T-duality and Poisson-Lie T-duality in generalized geometry

Peggy Pei-Wen Kao

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Principal Supervisor: Professor Peter Bouwknegt

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To my parents

Declaration

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

Peggy Kao

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Abstract

An important symmetry of string theory, T-duality relates string theory on different backgrounds and may be realized as a transformation between two-dimensional σ-models. A systematic method has been developed by Bouwknegt, Evslin and Mathai to analyze the global properties of T-duality in the presence of $NS - NS$ 3-form H-flux. Cavalcanti and Gualtieri subsequently realized that generalized geometry provides a natural setting to study global T-duality.

In the case when two T-dual σ-models with target spaces are principal circle bundles over a common base manifold, Cavalcanti and Gualtieri showed that T-duality can be viewed as an isomorphism between Courant algebroids. In this thesis, we generalize the result of Cavalcanti and Gualtieri to general principal torus bundles and show that a principal torus bundle E and its T-dual space $\hat{E}$ are related by T-duality as isomorphic Courant algebroids.

Next, we generalize the above construction to Poisson-Lie T-duality, which is a non-Abelian T-duality proposed by Klimčík and Ševera. For a Poisson-Lie group G and its dual group $\tilde{G}$, Poisson-Lie T-duality relates a pair of σ-models with targets being principal G and $\tilde{G}$-bundles E and $\hat{E}$, respectively. We then show that Poisson-Lie T-duality can be viewed as an isomorphism relating E and $\hat{E}$ as isomorphic Courant algebroids.

We also investigate the non-geometric flux compactifications proposed by Shelton, Taylor and Wecht. We show that the full gauge algebras arising from a Scherk-Schwarz compactification correspond to a Courant algebra, and the non-geometric fluxes proposed by Shelton, Taylor and Wecht correspond to the fluxes arising from the global T-duality formulated by Bouwknegt, Hannabuss and Mathai.

Contents

Chapter 1

Introduction

1.1 Introduction

String theory is without doubt the most active research area in Mathematical Physics at present. T-duality originally arises as a symmetry which relates type IIA and type IIB string theories. Geometrically, T-duality arises from compactifying a theory on a circle with radius R, and one can show that such a theory describes the same physics as a theory compactified on a circle with radius $1/R$. While T-duality is a symmetry of String theory, it relates different String backgrounds, i.e. classical solutions of the underlying low-energy effective theories. Thus it can be used, for example, to construct new solutions to Einstein's equations in General Relativity out of existing solutions.

In particular, T-duality relates String Theory on different backgrounds and may be realized as a transformation between two-dimensional σ-models [29]. A two-dimensional σ-model describes the world-sheet theory of a string propagating on a target manifold E equipped with a Riemannian metric g_{ij} and a locally defined antisymmetric B-field b_{ij}, with string background defined by $E_{ij} \equiv g_{ij} + b_{ij}$.

Locally the duality rules for T-duality with an Abelian isometry were constructed by Buscher [13] in 1987, and are known as the **Buscher rules**. To obtain the Buscher rules, E is required to have an Abelian isometry group which leaves the σ-model invariant. The dual model can then be obtained by gauging the isometry, with gauge fields being integrated out. Here, let us simply refer to this type of construction as **Abelian T-duality**.

However, since the B-field is often only locally defined, it is of great interest to obtain a global characterisation of T-duality in terms of some global objects.

One such object is the $NS-NS$ background H-flux, which is locally given by the exterior derivative of the B-field. Through examples in literature, it is argued that T-duality leads to a topology change of the underlying manifold [4, 34]. Many open questions regarding T-duality in the presence of background fluxes remain to be answered though. In order to understand the topology change under T-duality, we follow a systematic method that has been developed to study the global aspect of T-duality by Bouwknegt, Evslin and Mathai [6, 7].

In this construction, one considers a target space E that is a principal torus bundle over a base manifold M. σ-models are characterised topologically by (M, F, H), where the H-flux on E is a closed, integral, 3-form H on E and can be characterized by the four tuple $(H_{(3)}, H_{(2)}, H_{(1)}, H_{(0)})$, where $H_{(i)}$ is a vector valued i-form on M. This method of characterising $H \in \Omega^3(E)$ by forms on the base manifold is referred to as "dimensional reduction". F is the curvature 2-form on M. In the case when E is a principal circle bundle, $H_{(1)}$ and $H_{(0)}$ vanish and the T-dual space turns out to be another principal circle bundle with corresponding dual H-flux. Extending to the general principal torus bundle case, in which case the $H_{(1)}$ and (or) $H_{(0)}$ components can be non-vanishing, one arrives at some interesting results involving topology change. As argued in [62], if one considers a principal $\mathbb{T}^n$-bundle with $H_{(1)} \neq 0$ and $H_{(0)} = 0$, then the T-dual bundle surprisingly has noncommutative tori as fibres. Furthermore, if $H_{(0)} \neq 0$ then the T-dual bundle has non-associative tori as fibres.

In order to analyse the above mentioned topology change in more detail, we study T-duality in the framework provided by generalized geometry. Generalized geometry, first introduced by Hitchin [38] in 2002 and further developed by Gualtieri [32] and Cavalcanti [16], has emerged to provide a useful framework for studying string compactifications and T-duality. Generalized geometry is a geometry that doubles the original space, i.e. in generalized geometry, a vector space V is replaced by $V \oplus V^*$, here V^* is its dual space. In particular the object that we are interested in is the direct sum of the tangent and cotangent bundles $TE \oplus T^*E$. As first studied by Cavalcanti [16] and Gualtieri [33], considering the case when E is a principal S^1-bundle, there is a natural inner product and an antisymmetric bracket structure called the Courant bracket satisfying certain properties on $(TE \oplus T^*E)_{S^1}$ – the invariant sections of $TE \oplus T^*E$ – and making $(TE \oplus T^*E)_{S^1}$ into a Courant algebroid. They also showed that invariant structures on $(TE \oplus T^*E)_{S^1}$ can be transported to an invariant structure on its dual space $(T\hat{E} \oplus T^*\hat{E})_{\hat{S}^1}$, where $\hat{E}$ is the dual principal $\hat{S}^1$-bundle.

Thus, T-duality can be viewed as isomorphism between a pair of Courant algebroids:

$$\begin{array}{ccc} ((TE\oplus T^*E)_{inv}, \langle\ ,\ \rangle, [\![\ ,\ ]\!]_H) & \xrightarrow{\cong} & ((T\hat{E}\oplus T^*\hat{E})_{inv}, \langle\ ,\ \rangle, [\![\ ,\ ]\!]_{\hat{H}}) \\ & {\scriptstyle\pi}\searrow \qquad \swarrow{\scriptstyle\tilde{\pi}} & \\ & TM & \end{array}$$

We then extend Cavalcanti and Gualtieri's result of principal circle bundles to the general case – principal torus bundles – and generalist the Courant bracket on the invariant sections of $TE \oplus T^*E$, where E is a principal torus bundle and $\hat{E}$ is its dual space.

It it not apparent that the generalized Courant bracket in this case together with the non-degenerate bilinear form makes the space of invariant sections of $TE \oplus T^*E$ a Courant algebroid. But once we redefine the generalized Courant bracket in terms of the bracket on an object called **proto-bialgebroid** [55, 66], then the space of invariant sections of $TE \oplus T^*E$ is recognized as the double of a proto-bialgebroid, as a result $(TE \oplus T^*E)_{inv}$ can be interpreted as a Courant algebroid.

Any invariant structure on $TE \oplus T^*E$ can be transported to an invariant structure on $T^*\hat{E} \oplus T\hat{E}$, even when E and $\hat{E}$ has different topology. Therefore we use the setting of T-duality as an isomorphism between Courant algebroids and show that the topology change for principal torus bundles with nonvanishing $H_{(1)}$ and $H_{(0)}$ agrees with the result previously obtained by Bouwknegt, Hannabuss and Mathai [8, 9].

Since the Abelian T-duality described by the Buscher rules is so simple and beautifully symmetric, a naive question to ask is whether the Buscher rules can be extended to the case when the isometry is non-Abelian. With such a generalization, T-duality can then be further applied to σ-models with non-Abelian isometry groups.

A first attempt to construct T-duality with non-Abelian isometry was formulated by de la Ossa and Quevedo [20] in 1993. Inspired by Buscher's technique, they applied a T-duality transformation following Buscher's procedure using non-Abelian isometry groups. However it was soon realized by de la Ossa, Quevedo and other authors [4, 20, 27] that non-Abelian T-duality in this formalism suffered certain drawbacks, the most noticeable being that this technique is not symmetric, i.e. one can not in general recover the original theory by repeating the T-duality procedure.

In another attempt to construct non-Abelian T-duality, Klimčík and Ševera [49] abandoned the requirement of isometry as dualizability and proposed a generalization of T-duality in 1995, which has come to be known as **Poisson-Lie T-duality**. In this formalism of non-Abelian T-duality, instead of the requirement of an isometry, the crucial concept of Drinfel'd double are required to construct a pair of dual σ-models.

Since in the Abelian case, we have set up a framework to study T-dual spaces as isomorphic Courant algebroids, we are then attempted to set up a similar framework for Poisson-Lie T-duality. For a pair of Poisson-Lie groups G and $\tilde{G}$, Poisson-Lie T-duality relates a pair of dual σ-models on targets E and $\hat{E}$ being principal G and $\tilde{G}$-bundles. Thus generalizing the Abelian case, $TE \oplus T^*E$ and $T\hat{E} \oplus T^*\hat{E}$ can be viewed as isomorphic Courant algebroids related by Poisson-Lie T-duality.

We also investigate the non-geometric flux compactification proposed by Shelton, Taylor and Wecht [74]. In this formalism, they showed that performing T-duality on H-flux in one direction yields a twisted torus compactification, which is characterised by a 'geometric fluxes' $f_{ab}{}^c$. While performing another T-duality on a twisted torus in one direction yields a globally non-geometric space characterized by a 'non-geometric flux' $q^{ab}{}_c$. They further proposed that if one performs T-duality once more in one direction, one obtains a non-geometric string background characterized by a non-geometric flux r^{abc}. To be more precise, these fluxes arise as the charges of algebras of the full T-duality invariant gauge algebra from a Scherk-Schwarz compactification on the string background E. We show that the full gauge algebra arising from a Scherk-Schwarz compactification correspond to the Courant bracket on the invariant sections of the generalized tangent space $TE \oplus T^*E$, and the charges $f_{ab}{}^c$, $q^{ab}{}_c$ and r^{abc} correspond to the fluxes $F_{(2)}$, $F_{(1)}$ and $F_{(0)}$ which are the T-dual of $H_{(2)}$, $H_{(1)}$ and $H_{(0)}$.

1.2 Outline

This thesis is organized as follows.

In Chapter 2, we first review some basic concepts of Poisson geometry including the concepts of Poisson-Lie groups and Drinfel'd doubles, which is crucial to the study of Poisson-Lie T-duality. Then we introduce some algebraic objects such as Lie algebroids, Lie bialgebroids, Proto-bialgebroids and Courant algebroids, which will be needed later.

In chapter 3 we review the basic construction of Abelian T-duality, i.e. the Buscher rules. Followed by a review of a non-Abelian version of T-duality, the

Poisson-Lie T-duality.

Chapter 4 reviews the basic ingredients of generalized geometry, in particular we focus on the generalized tangent space – the direct sum of the tangent and cotangent bundle $TE \oplus T^*E$ – with an interpretation of a Courant algebroid.

Then in Chapter 5, we move on to setting up a framework to study T-duality using the framework of generalized geometry. We start by reviewing the global T-duality followed by a generalization of such setting using the framework of generalized geometry. At the end of this chapter we analyse the generalize Courant bracket in the principal torus bundle case and show that this bracket can be rewritten as the derived bracket of a proto-bialgebroid, thus the generalized tangent spaces of a principal torus bundle and its dual space are simply the double of a proto-bialgebroid, or as one expect, a Courant algebroid. This chapter is a joint work with Bouwknegt and Garretson.

In Chapter 6, we consider the non-Abelian version of T-duality – Poisson-Lie T-duality – in the framework of generalized geometry. We begin by introducing the Semenov-Tian-Shansky Poisson structure on a Poisson-Lie group G. We then generalize the (Abelian) T-duality and establish an isomorphism of Courant algebroids related by Poisson-Lie T-duality. This chapter is a joint work with Bouwknegt.

The goal of Chapter 7 is to investigate the non-geometric flux compactification and relate the non-geometric fluxes with the fluxes which appear in the global T-duality using the language of Courant algebroid. This chapter is a joint work with Bouwknegt and Garretson.

Chapter 2

Poisson geometry and Lie algebroid theory

2.1 Introduction and outline

As stated by Mackenzie [61], Poisson geometry has been developed over the years from three principal sources.

Firstly, Poisson geometry provides a more natural and convenient framework to study symplectic geometry. There is a canonical Poisson structure associated with any symplectic manifold (M, ω) such that the Poisson bracket makes $C^\infty(M)$ into a real Lie algebra. Symplectic structures naturally give rise to non-degenerate Poisson structures, and conversely the symplectic structure can be recovered from the Poisson bracket.

Secondly, Poisson geometry can be viewed as a semi-classical limit of modern quantum geometry. In particular, a class of interesting Poisson manifolds introduced by Drinfel'd called Poisson-Lie groups has emerged from the study of Quantum Groups. On the other hand, the discovery of Poison-Lie groups provides a large class of degenerate Poisson structures which makes the theory of Poisson geometry more interesting. Finally, Poisson geometry embodies a theory dual to Lie algebra theory, and more generally, to Lie algebroid theory. In this chapter we shall concentrate more on this observation, in particular the relation between Poisson geometry and Lie algebroid theory.

The third source arises from the fact that poisson brackets on a manifold M give a Lie algebra structure on the real vector space of smooth functions on M. In the case where M is symplectic, the condition that the symplectic two-form be closed is precisely what is needed for the Poisson bracket to satisfy the Jacobi identity. These structures are the famous Poisson-Lie structures. Poisson-Lie structures can be considered as canonical Poisson structures living on the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. Thus one can view these manifolds with Poisson-Lie structures as being dual to Lie algebras, and instead of the Lie algebras one can work entirely with these Poisson-Lie structures. One can also interpret the symplectic leaves of the Poisson-Lie structures as its coadjoint orbits, and the symplectic forms on the orbits in such a case was obtained by Kirillov [46].

The importance of Lie algebroid theory in Poisson geometry is based on the following observation. As an observation made by various authors (eg. [37]), it is well-known that the Poisson bracket of one-forms on a Poisson manifold (M, π) makes the cotangent bundle T^*M a Lie algebroid. The cotangent bundle T^*M as a Lie algebroid, called the cotangent Lie algebroid of M, is both a tool in understanding the Poisson geometry and an important source of examples of Lie algebroids.

Furthermore, the cotangent Lie algebroid together with the tangent bundle of a Poisson manifold (M, Π) turns out to be a Lie bialgebroid (TM, T^*M), with its double $TM \oplus T^*M$ being a Courant algebroid. This observation provides an important link between Poisson geometry and Courant algebroid theory.

This chapter is organized as follows.

Section 2.2 reviews the basic concept of Poisson geometry, starting with basic definitions of a Poisson manifold. Then a special class of Poisson manifolds called Poisson-Lie groups and their infinitesimal analogues, Lie bialgebras are introduced in Section 2.2.2. Lie bialgebras can be constructed out of its double, the Manin triples, which is reviewed in section 2.2.3. The concept of a Manin triple is important in the construction of Poisson-Lie T-duality which will be discussed in Chapter 3. In section 2.3, objects called Lie algebroids, Lie bialgebroids and Courant algebroids are introduced. Section 2.3.4 gives an alternative definition of Lie bialgebroid in terms of derived bracket due to Kosmann-Schwarzbach [56] and Roytenberg [67]. At the end of this section, the concept of an object called proto-bialgebroid is introduced. Since the double of a proto-bialgebroid gives rise to a Courant algebroid, it provides interesting examples to construct Courant algebroids which will play an important role in Chapter 5.

