REPRESENTATIONS OF THE SYMMETRIC GROUP

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Abstract. I will introduce the topic of representation theory of finite groups by investigating representations of S_3 and S_4 using character theory. Then I will generalize these examples by describing all irreducible representations of any symmetric group on n letters. Finally, I will briefly discuss how to discover irreducible representations of any group using Schur Functors, which are constructed using the irreducible representations of S_n. This paper assumes familiarity with group theory, FG-modules, linear algebra, and category theory.

Contents

1. Introduction

A *representation* of a group G is a homomorphism ρ from G into the general linear group of some vector space V. When such a map exists, if we write $g \cdot v$ rather than $\rho(g)(v)$ to denote the image of a vector v in V under the automorphism $\rho(g)$, then we can think of V as a $\mathbb{C}G$-module, since for any vectors v and w, any group elements g and h, and any scalar c, the following properties hold:

(1) $\rho(g)(v) = g \cdot v$ is in V.
(2) $\rho(1) = I \in GL(V)$ by the homomorphism property, so $\rho(1)(v) = I(v) = v$, which means $1 \cdot v = v$.
(3) $(\rho(g) \circ \rho(h))(v) = \rho(g)(\rho(h)(v))$, so $(gh) \cdot v = g \cdot (h \cdot v)$.
(4) $\rho(g)(cv) = c\rho(g)(v)$ by linearity, so $g \cdot cv = c(g \cdot v)$.
(5) $\rho(g)(v + w) = \rho(g)(v) + \rho(g)(w)$ by linearity, so $g \cdot (v + w) = g \cdot v + g \cdot w$.

When we know the particular $\mathbb{C}G$-module structure of V, we often call V itself a representation of G. A subspace W of V is a *subrepresentation* if it is invariant under the action of G. A representation V of G is *irreducible* if it has no proper nontrivial subrepresentations. Otherwise, it is *reducible*. As we will soon see, every representation of a finite group over a finite-dimensional complex vector space can be expressed as a direct sum of irreducible representations (over finite-dimensional

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complex vector spaces) of the group. Two main goals of representation theory are to describe all irreducible representations of a given group G and to find a sure-fire method of decomposing any arbitrary representation of G into a direct sum of irreducible representations. We will accomplish both of these goals for the symmetric group on n letters S_n.

2. Basic Definitions and Complete Reducibility

We begin by describing some common representations and a few ways of constructing new representations from known ones. Assume that all groups mentioned in this paper (except in Section 5) are finite and that all vector spaces mentioned are finite-dimensional over $\mathbb{C}$.

Definition 2.1. The *trivial representation* of a finite group G is $\mathbb{C}$ equipped with the trivial action of G: $gx = x$ for every x in $\mathbb{C}$ and for every g in G. Note that every finite group has the trivial representation, and since $\mathbb{C}$ has no proper nontrivial subspaces, it is irreducible, as is any one-dimensional representation.

Definition 2.2. Let X be any finite G-set. Let W be the vector space generated by the basis $\{e_x \mid x \in X\}$. Define the action of G on W by

$$g \cdot (a_1 e_{x_1} + a_2 e_{x_2} + \ldots + a_m e_{x_m}) = a_1 e_{gx_1} + a_2 e_{gx_2} + \ldots + a_m e_{gx_m}.$$

Such a W is called a *permutation representation* of G. Notice that the subspace spanned by the vector $e_{x_1} + e_{x_2} + \cdots + e_{x_m}$ is invariant under the action of G because each element of G simply "shuffles" the addends but does not change the sum. Thus, every permutation representation has a nontrivial subrepresentation and is therefore reducible.

Definition 2.3. In Definition 2.2, replace the arbitrary G-set X by G itself under the action of left multiplication. In this case, W is called the *regular representation* of G.

Definition 2.4. For a symmetric group S_n, the *alternating representation* is $\mathbb{C}$ equipped with the action

$$\sigma \cdot v = \begin{cases} v, & \text{if } \sigma \text{ is an even permutation} \\ -v, & \text{if } \sigma \text{ is an odd permutation} \end{cases}$$

or equivalently, $\rho(\sigma) = \mathrm{sgn}(\sigma) I$ for every σ in S_n. Note that any S_n where $n \geq 2$ has the alternating representation, and since this representation is one-dimensional, it is irreducible.

Definition 2.5. For any n, let $\{e_1, e_2, \ldots e_n\}$ be the standard basis for $\mathbb{C}^n$. Define the action of S_n on $\mathbb{C}^n$ to be

$$\sigma(a_1 e_1 + a_2 e_2 + \cdots + a_n e_n) = a_1 e_{\sigma(1)} + a_2 e_{\sigma(2)} + \cdots + a_n e_{\sigma(n)}.$$

This is a permutation representation of S_n. Again, notice that the one-dimensional subspace of $\mathbb{C}$ spanned by $e_1 + e_2 + \cdots + e_n$ is invariant under the action of S_n. Therefore, $\langle e_1 + e_2 + \cdots + e_n \rangle$ is a subrepresentation of $\mathbb{C}^n$. Its orthogonal complement $V = \{(x_1, x_2, \ldots, x_n) \mid x_1 + x_2 + \cdots + x_n = 0\}$ is also invariant and therefore a subrepresentation. V is called the *standard representation* of S_n.

Remark 2.6. It can be shown that given any representation V of a group G, its dual space, its symmetric powers, and its alternating powers are also representations of G. Furthermore, given two representations V and W of G, their tensor product and direct sum are also representations, as is the vector space Hom(V,W). In Section 5, we will see how we can construct new representations (of any group) from old using special functors.

The following results show that the irreducible representations of a given group G are the "building blocks" for all of its other representations.

