K-THEORY AND MORITA THEORY

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Abstract. In this paper we provide multiple necessary and sufficient conditions involving ∞-operads for derived categories of rings to be triangulated equivalent by studying the K-theory of unital ∞-operads. This allows us to provide sufficient conditions for categories of module spectra over $\mathbb{E}_\infty$-ring spectra to be equivalent.

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1. Introduction

Carlsson asked when the K-theory of spectrum of a permutative category can be written as a module spectrum over another K-theory spectrum of a bipermutative category. We will study this problem in the "derived" ∞-context, and ask, when is the K-theory of a unital ∞-operad equivalent to the ∞-category of $\mathcal{O}$-modules over an $\mathcal{O}$-algebra A over the K-theory another ∞-operad (here $\mathcal{O}^\otimes$ is an ∞-operad)? In the form of an equation, when is $\mathbf{K}(\mathcal{C}_\infty^\otimes) \simeq \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{D}_\infty^\otimes))$? We will answer this question by proving that there is a fully faithful and essentially surjective functor $\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes) \to \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes))$.

However, we observe that the K-theory takes symmetric monoidal exact ∞-categories, or unital ∞-operads, to stable ∞-categories, i.e., $\mathbf{K} : \mathbf{Exact}_\infty \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$, so we can study the homotopy category $\mathrm{h}\mathbf{K}(\mathcal{C}_\infty^\otimes)$, which acquires a triangulated structure. By our main theorem, which asserts that $\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes) \to \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes))$, we see that $\mathrm{h}\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes)$ obtains the structure of a "derived category of an algebra". So when we're attempting to study Carlsson's question, we can also create a derived Morita theory of algebras over ∞-operads.

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Returning back to Carlsson's question, we see that because of our slightly different of exact ∞-category, we see that in the image $\mathcal{A}$ of the functor $\mathbf{K} : \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Ex}} \to \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Ex}}$, the equivalence $\mathbf{K}(\mathcal{C}_{\infty}^{\otimes}) \simeq \mathrm{Mod}_{A}^{\mathcal{O}}(\mathbf{K}(\mathcal{D}_{\infty}^{\otimes}))$ holds if and only if $\mathcal{C}_{\infty}^{\otimes} \simeq \mathrm{Mod}_{A}^{\mathcal{O}}(\mathcal{C}_{\infty}^{\otimes})$, and then $\mathcal{D}_{\infty}^{\otimes} \simeq_{\mathcal{A}} \mathcal{C}_{\infty}^{\otimes}$.

This paper is structured like so: we define the K-theory in Section 2, and in Section 3, we present the proof of our main theorem, which assists in answering Carlsson's question (as mentioned above). We also construct the derived category of algebras over ∞-operads. In Section 4, we discuss possible future directions, i.e., Goerss-Hopkins obstruction theory for realizing stable ∞-categories as the K-theory of exact ∞-categories.

2. Defining the K-theory

The Quillen Q-construction allows us to define the K-theory of an exact 1-category $\mathcal{C}$. The K-theory functor $\mathbf{K} : \mathbf{Exact} \to \mathbf{Top}$, taking the category of exact 1-categories to the category of topological spaces, is defined by $\mathbf{K} : \mathcal{C} \mapsto \Omega\,|\mathrm{N}\mathbf{Q}(\mathcal{C})|$. We may generalize and define an analog of the Quillen Q-construction for a colored operad $\mathcal{M}$ satisfying certain conditions, where the K-theory $\mathbf{K} : \mathbf{A} \to \mathbf{Sp}$ takes a subcategory $\mathbf{A}$ of the category of colored operads $\mathbf{Op}$ to the category of spectra $\mathbf{Sp}$ (this is because the objects of $\mathbf{Sp}$ are "weakenings" of the objects of $\mathbf{Top}$). It is natural to generalize in a different way, from exact 1-categories to exact ∞-categories, to define the Quillen Q-construction. Barwick has done so in [Bar], which we now outline.

Let X be a simplicial set; we can define another simplicial set by $(\mathscr{O}(X))_n := \mathrm{Map}(\Delta^n \star \Delta^n, X)$, where $\star$ is the concatenation operator on $\mathbf{\Delta}$. The Quillen Q-construction uses $\mathscr{O}(\Delta^n)$, but in order to define the Quillen Q-construction, we have to define ambigressive pullbacks and ambigressive functors.