2.2 Poisson geometry

A Poisson manifold is a smooth manifold with a Poisson bracket defined on its function space. The classical Poisson bracket was first introduced by Poisson in the early 19th century in his study of the equations of motion in celestial mechanics. In 1835, Hamilton [35] revolutionized mechanics and reformulated the equation of motion using the Poisson brackets. Since then, Poisson brackets, or more precisely non-degenerate Poisson brackets coming from a canonical symplectic structure on a vector space, have attracted interests in Physics and have been exploited since the 19th century.

Later in 1980, Lie [58] examined some degenerate Poisson brackets and they turned out to be the famous Poisson-Lie structure on Poisson-Lie groups.

In the last few decades, the theory of Poisson geometry has undertaken tremendous developments due to its close connection with many fields in mathematics. Many independent researchers came across to the notion of "general Poisson manifolds". Among them, Kirillov [45] developed the notion of local Lie algebras which encompasses that of Poisson structure, while Lichnerowicz [57] introduced a precise definition of a Poisson manifold. A substantial contribution to establish local structure of a general Poisson manifold was made by Weinstein [80] in 1980.

In the last 15 years, a large class of interesting Poisson manifolds have emerged from the study of Quantum Group theory, these are Poisson-Lie groups. The notion of Poisson-Lie groups was first introduced by Drinfel'd in the early 1980s and studied by Semenov-Tian-Shansky, Kosmann-Schwarzbach, Weinstein, Lu and many others [14, 23, 70, 80]. The study of Poisson-Lie groups provides a large class of interesting nonlinear and degenerate Poisson brackets.

In this section we review the basic concepts of Poisson manifolds, Poisson-Lie groups, their infinitesimal analogues, the Lie bialgebras, and Manin triples which provide a rich source of examples for Lie bialgebras.

2.2.1 Poisson manifolds

Poisson structures

Definition 2.1. A **Poisson structure** on a smooth manifold M is an $\mathbb{R}$-bilinear Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ called the **Poisson bracket**, which satisfies the following conditions:

1. Antisymmetry, $\{f, g\} = -\{g, f\}$,

2. Jacobi-identity, $\{f,\{g,\{h\}\}+cyclic=0$,

3. Leibnitz identity, $\{f,gh\}=\{f,g\}h+g\{f,h\}$, where $f,g,h\in C^\infty(M)$.

Remark 2.2. 1. The skew-symmetry property together with the Jacobi-identity implies that the Poison bracket $\{\ ,\ \}$ is a Lie bracket on $C^\infty(M)$.

2. The Leibnitz identity implies that the Poisson bracket can be considered as a derivation in each variable for the associative multiplication on $C^\infty(M)$. A vector field X_f can be defined on M by

$$X_f(g)=\{f,g\},$$

and is called a **Hamiltonian vector field**.

3. The Jacobi-identity implies that the map $f\mapsto X_f$ is a homomorphism from the Lie algebra $C^\infty(M)$ of smooth functions under the Poisson bracket to the Lie algebra of smooth vector fields under the Lie bracket, i.e.

$$[X_f,X_g]=X_{\{f,g\}}.$$

One can define a bivector on M known as the Poisson bivector as follows.

Definition 2.3. The **Poisson bivector** is a map $\Pi:T^*M\oplus T^*M\to M\times\mathbb{R}$ corresponding to a 2-vector field $\Pi:\wedge^2T^*M\to\mathbf{R}$ defined by

$$\Pi(fdg,f'dg')=ff'\{g,g'\}.$$

Definition 2.4. The map Π induces the **sharp map** $\pi^\sharp:T^*M\to TM$ on (M,Π) defined by

$$(\pi^\sharp(\alpha))(f)=\imath_\alpha\Pi(f),\quad \alpha\in T^*M.$$

The sharp map $\pi^\sharp$ is a well-defined bundle map on T^*M, with the following properties:

(1) $\pi^\sharp$ is a bundle map on M, it induces a map on sections of T^*M via

$$\pi^\sharp:\Gamma(T^*M)\to\Gamma(TM),\quad \alpha\mapsto\imath_\alpha\Pi. \tag{2.2.1}$$

(2) On exact one-forms one has $\pi^\sharp(df)=X_f$. The Poisson two-tensor Π can be written in term of $\pi^\sharp$ as

$$\Pi(df,dg)=\langle\pi^\sharp(df),dg\rangle=\{f,g\}=\langle X_f,dg\rangle.$$

Definition 2.5. The Schouten-Nijenhuis bracket $[\ ,\ ]_{SN} : \Gamma(\wedge^p TM)\times\Gamma(\wedge^q TM) \to \Gamma(\wedge^{p+q-1}TM)$ is the natural extension of Lie bracket to multi-vector fields. It is defined explicitly as

$$\langle \omega, [X,Y]_{SN}\rangle = (-1)^{(p-1)(q-1)}\langle d(\imath_Y\omega), X\rangle - \langle d(\imath_X\omega), Y\rangle + (-1)^p\langle d\omega, X\wedge Y\rangle, \tag{2.2.2}$$

for all $X \in \Gamma(\wedge^p TM)$, $Y \in \Gamma(\wedge^q TM)$ and $\omega \in \Omega^{p+q-1}(M)$.

Here is a sufficient condition for a skew-symmetric two-vector field to define a Poisson structure:

Proposition 2.6. *The Poisson two-tensor is required to satisfy*

$$[\Pi,\Pi]_{SN} = 0, \tag{2.2.3}$$

here $[\ ,\ ]_{SN}$ *is the Schouten-Nijenhuis bracket on multi-vector fields defined by* (2.2.2).

Proof. Let $\omega \in \Omega^3(M)$. While $[\Pi,\Pi]_{SN} \in \Gamma(\wedge^3 TM)$, (2.2.2) gives rise to

$$\langle \omega, [\Pi,\Pi]_{SN}\rangle = -\langle d(\imath_\Pi\omega), \Pi\rangle - \langle d(\imath_\Pi\omega), \Pi\rangle + \langle d\omega, \Pi\wedge\Pi\rangle. \tag{2.2.4}$$

Let $\omega = df\wedge dg\wedge dh$ and $\{f,g\} = \langle df\wedge dg, \Pi\rangle$. Since $d(df\wedge dg\wedge dh) = 0$, the above equation reduces to

$$\begin{aligned}
\langle \omega, [\Pi,\Pi]_{SN}\rangle &= -2\langle d\imath_\Pi\omega, \Pi\rangle \\
&= -2\langle d\imath_\Pi(df\wedge dg\wedge dh), \Pi\rangle \\
&= -2\langle d(\{g,h\}df - \{f,h\}dg + \{f,g\}dh), \Pi\rangle \\
&= -2\langle d\{g,h\}\wedge df - d\{f,h\}\wedge dg + d\{f,g\}\wedge dh, \Pi\rangle \\
&= -2(\{\{g,h\},f\} - \{\{f,h\},g\} + \{\{f,g\},h\}) = 0. \qquad (2.2.5)
\end{aligned}$$

Therefore if Π is a Poisson structure then $[\Pi,\Pi]_{SN} = 0$. □

Note 1. Hence forth by abuse of notation, we will refer the Poisson bivector Π as the Poisson structure.

Finally, let us define a particular set of functions on M.

Definition 2.7. A function $f \in C^\infty(M)$ is called a **Casimir function** if $\{f,g\} = 0$ for all $g \in C^\infty(M)$.

Here are some examples of Poisson structures on a manifold M.

Example 2.8. Every manifold admits the *trivial* Poisson structure $\{f, g\} = 0$, in such a case the bundle map $\pi^\sharp = 0$.

Example 2.9. At the opposite extreme to the trivial Poisson structure, (M, ω) is a **symplectic manifold** if it is equipped with a non-degenerate closed 2-form ω.
A Hamiltonian vector field X_f can be defined by $\omega(X_f, \cdot) = \langle -df, \cdot\rangle$, i.e. $\imath_{X_f}\omega = -df$. An antisymmetric bracket on $C^\infty(M)$ is defined by $\{f, g\} := \omega(X_f, X_g) = -\omega(X_g, X_f) = -\{g, f\}$.
Then one can check that the bracket $\{\ ,\ \}$ satisfies the following properties and is a Poisson bracket:

- $\{\ ,\ \}$ is bilinear:
  $\{f_1 + f_2, g\} = \omega(X_{f_1+f_2}, X_g) = \omega(X_{f_1}, X_g) + \omega(X_{f_2}, X_g)$, since $X_{f_1+f_2} = X_{f_1} + X_{f_2}$.
- $\{\ ,\ \}$ satisfies the Jacobi-identity:
  One can easily show that $d\omega(X_f, X_g, X_h) = 3\mathrm{Jac}(f, g, h) = 0$.
- $\{\ ,\ \}$ satisfies the Leibnitz identity:
  $\{f, gh\} = \imath_{X_f}d(gh) = \imath_{X_f}(g(dh)+(dg)h) = g\imath_{X_f}(dh)+\imath_{X_f}(dg)h = g\{f, h\}+\{f, g\}h$.

Thus every symplectic structure gives rise to a Poisson structure.

Poisson manifolds

Definition 2.10. A smooth manifold M equipped with a Poisson structure Π is a **Poisson manifold**, (M, Π).

Here is an example of Poisson manifold [68]:

Example 2.11. There is a Poisson structure on $\mathbb{R}^3$ with Poisson bracket given by

$$\{X^i, X^j\} = \sum_{k=1}^{3} \epsilon^{ijk} X^k, \tag{2.2.6}$$

where ϵ^{ijk} is the completely antisymmetric three tensor.

One can show that this Poisson structure (2.2.6) is a degenerate Poisson structure, since it is well-known that Darboux coordinates can never exist on a manifold of odd dimension.

It can be shown that in this example, $\mathcal{C}_2 := \sum_{i=1}^{3} X^i X^i$ is a Casimir function.

In this example, the Poisson structure can be restricted constantly on two-spheres given by the choice of a constant value for $\mathcal{C}_2$, and the restricted Poisson structure turns out to be non-degenerate.

Generally speaking, any Poisson manifold with degenerate Poisson structure foliates into a family of lower dimensional manifolds, which are often called the symplectic leaves. Each symplectic leave is characterized by assigning a constant value to the Casimir functions and is equipped with a non-degenerate Poisson structure [68].

Poisson maps

Definition 2.12. Let (M_1, Π_1) and (M_2, Π_2) be Poisson manifolds. A smooth map $f : M_1 \to M_2$ is a **Poisson map** if

$$\{u \circ f, v \circ f\} = \{u, v\} \circ f, \qquad \forall u, v \in C^\infty(M_2).$$

Similarly, a map f is called **anti-Poisson** if

$$\{u \circ f, v \circ f\} = -\{u, v\} \circ f, \qquad \forall u, v \in C^\infty(M_2).$$

Example 2.13. Let $M = \mathbb{R}^4$ with basis $\{x_1, y_1, x_2, y_2\}$ with a Poisson structure given by

$$\{x_i, y_j\} = \delta_{ij}, \quad \textit{otherwise } 0. \tag{2.2.7}$$

Let $N = \mathbb{R}^2$ with basis $\{x_1, y_1\}$ with a Poisson structure $\{x_1, y_1\} = 1$.

Consider a map $\phi : M \to N$ which is defined by $(x_1, y_1, x_2, y_2) \mapsto (x_1, y_1)$. It is obvious that ϕ is a Poisson map.

Local description

Let (M, Π) be a Poisson manifold. In local coordinates $(x_1, \ldots, x_n)$ of M, the Poisson tensor Π is determined by the matrix

$$\Pi_{ij}(x) = \{x_i, x_j\}$$

If Π_{ij} is invertible at each x, then Π is called **nondegenerate** or **symplectic**.

When Π is symplectic, the local matrices $(\omega_{ij}) = (-\Pi_{ij})^{-1}$ defines a global two-form $\omega \in \Omega^2(P)$ and ω is called the Symplectic form on M. With ω a symplectic form, the Jacobi identity is equivalent to $d\omega = 0$ as we have seen previously in Example (2.9).

Here are some example of Poisson structures:

Example 2.14 (Constant Poisson structure)**.** Let $M = \mathbb{R}^n$ and Π_{ij} be constant. By a change of coordinates, we can choose coordinates $(q_1, \dots, q_k; p_1, \dots, p_k; e_1, \dots, e_l)$ on M, with $2k + l = n$. The splitting theorem [80] states that the Poisson structure is given by

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j=1}^{l} \varphi_{ij}(e) \frac{\partial}{\partial e_i} \wedge \frac{\partial}{\partial e_j}, \quad \varphi_{ij}(0) = 0. \tag{2.2.8}$$

When $l = 0$, the Poisson structure is then symplectic and Π can be defined as follows

$$\Pi = \sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}.$$

The Poisson bracket $\{\ ,\ \}$ can be expressed as

$$\{f, g\} = \sum_i (\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i})$$

We can recognize this as the original Poisson bracket in classical mechanics. Originally only non-degenerate Poisson structures are employed in classical mechanics. However in the 1970s, many physical models arising from mechanical systems with symmetry groups or constraints were discovered to have degenerate Poisson brackets (eg. [21]).

Example 2.15 (Poisson-Lie structure)**.** Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra with Lie bracket $[\ ,\ ]$, and $\mathfrak{g}^*$ be the dual algebra. The tangent space of $\mathfrak{g}^*$ at any point can be identified canonically with $\mathfrak{g}^*$ itself, so df of any smooth function f on $\mathfrak{g}^*$ is a map $d : \mathfrak{g}^* \to (\mathfrak{g}^*)^* \cong \mathfrak{g}$, and we can define a Poisson structure on $\mathfrak{g}$ as

$$\{f_1, f_2\}(\xi) = \langle [(df_1)_\xi, (df_2)_\xi], \xi \rangle, \quad \forall \xi \in \mathfrak{g}^*, \tag{2.2.9}$$

Note that if we take $\{T_a\}$ as a basis of $\mathfrak{g}$ and $\{\tilde{T}^a\}$ a basis of $\mathfrak{g}^*$, $\mathfrak{g}$ has a Lie bracket $[T_a, T_b] = f_{ab}{}^c T_c$. If one chooses $f_1 = \tilde{T}^a$ and $f_2 = \tilde{T}^b$, then by (2.2.9) we have

$$\{\tilde{T}^a, \tilde{T}^b\}(\tilde{T}^c) = \langle [T_a, T_b], \tilde{T}^c \rangle = \langle f_{ab}{}^c T_c, \tilde{T}^c \rangle = f_{ab}{}^c, \tag{2.2.10}$$

since $T_a(\tilde{T}^b) = \delta_a^b$ thus

$$\{\tilde{T}^a, \tilde{T}^b\} = f_{ab}{}^c T_c. \tag{2.2.11}$$

Therefore $\Pi = f_{ab}{}^c T_a \wedge T_b$ is a Poisson tensor on $\mathfrak{g}^*$.

As a result, the dual space of a Lie algebra is always equipped with a canonical Poisson structure called the **Poisson-Lie structure**.

2.2.2 Poisson-Lie groups and Lie bialgebras

A Poisson-Lie group is a Lie group and a Poisson manifold, the two structures being compatible as follows.

Definition 2.16. A **Poisson-Lie group** (G, Π) is a Lie group G equipped with a Poisson structure Π on the manifold G, such that the multiplication map $\mu : G \times G \to G$ $(\mu(g_1, g_2) = g_1 g_2)$ is a Poisson map.