Lemma 2.7. *Any representation V of a finite group G can be given a G-invariant inner product, meaning that for any h in G and for any v_1, v_2 in V, $\langle hv_1, hv_2\rangle = \langle v_1, v_2\rangle$.*

Proof. Let $\langle\cdot,\cdot\rangle_*$ be any positive-definite Hermitian inner product on V. Define a new Hermitian inner product on V in the following way:

$$\langle v_1, v_2\rangle = \frac{1}{|G|}\sum_{g\in G}\langle gv_1, gv_2\rangle_* \text{ for any } v_1, v_2 \in V.$$

Now see that for any h in G and for any v_1, v_2 in V, we have:

$$\begin{aligned}
\langle hv_1, hv_2\rangle &= \frac{1}{|G|}\sum_{g\in G}\langle g(hv_1), g(hv_2)\rangle_* \\
&= \frac{1}{|G|}\sum_{g\in G}\langle (gh)v_1, (gh)v_2\rangle_* \\
&= \frac{1}{|G|}\sum_{g\in G}\langle gv_1, gv_2\rangle_* \text{ because for every } g \in G,\ gh \text{ is in } G \text{ also} \\
&= \langle v_1, v_2\rangle.
\end{aligned}$$

□

Lemma 2.8. *Let V be a representation of a finite group G, and let W be a subrepresentation of V. Then $W^\perp$, the orthogonal complement to W inside V under the G-invariant inner product described above, is also a subrepresentation of V.*

Proof. Fix an x in $W^\perp = \{x \in V \mid \langle w, x\rangle = 0 \text{ for every } w \in W\}$. Because our inner product is G-invariant, $\langle w, gx\rangle = \langle w, x\rangle = 0$ for any g in G and for any w in W. Therefore, for any x in $W^\perp$, gx is in $W^\perp$ also. This shows that $W^\perp$ is invariant under the action of G. □

Exercise 2.9. Using Lemma 2.8, show that any representation of a finite group G is the direct sum of irreducible representations.

Definition 2.10. Let V and W be representations of a finite group G. A map between V and W as representations is a vector space map $\phi : V \to W$ such that $g\cdot\phi(v) = \phi(g\cdot v)$ for every v in V and for every g in G.

Proposition 2.11. *The kernel of ϕ is a subrepresentation of V, and the image of ϕ is a subrepresentation of W.*

Proof. Fix a g in G. Since ϕ is a map between representations, for any x in V we have $\phi(g\cdot x) = g\cdot(\phi(x)) = g\cdot(0) = \rho(g)(0) = 0$ by linearity of the transformation $\rho(g)$. Thus, kerϕ is invariant under the action of G.

Again, fix a g in G. Let z be any vector in the image of ϕ. Then $g \cdot z = g \cdot \phi(x)$ for some x in V. And since ϕ is a map between representations, we have $g \cdot \phi(x) = \phi(g \cdot x)$, which is in the image since for every x in V, $g \cdot x$ is also in V. Therefore, $\mathrm{im}\phi$ is also invariant under the action of G. □

Lemma 2.12. ***(Schur's Lemma)*** *Let X and Y be two distinct, irreducible representations of G, and let $\phi : X \to Y$ be a map of representations between them. Then ϕ is either an isomorphism or the zero map.*

Proof. By Proposition 2.11, $\ker\phi$ and $\mathrm{im}\phi$ are subrepresentations of X and Y, respectively. Since X is irreducible, it can have no proper nontrivial subrepresentations. Therefore, $\ker\phi$ is either $\{0\}$ or all of X. If $\ker\phi = \{0\}$, then ϕ is one-to-one. Since Y is also irreducible, again, either $\mathrm{im}\phi = \{0\}$ or $\mathrm{im}\phi = Y$. If the image is all of Y, then ϕ is onto, making it an isomorphism. If the image is $\{0\}$, then ϕ is the zero map. If, on the other hand, $\ker\phi = X$, then ϕ sends every vector in X to 0, making it the zero map. □

Theorem 2.13. *Any representation V of a finite group G can be written uniquely as a direct sum of the form*

$$V = V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where the V_i's are distinct irreducible representations of V and the multiplicity a_i of each V_i is unique.

Proof. Suppose V can also be expressed as

$$V = W_1^{\oplus b_1} \oplus W_2^{\oplus b_2} \oplus \cdots \oplus W_m^{\oplus b_m}.$$

Let $\phi : V \to V$ be the identity map. Then by Schur's Lemma, restricted to each irreducible component $V_i^{\oplus a_i}$, ϕ is an isomorphism between $V_i^{\oplus a_i}$ and the component $W_j^{\oplus b_j}$ for which V_i is isomorphic to W_j. □

3. Characters and Hands-on Examples Involving Them

This section introduces characters and how they can be used to find irreducible representations and decompositions of arbitrary representations.

Definition 3.1. Let V be a representation of a finite group G, and let ρ be the associated group homomorphism. The *character* χ_V of V is the function from G into $\mathbb{C}$ given by

$$\chi_V(g) = \mathrm{Tr}(\rho(g))$$

Remark 3.2. Since conjugate matrices have the same trace, for a fixed element g in G, $\mathrm{Tr}(\rho(g)) = \mathrm{Tr}(\rho(h)\rho(g)\rho(h^{-1})) = \mathrm{Tr}(\rho(hgh^{-1}))$ for any h in G. This follows from the linearity of ρ. Therefore, the character of any representation V of G is, in fact, a *class function*, which is a function that is constant on conjugacy classes.