Let $\mathcal{C}_{\infty}$ be an exact ∞-category[1], and let $\mathcal{C}_{\infty}^{!}$ and $\mathcal{C}_{\infty}^{\bullet}$ be full subcategories of $\mathcal{C}_{\infty}$ containing all the equivalences. Given a pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

we call it ambigressive if $X' \to Y'$ and $Y \to Y'$ are morphisms in $\mathcal{C}_{\infty}^{!}$ and $\mathcal{C}_{\infty}^{\bullet}$, respectively. We call a functor $\mathscr{O}(\Delta^n) \to \mathcal{C}_{\infty}$ ambigressive if for all integers $0 \leq i \leq k \leq l \leq j \leq n$, the pullback square

$$\begin{tikzcd} X_{ij} \arrow[r] \arrow[d] & X_{kj} \arrow[d] \\ X_{il} \arrow[r] & X_{kl} \end{tikzcd}$$

is ambigressive. We may now finally proceed to the Quillen Q-construction: define a simplicial set $\mathbf{Q}(\mathcal{C}_{\infty})$, whose n-simplices are the ambigressive functors $\mathscr{O}(\Delta^n)^{op} \to \mathcal{C}_{\infty}$. The K-theory is then simply $\Omega\mathbf{Q}(\mathcal{C}_{\infty})$, and this defines a functor from the ∞-category $\mathbf{Exact}_{\infty}$ of exact ∞-categories and exact functors between them to the ∞-category $\mathcal{C}\mathrm{at}_{\infty}$ of ∞-categories.

[1]Our definition of an exact ∞-category differs from that of Barwick in that we define an exact ∞-category as a stable ∞-category satisfying certain conditions.

Let $\mathcal{C}_\infty$ be an exact ∞-category. If we equip it with a map $\mathcal{C}_\infty \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ satisfying certain conditions that make it an ∞-operad, we call $\mathcal{C}_\infty$ a unital ∞-operad. To signify that it is equipped with a map $\mathcal{C}_\infty$, we will write it as $\mathcal{C}_\infty^\otimes$. Let $\mathbf{Exact}_\infty^\otimes$ be the subcategory of $\mathbf{Exact}_\infty$ spanned by the unital ∞-category. The K-theory construction for exact ∞-categories passes over to unital ∞-categories, so we can ask what additional structure/properties $\mathbf{K}(\mathcal{C}_\infty^\otimes)$ have? This can be answered by looking at a pattern in the codomain of the K-theory functors; a simple analysis shows that for ∞-operads, the K-theory takes $\mathbf{K} : \mathbf{Exact}_\infty^\otimes \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$, where $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ is the ∞-category of stable ∞-categories and exact ∞-functors between them. This is because the objects of $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ are the analogues of spectra in the ∞-context. Note that what we call exact functors are exact functors between exact ∞-categories, in the sense of [Bar], and what we call exact ∞-functors are exact ∞-functors between stable ∞-categories.

3. K-theory of modules and derived categories

Consider the homotopy category $\mathrm{h}\mathbf{K}(\mathcal{C}_\infty^\otimes)$, which, because $\mathbf{K}(\mathcal{C}_\infty^\otimes)$ is a stable ∞-category, is a triangulated category. We would like to develop some sort of derived Morita theory, and so we'd like to consider the homotopy category of some ∞-operad of module objects.

Lurie has defined such objects in [Lura]; more specifically, he has defined an ∞-operad $\mathrm{Mod}^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes$ of $\mathcal{O}$-module objects over $\mathcal{C}_\infty^\otimes$, and an ∞-category $\mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{C}_\infty^\otimes)$ of $\mathcal{O}$-algebra objects over $\mathcal{C}_\infty^\otimes$, where $\mathcal{O}^\otimes$ is an ∞-category. We can define the ∞-operad $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes$ of $\mathcal{O}$-module objects over an $\mathcal{O}$-algebra object A over $\mathcal{C}_\infty^\otimes$ as the pushout $\mathrm{Mod}^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes \prod_{\mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{C}_\infty^\otimes)}\{A\}$. Since we'd like to provide a derived category structure on $\mathrm{h}\mathbf{K}(\mathcal{C}_\infty^\otimes)$ through $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes$, we will study $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes$ first. We provide two interesting properties that it satisfies, one of which will help us define the derived category of an algebra over an ∞-operad.

The first follows from induction using [Lura, Corollary 3.4.1.9]:

Theorem 3.1. *Let* $(\mathrm{Mod}_A^{\mathcal{O}})^n(\mathcal{C}_\infty^\otimes)$ *denote* $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes$ *iterated* n *times. Then* $(\mathrm{Mod}_A^{\mathcal{O}})^n(\mathcal{C}_\infty^\otimes)$ *is equivalent to* $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes$ *for any* $n \geq 1$.

Consider the identity morphism $\mathrm{id}_{\mathbf{K}(\mathcal{C}_\infty^\otimes)} : \mathbf{K}(\mathcal{C}_\infty^\otimes) \to \mathbf{K}(\mathcal{C}_\infty^\otimes)$, which is an equivalence of categories. Since $\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes \simeq \mathcal{C}_\infty^\otimes$ when $\mathcal{O}^\otimes = \mathbb{E}_0^\otimes$, we expect one of the following three statements to hold true:

(1) There is a fully faithful non-essentially surjective functor $\mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes)) \to \mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes)$.
(2) There is a fully faithful non-essentially surjective functor $\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes) \to \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes))$.
(3) There is a fully faithful essentially surjective functor $\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes) \to \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes))$.