Equivalently, Π is a Poisson-Lie structure on G if Π satisfies the equation:

$$\Pi(gh) = (L_g)_* \Pi(h) + (R_h)_* \Pi(g), \quad \forall g, h \in G, \tag{2.2.12}$$

where $L_g : h \mapsto gh$ and $R_g : h \mapsto hg$.

Example 2.17. Any Lie group G with the trivial Poisson structure $\Pi = 0$ is a Poisson-Lie group.

Example 2.18. A finite dimensional real Lie algebra $\mathfrak{g}$ with a Poisson structure defined in example (2.15) is a Poisson-Lie group when $\mathfrak{g}$ is regarded as an Abelian Lie group under addition.

Now, let us turn our attention to the infinitesimal analogues of Poisson-Lie groups, the Lie bialgebras.

Definition 2.19. A **Lie bialgebra** is a vector space $\mathfrak{g}$ together with a Lie bracket $[\ ,\ ] : \wedge^2 \mathfrak{g} \to \mathfrak{g}$ and a Lie cobracket $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ satisfying
(a) $\mathfrak{g}$ together with $[\ ,\ ]$ is a Lie algebra,
(b) $\mathfrak{g}^*$ together with $[\ ,\ ]_* = \delta^* : \wedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie algebra.
(c) A compatibility condition between $[\ ,\ ]$ and δ is satisfied:

$$\delta[x, y] = (ad_x \otimes 1 + 1 \otimes ad_x)\delta(y) - (ad_y \otimes 1 + 1 \otimes ad_y)\delta(x), \tag{2.2.13}$$

for all $x, y \in \mathfrak{g}$. The adjoint representation ad_x on $\mathfrak{g}$ is given by the commutator relation

$$ad_x y = [x, y], \quad x, y \in \mathfrak{g}. \tag{2.2.14}$$

Examples 2.20. (1) Any Lie algebra with $\delta = 0$ is a Lie bialgebra.
(2) Let $\mathfrak{g}$ be an Abelian Lie bialgebra. For δ^* satisfies the Jacobi-identity, the cobracket δ gives $\mathfrak{g}$ a Lie bialgebra structure.

There is a direct correspondence between Poisson-Lie groups and Lie bialgebras.

Theorem 2.21 (Drinfel'd [23]). *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If G is a Poisson-Lie group, then $\mathfrak{g}$ has a natural Lie bialgebra structure called the **tangent Lie bialgebra** of G.*

Conversely, if G is connected and simply connected and its Lie algebra $\mathfrak{g}$ is a Lie bialgebra, then there is a unique Poisson structure on G which makes G into a Poisson-Lie group.

Proof. Let $\xi_1, \xi_2 \in \mathfrak{g}^*$ and $f_1, f_2 \in C^\infty(G)$ such that $(df_i)|_e = \xi_i$, there is a canonical Lie algebra structure on $\mathfrak{g}^*$

$$[\xi_1, \xi_2]_* = (d\{f_1, f_2\})_e \tag{2.2.15}$$

such that δ and the Lie bracket on $\mathfrak{g}$ satisfy the compatibility condition (2.2.13), thus the Lie algebra of a Poisson-Lie group has a natural Lie bialgebra structure.

For the converse, consider a Poisson-Lie group (G, Π) with Lie algebra $(\mathfrak{g}, \delta)$, then $(\mathfrak{g}^*, [\cdot,\cdot]_*)$ is a Lie bialgebra. It integrates to an unique connected, simply connected Poisson-Lie group $\tilde{G}$. We will refer $\tilde{G}$ as the **dual Poisson-Lie group** of G. □

2.2.3 Drinfel'd doubles and Manin triples

In this section we introduce the double of a Lie bialgebra, the Manin triple. The definition of Manin triple conceals the fact that for a given Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, $\mathfrak{g}$ and $\mathfrak{g}^*$ play a symmetric role. Manin triples also provide a rich source of examples for Lie bialgebras.

Let D be a Lie group with Lie algebra $\mathcal{D}$. Now we define a symmetric, nondegenerate bilinear form $\langle\ ,\ \rangle$ on $\mathcal{D}$, that is also ad-invariant, i.e.

$$ad_a\langle b, c\rangle = \langle [a, b], c\rangle + \langle b, [a, c]\rangle = 0, \quad \forall a, b, c \in \mathcal{D}. \tag{2.2.16}$$

Definition 2.22. $\mathfrak{g}$ is **isotropic** if for all $x, y \in \mathfrak{g}$, $\langle x, y\rangle = 0$. **Maximally isotropic** means that the space cannot be enlarged while preserving the property of isotropy.

Definition 2.23. Let D be a Lie group with Lie algebra $\mathcal{D}$. A **Manin triple** is a triple of Lie algebras $(\mathcal{D}, \mathfrak{g}_+, \mathfrak{g}_-)$ together with an ad-invariant, non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ on $\mathcal{D}$, such that
(i) $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are subalgebras of $\mathcal{D}$,
(ii) $\mathcal{D} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector spaces,
(iii) $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are maximally isotropic with respect to $\langle\cdot,\cdot\rangle$.

The pair $(\mathfrak{g}_+, \mathfrak{g}_-)$ is called a **Drinfel'd double**.

Remark 2.24. $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are maximally isotropic with respect to $\langle\ ,\ \rangle$ implies that

(1) $\mathfrak{g}_+ \cong \mathfrak{g}_-^*, \quad \mathfrak{g}_- \cong \mathfrak{g}_+^*$.
(2) $\dim(\mathfrak{g}_+) = \dim(\mathfrak{g}_-)$.

It follows from the next theorem that a Manin triple gives rise to Lie bialgebras, and conversely the double of a Lie bialgebra is a Drinfel'd double:

Theorem 2.25. *(1) Suppose* $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ *is a Manin triple, and let* $[\ ,\ ] : \wedge^2\mathfrak{g} \to \mathfrak{g}$ *be a Lie bracket on* $\mathfrak{g}$*, and* $[\ ,\ ]_+$ *and* $[\ ,\ ]_-$ *are the restriction of* $[\ ,\ ]$ *on* $\wedge^2\mathfrak{g}_+$ *and* $\wedge^2\mathfrak{g}_-$ *respectively. If one defines the dual operations*

$$\delta_+ : (\mathfrak{g}_-)^* = \mathfrak{g}_+ \to \wedge^2\mathfrak{g}_+, \quad \delta_- : (\mathfrak{g}_+)^* = \mathfrak{g}_- \to \wedge^2\mathfrak{g}_-, \tag{2.2.17}$$

then $(\mathfrak{g}_+, \delta_+)$ *and* $(\mathfrak{g}_-, \delta_-)$ *are Lie bialgebras.*

(2) Let $(\mathfrak{g}, \delta)$ *define a Lie bialgebra. One can define a bracket* $[\ ,\ ]$ *and a bilinear form* $\langle\ ,\ \rangle$ *on* $\mathfrak{g} \oplus \mathfrak{g}^*$ *by*

$$\begin{aligned} \langle x+\xi, y+\eta\rangle &= \xi(y) + \eta(x) \\ [x+\xi, y+\eta] &= [x,y] + \mathrm{ad}^*_x\eta - \mathrm{ad}^*_y\xi + [\xi,\eta] + \mathrm{ad}^*_\xi y - \mathrm{ad}^*_\eta x, \end{aligned} \tag{2.2.18}$$

where $x, y \in \mathfrak{g}$ *and* $\xi, \eta \in \mathfrak{g}^*$.

$\mathfrak{g} \oplus \mathfrak{g}^*$ *with the above defined* $[\ ,]$ *and* $\langle\ ,\ \rangle$ *is a Manin triple.*

Proof. The coadjoint representation on $\mathfrak{g}^*$ is given by

$$\langle ad_x y, \xi\rangle = \langle y, -ad^*_x\xi\rangle, \quad x, y \in \mathfrak{g}, \xi \in \mathfrak{g}^*. \tag{2.2.19}$$

Thus the cocycle condition of a Lie bialgebra (2.2.13) can be rewritten as

$$\begin{aligned} &\langle\delta([x,y]), \xi\otimes\eta\rangle = \langle(ad_x\otimes 1 + 1\otimes ad_x)\delta(y) - (ad_y\otimes 1 + 1\otimes ad_y)\delta(x), \xi\otimes\eta\rangle \\ &= -\langle\delta(y), ad^*_x(\xi)\otimes\eta + \xi\otimes ad^*_x\eta\rangle + \langle\delta(x), ad^*_y(\xi)\otimes\eta + \xi\otimes ad^*_y\eta\rangle \\ &= \langle y, [ad^*_x\xi, \eta] + [\xi, ad^*_x\eta]\rangle - \langle x, [ad^*_y\xi, \eta] + [\xi, ad^*_y\eta]\rangle \\ &= \langle ad^*_\eta y, ad^*_x\xi\rangle - \langle ad^*_\xi y, ad^*_x\eta\rangle - \langle ad^*_\eta x, ad^*_y\xi\rangle + \langle ad^*_\xi x, ad^*_y\eta\rangle. \end{aligned} \tag{2.2.20}$$

The invariance of the bilinear form implies

$$[\xi, x] = ad^*_\xi x - ad^*_x\xi, \tag{2.2.21}$$

and

$$\langle[\xi,x],\eta\rangle = \langle\xi,[x,\eta]\rangle = -\langle ad^*_x\xi,\eta\rangle, \quad \langle[\xi,x],y\rangle = \langle\xi,[x,y]\rangle = -\langle ad^*_x\xi, y\rangle. \tag{2.2.22}$$

Let us rewrite $\langle\delta([x,y]),\xi\otimes\eta\rangle$ as

$$\langle\delta([x,y]),\xi\otimes\eta\rangle=\langle[x,y],[\xi,\eta]\rangle,\quad x,y\in\mathfrak{g}_+,\xi,\eta\in\mathfrak{g}_-. \tag{2.2.23}$$

Then

$$\begin{aligned}&\langle\delta[x,y],\xi\otimes\eta\rangle=\langle[x,y],[\xi,\eta]\rangle=-\langle x,[y,[\xi,\eta]]\rangle\\&=-\langle x,[\eta,[y,\xi]]+[\xi,[\eta,y]]\rangle\quad(\text{by Jacobi}-\text{identity})\\&=\langle x,[\eta,ad^*_y\xi-ad^*_\eta y]+[\xi,ad^*_\eta y-ad^*_y\eta]\rangle\\&=\langle ad^*_\eta y,ad^*_x\xi\rangle-\langle ad^*_\xi y,ad^*_x\eta\rangle-\langle ad^*_\eta x,ad^*_y\xi\rangle+\langle ad^*_\xi x,ad^*_y\eta\rangle\end{aligned} \tag{2.2.24}$$

which agrees with (2.2.20).

The bracket in (2.2.18) simply follows from (2.2.21). □

One can choose a basis in each subalgebra $T_a\in\mathfrak{g}$, $\tilde{T}^a\in\mathfrak{g}^*$ such that

$$\begin{aligned}\langle T_a,T_b\rangle&=\langle\tilde{T}^a,\tilde{T}^b\rangle=0\\\langle T_a,\tilde{T}^b\rangle&=\delta^b_a.\end{aligned} \tag{2.2.25}$$

The ad-invariance of the bilinear form $\langle\cdot,\cdot\rangle$ given by (2.2.16) implies that the brackets on the Drinfel'd double $\mathcal{D}$ is given by

$$\begin{aligned}&[T_a,T_b]=f_{ab}{}^cT_c,\\&[\tilde{T}^a,\tilde{T}^b]=\tilde{f}^{ab}{}_c\tilde{T}^c,\\&[T_a,\tilde{T}^b]=f_{ca}{}^b\tilde{T}^c+\tilde{f}^{bc}{}_aT_c,\end{aligned} \tag{2.2.26}$$

where $f_{ab}{}^c$ and $\tilde{f}^{ab}{}_c$ are the structure constants of $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively.

It follows from the Jacobi-identity on the Drinfel'd double that the structure constants are constrained to satisfy

$$f_{ab}{}^e\tilde{f}^{cd}{}_e=f_{ae}{}^c\tilde{f}^{ed}{}_b+f_{ae}{}^d\tilde{f}^{ce}{}_b-f_{bd}{}^c\tilde{f}^{ed}{}_a-f_{be}{}^d\tilde{f}^{ce}{}_a. \tag{2.2.27}$$

Example 2.26 (Abelian double)**.** Let $D=U(1)^n\times\tilde{U}(1)^n$, and denote the generators of the Lie algebras of $U(1)^n$ and $\tilde{U}(1)^n$ by $\{T_a\},\{\tilde{T}^a\}$, for $a=1,\cdots,n$, respectively. In this case all brackets vanish

$$[T_a,T_b]=[\tilde{T}^a,\tilde{T}^b]=[T_a,\tilde{T}^b]=0.$$

Thus D is a Drinfel'd double.

Example 2.27 ($O(2,2)$ double)**.** Consider the Lie algebra $sl(2,\mathbb{R})$ with generators H, E_+, E_- and commutators

$$[H, E_\pm] = \pm 2E_\pm, \quad [E_+, E_-] = H, \tag{2.2.28}$$

equipped with the non-degenerate, symmetric, invariant bilinear form

$$\langle E_+, E_- \rangle = 1, \quad \langle H, H \rangle = 2. \tag{2.2.29}$$

There exists a Drinfel'd double D called the $O(2,2)$ double such that its Lie algebra $\mathcal{D}$ is the direct sum of two copies of $sl(2,\mathbb{R})$

$$\mathcal{D} = sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}) \tag{2.2.30}$$

with the following bilinear form

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle, \quad (x_1, y_1), (x_2, y_2) \in \mathcal{D}. \tag{2.2.31}$$

$\mathcal{D}$ can be decomposed into a pair of maximally isotropic subalgebras $(sl(2,\mathbb{R}), b2)$ such that $sl(2,\mathbb{R})$ is generated by

$$T_1 = \frac{1}{2}(H, H), \quad T_2 = (E_+, E_+), \quad T_3 = (E_-, E_-), \tag{2.2.32}$$

and $b2$ has generators

$$\tilde{T}^1 = \frac{1}{2}(H, -H), \quad \tilde{T}^2 = (0, -E_-), \quad \tilde{T}^3 = (E_+, 0). \tag{2.2.33}$$

These two sets of generators satisfy (2.2.25) and (2.2.26), therefore the pair $(sl(2,\mathbb{R}), b2)$ is a Drinfel'd double.

2.3 Lie algebroid and Courant algebroid theory

For a general Poisson manifold (M, Π), it was noticed by Weinstein, Mackenzie and Liu [61] that the concept of a Lie algebroid captures more closely the inherent nature of a Poisson manifold than that of a Lie bialgebra. A Poisson structure on M induces a Lie algebroid structure on the cotangent bundle T^*M, thus both TM and T^*M have Lie algebroid structures and one can extend the notion of Lie bialgebra to Lie bialgebroid in the sense of Mackenzie and Xu [83].

In the previous section, we noted that Lie bialgebras can be defined in terms of Manin triples. It is then natural to ask if the notion of Manin triples can be extended to Lie bialgebroids. This extension of the theory of Manin triples to Lie bialgebroids is constructed by Liu, Weinstein and Wu [59].

For a Lie bialgebroid (A, A^*) over M, there is a natural Courant algebroid structure on the double of the Lie bialgebroid, which is analogous to the Drinfel'd double of a Lie bialgebra when M is a point. And conversely, for a pair of complementary isotropic subbundles (A, A^*) of a Courant algebroid E, closed under the bracket structure defined on a Courant algebroid, there is a natural Lie bialgebroid structure on (A, A^*) such that its double is isomorphic to E. Therefore the theory of Manin triples of Lie algebras extends to Lie algebroids.