It turns out that computing the characters of permutation representations is very easy if we use the next theorem:

Theorem 3.3. ***(The Fixed Point Formula)*** *Let G be a finite group, and let X be a finite G-set. Let V be the associated permutation representation of G as described in Definition 2.2. Then for every element g in G, $\chi_V(g)$ is the number of elements in X left fixed by the action of g.*

Proof. The matrix M associated with the action of g is a permutation matrix. For example, suppose that X has four elements x_1, x_2, x_3, and x_4 and that $\rho(g)$ permutes the basis vectors of V by sending e_{x_1} to e_{x_3}, e_{x_2} to itself, e_{x_3} to e_{x_1}, and e_{x_4} to itself. Then M is the 4×4 matrix

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In general, if $ge_{x_i} = e_{x_j}$ for some x_i in X, then M will have a 1 in the i-th column and j-th row, and zeroes in all other entries of that column. In particular, if $gx_i = x_i$, then $ge_{x_i} = e_{gx_i} = e_{x_i}$, meaning that M has a 1 in the i-th row and i-th column. Therefore, the trace of M is the number of 1's along the diagonal, which is exactly the number of points left fixed by g. □

Proposition 3.4. *For any representations V and W of a group G, $\chi_{V \oplus W} = \chi_V + \chi_W$ and $\chi_{V \otimes W} = \chi_V \cdot \chi_W$.*

Proof. This is left as an exercise, but it simply involves counting the number of eigenvalues of any transformation $\rho(g)$. □

The set of class functions on a finite group G is a vector space. We can define the following inner product on this space by:

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g)$$

Theorem 3.5 and Corollaries 3.6 through 3.9 below allow us to decompose any representation of a finite group by applying this inner product to characters. Proofs can be found in [3] on pages 16, 17, and 22 and in [4] on pages 137 through 144.

Theorem 3.5. *The set of character functions of the irreducible representations of G is orthonormal with respect to this inner product.*

Corollary 3.6. *Any representation of G is determined by its character.*

Corollary 3.7. *A representation V of G is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.*

Corollary 3.8. *Let W be any representation of G. As we know, W can be written uniquely as a direct sum of irreducible representations in the form*

$$W = V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus \cdots \oplus V_k^{\oplus a_k}.$$

In this decomposition,

- $a_i = \langle \chi_W, \chi_{V_i} \rangle$ *for every* i,
- $\langle \chi_W, \chi_W \rangle = \sum_{i=1}^k a_i^2$,
- $\sum_{i=1}^k (dim(V_i))^2 = |G|$, *and*
- *for any g in G that is not the identity,* $\sum_{i=1}^k (dim V_i) \cdot \chi_{V_i}(g) = 0$.

Corollary 3.9. *The number of irreducible representations of a finite group G is equal to the number of conjugacy classes in G.*

Example 3.10. Let's see what these results can tell us about the group S_3. We know right away from Corollary 3.9 that S_3 has exactly three irreducible representations. This is because in a symmetric group, each equivalence class of permutations of a certain cycle type constitutes a conjugacy class, and S_3 has three cycle-types; namely, the identity, transpositions, and 3-cycles. The trivial and alternating representations (let's denote them by U and U', respectively) of S_3 are irreducible because they are one-dimensional. Since in the trivial representation $\rho(g)$ is the 1×1 identity matrix [1] for every g, we know that $\chi_U(g) = \mathrm{Tr}[1] = 1$ for every g. In the alternating representation, we have $\chi_{U'}(\iota) = \mathrm{Tr}[1] = 1$, $\chi_{U'}(12) = \mathrm{Tr}[-1] = -1$, and $\chi_{U'}(123) = \mathrm{Tr}[1] = 1$. There now remains only one more irreducible representation for us to find. It cannot be any sort of permutation representation, because those are all reducible. Thus, it is worth checking the standard representation $V = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$. By definition, $\mathbb{C}^3 = \langle e_1 + e_2 + e_3\rangle \oplus V$. Note that $\langle e_1 + e_2 + e_3\rangle$ under the permutation action of S_3 is isomorphic to $\mathbb{C}$ under the trivial action of S_3, so it is the trivial representation U. Since S_3 acts on $\mathbb{C}^3$ by permuting the three standard basis vectors, by the Fixed-Point Formula we have $\chi_{\mathbb{C}^3}(\iota) = 3$, $\chi_{\mathbb{C}^3}(12) = 1$, and $\chi_{\mathbb{C}^3}(123) = 0$. Also, because $\mathbb{C}^3 \cong U \oplus V$, by Proposition 3.4 we have $\chi_{\mathbb{C}^3} = \chi_U + \chi_V$. Therefore, $\chi_{\mathbb{C}^3}$ has values $3 - 1 = 2$, $1-1 = 0$, and $0-1 = -1$ on the conjugacy classes of ι, (12) and (123), respectively. Notice that

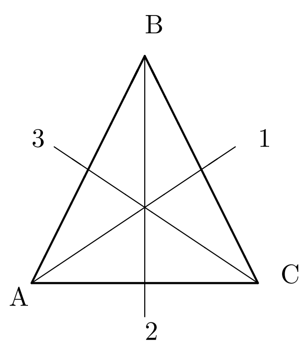
$$\langle \chi_{\mathbb{C}^3}, \chi_{\mathbb{C}^3}\rangle = \frac{1}{|S_3|}(2^2 \cdot 1 + 0^2 \cdot 3 + (-1)^2 \cdot 2) = \frac{1}{6} \cdot 6 = 1.$$

Therefore, by Corollary 3.7, V is irreducible. We found it! We can summarize the work we did into the following simple array called the *character table* for S_3:

Conjugacy Class Sizes	1	3	2
Representatives	ι	(12)	(123)
Trivial Representation U	1	1	1
Alternating Representation U'	1	-1	1
Standard Representation V	2	0	-1

Remark 3.11. Notice that for any representation W of a finite group G, $\chi_W(1) = \mathrm{Tr}(\rho(1)) = \mathrm{Tr}(I)$. This is simply the number of columns of I, which is the dimension of W. Thus, the first column of the character table above shows the dimension of each irreducible representation. We can use this and other handy information from the table to decompose any other representation of S_3.