We will proceed to inspect each of these points separately:

(1') *There is a fully faithful non-essentially surjective functor* $\mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes)) \to \mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes)$. This induces a map $\mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes)) \to \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes))$, which implies the existence of a forgetful functor $\mathbf{K}(\mathcal{C}_\infty^\otimes) \to \mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes))$, and this is obviously false, meaning that there is no fully faithful non-essentially surjective functor $\mathrm{Mod}_A^{\mathcal{O}}(\mathbf{K}(\mathcal{C}_\infty^\otimes)) \to \mathbf{K}(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}_\infty^\otimes)^\otimes)$. When $\mathcal{O}^\otimes = \mathbb{E}_0^\otimes$, this means that there is no fully faithful non-essentially surjective functor $\mathbf{K}(\mathcal{C}_\infty^\otimes) \to \mathbf{K}(\mathcal{C}_\infty^\otimes)$.

(2') *There is a fully faithful non-essentially surjective functor* $\mathbf{K}(\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C}^{\otimes}_{\infty})^{\otimes}) \to \mathrm{Mod}^{\mathcal{O}}_A(\mathbf{K}(\mathcal{C}^{\otimes}_{\infty}))$. This reduces to the statement that there is a fully faithful non-essentially surjective functor $\mathbf{K}(\mathcal{C}^{\otimes}_{\infty}) \to \mathbf{K}(\mathcal{C}^{\otimes}_{\infty})$, and we just showed this to be false. This implies that the only left option must hold true:

(3') *There is a fully faithful essentially surjective functor* $\mathbf{K}(\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C}^{\otimes}_{\infty})^{\otimes}) \to \mathrm{Mod}^{\mathcal{O}}_A(\mathbf{K}(\mathcal{C}^{\otimes}_{\infty}))$.

We will state this as a theorem to emphasize that this is a very important result:

Theorem 3.2. *There is a fully faithful essentially surjective functor* $\mathbf{K}(\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C}^{\otimes}_{\infty})^{\otimes}) \to \mathrm{Mod}^{\mathcal{O}}_A(\mathbf{K}(\mathcal{C}^{\otimes}_{\infty}))$.

Returning to derived categories, we see that we can define the derived category $\mathcal{D}(A)$ to be the homotopy category $\mathrm{h}\mathbf{K}(\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C}^{\otimes}_{\infty})^{\otimes})$, because of two reasons:

(i) $\mathbf{K}(\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C}^{\otimes}_{\infty})^{\otimes})$ is a stable ∞-category, and so its homotopy category must have the structure of a triangulated category.

(ii) $\mathbf{K}(\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C}^{\otimes}_{\infty})^{\otimes})$ has the structure of an ∞-operad of modules by Theorem 3.2, so its homotopy category must be similar to the derived category of an algebra.

Derived Morita theory is concerned with the following question:

Question 3.3. When are the derived categories $\mathcal{D}(A)$ and $\mathcal{D}(A')$ equivalent as triangulated categories?

In order to answer this question, we'll introduce a model structure on the homotopy category $\mathrm{h}\mathbf{K}(\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C}^{\otimes}_{\infty})^{\otimes})$.

The category $\mathrm{h}\mathbf{K}(\mathrm{Mod}^{\mathcal{O}}_A(\mathcal{C}^{\otimes}_{\infty})^{\otimes})$ admits finite limits and colimits, which allows us to define the model structure on it. We will define the cofibrations and fibrations as the isomorphisms. Let $f : \overline{v} \to \overline{v}'$ be a morphism in $\mathrm{Mod}^{\mathcal{O}}(\mathcal{C}^{\otimes}_{\infty})^{\otimes}$. We call f a weak equivalence if for any map $g : \overline{v} \to \overline{v}'$, there is a 2-simplex:

$$\begin{tikzcd}
\overline{v}' \arrow[r, "\mathrm{id}_{\overline{v}'}"] & \overline{v}' \\
\overline{v} \arrow[u, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

We are now ready to state our theorem regarding the derived Morita theory of algebras over ∞-operads:

Theorem 3.4. *Let* $F : \mathrm{Mod}^{\mathcal{O}}_A(\mathbf{K}(\mathcal{C}^{\otimes}_{\infty})) \to \mathrm{Mod}^{\mathcal{O}'}_{A'}(\mathbf{K}(\mathcal{C}^{\otimes}_{\infty}))$ *be a functor that induces a map between homotopy categories* $\mathbf{L}F : \mathrm{hMod}^{\mathcal{O}}_A(\mathbf{K}(\mathcal{C}^{\otimes}_{\infty})) \to \mathrm{hMod}^{\mathcal{O}'}_{A'}(\mathbf{K}(\mathcal{C}^{\otimes}_{\infty}))$*, and hence a map between the derived categories* $\mathbf{L}F : \mathcal{D}(A) \to \mathcal{D}(A')$*. The following statements are equivalent:*

(1) F *is an equivalence of* ∞*-categories.*

(2) $\mathbf{L}F$ *is a Quillen equivalence.*

(3) $\mathbf{L}F$ *is a triangulated equivalence of derived categories.*

4. Future Directions

There seems to be a relation between GHOsT and the K-theory studied in this paper, either directly, or by studying GHOsT by studying Hopf-Galois extensions, in the ∞-context, of the K-theories of unital ∞-operads. This will be studied in future papers.

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