In this section we review the definitions of a Lie algebroid, a Lie bialgebroid, a Courant algebroid and an exact Courant algebroid, and we also review an alternative construction of a Lie bialgebroid using the definition of a derived bracket constructed by Kosmann-Schwarzbach [56] and Roytenberg [67]. At the end of this section, we introduce an object called the proto-bialgebroid which is a "quasi"-version of Lie bialgebroids [56].

2.3.1 Lie algebroids and Lie bialgebroids

In this section we review the notion of a Lie algebroid and a Lie bialgebroid, which are important for the construction of Courant algebroids.

Definition 2.28. A **Lie algebroid** $(A, M, \rho, [\cdot,\cdot])$ is a vector bundle A over a manifold M together with an anchor map $\rho : A \to TM$ and a Lie bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$ satisfying the following conditions:

(1) $\rho[X, Y]_A = [\rho X, \rho Y]$.

(2) $[X, fY]_A = f[X, Y]_A + (\rho(X)f)Y$,

where $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$.

Example 2.29. Any vector bundle with zero anchor map and Lie bracket is a Lie algebroid.

Example 2.30. A Lie algebra regarded as a vector bundle over a point is a Lie algebroid.

Example 2.31. The tangent bundle TM of a manifold M with the usual Lie bracket of vector fields and ρ the identity map is a Lie algebroid.

Example 2.32 (Cotangent Lie algebroid)**.** Let (M, Π) be a Poisson manifold, and $\pi^\sharp : T^*M \to TM$ be the bundle map defined by $\langle \beta, \pi^\sharp\alpha \rangle = \Pi(\alpha, \beta)$ for all differential one-forms α and β. For every Poisson manifold M there exists a Lie algebroid structure on T^*M, with anchor $\rho = \pi^\sharp$ and the associated bracket on $\Gamma(T^*M)$ being the **Koszul bracket** of differential forms defined by

$$[\alpha, \beta]_\Pi = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - d(\Pi(\alpha, \beta)). \tag{2.3.1}$$

Here the Koszul bracket has the following properties:
(1) It is a Lie bracket $[\ ,\ ]_\Pi$ satisfying

$$[df, dg]_\Pi = d\{f, g\}, \quad \forall f, g \in C^\infty(P),$$

where $\{\cdot, \cdot\}$ is the Poisson bracket of functions defined by Π.
(2) It satisfies

$$[\alpha, f\beta]_\Pi = f[\alpha, \beta]_\Pi + (\pi^\sharp(\alpha)f)\beta, \quad \forall \alpha, \beta \in \Omega^1(P), f \in C^\infty(P),$$

(3) The bundle map $\pi^\sharp$ is a Lie algebra homomorphism:

$$[\pi^\sharp(\alpha), \pi^\sharp(\beta)] = \pi^\sharp[\alpha, \beta]_\Pi. \tag{2.3.2}$$

And the associated differential on $\Gamma(\wedge^\bullet TM)$ is defined by $d_\Pi = [\Pi, \cdot]_{SN}$, here the bracket $[\ ,\ ]_{SN}$ is the Schouten bracket of multi-vector fields.

The Lie algebroid in the previous example is called a **cotangent Lie algebroid**, it provides an important link between Poisson geometry and Lie algebroid theory.

Definition 2.33. Let $(A, M, \rho, [\ ,\ ])$ and the dual bundle $(A^*, M, \rho_*, [\ ,\ ]_*)$ both be vector bundles equipped with Lie algebroid structures. A **Lie bialgebroid** is a dual pair (A, A^*) such that the differential d_* on $\Gamma(\wedge^\bullet A)$ coming from the structure on A^* is a derivation of the bracket $[\ ,\ ]$ on $\Gamma(A)$, i.e. if for all $X, Y \in \Gamma(A)$, the following condition is satisfied:

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y]. \tag{2.3.3}$$

Remark 2.34. The condition (2.3.3) is equivalent to having the differential d on $\Gamma(\wedge^\bullet A^*)$ dual to d_* with respect to the natural pairing $\langle\cdot, \cdot\rangle$ on A and A^*, such that d is a derivation of the bracket $[\ ,\ ]_*$ on $\Gamma(A^*)$, i.e.

$$d[\phi, \psi]_* = [d\phi, \psi]_* + [\phi, d\psi]_*$$

for all $\phi, \psi \in \Gamma(A^*)$.

Example 2.35. A Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ regarded as a pair of vector bundles over a point is a Lie bialgebroid.

Example 2.36. Let (M, Π) be a Poisson manifold, one can verify that (TM, T^*M) is a Lie bialgebroid, where the Lie algebroid $(TM, [\ ,\ ], \rho, d)$ is equipped with the standard Lie bracket $[\ ,\ ]$, identity anchor map $\rho = id$, and the differential d on $\Gamma(\wedge T^*M)$ is given by the de Rham differential d. While $(T^*M, [\ ,\ ]_\Pi, \rho_* = \pi^\sharp, d_\Pi)$ is a Lie algebroid with the structures as given by example 2.32.

Not only does a Poisson structure induce a Lie algebroid structure on the cotangent bundle T^*M of a Poisson manifold M, a Lie algebroid structure also induces a Poisson structure on its base manifold, by the following lemmas.

Let (A, A^*) be a Lie bialgebroid over a Poisson manifold M with Lie algebroid structures $(A, [\ ,\ ], \rho, d)$ and $(A^*, [\ ,\ ]_*, \rho_*, d_*)$, then [53]

Lemma 2.37. *for all $f, g \in C^\infty(M)$,*

$$d\{f, g\} = [df, dg]_*, \qquad d_*\{f, g\} = -[d_* f, d_* g]. \tag{2.3.4}$$

Lemma 2.38. *The bracket on $C^\infty(M)$ defined by*

$$\{f, g\} = \langle df, d_* g\rangle \tag{2.3.5}$$

is a Poisson structure on M.

2.3.2 Courant algebroids

The Courant bracket was first introduced by Courant and Weinstein [18, 19] as an extension of the Lie bracket of vector fields on TM to sections of $TM \oplus T^*M$.

In order to extend the theory of Manin triples from Lie bialgebras to Lie bialgebroids, Liu, Weinstein and Xu [59] axiomatized the properties of the Courant bracket to those of a Courant algebroid. Under this construction, the double of a Lie bialgebroid carries a Courant algebroid structure, and conversely a Courant algebroid gives rise to a Lie bialgebroid. Thus a Courant algebroid can be naturally constructed from a given Lie bialgebroid.

It was also realized that an **exact Courant algebroid** provides the natural setting to study generalized complex structures in generalized geometry.

The following definition of Courant algebroids is due to Kosmann-Schwarzbach [55]:

Definition 2.39. A **Courant algebroid** $\mathcal{C} = (E, M, \circ, \rho)$ is a vector bundle $E \to M$ equipped with a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$, a Dorfmann bracket $\circ$ on $\Gamma(E)$ which is an $\mathbb{R}$-linear map satisfying the Jacobi-identity,

$$A \circ (B \circ C) = (A \circ B) \circ C + B \circ (A \circ C), \quad A, B, C \in \Gamma(E), \tag{2.3.6}$$

a bundle map $\rho : E \to TM$ called the anchor satisfying the following conditions for all $A, B, C \in \Gamma(E)$:

$$\begin{aligned} &(1) \qquad \rho(A)\langle B, C\rangle = \langle A, B \circ C + C \circ B\rangle, \\ &(2) \qquad \rho(A)\langle B, C\rangle = \langle A \circ B, C\rangle + \langle B, A \circ C\rangle. \end{aligned} \tag{2.3.7}$$

The Dorfmann bracket in a Courant algebroid $\mathcal{C}$ is not skew-symmetric. The skew symmetrization of the Dorfmann bracket of $\mathcal{C}$ is known as the Courant bracket

$$[\![A, B]\!] = \frac{1}{2}(A \circ B - B \circ A). \tag{2.3.8}$$

Two main theorems for Lie bialgebroids in [59] show that a Courant algebroid can be constructed from a Lie bialgebroid and conversely a Courant algebroid gives rise to a Lie bialgebroid.

Theorem 2.40. *Let (A, A^*) be a Lie bialgebroid. Both A and A^* are Lie algebroids over the base manifold M, with anchor maps ρ and ρ_*, respectively. (A, A^*) is equipped with two natural nondegenerate bilinear forms, one symmetric and one antisymmetric*

$$\langle x_1 + \xi_1, x_2 + \xi_2\rangle_\pm = \xi_1(x_2) \pm \xi_2(x_1), \tag{2.3.9}$$

and a Courant bracket on $\Gamma(A \oplus A^*)$

$$\begin{aligned} [\![e_1, e_2]\!] \;&=\; ([x_1, x_2] + \mathcal{L}^*_{\xi_1} x_2 - \mathcal{L}^*_{\xi_2} x_1 - \frac{1}{2} d_* \langle e_1, e_2\rangle_-) \\ &\quad + ([\xi_1, \xi_2]_* + \mathcal{L}_{x_1}\xi_2 - \mathcal{L}_{x_2}\xi_1 + \frac{1}{2} d\langle e_1, e_2\rangle_-), \end{aligned} \tag{2.3.10}$$

where $e_1 = x_1 + \xi_1, e_2 = x_2 + \xi_2$, $x_1, x_2 \in \Gamma(A), \xi_1, \xi_2 \in \Gamma(A^*)$ and $\mathcal{L}^*_\xi = \imath_\xi d_* + d_* \imath_\xi$.

There is an anchor map $\rho_E : A \oplus A^* \to TM$ defined by $\rho_E = \rho + \rho_*$, i.e.

$$\rho_E(x + \xi) = \rho(x) + \rho_*(\xi), \quad \forall x \in \Gamma(A), \xi \in \Gamma(A^*). \tag{2.3.11}$$

Then $E = A \oplus A^*$ together with $([\![\ ,\ ]\!], \rho_E, \langle\ ,\ \rangle_+)$ is a Courant algebroid.

Conversely, we have

Theorem 2.41. *In a Courant algebroid $(E, \rho, [\cdot,\cdot], \langle\cdot,\cdot\rangle)$, suppose that L_1 and L_2 are Dirac subbundles transversal to each other, i.e., $E = L_1 \oplus L_2$. Then (L_1, L_2) is a Lie bialgebroid, where L_2 is considered as the dual bundle of L_1 under the pairing $\langle\ ,\ \rangle$.*

2.3.3 Exact Courant algebroids

Definition 2.42. A Courant algebroid E over M is an **exact Courant algebroid** if the following sequence

$$0 \to T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \to 0 \tag{2.3.12}$$

is exact. Here ρ is the anchor map of E.

Definition 2.43. A **connection** on E is a map $A : TM \to E$ such that it is an isotropic splitting - it splits the anchor map ρ, i.e. $\rho \circ A = id_{TM}$, and is isotropic, i.e. for $x, y \in \Gamma(TM)$, $\langle A(x), A(y)\rangle = 0$.

Remark 2.44. The space of connections is an affine space under $\Omega^2(E)$. Let $\omega \in \Omega^2(E)$ be a two-form. If one adds ω to the original connection, one sees that $A + \omega : x \mapsto A(x) + \imath_x\omega$ is also a connection – the new connection splits ρ and is isotropic.

One defines the curvature of the connection A as the antisymmetric map $F : TM \times TM \to E$ given by

$$F(x, y) = [\![A(x), A(y)]\!] - A([x, y]). \tag{2.3.13}$$

The **curvature** 3-**form** H for the connection A is defined by $H(x, y, z) = \imath_z F(x, y)$. Here $\imath_z$ acts on $\Gamma(E)$ via the natural pairing $\langle\ ,\ \rangle$ on E, that is, $\imath_z F(x, y) = \langle z, F(x, y)\rangle$. Thus the curvature 3-form is given by

$$H(x, y, z) = \langle A(x) \circ A(y), A(z)\rangle \tag{2.3.14}$$

and H is closed.

Proof. The closedness of H follows from the Jacobi-identity

$$Jac(A(x), A(y), A(z)) = -\imath_x\imath_y\imath_z dH, \tag{2.3.15}$$

i.e. $dH = 0$. □

Thus given an exact Courant algebroid E with an isotropic splitting A, such a splitting is characterized by a curvature 3-form H given by (2.3.14). The splitting A give rises to the bundle isomorphism $A+\rho^* : TM\oplus T^*M \to E$, and one can use such a bundle map to transport the Courant algebroid structures onto $TM\oplus T^*M$. The Courant algebroid structure on $TM \oplus T^*M$ has a natural pairing as before:

$$\langle x + \xi, y + \eta\rangle = \frac{1}{2}(\xi(y) + \eta(x)), \tag{2.3.16}$$

for $x+\xi, y+\eta \in \Gamma(TM \oplus T^*M)$. While the Courant bracket on $TM \oplus T^*M$ is twisted by the curvature 3-form H, i.e. it is equipped with the **H-twisted Courant bracket** on $TM \oplus T^*M$:

$$[\![x+\xi, y+\eta]\!]_H = [\![x+\xi, y+\eta]\!] + \imath_x \imath_y H. \tag{2.3.17}$$

Therefore an exact Courant algebroid E equipped with a natural pairing and a Courant bracket is equivalent to $TM \oplus T^*M$ equipped with the same natural pairing and a twisted Courant bracket. The notion of a generalized tangent space $TM \oplus T^*M$ will be discussed in Chapter 4.

If one chooses an isotropic splitting (a connection A) of the exact sequence, i.e. $E \cong TM \oplus T^*M$, one can add to A a 2-form $b \in \Omega^2(M)$ to obtain a new connection. Adding b changes H to $H + db$. Thus an H-twisted Courant algebroid $(TM \oplus T^*M)_H$ is isomorphic to an $H' = H + db$-twisted Courant algebroid $(TM \oplus T^*M)_{H'}$, with the isomorphic map $(x, \xi) \mapsto (x, \xi + \imath_x b)$. So we have the following lemma:

Lemma 2.45. *An exact Courant algebroid E is characterized by $[H] \in H^3(M, \mathbb{R})$.*

This classification of exact Courant algebroids is due to Ševera [72] and $[H]$ is often referred to as the **Ševera class** or **characteristic class**.

2.3.4 An alternative definition of a Lie bialgebroid

There is an elegant way to define a Lie bialgebroid using the concept of a derived bracket in the sense of Kosmann-Schwarzbach [55] (originally introduced by Roytenberg [67].) This construction of a Lie bialgebroid provides a convenient setting to extend the notion of a Lie bialgebroid to a proto-bialgebroid defined by Kosmann-Schwarzbach [56].

In this section we start by introducing the derived brackets, then the general construction of a Lie bialgebroid using derived brackets. At the end of this section we introduce an object called proto-bialgebroid.

Derived bracket

Definition 2.46. If $(V, [\cdot,\cdot], D)$ is a graded Lie algebra over $\mathbb{R}$ with degree n and a derivation D, the **derived bracket** of $[\ ,\ ]$ by D is a bilinear map $[\ ,\ ]_D : V \otimes V \to V$ defined by

$$[a,b]_D = (-1)^{n+|a|+1}[Da, b], \tag{2.3.18}$$

for $a, b \in V$.

Properties of the derived bracket:
(1) The derived bracket satisfies the graded Jacobi-identity

$$[a,[b,c]_D]_D = [[a,b]_D,c]_D + (-1)^{(n+|a|+1)(n+|b|+1)}[b,[a,c]_D]_D. \tag{2.3.19}$$

(2) With the graded Leibnitz rule

$$D[a,b]_D = [Da,b]_D + (-1)^{|a|+n+1|}[a,Db]_D. \tag{2.3.20}$$

The graded Leibnitz rule can also be expressed as

$$[a,b]_D = [a,Db] + (-1)^{n+|a|+1}D[a,b], \tag{2.3.21}$$

$\forall a,b,c \in V$.