Example 3.12. Since S_3 acts on the vertices of the equilateral triangle by permuting its three diagonals, there is a permutation representation associated with the resulting action on the set of vertices. If we label our triangle as shown,



then we see that the transposition (12) acts on the triangle by interchanging diagonals 1 and 2. This is a reflection about diagonal 3, and it interchanges vertices

A and B but leaves vertex C fixed. Similarly, the other transpositions in S_3 each leave exactly one vertex fixed. The element (123) rotates the triangle by 120 degrees clockwise and therefore leaves no vertices fixed. Finally, the identity element leaves all three vertices fixed. By the Fixed Point Formula, then, the character of this representation has values 3, 1, and 0 on the conjugacy classes consisting of the identity, transpositions, and 3-cycles, respectively. Therefore, we have

$$\langle \chi_W, \chi_W \rangle = \frac{1}{|S_3|}(3^2 \cdot 1 + 1^2 \cdot 3 + 0^2 \cdot 2) = \frac{1}{6}(9+3+0) = \frac{1}{6} \cdot 12 = 2.$$

By Corollary 3.8, $\langle \chi_W, \chi_W \rangle$ is the sum of the squares of the multiplicities of the irreducible representations in the decomposition of W. Since $2 = 1^2 + 1^2$, and this is the only way to write 2 as the sum of squares of nonnegative integers, it must be that W is the direct sum of exactly two distinct irreducible representations of S_3. To find out which two, we simply take the product of χ_W with the character of each irreducible representation as follows:

$$\langle \chi_W, \chi_U \rangle = \frac{1}{6}(3 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 3 + 0 \cdot 1 \cdot 2) = 1,$$

$$\langle \chi_W, \chi_{U'} \rangle = \frac{1}{6}(3 \cdot 1 \cdot 1 + 1 \cdot (-1) \cdot 3 + 0 \cdot 1 \cdot 2) = 0,$$

$$\langle \chi_W, \chi_V \rangle = \frac{1}{6}(3 \cdot 2 \cdot 1 + 1 \cdot 0 \cdot 3 + 0 \cdot (-1) \cdot 2) = 1$$

By Corollary 3.8, the trivial representation U appears in the decomposition of W once, the standard representation V appears once, and the alternating representation U' does not appear at all. Therefore, we have the decomposition $W \cong U \oplus V$.

Example 3.13. As mentioned before, any tensor power of a known representation of a group is again a representation. Therefore, the n-th tensor power of the standard representation V of S_3 is another representation of S_3. To find its decomposition, first note that Proposition 3.4 implies $\chi_{V^{\otimes n}} = (\chi_V)^n$. Therefore, we have $\chi_{V^{\otimes n}}(\iota) = 2^n, \chi_{V^{\otimes n}}((12)) = 0^n = 0$, and $\chi_{V^{\otimes n}}((123)) = (-1)^n$. Take the product of $\chi_{V^{\otimes n}}$ with the character of each irreducible:

$$\langle \chi_{V^{\otimes n}}, \chi_U \rangle = \frac{1}{6}(2^n \cdot 1 \cdot 1 + 0 \cdot 1 \cdot 3 + (-1)^n \cdot 1 \cdot 2) = \frac{1}{6}(2^n + 2(-1)^n) = \frac{1}{3}(2^{n-1} + (-1)^n),$$

$$\langle \chi_{V^{\otimes n}}, \chi_{U'} \rangle = \frac{1}{6}(2^n \cdot 1 \cdot 1 + 0 \cdot (-1) \cdot 3 + (-1)^n \cdot 1 \cdot 2) = \frac{1}{6}(2^n + 2(-1)^n) = \frac{1}{3}(2^{n-1} + (-1)^n),$$

$$\langle \chi_{V^{\otimes n}}, \chi_V \rangle = \frac{1}{6}(2^n \cdot 2 \cdot 1 + 0 \cdot 0 \cdot 3 + (-1)^n \cdot (-1) \cdot 2) = \frac{1}{6}(2^{n+1} + 2(-1)^{n+1}) = \frac{1}{3}(2^n + (-1)^{n+1}).$$

This gives us the decomposition

$$V^{\otimes n} \cong U^{\frac{1}{3}(2^{n-1}+(-1)^n)} \oplus U'^{\frac{1}{3}(2^{n-1}+(-1)^n)} \oplus V^{\frac{1}{3}(2^n+(-1)^{n+1})}.$$

Example 3.14. Now let's compute the character table of S_4. Since there are five cycle-types (i.e., conjugacy classes) in S_4, we know right away that S_4 has exactly five irreducible representations. The trivial and alternating representations U and U' are irreducible, and we know their characters. The next easy candidate to check is the standard representation V. It is left as an exercise to show that V and the representation $V' = V \otimes U$ are both irreducible. (You will need Proposition 3.4 and Corollary 3.7.) It remains now to find the character values of the final irreducible representation, which we will call W. Let a, b, c, d, and e denote the values of χ_W on the conjugacy classes of $\iota, (12), (123), (1234)$ and $(12)(34)$, respectively. Corollary 3.8 and the entries in the first column that we know so far give us

$$\sum_{i=1}^{5}(\dim(V_i))^2 = |S_4| \Longrightarrow 1^2+1^2+3^2+3^2+a^2 = 24 \Longrightarrow a^2 = 4 \Longrightarrow a = 2.$$

Now we can use the last part of Corollary 3.8 to compute b:

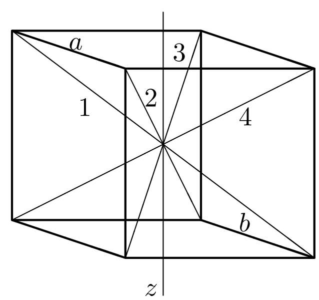
$$\sum_{i=1}^{5}(\dim\, V_i)\cdot\chi_{V_i}(g) = 0 \Longrightarrow 1\cdot 1+(-1)\cdot 1+1\cdot 3+(-1)\cdot 3+b\cdot 2 = 0 \Longrightarrow 2b = 0 \Longrightarrow b = 0.$$

We can compute c, d and e in the same way, giving us the complete character table:

	1	6	8	6	3
S_4	ι	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1
V'	3	-1	0	1	-1
W	2	0	-1	0	2

Although for now the only thing we know about W is its character, in Section 4 we will learn the tools for explicitly describing the vector space W itself.

