $f_i = f_{i-1}$ outside compact subset of W_i
  - f_i transverse to B throughout $K_1 \cup \cdots \cup K_i$
  - $\xi \circ f_0 = \xi \circ f_i$ on $f_0^{-1}(T - t_0)$

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \dots, W_k$, images contained in local trivialization of vector bundle ξ
- Cover $f_0^{-1}(B)$ by compacts $K_i \subset W_i$ $(1 \leq i \leq k)$ such that $f_0^{-1}(B) \subset \operatorname{int}(K_1 \cup \cdots \cup K_k)$. $\implies$ $|f_0(x)| \geq c > 0$ outside $K_1 \cup \cdots \cup K_k$
- Obtain $\hat{f} = f_k$ by inductively define maps $f_i \colon S^m \to T(\xi)$ smooth throughout $f_i^{-1}(T - t_0) = f_0^{-1}(T - t_0)$ such that
  - $f_i = f_{i-1}$ outside compact subset of W_i
  - f_i transverse to B throughout $K_1 \cup \cdots \cup K_i$
  - $\xi \circ f_0 = \xi \circ f_i$ on $f_0^{-1}(T - t_0)$
  - $|f_i(x) - f_{i-1}(x)| < c/k$

Thom's Transversality Theorem

Existence.

- Approximate f by f_0, smooth throughout $f_0^{-1}(T - t_0)$
- Cover $f_0^{-1}(B)$ by opens $W_1, \dots, W_k$, images contained in local trivialization of vector bundle ξ
- Cover $f_0^{-1}(B)$ by compacts $K_i \subset W_i$ $(1 \leq i \leq k)$ such that $f_0^{-1}(B) \subset \operatorname{int}(K_1 \cup \dots \cup K_k)$. $\implies |f_0(x)| \geq c > 0$ outside $K_1 \cup \dots \cup K_k$
- Obtain $\hat{f} = f_k$ by inductively define maps $f_i \colon S^m \to T(\xi)$ smooth throughout $f_i^{-1}(T - t_0) = f_0^{-1}(T - t_0)$ such that
    - $f_i = f_{i-1}$ outside compact subset of W_i
    - f_i transverse to B throughout $K_1 \cup \dots \cup K_i$
    - $\xi \circ f_0 = \xi \circ f_i$ on $f_0^{-1}(T - t_0)$
    - $|f_i(x) - f_{i-1}(x)| < c/k$
- Use coordinates $U_i \times \mathbb{R}^k \cong \xi^{-1}(U_i) \supset f_0(W_i)$: Need to construct map $f_i|_{W_i} \colon W_i \to U_i \times \mathbb{R}^k$ transversal to U_i throughout $(K_1 \cup \dots \cup K_{i-1}) \cup K_i$. First coordinate given by third condition. Second coordinate given by lemma.

Independence of $\hat{f}^{-1}(B)$ representative.

□

Thom's Transversality Theorem

Independence of $\hat{f}^{-1}(B)$ representative.

- Given two homotopic maps $\hat{f}_1$ and $\hat{f}_2$, choose homotopy $h_0 \colon S^m \times [0,3] \to T(\xi)$, smooth throughout $h_0^{-1}(T - t_0)$, $h_0(x,t) = \hat{f}_1(x)$ for $t \leq 1$ and $h_0(x,t) = \hat{f}_2(x)$ for $t \geq 2$.

□

Independence of $\hat{f}^{-1}(B)$ representative.

- Given two homotopic maps $\hat{f}_1$ and $\hat{f}_2$, choose homotopy $h_0\colon S^m \times [0,3] \to T(\xi)$, smooth throughout $h_0^{-1}(T - t_0)$, $h_0(x,t) = \hat{f}_1(x)$ for $t \leq 1$ and $h_0(x,t) = \hat{f}_2(x)$ for $t \geq 2$.
- Construct new homotopy $h\colon S^m \times [0,3] \to T(\xi)$ which coincides with h_0 outside compact subset of $S^m \times (0,3)$, transverse to B.

□

Thom's Transversality Theorem

Independence of $\hat{f}^{-1}(B)$ representative.