Definition 2.47. An interior derivation D by an element $d \in V$ is a map $D : a \mapsto [d,a]$ for $a \in V$, where d is an element of square 0 in $(V,[\ ,\ ])$.

Theorem 2.48 (Kosmann-Schwarzbach[55])**.** *Let D be the interior derivation of $(V,[\ ,\ ])$ by an element $d \in V$, with $|d|$ the degree of d and n the degree of the bracket. If $|d|+n$ is odd and $[d,d]=0$, then the derived bracket is*

$$[a,b]_d = [[d,a],b], \tag{2.3.22}$$

for $a,b \in V$.

Remark 2.49. For example, the Cartan formulae $[\mathcal{L}_x, \imath_y] = \imath_{[x,y]}$ can be rewritten as

$$\imath_{[x,y]} = [[d,\imath_x],\imath_y],$$

As a result, the Lie bracket of vector fields is a derived bracket.

Interpreting a Lie bialgebroid using the derived bracket

This approach defines Lie algebroids in terms of functions on the supermanifolds and is developed by Vaintrob [76], Roytenberg [66, 67] and Kosmann-Schwarzbach [55].

Let A be a vector bundle over M. Equivalently, a Lie algebroid structure can be defined in three ways:

Firstly, recall in Section 2.3.1 that a Lie algebroid structure on A is given by a Lie algebra structure $[\ ,\ ]$ on ΓA and an anchor map ρ satisfying the axioms in Definition 2.28. Or equivalently the Lie algebroid structure is given by a derivation d on $\Gamma(A^*)$.

Alternatively, one can view $\Gamma(A^*)$ as the algebra of funcitons on the supermanifold ΠA, where Π denotes the change of parity functors applied to each fibre.

Let x^i be local coordinates on M, $\{e_a\}$ be a local basis of $\Gamma(A)$ and (x^i, y^a) be the local coordinates on A. The anchor ρ and the Lie bracket on A is given by

$$\rho(e_a) = A^i_a(x)\frac{\partial}{\partial x^i},$$
$$[e_a, e_b] = f_{ab}{}^c e_c. \tag{2.3.23}$$

Let $(x^i, \tilde{y}^a)$ be the local coordinates on ΠA, then the homological vector field d_A has the local expression

$$d_A = \tilde{y}^a A^i_a(x)\frac{\partial}{\partial x^i} + \frac{1}{2} f_{ab}{}^c \tilde{y}^a \tilde{y}^b \frac{\partial}{\partial \tilde{y}^c}. \tag{2.3.24}$$

One can also view a Lie algebroid structure on A as follows [55, 66, 67]:

Definition 2.50. A Lie algebroid structure on A is a degree three function μ on the supermanifold $T^*(\Pi A^*)$. μ is required to satisfy

$$\{\mu, \mu\} = 0, \tag{2.3.25}$$

where $\{\ ,\ \}$ is the canonical Poisson bracket of $T^*(\Pi A^*)$.

Let (x^i, ξ^*_a) be the local coordinates on ΠA^* dual to $(x^i, \tilde{y}^a)$, and $(x^i, \xi^*_a, x^*_i, \xi^a)$ be the associated coordinates on $T^*(\Pi A^*)$, then locally μ is defined by

$$\mu = x^*_i A^i_a(x)\xi^a + \frac{1}{2} f_{ab}{}^c(x)\xi^*_c \xi^a \xi^b. \tag{2.3.26}$$

Since $T^*(\Pi A^*)$ is $\mathbb{Z}^2$-graded, we assign bi-degree $(0,1)$ to variables ξ^a, $(1,0)$ to variables ξ^*_a, and $(1,1)$ to variables x^a and x^*_a. The degree of a function on $T^*(\Pi A^*)$ with bi-degree (p,q) is $p+q$.

The canonical Poisson structure on $C^\infty(T^*\Pi A)$ is defined as follows and has the following properties:

1. The bracket is uniquely determined by the relations:

$$\{x^a, x^b\} = \{x^*_a, x^*_b\} = \{\xi^*_a, \xi^*_b\} = \{\xi^a, \xi^b\} = 0,$$
$$\{x^*_a, x^b\} = \{\xi^*_a, \xi^b\} = \delta^b_a. \tag{2.3.27}$$

2. The Poisson bracket is skew symmetric:

$$\{e_1, e_2\} = -(-1)^{kl}\{e_2, e_1\}, \tag{2.3.28}$$

where e_1 and e_2 are degree k and degree l functions on $T^*\Pi A^*$, respectively.

3. For e a degree k function on $T^*\Pi A$, the bracket is a derivation of degree $k-2$, i.e.
$$\{e, e_1e_2\} = \{e, e_1\}e_2 + (-1)^{k-2}e_1\{e, e_2\}. \tag{2.3.29}$$

4. And the bracket $\{\ ,\ \}$ satisfies a graded Jacobi-identity:
$$\{e_1, \{e_2, e_3\}\} = \{\{e_1, e_2\}, e_3\} + (-1)^{kl}\{e_2, \{e_1, e_3\}\}, \tag{2.3.30}$$
where e_1, e_2, e_3 are degree k, l, m functions on $T^*\Pi A$, respectively.

Definition 2.51. A Lie algebroid structure on A is the supermanifold ΠA together with a homological vector field of degree 1, i.e. a derivation d_A on $\Gamma(\wedge^\bullet A^*)$ increasing degree by 1 and satisfying $d_A^2 = 0$.

The Lie algebroid brackets can be defined in terms of the Poisson bracket on $T^*(\Pi A^*)$ by [56]:

Theorem 2.52. *A Lie algebroid bracket* $[\ ,\ ]_A$ *on* A *is given by the derived bracket*
$$[x, y]_A = \{\{x, \mu\}, y\}, \quad x, y \in C^\infty(\Pi A^*) = \Gamma(\wedge^\bullet A), \tag{2.3.31}$$
while the anchor is given by
$$\rho_A(x)f = \{\{x, \mu\}, f\}, \quad x \in C^\infty(\Pi A^*), f \in C^\infty(M). \tag{2.3.32}$$

A Lie bialgebroid (A, A^*) can be defined as follows:

Definition 2.53. A Lie bialgebroid (A, A^*) is a pair of Lie algebroids in duality. The Lie algebroids $(A, [\ ,\ ]_A, \rho_A)$ and $(A^*, [\ ,\ ]_{A_*}, \rho_{A_*})$ have Lie algebroid structures correspond to functions μ and γ on the same supermanifold $T^*(\Pi A)$, respectively. The Lie bialgebroid condition (2.3.3) is equivalent to placing the following condition on structures μ and γ
$$\{\mu + \gamma, \mu + \gamma\} = 0. \tag{2.3.33}$$

The Lie bialgebroid (A, A^*) has the associated differential d_A and d_{A^*} on $\Gamma(\wedge^\bullet A^*)$ and $\Gamma(\wedge^\bullet A)$ given by
$$d_A = \{\mu, \cdot\}, \quad d_{A^*} = \{\gamma, \cdot\}, \tag{2.3.34}$$
which satisfy
$$d_A^2 = d_{A^*}^2 = 0. \tag{2.3.35}$$

Proto bialgebroid

The notion of a Lie bialgebroid discussed previously in this section can be generalized to a *proto-bialgebroid* [55, 67].

Definition 2.54. A **Proto-bialgebroid** (A, A^*) is a supermanifold $T^*\Pi A$ with a quadruple $(\mu, \gamma, \varphi, \psi)$ such that μ, γ, φ and ψ are bi-degree $(1,2)$, $(2,1)$, $(3,0)$ and $(0,3)$ functions on $T^*\Pi A^*$ satisfying

$$\{\theta, \theta\} = 0, \tag{2.3.36}$$

where $\theta = \mu + \gamma + \varphi + \psi$.

In local coordinates on $T^*\Pi A$, these structures are given by [67]

$$\begin{aligned}
\mu &= \xi^a A^i_a(x) x^*_i - \frac{1}{2} f_{ab}{}^c(x) \xi^a \xi^b \xi^*_c, \\
\gamma &= \bar{A}^{ai}(x) x^*_i \xi^*_a - \frac{1}{2} q^{ab}{}_c(x) \xi^*_a \xi^*_b \xi^c, \\
\varphi &= \frac{1}{6} \varphi^{abc}(x) \xi^*_a \xi^*_b \xi^*_c, \\
\psi &= \frac{1}{6} \psi_{abc}(x) \xi^a \xi^b \xi^c.
\end{aligned} \tag{2.3.37}$$

On a proto-bialgebroid (A, A^*), we have the following structures:

- Lie algebroid brackets $[\ ,\ ]_A$ and $[\ ,\ ]_{A^*}$ are given by

$$\begin{aligned}
[\cdot, \cdot]_A &= \{\{\cdot, \mu + \psi\}, \cdot\}, \\
[\cdot, \cdot]_{A^*} &= \{\{\cdot, \gamma + \varphi\}, \cdot\}.
\end{aligned} \tag{2.3.38}$$

- Anchor maps ρ_A and ρ_{A^*} given by

$$\begin{aligned}
\rho_A(x)(f) &= \{\{x, \mu\}, f\} = A^i_a(x) \partial_i f, \\
\rho_{A^*}(\xi)(f) &= \{\{\xi, \gamma\}, f\} = \bar{A}^{ai}(x) \partial_i f,
\end{aligned} \tag{2.3.39}$$

  where $x \in \Gamma(A)$ and $\xi \in \Gamma(A^*)$.

- Quasi-differentials d_A and d_{A^*} on $\Gamma(\wedge^\bullet A^*)$ and $\Gamma(\wedge^\bullet A)$ respectively,

$$d_A = \{\mu, \cdot\}, \quad d_{A^*} = \{\gamma, \cdot\}, \tag{2.3.40}$$

  satisfying

$$(d_A)^2 + \{d_{A^*}\psi, \cdot\} = 0, \quad (d_{A^*})^2 + \{d_A \varphi, \cdot\} = 0. \tag{2.3.41}$$

Recall that the condition for the structures $(\mu, \gamma, \psi, \varphi)$ to define a proto-bialgebroid (A, A^*) is that it obeys the structure equation (2.3.36). Splitting the degree 3-function $\theta = \mu + \gamma + \psi + \varphi$ into components according to the bi-grading, the above condition is equivalent to a set of five conditions:

$$\begin{cases} \frac{1}{2}\{\mu, \mu\} + \{\gamma, \psi\} = 0, \\ \{\mu, \gamma\} + \{\varphi, \psi\} = 0, \\ \frac{1}{2}\{\gamma, \gamma\} + \{\mu, \varphi\} = 0, \\ \{\mu, \psi\} = 0, \\ \{\gamma, \varphi\} = 0. \end{cases} \tag{2.3.42}$$

Proposition 2.55. *The double of a proto-bialgebroid $A \oplus A^*$ is a Courant algebroid, with the Dorfmann bracket defined by the derived bracket*

$$(x + \xi) \circ (y + \eta) = \{\{x + \xi, \theta\}, y + \eta\}, \tag{2.3.43}$$

where $x, y \in \Gamma(A)$ and $\xi, \eta \in \Gamma(A^*)$. And the anchor map ρ is given by

$$\rho(x + \xi)(f) = \rho_A(x)(f) + \rho_{A^*}(\xi)(f) = \{\{x + \xi, \theta\}, f\}, \tag{2.3.44}$$

where $f \in C^\infty(M)$.

Proof. We need to check the properties (2.3.7) for $A \oplus A^*$ to define a Courant algebroid. For simplicity, let us identify $A, B, C \in \Gamma(A \oplus A^*)$ with their images in $C^\infty(T^*\Pi A)$. We will also use the Jacobi-identity (2.3.30) of the Poisson bracket $\{\ ,\ \}$ of $C^\infty(T^*\Pi A)$.

We first prove property (1) of (2.3.7). In terms of functions on $C^\infty(T^*\Pi A)$ and the canonical Poisson bracket, this property can be rewrite as

$$\{\{A, \Theta\}, \{B, C\}\} = \{A, \{\{B, \Theta\}, C\}\} + \{A\{\{C, \Theta\}, B\}\}$$

$$\begin{aligned} RHS \;=\; & \{A, \{B, \{\Theta, C\}\}\} + \{A, \{C, \{\Theta, B\}\}\} + \{A, \{\Theta, \{B, C\}\}\} \\ & + \{A, \{\Theta, \{C, B\}\}\} \\ & = \{A, \{\{B, \Theta\}, C\}\} + \{A, \{\{C, \Theta\}, B\}\} \\ & = \{A, \{\{B, \Theta\}, C\}\} + \{A, \{C, \{\Theta, B\}\}\} + \{A, \{, \Theta, \{C, B\}\}\} \\ & = \{\{A, \Theta\}, \{B, C\}\} = LHS. \end{aligned} \tag{2.3.45}$$

Property (2) of (2.3.7) can similarly be written as

$$\{\{A, \Theta\}, \{B, C\}\} = \{A, \Theta\}, B\}, C\} + \{B, \{\{A, \Theta\}, C\}\}.$$

$$\begin{aligned} RHS &= \{\{A,\Theta\},\{B,C\}\} - \{B,\{\{A,\Theta\},C\}\} + \{B,\{\{A,\Theta\},C\}\} \\ &= \{\{A,\Theta\},\{B,C\}\} = LHS. \end{aligned} \tag{2.3.46}$$

□

Chapter 3

Abelian T-duality and Poisson-Lie T-duality

3.1 Introduction and outline

T-duality in string theory plays an important role as it relates String theory on different backgrounds and can be realized as a transformation between two-dimensional σ-models [29]. A two-dimensional σ-model describes the world-sheet theory of a string propagating on a target manifold M equipped with a Riemannian metric g_{ij} and an antisymmetric B-field b_{ij}, with string background defined by $E_{ij} \equiv g_{ij} + b_{ij}$.

The rules for T-duality with an Abelian isometry were first constructed by Buscher [13] in 1987, and these rules are known as the **Buscher rules**. To obtain the Buscher rules, M is required to have some Abelian isometry group which leaves the σ-model invariant. The dual model can then be obtained by gauging the isometry, with gauge fields being integrated out. Here, let us simply refer to this type of construction as **Abelian T-duality**.

Since the Buscher rules are so simple and beautifully symmetric, a naive question to ask is whether the Buscher rules can be extended to the case when the isometry is non-Abelian.

A first attempt to construct a version of T-duality with respect to a non-Abelian isometry was done by de la Ossa and Quevedo [20] in 1993. Inspired by Buscher's technique, they applied a T-duality transformation following Buscher's procedure using non-Abelian isometry groups. However it was soon realized by de la Ossa, Quevedo and other authors [4, 20, 27] that non-Abelian T-duality in this formalism suffered certain drawbacks, the most noticeable being that this technique is not symmetric, i.e. one does not in general recover the original theory by repeating the T-duality procedure.

In another attempt to construct non-Abelian T-duality, Klimčík and Ševera [49] abandoned the requirement of isometry as dualizability and proposed a generalization of T-duality in 1995, which has come to be known as the **Poisson-Lie T-duality**. In this formalism of non-Abelian T-duality, the requirement of an isometry is replaced by the Poisson-Lie condition, which places a restriction on the backgrounds of a dual pair of σ-models. The Poisson-Lie condition is necessary for the existence of the dual worldsheet. Also the relevant structure underlying non-Abelian T-duality is a Drinfel'd double. With a given Drinfel'd double, a dual pair of σ-models with backgrounds satisfying the Poisson-Lie condition can be constructed.

The structure of this chapter is organized as follows. Section 3.2 gives a brief account of the basic concept of Abelian T-duality, i.e. the Buscher rules. In section 3.3, we review the construction of Poisson-Lie T-duality due to Klimčík and Ševera [47, 49].