Example 3.15. S_4 acts on the faces, vertices and edges of the cube by permuting the four long diagonals inside the cube and thus has a permutation representation associated with each action. We will decompose each representation. Label the cube as shown:



Faces: Call this permutation representation X. The identity permutation does not move the cube at all, leaving the 6 faces fixed. The transposition (12) rotates the cube 180 degrees about the axis connecting the midpoints of edges a and b. This motion moves every face to a new location and therefore fixes no faces. The 3-cycle (123) rotates the cube 120 degrees about long diagonal number 4 and also fixes no faces. The 4-cycle (1234) rotates the cube 90 degrees about the axis z, leaving the top and bottom faces fixed but moving the other four around. Finally, (12)(34) rotates the cube 180 degress about z, again leaving only the top and bottom faces fixed. Therefore, by the Fixed Point Formula, χ_X has the values 6, 0, 0, 2, 2 on the conjugacy classes of ι, (12), (123), (1234), and (12)(34), respectively. This gives us

$$\langle\chi_X,\chi_X\rangle = \frac{1}{|S_4|}(6^2\cdot 1+0^2\cdot 6+0^2\cdot 8+2^2\cdot 6+2^2\cdot 3) = \frac{1}{24}(36+24+12) = \frac{1}{24}\cdot 72 = 3.$$

Since the only way to express 3 as the sum of squares is $3 = 1^2 + 1^2 + 1^2$, we know that X is the direct sum of exactly three irreducible representations. See that

$$\langle \chi_X, \chi_U \rangle = \frac{1}{|S_4|}(6{\cdot}1{\cdot}1{+}0{\cdot}1{\cdot}6{+}0{\cdot}1{\cdot}8{+}2{\cdot}1{\cdot}6{+}2{\cdot}1{\cdot}3) = \frac{1}{24}(6{+}12{+}6) = \frac{1}{24}{\cdot}24 = 1,$$

$$\langle \chi_X, \chi_{U'} \rangle = \frac{1}{|S_4|}(6{\cdot}1{\cdot}1{+}0{\cdot}(-1){\cdot}6{+}0{\cdot}1{\cdot}8{+}2{\cdot}(-1){\cdot}6{+}2{\cdot}1{\cdot}3) = \frac{1}{24}(6{-}12{+}6) = \frac{1}{24}{\cdot}0 = 0,$$

$$\langle \chi_X, \chi_V \rangle = \frac{1}{|S_4|}(6{\cdot}3{\cdot}1{+}0{\cdot}1{\cdot}6{+}0{\cdot}0{\cdot}8{+}2{\cdot}(-1){\cdot}6{+}2{\cdot}(-1){\cdot}3) = \frac{1}{24}{\cdot}(18{-}12{-}6) = \frac{1}{24}{\cdot}0 = 0,$$

$$\langle \chi_X, \chi_{V'} \rangle = \frac{1}{|S_4|}(6{\cdot}3{\cdot}1{+}0{\cdot}(-1){\cdot}6{+}0{\cdot}0{\cdot}8{+}2{\cdot}1{\cdot}6{+}2{\cdot}(-1){\cdot}3) = \frac{1}{24}{\cdot}(18{+}12{-}6) = \frac{1}{24}{\cdot}24 = 1.$$

At this point, we know that X must contain a copy of W. Therefore, we have the decomposition $X = U \oplus V' \oplus W$.

Vertices: Call this permutation representation Y. The identity permutation fixes all 8 vertices. Transpositions fix no vertices. Each 3-cycle fixes 2 vertices. Each 4-cycle fixes no vertices. Finally, elements in the conjugacy class of (12)(34) do not fix any vertices, either. Therefore, χ_Y has the values 8, 0, 2, 0, 0 on the respective conjugacy classes, which gives us

$$\langle \chi_Y, \chi_Y \rangle = \frac{1}{|S_4|}(8^2 \cdot 1 + 0^2 \cdot 6 + 2^2 \cdot 8 + 0^2 \cdot 6 + 0^2 \cdot 3) = \frac{1}{24}(64 + 32) = \frac{1}{24} \cdot 96 = 4.$$

Since the possible ways of writing 4 as the sum of squares of nonnegative integers are $4 = 2^2$ and $4 = 1^2 + 1^2 + 1^2 + 1^2$, Y is either the direct sum of four irreducible representations or the direct sum of two copies of one irreducible representation. To find out which case it is, we simply compute:

$$\langle \chi_Y, \chi_U \rangle = \frac{1}{|S_4|}(8{\cdot}1{\cdot}1{+}0{\cdot}1{\cdot}6{+}2{\cdot}1{\cdot}8{+}0{\cdot}1{\cdot}6{+}0{\cdot}1{\cdot}3) = \frac{1}{24}(8{+}16) = \frac{1}{24}{\cdot}24 = 1,$$

$$\langle \chi_Y, \chi_{U'} \rangle = \frac{1}{|S_4|}(8{\cdot}1{\cdot}1{+}0{\cdot}(-1){\cdot}6{+}2{\cdot}1{\cdot}8{+}0{\cdot}(-1){\cdot}6{+}0{\cdot}1{\cdot}3) = \frac{1}{24}(8{+}16) = \frac{1}{24}{\cdot}24 = 1,$$