- Given two homotopic maps $\hat{f}_1$ and $\hat{f}_2$, choose homotopy $h_0 \colon S^m \times [0,3] \to T(\xi)$, smooth throughout $h_0^{-1}(T - t_0)$, $h_0(x,t) = \hat{f}_1(x)$ for $t \leq 1$ and $h_0(x,t) = \hat{f}_2(x)$ for $t \geq 2$.
- Construct new homotopy $h \colon S^m \times [0,3] \to T(\xi)$ which coincides with h_0 outside compact subset of $S^m \times (0,3)$, transverse to B.
- $h^{-1}(B)$ is cobordism between $\hat{f}_1^{-1}(B)$ and $\hat{f}_2^{-1}(B)$

□

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha \colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta \colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof.

□

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{Gr}}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof.

- Clear: $\beta \circ \alpha = \mathrm{id}$ (recall motivation for map β)

□

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof.

- Clear: $\beta \circ \alpha = \mathrm{id}$ (recall motivation for map β)
- Let $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ represent element of $\pi_{n+k}(T(\tilde{\gamma}^k), t_0)$. Assume that f is transverse to $\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty)$.

□

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof.

- Clear: $\beta \circ \alpha = \mathrm{id}$ (recall motivation for map β)
- Let $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ represent element of $\pi_{n+k}(T(\tilde{\gamma}^k), t_0)$. Assume that f is transverse to $\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty)$.
- Choose tubular neighborhood U of $M = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ and disc subbundle $D \subset U$.

□

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof.

- Clear: $\beta \circ \alpha = \mathrm{id}$ (recall motivation for map β)
- Let $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ represent element of $\pi_{n+k}(T(\tilde{\gamma}^k), t_0)$. Assume that f is transverse to $\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty)$.
- Choose tubular neighborhood U of $M = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ and disc subbundle $D \subset U$.
- **Input 1:** f transverse to $B \implies f$ homotopic to a map $\Phi\colon S^{n+k} \to T(\tilde{\gamma}^k)$ which restricts to a bundle map on D

□

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof.

- Clear: $\beta \circ \alpha = \mathrm{id}$ (recall motivation for map β)
- Let $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ represent element of $\pi_{n+k}(T(\tilde{\gamma}^k), t_0)$. Assume that f is transverse to $\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty)$.
- Choose tubular neighborhood U of $M = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ and disc subbundle $D \subset U$.
- **Input 1:** f transverse to $B \implies f$ homotopic to a map $\Phi\colon S^{n+k} \to T(\tilde{\gamma}^k)$ which restricts to a bundle map on D
- $\Phi|_D\colon D \to E(\tilde{\gamma}^k)$ is differential of f at M. Homotopic to $f|_D$ via homotopy $h_t(x) = f(tx)/t$.

□

Proof of Thom's Theorem

Theorem

The Thom-Pontryagin construction $\alpha\colon \Omega_n \to \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ *and* $\beta\colon \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to \Omega_n, f \mapsto \hat{f}^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ *are mutually inverses.*

Proof.

- Clear: $\beta \circ \alpha = \mathrm{id}$ (recall motivation for map β)
- Let $f\colon S^{n+k} \to T(\tilde{\gamma}^k)$ represent element of $\pi_{n+k}(T(\tilde{\gamma}^k), t_0)$. Assume that f is transverse to $\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty)$.
- Choose tubular neighborhood U of $M = f^{-1}(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$ and disc subbundle $D \subset U$.
- **Input 1:** f transverse to $B \implies f$ homotopic to a map $\Phi\colon S^{n+k} \to T(\tilde{\gamma}^k)$ which restricts to a bundle map on D
- $\Phi|_D\colon D \to E(\tilde{\gamma}^k)$ is differential of f at M. Homotopic to $f|_D$ via homotopy $h_t(x) = f(tx)/t$.
- **Input 2:** The Thom-Pontryagin collapse map and Φ agree on D and they map $S^{n+k} - \mathrm{int}\,(D)$ to the contractible space $T(\tilde{\gamma}^k) - \tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty) \implies$ they are homotopic □

Topology of the Thom space

Lemma

If the base space B of ξ admits a CW-structure, then $T(\xi)$ admits a $(k-1)$-connected CW-structure where the $(n+k)$-cells correspond one-to-one to n-cells of B (and one additional base point).

Topology of the Thom space

Lemma

If the base space B of ξ admits a CW-structure, then $T(\xi)$ admits a $(k-1)$-connected CW-structure where the $(n+k)$-cells correspond one-to-one to n-cells of B (and one additional base point).