3.2 Abelian T-duality

3.2.1 T-duality with a $U(1)^n$ isometry

In string theory, a string propagates in d-dimensional space-time E sweeping out a two-dimensional worldsheet with coordinates z and $\bar{z}$, and the action of such a string is described by the two-dimensional σ-model action:

Definition 3.1. The **two-dimensional σ-model action** is described by a metric g_{ij} and a locally defined two-form b_{ij} on the d-dimensional target manifold E with the following action

$$S = \frac{1}{2\pi}\int d^2z(g_{ij} + b_{ij})\partial x^i \bar{\partial} x^j = \frac{1}{2\pi}\int d^2z E_{ij}(x)\partial x^i \bar{\partial} x^j, \qquad (3.2.1)$$

here $x^i, i = 1, \dots, d$ denote target space coordinates and ∂x^i (resp. $\bar{\partial} x^i$) are derivatives with respect to the world-sheet coordinates z (resp. $\bar{z}$).

Roughly speaking, this action integrates over the two-dimensional world-sheet of a string.

Starting with a σ-model action (3.2.1) with a $U(1)^n$ isometry, let us choose coordinates $\{x^i\} = \{x^a, x^\mu\}, a = 1, \dots, n,\ \mu = n{+}1, \dots, d$, such that the isometry acts by translation of x^a. The string background E_{ij} can be decomposed as

$$E_{ij} = \begin{pmatrix} E_{ab} & E_{a\nu} \\ E_{\mu b} & E_{\mu\nu} \end{pmatrix}, \tag{3.2.2}$$

and the σ-model action becomes

$$S = \frac{1}{2\pi}\int d^2z(E_{ab}\partial x^a\bar{\partial}x^b + E_{a\nu}\partial x^a\bar{\partial}x^\nu + E_{\mu b}\partial x^\mu\bar{\partial}x^b + E_{\mu\nu}\partial x^\mu\bar{\partial}x^\nu). \tag{3.2.3}$$

To obtain the dual theory, let us first gauge the $U(1)^n$ isometry $x^a \mapsto x^a + \epsilon^a$ by replacing

$$\partial x^a \mapsto Dx^a = \partial x^a + A^a, \quad \bar{\partial}x^a \mapsto \bar{D}x^a = \bar{\partial}x^a + \bar{A}^a, \tag{3.2.4}$$

where A^a are connection one forms on M.

Adding a Lagrangian multiplier term

$$\frac{1}{2\pi}\int d^2z\ \tilde{x}^a(\partial\bar{A}^a - \bar{\partial}A^a) \tag{3.2.5}$$

we then end up with a σ-model action

$$\begin{aligned} S^{d+1} &= \frac{1}{2\pi}\int d^2z[E_{ab}Dx^a\bar{D}x^b + (E_{a\nu})Dx^a\bar{\partial}x^\nu + (E_{\mu b})\partial x^\mu\bar{D}x^b + (E_{\mu\nu})\partial x^\mu\bar{\partial}x^\nu \\ &+ \tilde{x}^a(\partial\bar{A}^a - \bar{\partial}A^a)]. \end{aligned} \tag{3.2.6}$$

This action has a gauge symmetry

$$x^a \mapsto x^a + \epsilon^a, \quad A^a \mapsto A^a - \partial\epsilon^a, \quad \bar{A}^a \mapsto \bar{A}^a - \bar{\partial}\epsilon^a, \tag{3.2.7}$$

Integrating out the Lagrange multipliers $\tilde{x^a}$ requires the field strength $F^a = \partial\bar{A}^a - \bar{\partial}A^a$ to vanish, which imposes pure gauge conditions on A^a and $\bar{A}^a$, i.e. $A^a = \partial\tilde{x}^a, \bar{A}^a = \bar{\partial}\tilde{x}^a$.

To retrieve the original σ-model action, we gauge fix S^{d+1} by either choosing $x^a = 0$ or $\tilde{x}^a = 0$. On the other hand, to obtain the dual σ-model we integrate out the gauge fields A^a and $\bar{A}^a$ and the resulting σ-model action with the dual metric and B-fields is given in terms of the dual coordinates $(\tilde{x}^a, x^\mu)$. It follows that the dual string background $\hat{E}_{ij}$ is related to the original string background E_{ij} via:

Theorem 3.2. *The T-duality transformation rules of the metric g and the B-field b with a $U(1)^n$ isometry are given by a set of rules called the* ***Buscher rules****:*

$$\begin{aligned} \hat{E}_{ab} &= (E^{-1})_{ab}\,, & \hat{E}_{a\nu} &= (E^{-1})_a{}^b E_{b\nu}\,, \\ \hat{E}_{\mu b} &= -E_{\mu a}(E^{-1})^a{}_b\,, & \hat{E}_{\mu\nu} &= E_{\mu\nu} - E_{\mu a}(E^{-1})^{ab}E_{b\nu}\,, \end{aligned} \tag{3.2.8}$$

As an example, let us consider the $n = 1$ case, when a two-dimensional σ-model has an Abelian isometry corresponding to a compact $U(1)$ group.

One can now choose coordinates $\{x^i\} = \{x^0, x^\mu\}$ such that the isometry acts by translation of the coordinate $x^0 = \theta$. In this case the metric g and B-field b are transformed to $\hat{g}$ and $\hat{b}$ under the T-duality transformation given by the Buscher rules:

$$\begin{aligned} &\hat{g}_{\mu\nu} = g_{\mu\nu} - \tfrac{1}{g_{00}}\left(g_{\mu0}g_{\nu0} - b_{\mu0}b_{\nu0}\right), \quad \hat{g}_{00} = \tfrac{1}{g_{00}}, \quad \hat{g}_{\mu0} = \tfrac{b_{\mu0}}{g_{00}}, \\ &\hat{b}_{\mu\nu} = b_{\mu\nu} - \tfrac{1}{g_{00}}\left(g_{\mu0}b_{\nu0} - g_{\mu0}b_{\nu0}\right), \quad \hat{b}_{\mu0} = \tfrac{g_{\mu0}}{g_{00}}. \end{aligned} \tag{3.2.9}$$

3.2.2 $O(n,n)$ T-duality group

It turns out that the T-duality symmetry of $R \to 1/R$ when, generalized to arbitrary n-dimensional toroidal compactifications, is generated by an element of the T-duality group $O(n,n;\mathbb{Z})$ [28, 73].

Consider string theory compactified on a n-torus.

Let T be an element of $O(n,n;\mathbb{Z})$ defined by

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.2.10}$$

where a, b, c, d are $n \times n$ matrices.

Then T preserves the form

$$J = \begin{pmatrix} 0 & \mathbb{I}_d \\ \mathbb{I}_d & 0 \end{pmatrix},$$

such that

$$T^t J T = J. \tag{3.2.11}$$

In terms of (3.2.10) we have

$$\begin{aligned} a^t c + c^t a &= 0 \\ b^t d + d^t b &= 0 \\ a^t d + c^t b &= 1. \end{aligned} \tag{3.2.12}$$

The T-duality group acts on a string background $E_{ab} = g_{ab} + b_{ab}$ $(a, b = 1, \ldots, n)$ on the n-torus via

$$T(E) = (a(E) + b)(c(E) + d)^{-1}. \tag{3.2.13}$$

Example 3.3. Consider a particular element $T \in O(d, d; \mathbb{Z})$ given by

$$T = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}. \tag{3.2.14}$$

The T-duality action of T on $E = g + b$ gives us the dual string background $\hat{E}$ on the n-torus as follows

$$T(E_{ab}) = \hat{E}_{ab} = (E^{-1})_{ab} = \hat{g}_{ab} + \hat{b}_{ab}, \tag{3.2.15}$$

which is consistent with the Buscher rules given by Theorem 3.2. And it follows that the dual metric and B-field transform as

$$\begin{aligned} \hat{g} &= (g - bg^{-1}b)^{-1} \\ \hat{b} &= (b - gb^{-1}g)^{-1}. \end{aligned} \tag{3.2.16}$$

In the case when $n = 1$, (3.2.16) reduces to $\hat{g}_{00} = {g_{00}}^{-1}$ and obeys the Buscher rules (3.2.9) for T-duality with $U(1)$-isometry, i.e. this is the standard $R \to 1/R$ duality.

3.3 Poisson-Lie T-duality

This section is organized as follows. In Section 3.3.1 we review the Poisson-Lie condition which is necessary for the existence of a dual σ-model. In Section 3.3.2 we show that one can solve the Poisson-Lie condition using the concept of the Drinfel'd double. Section 3.3.3 gives three examples for different types of Poisson-Lie T-duality.

3.3.1 The Poisson-Lie condition and Poisson-Lie symmetry

Poisson-Lie T-duality is a generalization of the non-Abelian T-duality proposed by Klimčík and Ševera [49], which allows the duality to be performed on a target space E without the requirement of an isometry.

However, the background does need to satisfy a certain condition which we will refer to as the Poisson-Lie condition. The Poisson-Lie condition is essential for the existence of a well-defined dual world-sheet.

Before going into Poisson-Lie T-duality, we need the concept of Noetherian currents.

Consider a σ-model with a group G acting freely on its target manifold E with $v_a(x)$ the left-invariant vector fields corresponding to the action of G on E.

Definition 3.4. Let G be a Lie group acting on a manifold E. A left invariant vector field on G is a section v_h of TG such that

$$(L_g)_* v_h = v_{gh}, \quad \forall g \in G. \tag{3.3.1}$$

where (L_g) is the left translation given by $(L_g)h = gh$ and $(L_g)_*$ is the induced map on the tangent spaces, i.e. $(L_g)_* : T_hG \to T_{gh}G$.

Definition 3.5. Noetherian currents J_a are quantities corresponding to symmetries of the Lagrangian, which are defined by the variation of the σ-model action with respect to the free action of G on the target manifold E, and are given by

$$\delta S = S(x + \epsilon^a v_a) - S(x) = \int d\epsilon^a \wedge J_a + \int \epsilon^a \mathcal{L}_{v_a}(L). \tag{3.3.2}$$

where $g \in G$ and ϵ is in the Lie algebra of G and $\mathcal{L}_{v_a}$ is the Lie derivative with respect to the left-invariant vector fields v_a.

Now consider the 2-dimensional σ-model described by a metric g_{ij} on the target manifold E and a locally defined 2-form b_{ij} on E with the action given by (3.2.1).

We can associate to the action of G on E the Noetherian current one forms on the world-sheet:

Lemma 3.6. *The Noether current* 1*-forms associated with the action of* G *on* E *are given by*

$$J_a = v_a^i(x) E_{ij} \bar{\partial} x^j d\bar{z} - v_a^i(x) E_{ji} \partial x^j dz. \tag{3.3.3}$$

Proof. Using the variational principle, δS is given by

$$\begin{aligned}
\delta(S) &= S(x + \epsilon^a v_a) - S(x) \\
&= \int dz^2 E_{ij} \partial(x^i + \epsilon^a v_a^i) \bar{\partial}(x^j + \epsilon^a v_a^j) - \int dz^2 E_{ij} \partial x^i \bar{\partial} x^j \\
&= \int d\epsilon^a \wedge (v_a^i E_{ij} \bar{\partial} x^j d\bar{z} - v_a^i E_{ji} \partial x^j dz) + \int \epsilon^a \mathcal{L}_{v_a}(L) \\
&= \int d\epsilon^a \wedge J_a + \int \epsilon^a \mathcal{L}_{v_a}(L). \qquad (3.3.4)
\end{aligned}$$

□

Let us consider the following two cases:

Case I : If the action of G is an isometry, i.e. $\mathcal{L}_{v_a}E = 0$, then [50]

$$0 = \delta S = \int d\epsilon^a \wedge J_a \quad \rightarrow dJ = 0, \tag{3.3.5}$$

i.e. the Noether currents given by (3.3.3) are closed one forms on the world-sheets of extremal strings.

Case II : If the action of G is not an isometry but a free action on E such that $\delta S = 0$, then it follows from (3.3.2) that on the extremal string surfaces we have

$$\mathcal{L}_{v_a}(L) = -dJ_a. \tag{3.3.6}$$

Although these forms J_a are no longer closed, one simply requires that on the extremal surfaces the forms J_a satisfy the Maurer-Cartan equation so they are still integrable

$$dJ_a = -\frac{1}{2}\tilde{f}^{bc}{}_a J_b \wedge J_c, \tag{3.3.7}$$

where $\tilde{f}_a{}^{bc}$ are the structure constants of some Lie algebra $\tilde{\mathfrak{g}}$ of a Lie group $\tilde{G}$.

Definition 3.7. If the Noether currents (3.3.3) with respect to the action of a group G obey the condition (3.3.7) on the extremal surfaces, then such a σ-model is said to have G**-Poisson-Lie symmetry** with respect to the group $\tilde{G}$.

Proposition 3.8. *The condition*

$$\boxed{\mathcal{L}_{v_a}(E_{ij}) = \tilde{f}_a{}^{bc} v_b^m v_c^n E_{mj} E_{in}} \tag{3.3.8}$$

on the string backgrounds is referred to as the ***Poisson-Lie condition****. It is the sufficient condition for a σ-model to possess a Poisson-Lie dual.*

Proof. It follows from (3.3.3) and (3.3.6) that (3.3.7) becomes

$$\begin{aligned}
\mathcal{L}_{v_a}(L) &= -\frac{1}{2}\tilde{f}^{bc}{}_a J_b \wedge J_c \\
&= -\frac{1}{2}\tilde{f}^{bc}{}_a (v_b^m E_{mj}\bar{\partial}x^j d\bar{z} - v_b^m E_{jm}\partial x^j dz) \wedge (v_c^n E_{ni}\bar{\partial}x^i d\bar{z} - v_c^n E_{in}\partial x^i dz), \\
&= \tilde{f}^{bc}{}_a v_b^m v_c^n E_{mj} E_{in} dx^i dx^j \\
&= \mathcal{L}_{v_a}(E_{ij}) dx^i dx^j,
\end{aligned} \tag{3.3.9}$$

thus $\mathcal{L}_{v_a}E_{ij} = \tilde{f}^{bc}{}_a v_b^m v_c^n E_{mj} E_{in}$. □

When the string background satisfies the Poisson-Lie condition (3.3.8), then following from (3.3.7) J_a can be explicitly expressed as

$$\tilde{T}^a \cdot J_a = -d\tilde{g}\tilde{g}^{-1} \tag{3.3.10}$$

or

$$\tilde{g} = P \exp \int_\gamma J_a \tilde{T}^a, \tag{3.3.11}$$

where $\tilde{g} \in \tilde{G}$, $\tilde{T}^a$ are the generators of the Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$, and P means the path-ordered exponential.

At this point a natural question to ask is, how is the group $\tilde{G}$ related to G?

It turns out that the Poisson-Lie condition (3.3.8) requires a certain compatibility condition on the structure constants of the Lie algebras of G and $\tilde{G}$.