$$\langle \chi_Y, \chi_V \rangle = \frac{1}{|S_4|}(8 \cdot 3 \cdot 1 + 0 \cdot 1 \cdot 6 + 2 \cdot 0 \cdot 8 + 0 \cdot (-1) \cdot 6 + 0 \cdot (-1) \cdot 3) = \frac{1}{24} \cdot 24 = 1,$$

$$\langle \chi_Y, \chi_{V'} \rangle = \frac{1}{|S_4|}(8 \cdot 3 \cdot 1 + 0 \cdot (-1) \cdot 6 + 2 \cdot 0 \cdot 8 + 0 \cdot 1 \cdot 6 + 0 \cdot (-1) \cdot 3) = \frac{1}{24} \cdot 24 = 1.$$

Again, we don't need to compute $\langle \chi_Y, \chi_W \rangle$. We already know the decomposition: $Y = U \oplus U' \oplus V \oplus V'$.

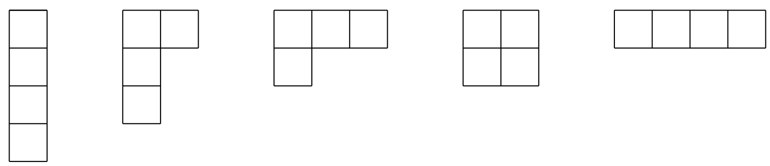
Edges: Call this permutation representation Z. By following the same process as described above, we can find that χ_Z has the values 12, 2, 0, 0, 0 on the respective conjugacy classes, and therefore Z decomposes as $Z = U \oplus V^{\oplus 2} \oplus V' \oplus W$.

4. Representations of S_n

Recall that S_n has exactly as many irreducible representations as it does conjugacy classes, and each conjugacy class is a cycle-type equivalence class. Now, there is a one-to-one correspondence between the set of cycle-types and the ways to write n as the sum of positive integers. For example, S_4 has the following five cycle-types:

Permutation of the Form	Can Also be Written As	Corresponds to the Sum
ι	(1)(2)(3)(4)	1 + 1 + 1 + 1
(12)	(12)(3)(4)	2 + 1+ 1
(123)	(123)(4)	3 + 1
(12)(34)	(12)(34)	2 + 2
(1234)	(1234)	4

A fundamental tool for studying the representations of S_n is the *Young Diagram*, which is an array of boxes: for a given partition $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ (which we denote by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$) where $\lambda_i \geq \lambda_{i+1}$ for every $1 \leq i \leq k$, we draw a row of λ_1 boxes. Beneath it, we draw a row of λ_2 boxes, and so on so that the last row contains λ_k boxes and each row is as long as or shorter than the one above it. Thus, the Young Diagrams corresponding to the partitions of 4 are:



Now that we see how each conjugacy class of S_n corresponds to a Young Diagram, let's look at an algorithm that actually generates all of the irreducible representations of S_n. (For a proof of why this method works, see Section 7.2 of [2] and Section 4.2 of [3].) First, let's pick a particular partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n and label the boxes of the corresponding Young Diagram in order, from left to right and from the top down, with the integers 1 through n. For example, if we were working in S_6 and had chosen to find the irreducible representation associated with the partition $\lambda = (3, 2, 1)$, then we would number our boxes like this:

$$\begin{array}{ll} \lambda_1 = 3 & \boxed{1}\boxed{2}\boxed{3} \\ \lambda_2 = 2 & \boxed{4}\boxed{5} \\ \lambda_3 = 1 & \boxed{6} \end{array}$$

Define the following two sets, which are, in fact, subgroups of S_n:

$$\begin{aligned} P_\lambda &= \{\sigma \in S_n \mid \sigma \text{ preserves the set of numbers in each row}\} \text{ and} \\ Q_\lambda &= \{\sigma \in S_n \mid \sigma \text{ preserves the set of numbers in each column}\}. \end{aligned}$$

For our particular numbered diagram above, we have:

$$\begin{aligned} P_\lambda &= \{\iota, (12), (23), (13), (123), (132), (45), (12)(45), (23)(45), (13)(45), (123)(45), (132)(45)\}, \\ Q_\lambda &= \{\iota, (14), (16), (46), (146), (164), (25), (14)(25), (16)(25), (46)(25), (146)(25), (164)(25)\}. \end{aligned}$$

Before we proceed, we need:

Definition 4.1. Let F be any field, and let G be a group. The *group algebra* of G over F is the vector space over F generated by the basis $\{e_x \mid x \in G\}$ with multiplication defined by

$$\left(\sum_{x\in G} c_x x\right)\left(\sum_{y\in G} d_y y\right) = \sum_{x,y\in G} c_x d_y xy \ ,$$

where the c_x's and d_y's are scalars in F. Note that if we let $F = \mathbb{C}$, then $\mathbb{C}G$ becomes a permutation representation under the group action defined by $g \cdot e_x = e_{gx}$.

Now define the following three elements of the group algebra $\mathbb{C}S_n$:

$$\begin{aligned} a_\lambda &= \sum_{\sigma\in P_\lambda} e_\sigma, \\ b_\lambda &= \sum_{\tau\in Q_\lambda} \operatorname{sgn}(\tau) e_\tau, \text{ and} \\ c_\lambda &= a_\lambda b_\lambda. \end{aligned}$$

It turns out that the subspace $\mathbb{C}S_n \cdot c_\lambda$ is an irreducible representation of S_n, and distinct partitions of λ correspond to distinct irreducible representations. It is important to know that although the number of conjugacy classes of any finite group will always equal the number of its irreducible representations, very rarely do we see an explicit bijection between these sets as we do here.