Proof.

Preimage of open n-cells in B under ξ are open $(n+k)$-cells in E. □

Homotopy and Homology groups modulo $\mathcal{C}$

Definition

Let $\mathcal{C} \subset \mathrm{Ab}$ denote the class of all finite abelian groups. A map $f : A \to B$ of abelian groups is a *$\mathcal{C}$-isomorphism* if $\ker(f) \in \mathcal{C}$ and $\operatorname{coker}(f) \in \mathcal{C}$.

Homotopy and Homology groups modulo $\mathcal{C}$

Definition

Let $\mathcal{C} \subset \mathrm{Ab}$ denote the class of all finite abelian groups. A map $f : A \to B$ of abelian groups is a *$\mathcal{C}$-isomorphism* if $\ker(f) \in \mathcal{C}$ and $\mathrm{coker}(f) \in \mathcal{C}$.

Theorem

Let X be finite $(k-1)$-connected CW-complex for an integer $k \geq 2$. The Hurewicz morphism $\pi_n(X, x_0) \to H_n(X)$ is a $\mathcal{C}$-isomorphism for $n < 2k - 1$.

$\mathcal{C}$-isomorphism $\pi_n(T(\xi), t_0) \to H_{n-k}(B(\xi))$

Corollary

There is a $\mathcal{C}$-isomorphism: $\pi_{n+k}(T(\xi), t_0) \to H_n(B(\xi))$ in degree $n < k - 1$.

$\mathcal{C}$-isomorphism $\pi_n(T(\xi), t_0) \to H_{n-k}(B(\xi))$

Corollary

There is a $\mathcal{C}$-isomorphism: $\pi_{n+k}(T(\xi), t_0) \to H_n(B(\xi))$ in degree $n < k - 1$.

Proof.

- Generalized Hurewicz: There is $\mathcal{C}$-isomorphism $\pi_{n+k}(T(\xi), t_0) \to H_{n+k}(T(\xi))$
- Let T_0 denote the complement of the zero-section in $T(\xi)$. Since T_0 is contractible: $H_{n+k}(T(\xi)) \cong H_{n+k}(T(\xi), T_0)$. By Excision: $H_{n+k}(T(\xi), T_0) \cong H_{n+k}(E(\xi), E_0)$. Thom isomorphism: $H_{n+k}(E(\xi), E_0) \cong H_n(B(\xi))$.

□

Description of Ω_n

Theorem (Thom, 1954)

The oriented cobordism group Ω_n is finite for $4 \nmid n$ and finitely generated of rank $p(r)$ (numbers of partitions of r) if $n = 4r$.

Proof.

- We know that $\Omega_n \cong \pi_{n+k}(T(\tilde{\gamma}^k), t_0)$ for $k \gg 0$
- There is a $\mathcal{C}$-isomorphism $\pi_{n+k}(T(\tilde{\gamma}^k), t_0) \to H_n(\tilde{\mathrm{G}}\mathrm{r}_k(\mathbb{R}^\infty))$.
- This group is finite for $4 \nmid n$ and finitely generated of rank $p(r)$ (number of partitions) if $n = 4r$.

□

Corollary

The graded ring $\Omega_* \otimes \mathbb{Q}$ is a polynomial algebra over $\mathbb{Q}$ with linearly independent generators $\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \ldots$.

Classification of oriented boundaries

Corollary

The multiple of an n-dimensional manifold M is diffeomorphic to an oriented boundary if and only if all Pontrjagin numbers vanish.

Theorem (Wall, 1960)

An n-dimensional manifold M is an oriented boundary if and only if all Pontrjagin numbers and all Stiefel-Whitney classes vanish.

References

Morris W. Hirsch. *Differential topology*. Vol. 33. Graduate Texts in Mathematics. Corrected reprint of the 1976 original. Springer-Verlag, New York, 1994, pp. x+222.

John W. Milnor and James D. Stasheff. *Characteristic classes*. Vol. 76. Annals of Mathematics Studies. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974, pp. vii+331.

Edwin H. Spanier. *Algebraic topology*. Corrected reprint. Springer-Verlag, New York-Berlin, 1981, pp. xvi+528.

René Thom. "Quelques propriétés globales des variétés différentiables". In: *Comment. Math. Helv.* 28 (1954), pp. 17–86.