Proposition 3.9. *Let $f_{ab}{}^c$ be the structure constants of $\mathfrak{g}$, the Lie algebra of G. The integrability condition on* (3.3.8) *requires the following constraint*

$$\tilde{f}^{ac}{}_k f_{fa}{}^e - \tilde{f}^{ae}{}_k f_{fa}{}^c - \tilde{f}^{ac}{}_f f_{ka}{}^e + \tilde{f}^{ae}{}_f f_{ka}{}^c - \tilde{f}^{ea}{}_a f_{fk}{}^a = 0. \tag{3.3.12}$$

Proof. It follows from $[v_a, v_b] = f_{ab}{}^c v_c$ that $\mathcal{L}_{v_a}$ satisfies the following relation

$$[\mathcal{L}_{v_a}, \mathcal{L}_{v_b}] = f_{ab}{}^c \mathcal{L}_{v_c}. \tag{3.3.13}$$

Using the above identity on E_{ij} and substituting the Poisson-Lie condition (3.3.8) into the above equation, we have

$$\begin{aligned}
f_{ab}{}^c \mathcal{L}_{v_c} E_{ij} &= f_{ab}{}^c \tilde{f}^{ed}{}_c v_e^m v_d^n E_{mj} E_{in} \\
&= [\mathcal{L}_{v_a}, \mathcal{L}_{v_b}] E_{ij} = \mathcal{L}_{v_a} \mathcal{L}_{v_b} E_{ij} - \mathcal{L}_{v_b} \mathcal{L}_{v_a} E_{ij} \\
&= \mathcal{L}_{v_a} (\tilde{f}^{cd}{}_b v_c^m v_d^n E_{mj} E_{in}) - \mathcal{L}_{v_b} (\tilde{f}^{cd}{}_a v_c^m v_d^n E_{mj} E_{in}) \\
&= (\tilde{f}^{cd}{}_b f_{ac}{}^e + \tilde{f}^{ef}{}_b f_{af}{}^d - \tilde{f}^{cd}{}_a f_{bc}{}^e - \tilde{f}^{ef}{}_a f_{bf}{}^d) v_e^m v_d^n E_{mj} E_{in} \\
&\quad + (\tilde{f}^{cd}{}_b \tilde{f}^{fg}{}_a - \tilde{f}^{fg}{}_a \tilde{f}^{cd}{}_b) v_c^m v_d^n v_f^k v_g^l E_{kj} E_{ml} E_{in} \\
&\quad + (\tilde{f}^{cd}{}_b \tilde{f}^{fg}{}_a - \tilde{f}^{fg}{}_a \tilde{f}^{cd}{}_b) v_c^m v_d^n v_f^k v_g^l E_{kn} E_{mj} E_{il} \\
&= (\tilde{f}^{cd}{}_b f_{ac}{}^e + \tilde{f}^{ef}{}_b f_{af}{}^d - \tilde{f}^{cd}{}_a f_{bc}{}^e - \tilde{f}^{ef}{}_a f_{bf}{}^d) v_e^m v_d^n E_{mj} E_{in},
\end{aligned}$$

therefore we obtain the following constraint:

$$\tilde{f}^{cd}{}_b f_{ac}{}^e + \tilde{f}^{ef}{}_b f_{af}{}^d - \tilde{f}^{cd}{}_a f_{bc}{}^e - \tilde{f}^{ef}{}_a f_{bf}{}^d - \tilde{f}^{ed}{}_c f_{ab}{}^c = 0. \tag{3.3.14}$$

□

It is obvious that the relation (3.3.12) is the standard relation which is obeyed by the structure constants of a Lie bialgebra, i.e. this is exactly the relation (2.2.27). Therefore the pair $(\mathfrak{g}, \tilde{\mathfrak{g}})$ forms a Drinfel'd double.

Condition (3.3.12) is manifestly dual, i.e. the condition (3.3.12) is invariant with respect to the exchange of structure constants $f \leftrightarrow \tilde{f}$.
This suggests that there exists an equivalent dual σ-model where the roles of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are exchanged. The following Poisson-Lie condition is required for the dual σ-model:

$$\boxed{\mathcal{L}_{\tilde{v}_a}(\hat{E}_{ij}) = f_a^{bc}\tilde{v}_b^m\tilde{v}_c^n\hat{E}_{mi}\hat{E}_{jn}.} \tag{3.3.15}$$

That is, there exists a dual σ-model on a target manifold $\hat{E}$ with $\tilde{G}$ acting freely on $\hat{E}$.

To see that this σ-model is dual to the σ-model with string background E satisfying (3.3.8), it follows from the fact that the dual Noetherian form for $\tilde{G}$ acting freely on $\hat{E}$ can be organized in the $\mathfrak{g}^*$-valued form $\tilde{J} = \tilde{J}_a\tilde{T}^a$ and the whole procedure can be repeated to retrieve the original σ model on E.

Both the target manifold E and its dual manifold $\hat{E}$ are embedded into an extended manifold $\tilde{E}$ which has a natural structure of a fibre bundle over base manifold $M = E/G = \hat{E}/\tilde{G}$ with the fibre being the Drinfel'd double.

As a result, **dual σ-models are naturally constructed using Drinfel'd doubles.**

As a conclusion, for a pair of Lie groups $(G, \tilde{G})$ whose Lie algebras constitute a Drinfel'd double, every σ-model such that a group G acts freely on its target space and its action is Poisson-Lie symmetric with respect to $\tilde{G}$ has a dual counterpart, such that $\tilde{G}$ acts freely on the dual target space $\hat{E}$ and its action is Poisson-Lie symmetric with respect to G.

3.3.2 Solutions to the Poisson-Lie condition

In this section we consider the case when G acts on the target E not only freely but also transitively, i.e. the target manifold can be identified with the group manifold. In this case solutions of (3.3.8) and (3.3.15) can be solved using the concept of the Drinfel'd double.

First, the action (3.2.1) can be rewritten in terms of group elements $g \in G$

$$S = \int d^2z w_i^a E_{ab}(g) w_j^b \partial x^i \bar{\partial} x^j, \tag{3.3.16}$$

where $w_i^a\partial x^i = (g^{-1}\partial g)^a$ and $w_i^a\bar{\partial}x^i = (g^{-1}\bar{\partial}g)^a$ are the left-invariant one-forms, and

$$E_{ab}(g) = w^i_a E_{ij}(g) w^j_b, \tag{3.3.17}$$

where $E_{ij}(g)$ is the string background given by (3.2.1).

Now, let D be a Drinfel'd double containing the groups G and $\tilde{G}$ with Lie algebra double $\mathcal{D}$. $\mathcal{D}$ can be decomposed as $\mathfrak{g} \oplus \tilde{\mathfrak{g}}$, where $\mathfrak{g}$ is the Lie algebra of G and $\tilde{\mathfrak{g}}$ is the Lie algebra of $\tilde{G}$.

Let us choose a basis $\{T_a, \tilde{T}^a\}$ of $\mathcal{D}$ such that $\{T_a\}$ are the generators of $\mathfrak{g}$ while $\{\tilde{T}^a\}$ are the generators of $\tilde{\mathfrak{g}}$, where $\{T_a\}$ and $\{\tilde{T}^a\}$ are dual with respect to the nondegenerate bilinear form $\langle\ ,\ \rangle$ on $\mathcal{D}$.

Lemma 3.10. *The adjoint representation of G on $\mathcal{D}$ can be written in terms of the matrices $a(g)$, $c(g)$, $d(g)$ defined as the coefficients in the expansion*

$$\boxed{g^{-1}T_a g \equiv a(g)_a{}^b T_b, \quad g^{-1}\tilde{T}^a g \equiv c(g)^{ab} T_b + d(g)^a{}_b \tilde{T}^b.} \tag{3.3.18}$$

Proof. Since $(\mathfrak{g}, \tilde{\mathfrak{g}})$ is a Drinfel'd double, T_a and $\tilde{T}^a$ follow relations (2.2.26) can be chosen to satisfy the orthogonality conditions (2.2.25) with respect to a non-degenerate bilinear form on $\mathcal{D}$, invariant under the adjoint action of D, i.e.

$$\langle Ad_l V, Ad_l W\rangle = \langle lVl^{-1}, lWl^{-1}\rangle = \langle V, W\rangle, \quad V, W \in \mathcal{D}, l \in D. \tag{3.3.19}$$

Then using (3.3.18) and $g \in G$, the orthogonality conditions give rise to the following constraints

$$\begin{aligned} a(g)^T &= d(g)^{-1}, \\ c(g)d(g)^T &= -d(g)c(g)^T. \end{aligned} \tag{3.3.20}$$

□

Remark 3.11. Let us express Ad_g as

$$\begin{pmatrix} a(g) & 0 \\ c(g) & d(g) \end{pmatrix}.$$

According to (3.3.20), $a(g)$, $c(g)$ and $d(g)$ satisfy the conditions (3.2.12) when one puts $a = a(g)$, $b = 0$, $c = c(g)$ and $d = d(g)$, in other words, Ad_g is an element of $O(n, n, \mathbb{Z})(n = \dim(\mathfrak{g}))$.

Next, we will follow a procedure cleverly constructed by Klimčík and P. Ševera [47] such that $E_{ab}(g)$ is found by translating a general g independent reference field $E(e)$ from the identity $e \in G$ to the point $g \in G$ by left action of G on itself.

Let R_+ be an n-dimensional subspace of the $2n$-dimensional $\mathcal{D}$ and R_- be its orthogonal complement such that $R_\pm$ span the whole algebra $\mathcal{D}$. $R_\pm$ are precisely the graph of an arbitrary matrix $E(e)$ in $\mathcal{D}$:

$$R_+ = \text{Span}\{(t+E(e)(t,\cdot)), t\in\tilde{\mathfrak{g}}\}, \quad R_- = \text{Span}\{(t-E(e)(\cdot,t)), t\in\tilde{\mathfrak{g}}\}. \quad (3.3.21)$$

Consider the Lagrangian L and $\hat{L}$ corresponding to the Drinfel'd double $(G,\tilde{G})$ as follows.

Lemma 3.12. *The Lagrangians L and $\hat{L}$ satisfying* (3.3.8) *and* (3.3.15) *can be deduced from an equation of motion on the Drinfel'd double. Let $l:\Sigma\to D$ such that*

$$\langle(\partial l)l^{-1}, R_+\rangle = \langle(\bar{\partial}l)l^{-1}, R_-\rangle = 0. \quad (3.3.22)$$

According to Drinfel'd, any arbitrary element of D can be decomposed as the product of elements $g\in G$ and $\tilde{h}\in\tilde{G}$, i.e.

$$l = g\tilde{h} = \tilde{g}h, \quad g,h\in G, \tilde{g},\tilde{h}\in\tilde{G}. \quad (3.3.23)$$

This decomposition is generally not unique. However, according to Drinfel'd [22], there exists the unique decomposition in the vicinity of the unit element of D as the product of elements from G and $\tilde{G}$.

Any two decompositions give rise to a pair of equivalent σ-models. The possibility to decompose a Drinfel'd double into two (or more) Manin triples enables one to construct two (or more than two) equivalent σ-models on G and $\tilde{G}$ (or equivalently on others groups too) from the decompositions. This property is called ***Poisson-Lie T-plurality*** *[77].*

So there is a dual pair of σ-models on D and the string backgrounds $E_{ab}(g)$ and $\hat{E}^{ab}(\tilde{g})$ are defined by $R_\pm$ via

$$\begin{aligned} g^{-1}R_+g &= \text{Span}(T_a + E_{ab}(g)\tilde{T}^b), \quad (3.3.24)\\ g^{-1}R_-g &= \text{Span}(T_a - E_{ba}(g)\tilde{T}^b), \end{aligned}$$

and

$$\begin{aligned} \tilde{g}^{-1}R_+\tilde{g} &= \text{Span}(\tilde{T}^a + \hat{E}^{ab}(\tilde{g})T_b), \quad (3.3.25)\\ \tilde{g}^{-1}R_-\tilde{g} &= \text{Span}(\tilde{T}^a - \hat{E}^{ba}(\tilde{g})T_b). \end{aligned}$$

Proof. Inserting (3.3.24) into (3.3.22) and imposing conditions (3.3.22), it follows that

$$\begin{aligned} -(\partial\tilde{g}\tilde{g}^{-1})_a &= E_{ab}(g)(g^{-1}\partial g)^b \equiv A_a^+(g)\\ -(\bar{\partial}\tilde{g}\tilde{g}^{-1})_a &= -E_{ba}(g)(g^{-1}\bar{\partial}g)^b \equiv A_a^-(g). \quad (3.3.26) \end{aligned}$$

Eliminating $\tilde{g}$, we arrive at the following set of equations

$$\partial A_a^-(g) - \bar{\partial} A_a^+ - \tilde{f}_a{}^{bc} A_b^-(g) A_c^+(g) = 0, \tag{3.3.27}$$

where the $\tilde{f}_a{}^{bc}$ are the structure constants of the Lie algebra $\tilde{\mathfrak{g}}$.

It can then be checked that the the above set of equations (3.3.27) are the field equations of the σ-model action (3.3.16).

Similarly, the field equations of the dual σ-model action can be obtained by eliminating g following the above arguments. Thus we conclude that the σ-model Lagrangian L and the dual Lagrangian $\hat{L}$ can be deduced from (3.3.22). □

Proposition 3.13. *The σ-model background $E_{ab}(g)$ can be written conveniently as*

$$\boxed{E_{ab}(g) = ([a(g) + E(e)c(g)]^{-1})_a{}^c E_{cd}(e) d(g)^d{}_b} \tag{3.3.28}$$

Proof. Starting with $g = e$ the identity element

$$R_+ = \mathrm{Span}(T_a + E_{ab}(e)\tilde{T}^b),$$

the explicit dependence of E_{ab} on g is given by the matrices of the adjoint representation of D and is given by

$$\begin{aligned} g^{-1} R_+ g &= \mathrm{Span}\{g^{-1}(T_a + E_{ab}(e)\tilde{T}^b)g\} \\ &= \mathrm{Span}[(a(g)_a{}^c + E_{ab}(e)c(g)^{bc})T_c + E_{ab}(e)d(g)^b{}_c\tilde{T}^c]. \end{aligned} \tag{3.3.29}$$

Comparing (3.3.24) and (3.3.29) the matrix $E(g)$ is given by

$$E_{ab}(g) = ([a(g) + E(e)c(g)]^{-1})_a{}^c E_{cd}(e) d(g)^d{}_b. \tag{3.3.30}$$

□

Alternatively, $E_{ab}(g)$ can be defined equivalently as follows.

Proposition 3.14. *Let $\Pi^{ab}(g)$ be defined by*

$$\boxed{\Pi^{ab}(g) \equiv c(g)^{ac}(a(g)^{-1})_c{}^b,} \tag{3.3.31}$$

then up to a similarity transformation, $E_{ab}(g)$ can be expressed as

$$E_{ab}(g) = ([E(e)^{-1} + \Pi(g)]^{-1})_{ab} \tag{3.3.32}$$

Proof. Starting with (3.3.29), the matrix $E(g)$ given by (3.3.30) can be rewritten as

$$g^{-1}R_+g = \text{Span}[(a(g)_a{}^c + E_{ab}(e)c(g)^{bc})T_c + E_{ab}(e)d(g)^b{}_c\tilde{T}^c]. \tag{3.3.33}$$

Consider a basis transformation of $\{T_a\}$ of $\mathfrak{g}$ and $\{\tilde{T}^a\}$ of $\mathfrak{g}^*$ via

$$T_a \mapsto (a(g)_a{}^b)^{-1}T_b \quad \tilde{T}^a \mapsto (d(g)^a{}_b)^{-1}\tilde{T}^b, \tag{3.3.34}$$

then (3.3.33) becomes

$$\begin{aligned} g^{-1}R_+g &= \text{Span}[(\mathbb{I} + E_{ab}(e)c(g)^{bc}(a(g)_c{}^d)^{-1})T_d + E_{ab}(e)\tilde{T}^b] \\ &= \text{Span}(T_a + E_{ab}(g)\tilde{T}^b), \end{aligned} \tag{3.3.35}$$

thus up to a similarity transformation, $E(g)$ can be expressed as

$$E(g) = [E(e)^{-1} + c(g)a(g)^{-1}]^{-1} = [E(e)^{-1} + \Pi]^{-1}. \tag{3.3.36}$$

□

Remark 3.15. As we will show in Section 6.2, $\Pi^{ab}(g) = c(g)^{ac}(a(g)^{-1})_c{}^b$ is a natural Poisson structure on G.