Example 4.2. To see this in action, let's find all irreducible representations of S_3 again. There are three partitions of 3 and therefore three corresponding Young Diagrams. We will number each one in the manner described above:

$\lambda = (3)$ $\quad$ $\mu = (2,1)$ $\quad$ $\nu = (1,1,1)$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

In the first diagram, 1, 2 and 3 are all sitting in one row, so any reshuffling of these numbers will preserve the row. The only permutation that will preserve the columns, however, is the identity. Therefore, $P_\lambda = S_3$ and $Q_\lambda = \{\iota\}$. This gives us

$$\begin{aligned} a_\lambda &= e_\iota + e_{(12)} + e_{(23)} + e_{(13)} + e_{(123)} + e_{(132)}, \\ b_\lambda &= e_\iota, \\ c_\lambda &= (e_\iota + e_{(12)} + e_{(23)} + e_{(13)} + e_{(123)} + e_{(132)})e_\iota \\ &= e_\iota + e_{(12)} + e_{(23)} + e_{(13)} + e_{(123)} + e_{(132)}. \end{aligned}$$

The associated irreducible representation, then, is $\mathbb{C}S_3 \cdot c_\lambda = \mathbb{C} \cdot c_\lambda = \langle c_\lambda \rangle$ because multiplying by any element in the basis of $\mathbb{C}S_3$ will simply rearrange the addends of c_λ but not change the sum. Notice that the subspace generated by c_λ is one-dimensional. Furthermore, because $\sigma \cdot rc_\lambda = rc_\lambda$ for any σ in S_3 and for any scalar r, the action of every σ leaves every vector in $\langle c_\lambda \rangle$ fixed, which means $\langle c_\lambda \rangle$ is the trivial representation.

In the second diagram, $P_\mu = \{\iota, (12)\}$ and $Q_\mu = \{\iota, (13)\}$. Therefore, we have

$$\begin{aligned} a_\mu &= e_\iota + e_{(12)}, \\ b_\mu &= e_\iota - e_{(13)}, \\ c_\mu &= (e_\iota + e_{(12)})(e_\iota - e_{(13)}) \\ &= e_\iota - e_{(13)} + e_{(12)} - e_{(132)}. \end{aligned}$$

According to our algorithm, the associated irreducible representation is $\mathbb{C}S_3 \cdot c_\mu$. To find out what this subspace is, we multiply c_μ by the basis elements of $\mathbb{C}S_3$:

$$\begin{aligned} e_\iota(e_\iota - e_{(13)} + e_{(12)} - e_{(132)}) &= e_\iota - e_{(13)} + e_{(12)} - e_{(132)}, \\ e_{(12)}(e_\iota - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(12)} - e_{(132)} + e_{(\iota)} - e_{(13)}, \\ e_{(13)}(e_\iota - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(13)} - e_\iota + e_{(123)} - e_{(23)}, \\ e_{(23)}(e_\iota - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(23)} - e_{(123)} + e_{(132)} - e_{(12)}, \\ e_{(123)}(e_\iota - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(123)} - e_{(23)} + e_{(13)} - e_{(\iota)}, \\ e_{(132)}(e_\iota - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(132)} - e_{(12)} + e_{(23)} - e_{(123)}. \end{aligned}$$

This set is spanned by the first and third vectors, showing that $\mathbb{C}S_3 \cdot c_\mu$ is the subspace $\langle e_\iota - e_{(13)} + e_{(12)} - e_{(132)}, e_{(13)} - e_\iota + e_{(123)} - e_{(23)} \rangle$. This must be the standard representation, which is the only two-dimensional representation of S_3.

In the third diagram, any permutation in S_3 will preserve the column, but only the identity will fix the rows. Therefore, we have $P_\nu = \{\iota\}$ and $Q_\nu = S_3$. This gives us

$$\begin{aligned} a_\nu &= e_\iota, \\ b_\nu &= e_\iota - e_{(12)} - e_{(23)} - e_{(13)} + e_{(123)} + e_{(132)}, \\ c_\nu &= e_\iota(e_\iota - e_{(12)} - e_{(23)} - e_{(13)} + e_{(123)} + e_{(132)}) \\ &= e_\iota - e_{(12)} - e_{(23)} - e_{(13)} + e_{(123)} + e_{(132)}. \end{aligned}$$

Again, the associated irreducible representation is $\mathbb{C}S_3 \cdot c_\nu = \mathbb{C} \cdot c_\nu = \langle c_\nu \rangle$ because multiplying by any element in the basis of $\mathbb{C}S_3$ will rearrange the addends of c_ν and negate their signs. This subspace is also one-dimensional. Furthermore, for any σ in S_3 and for any scalar r, $\sigma \cdot rc_\nu$ is rc_ν if σ is even and $-rc_\nu$ if σ is odd. Thus, $\langle c_\nu \rangle$ is the alternating representation.

Not only do we have a straightforward way of constructing every irreducible representation of S_n, but we also have an explicit formula for computing the character of each one: For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n, let C_i denote any conjugacy class of S_n. Let j be an index that runs from 1 through n. If we write any element in C_i as the product of disjoint cycles, then define i_j to be the number of cycles of length j in this product. Introduce k independent variables $x_1, x_2, \ldots, x_k$. Define the jth *power sum* to be $P_j(x) = x_1^j + x_2^j + \cdots + x_k^j$. Define the *discriminant* of $\{x_1, x_2, \ldots, x_k\}$ to be

$$\Delta(x) = \begin{vmatrix} 1 & x_k & \cdots & x_k^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_1 & \cdots & x_1^{k-1} \end{vmatrix}.$$

Let $\ell_s = \lambda_s + k - s$ for every $1 \leq s \leq k$. Finally, if $f(x)$ is some polynomial function of $x_1, x_2, \ldots, x_k$, let $[f(x)]_{(\ell_1, \ell_2, \ldots, \ell_k)}$ denote the coefficient on the term $x_1^{\ell_1} x_2^{\ell_2} \cdots x_k^{\ell_k}$. Then:

Theorem 4.3. ***(The Frobenius Formula)*** *The character of the irreducible representation of S_n associated with λ is given by*

$$\chi_\lambda(C_i) = \left[\Delta(x) \cdot \prod_j P_j(x)^{i_j}\right]_{(\ell_1,\ell_2,\ldots,\ell_k)}.$$

Proof. See Section 4.3 of [3]. □

Example 4.4. Let's compute $\chi_{(2,1)}$ for the conjugacy class of transpositions in S_3. Here $k = 2$, so we introduce two independent variables x_1 and x_2. Each transposition can be written as a product of a 2-cycle and a 1-cycle, so we have $i_1 = 1, i_2 = 1, i_3 = 0$. This means the polynomial inside the brackets of the formula above is $(x_2 - x_1)(x_1^1 + x_2^1)(x_1^2 + x_2^2)) = x_2^4 - x_1^4$. Therefore, the coefficient on the term $x_1^2 x_2$ is 0. Notice that this agrees with the value for $\chi_V(12)$ (where V is the standard representation) that we computed when we were constructing the character table of S_3.

5. Schur Functors

In this section, we will see how, given a representation of a group (finite or not), we can use the irreducible representations of S_n to discover new representations of that group.

Definition 5.1. If V and V' are representations of groups H and H', respectively, then the *external tensor product* $V \boxtimes V'$ is the representation $V \otimes V'$ of $H \times H'$ where the group action is given by $(h, h') \cdot (v \otimes v') = hv \otimes h'v'$ for every (h, h') in $H \times H'$ and every $v \otimes v'$ in $V \otimes V'$.

Definition 5.2. If $\rho : GL_n \to GL_k(W)$ is a representation of GL_n, then ρ sends matrices to matrices:

$$\rho \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} f_{11}(a_{11}, \ldots, a_{nn}) & \cdots & f_{1n}(a_{11}, \ldots, a_{nn}) \\ \vdots & \ddots & \vdots \\ f_{n1}(a_{11}, \ldots, a_{nn}) & \cdots & f_{nn}(a_{11}, \ldots, a_{nn}) \end{pmatrix}$$

If the f_{ij}'s in the image matrix are polynomial functions, we say that W is a *polynomial representation.*

The group $GL(V)$ has a natural representation: V equipped with the group action given by $T \cdot v = T(v)$ for every T in $GL(V)$. Just as with finite groups, any tensor power of V is also a representation of $GL(V)$. The group action here is

$$T \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = T(v_1) \otimes T(v_2) \otimes \cdots \otimes T(v_n)$$

$V^{\otimes n}$ is also naturally a representation of S_n under the group action given by

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\sigma = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$$

Notice that these actions commute. We can therefore compose them to obtain a new group action on $V^{\otimes n}$ of $GL(V) \times S_n$, making $V^{\otimes n}$ into a representation of $GL(V) \times S_n$.

Definition 5.3. As shown in Section 4, any partition λ of n corresponds to a Young symmetrizer c_λ. The image of the action of c_λ on $V^{\otimes n}$, which we denote by $\mathbb{S}_\lambda(V)$, is another representation of $GL(V)$. We call the functor (between the category of representations of G and itself) that sends a representation V to $\mathbb{S}_\lambda(V)$ the *Schur Functor* associated with the partition λ and denote it by F_λ.

Theorem 5.4. *$V^{\otimes n}$ can be decomposed as*

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} \mathbb{S}_\lambda(V)^{\otimes m_\lambda}$$

where each $\mathbb{S}_\lambda(V)$ is an irreducible representation of $GL(V)$ and m_λ is the dimension of V_λ, the irreducible representation of S_n associated with the partition λ of n.

Another way of saying this is that $V^{\otimes n}$ can be decomposed as $V^{\otimes n} = \bigoplus V_i$ where each i corresponds to a partition of n and each V_i has the form $M_i \boxtimes W_i$, and M_i and W_i are irreducible representations of S_n and $GL(V)$, respectively. Furthermore, if the dimension of V is greater than or equal to n, then every irreducible representation of S_n occurs exactly once in the decomposition, and every polynomial irreducible representation of $GL(V)$ occurs in the decomposition of exactly one $V^{\otimes n}$ as n varies from 1 to ∞. For a given n, we would like to find out exactly what the W_i's are in the decomposition of $V^{\otimes n}$. Let $m_i = \dim(M_i)$. If we ignore the $\mathbb{C}S_n$-module structure on S_n, we have

$$V^{\otimes n} = \bigoplus_i C^{m_i} \otimes W_i = \bigoplus_i (W_i)^{m_i}.$$

It turns out that we can recover each W_i by applying the Schur Functor associated with the partition i to $V^{\otimes n}$.

By starting with just one representation of $GL(V)$ (namely, V), we can recover all irreducible representations of $GL(V)$. The same method can be used to find many, if not all, of the irreducible representations of other groups besides $GL(V)$. There is also an explicit formula for computing the characters of these resulting representations called the *Weyl Character Formula*. It would be an interesting topic for further study.

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References

[1] David S. Dummit and Richard M. Foote. Abstract Algebra. John Wiley and Sons, New Jersey. 2004.

[2] William Fulton. Young Tableaux with Applications to Representation Theory and Geometry. London Mathematical Society Student Texts 35. Cambridge University Press, New York, NY. 1997.

[3] William Fulton and Joe Harris. Representation Theory: A First Course. Springer, New York. 2004.

[4] Gordon James and Martin Liebeck. Representations and Characters of Groups. Second Edition. Cambridge University Press, Cambridge, UK. 2001.