The dual background $\hat{E}_{ab}$ is found by transporting $\tilde{e} \in \tilde{G}$ to any $\tilde{h} \in \tilde{G}$ by the action of $\tilde{G}$ on itself.

Following the previous construction for the dual σ-model with target $\tilde{G}$, we have the following result:

Theorem 3.16. *The matrices $\tilde{a}(\tilde{g}), \tilde{c}(\tilde{g})$ and $\tilde{d}(\tilde{g})$ are defined in a similar way to (3.10)*

$$\tilde{g}^{-1}\tilde{T}^a\tilde{g} \equiv \tilde{a}(\tilde{g})^a{}_b\tilde{T}^b, \quad \tilde{g}^{-1}T_a\tilde{g} \equiv \tilde{c}(\tilde{g})_{ab}\tilde{T}^b + \tilde{d}(\tilde{g})_a{}^bT_b, \tag{3.3.37}$$

with the dual background found to be

$$\hat{E}^{ab}(\tilde{g}) = \tilde{d}(\tilde{g})^a{}_c\hat{E}^{cd}(\tilde{e})([\tilde{a}(\tilde{g}) + \tilde{c}(\tilde{g})\hat{E}(\tilde{e})]^{-1})_d{}^b. \tag{3.3.38}$$

Equivalently, in terms of the dual Poisson structure $\tilde{\Pi}_{ab}(\tilde{g}) \equiv \tilde{c}(\tilde{g})^{ac}(\tilde{a}(\tilde{g})^{-1})_c{}^b$, the dual background can be written as

$$\hat{E}^{ab}(\tilde{g}) = ([\hat{E}(\tilde{e})^{-1} + \tilde{\Pi}(\tilde{g})]^{-1})^{ab}. \tag{3.3.39}$$

The following lemma relates the original σ-model and the dual σ-model:

Lemma 3.17. *At the origin of the group, the matrices $E(e)$ and $\hat{E}(\tilde{e})$ are related by*

$$E(e)\hat{E}(\tilde{e}) = \hat{E}(\tilde{e})E(e) = 1 \tag{3.3.40}$$

Proof. $R_\pm = \tilde{R}_\pm$ is the crucial choice. Thus

$$\begin{aligned} R_+ &= \text{Span}\{T_a + E(e)_{ab}\tilde{T}^b\} \\ &= \tilde{R}_+ = \text{Span}\{\tilde{T}^a + \hat{E}(\tilde{e})^{ab}T_b\} = \text{Span}\{T_a + (\hat{E}(\tilde{e})^{-1})_{ab}\tilde{T}^b\}, \end{aligned} \tag{3.3.41}$$

therefore $E(e)_{ab} = (\hat{E}(\tilde{e})_{ab})^{-1}$. □

That is, the Poisson-Lie T-duality is a generalization of the standard Abelian T-duality, i.e. the $R \to 1/R$ symmetry. An example for Abelian T-duality is constructed in Section 3.3.3.

3.3.3 Classification of Poisson-Lie T-duality

Poisson Lie T-duality can be classified by the following types of underlying Drinfeld doubles:

1. Abelian doubles, which correspond to standard Abelian T-duality.
2. Semi-Abelian doubles, which correspond to the non-Abelian T-duality between a G-isometric σ-model with the target manifold being the group G, and a σ-model on $\tilde{G}$ viewed as the Abelian group. This is non-Abelian T-duality in the sense of de la Ossa and Quevedo [20].
3. Non-Abelian doubles which correspond to non-trivial Poisson-Lie T-duality where none of the σ-models of the dual pair is isometric with respect to the action of the group.

Poisson-Lie T-duality with Abelian double

In the Abelian double case, let us take the Drinfel'd double $D = U(1)^n \times U(1)^n$.

Starting with a σ-model with a free $G = U(1)^n$ action on the target manifold E, we choose coordinates y^μ ($\mu = 1, \dots, n$) for the orbits of $U(1)^n$ on E. The matrix of the σ-model, E_{ij}, has both types of indices corresponding to y^μ and $g \in U(1)^n$, and the Lagrangian can be decomposed as

$$\begin{aligned} L &= E(y)_{\mu\nu}\partial y^\mu \bar{\partial} y^\nu + E(y,g)_{\mu b}\partial y^\mu (g^{-1}\bar{\partial} g)^b \\ &+ E(y,g)_{a\nu}(g^{-1}\partial g)^a \bar{\partial} y^\nu + E(y,g)_{ab}(g^{-1}\partial g)^a (g^{-1}\bar{\partial} g)^b. \end{aligned} \tag{3.3.42}$$

Or in matrix form, let us write

$$E(y,g)_{ij} = \begin{pmatrix} E(y)_{\mu\nu} & E(y,g)_{\mu b} \\ E(y,g)_{a\nu} & E(y,g)_{ab} \end{pmatrix}. \tag{3.3.43}$$

The dependence of E_{ij} on g is fixed by the Poisson-Lie condition (3.3.8) and is given by (3.3.28), i.e.

$$E_{ab}(y,g) = [(a(g) + E(e)c(g))^{-1}]_a{}^c E(e)_{cd} d(g)^d{}_b,$$

hence it follows that the σ-model matrix E (3.3.43) is given by

$$E(y,g) = \left[\begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & a(g) \end{pmatrix} + E(y,e) \begin{pmatrix} 0 & 0 \\ 0 & c(g) \end{pmatrix} \right]^{-1} E(y,e) \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & d(g) \end{pmatrix}, \tag{3.3.44}$$

where e is the unit element of G, and $a(g)$, $c(g)$ and $d(g)$ are the matrices given by (3.3.18). Since $U(1)^n$ is Abelian, thus

$$a(g) = d(g) = Id_n, \quad c(g) = 0. \tag{3.3.45}$$

Comparing (3.3.43) and (3.3.44), we have made the choice $E(y,e) = E(y,g)$ in the adaptive coordinate (y,g).

The dual model $\hat{E}$ is defined similarly with the choice of coordinates $(y,\tilde{g})$ and the dependence of $\hat{E}$ on $\tilde{g}$ is given by (3.3.38), i.e.

$$\hat{E}(y,\tilde{g}) = \begin{pmatrix} Id_n & 0 \\ 0 & \tilde{d}(\tilde{g}) \end{pmatrix} \hat{E}(y,\tilde{e}) \left[\begin{pmatrix} 0 & 0 \\ 0 & \tilde{a}(\tilde{g}) \end{pmatrix} + \begin{pmatrix} Id_n & 0 \\ 0 & \tilde{c}(\tilde{g}) \end{pmatrix} \hat{E}(y,\tilde{e}) \right]^{-1}. \tag{3.3.46}$$

According to Section 3.2.2, $\hat{E}(y,\tilde{e})$ can be chosen as

$$\begin{aligned} &\hat{E}(y,\tilde{e}) \\ &= \left(\begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & a \end{pmatrix} E(y,e) + \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \right) \left(\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} E(y,e) + \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & d \end{pmatrix} \right)^{-1} \\ &= \left(\begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & 0 \end{pmatrix} E(y,e) + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_n \end{pmatrix} \right) \left(\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_n \end{pmatrix} E(y,e) + \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & 0 \end{pmatrix} \right)^{-1} \\ &= \begin{pmatrix} E_{\mu\nu} - E_{\mu a}(E^{-1})^{ab} E_{b\nu} & -E_{\mu a}(E^{-1})^a{}_b \\ (E_{-1})_a{}^b E_{b\nu} & E_{ab}{}^{-1} \end{pmatrix}, \end{aligned} \tag{3.3.47}$$

where we have chosen $a = d = 0$, and $b = c = \mathbb{I}_n$.

Since the dual group is $\tilde{G} = U(1)^n$, the matrices $\tilde{a}, \tilde{c}$ and $\tilde{d}$ are given by

$$\tilde{a}(\tilde{g}) = \tilde{d}(\tilde{g}) = Id_n, \quad \tilde{c}(\tilde{g}) = 0. \tag{3.3.48}$$

The dual model $\hat{E}(y, \tilde{g})$ follows from (3.3.46), (3.3.47) and (3.3.48) is thus

$$\hat{E}(y, \tilde{g}) = \begin{pmatrix} E_{\mu\nu} - E_{\mu a}(E^{-1})^{ab} E_{b\nu} & -E_{\mu a}(E^{-1})^a{}_b \\ (E_{-1})_a{}^b E_{b\nu} & E_{ab}{}^{-1} \end{pmatrix}. \tag{3.3.49}$$

This is exactly the Buscher rules with a $U(1)^n$ isometry (3.2.8), as given previously in Section 3.2.1.

Poisson-Lie T-duality with semi-Abelian double

Non-Abelian T-duality originally introduced by de la Ossa and Quevedo [20] is a special case of Poisson-Lie T-duality, such that the double corresponding to a Poisson-Lie symmetry is the so called semi-Abelian double.

Semi-Abelian doubles correspond to non-Abelian T-duality between a G-isometric σ-model with a G-target and a non-isometric σ-model with the target $\tilde{G}$ which can be considered as an Abelian Lie group.

In this case the double D is simply the cotangent bundle T^*G which is the semi-direct product of the non-Abelian group G and the Abelian group $U(1)^n$, where $n = \dim(G)$, i.e. $D = G \ltimes U(1)^n$.

Example 3.18. [36] Let $\{\tilde{T}^a\}$, $a = 1, 2$ be the generators for the Lie algebra of the Abelian group $\tilde{G} = U(1)^2$. And let $\{T_1, T_2\}$ be a basis of G such that

$$[T_1, T_2] = T_2, \tag{3.3.50}$$

then the mix-algebra relation is given by

$$[T_1, \tilde{T}^2] = -\tilde{T}^2, \quad [T_2, \tilde{T}^2] = \tilde{T}^1. \tag{3.3.51}$$

I.e. $(G, \tilde{G})$ is a Drinfel'd double.

Now let $(\tilde{\theta}, \tilde{\varphi})$ and (θ, φ) be group coordinates of $\tilde{G} = U(1)^2$ and G, respectively.

Computing the matrices $a(g)$, $c(g)$ and $d(g)$ of (3.3.18) we have

$$a(g) = \begin{pmatrix} 1 & 0 \\ -\varphi & e^{\theta} \end{pmatrix}, \quad c(g) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad d(g) = \begin{pmatrix} 1 & \varphi\, e^{-\theta} \\ 0 & e^{-\theta} \end{pmatrix}, \tag{3.3.52}$$

and similarly, $\tilde{a}(\tilde{g})$, $\tilde{c}(\tilde{g})$ and $\tilde{d}(\tilde{g})$ are given by

$$\tilde{a}(\tilde{g}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{c}(\tilde{g}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{d}(\tilde{g}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.3.53}$$

Let us choose the constant matrix $E(e)$ and $\hat{E}(\tilde{e})$ as follows

$$E(e) = \begin{pmatrix} x & y \\ u & v \end{pmatrix}, \quad \hat{E}(\tilde{e}) = \begin{pmatrix} \tilde{x} & \tilde{y} \\ \tilde{u} & \tilde{v} \end{pmatrix}, \tag{3.3.54}$$

such that $E(e)$ and $\hat{E}(\tilde{e})$ satisfy the relation $E(e)\hat{E}(\tilde{e}) = \hat{E}(\tilde{e})E(e) = \mathrm{Id}$.

Then the string backgrounds $E(\theta, \varphi)$ and $\hat{E}(\tilde{\theta}, \tilde{\varphi})$ are given by

$$\begin{aligned} E(\theta,\varphi) &= \frac{1}{\tilde{v}\tilde{x} - \tilde{u}\tilde{y}} \begin{pmatrix} -e^{-\theta}\varphi(u - \varphi + \varphi x + y) + v & -e^{-\theta}y + \varphi x \\ -e^{-\theta}u + e^{-2\theta}\varphi x & e^{2\theta}x \end{pmatrix}, \\ \hat{E}(\tilde{\theta},\tilde{\varphi}) &= \frac{1}{1 + \tilde{\varphi}\tilde{u} + \tilde{\varphi}^2\tilde{v}\tilde{x} - \tilde{\varphi}\tilde{y} - \tilde{\varphi}^2\tilde{u}\tilde{y}} \begin{pmatrix} 1 - \tilde{\varphi}\tilde{y} & -\tilde{\varphi}\tilde{v} \\ \tilde{\varphi}\tilde{x} & 1 + \tilde{\varphi}\tilde{u} \end{pmatrix}. \end{aligned} \tag{3.3.55}$$

Poisson-Lie T-duality with non-Abelian double

The following example is an example of Poisson-Lie T-duality with a non-Abelian double, which was first worked out explicitly in [48].

Example 3.19 (Borelian double)**.** The simplest non-Abelian double is the $D = GL(2,\mathbb{R})$ group with Lie algebra $\mathcal{D} = gl(2,\mathbb{R})$ and is called the Borelian double. The Borel group G with Lie algebras $\mathfrak{g}$ has the basis

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.3.56}$$

while the dual group $\tilde{G}$ with algebra $\tilde{\mathfrak{g}}$ has the following basis

$$\tilde{T}^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{T}^2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{3.3.57}$$

There is a symmetric bilinear pairing $\langle\ ,\ \rangle$ on $\mathcal{D}$ defined by

$$\langle x, y\rangle = \mathrm{Det}(x + y), \quad x, y \in \mathcal{D}. \tag{3.3.58}$$

It is obvious that $\langle T_a, T_b\rangle = \langle \tilde{T}^a, \tilde{T}^b\rangle = 0$ and $\langle T_a, \tilde{T}^b\rangle = \delta_a^b$. One can also show that $\langle\ ,\ \rangle$ is ad-invariant, i.e. it satisfies

$$\langle [x, y], z\rangle - \langle y, [x, z]\rangle = 0.$$

Thus both sets of generators span a Borel subalgebra of the algebra $\mathcal{D} = gl(2, R)$ and $\mathcal{D}$ is a Drinfel'd double. And an easy computation shows that the commutation relations is of the mixed type satisfying (2.2.26).

Now consider a pair of σ-models in duality with targets being G and $\tilde{G}$, respectively.

The elements $g \in G$ have a parametrization

$$g = \begin{pmatrix} e^{\chi} & \theta \\ 0 & 1 \end{pmatrix}.$$

Then according to Eqn. (3.3.18), we find the matrices $a(g)$, $c(g)$ and $d(g)$ to be

$$a(g) = \begin{pmatrix} 1 & e^{-\chi}\theta \\ 0 & e^{-\chi} \end{pmatrix}, \quad c(g) = \begin{pmatrix} 0 & -e^{-\chi}\theta \\ \theta & e^{-\chi}\theta^2 \end{pmatrix}, \quad d(g) = \begin{pmatrix} 1 & 0 \\ -\theta & e^{\chi} \end{pmatrix}. \tag{3.3.59}$$

Let us define the σ-model matrix $E(e)$ at the unit element of G by

$$E(e) = \begin{pmatrix} x & y \\ u & v \end{pmatrix}. \tag{3.3.60}$$

Substituting (3.3.59) and (3.3.60) into (3.3.28), the string background $E_{ab}(g)$ is given by

$$E(g) = A \begin{pmatrix} x - \theta(u+y) + \theta^2 v & e^{\chi}(y + \theta(v(x-1) - uy)) \\ e^{\chi}(u - \theta v(1+x) + \theta uy) & e^{2\chi} v \end{pmatrix}, \tag{3.3.61}$$

where $A = (1 + \theta(y-u) + \theta^2(vx - uy))^{-1}$.

Similarly the element of $\tilde{G}$ as the group manifold of the dual σ-model with Lie algebra $\tilde{\mathfrak{g}}$ given by (3.3.57) can be parameterized as

$$\tilde{g} = \begin{pmatrix} 1 & 0 \\ -\rho & e^{\sigma} \end{pmatrix}. \tag{3.3.62}$$

Then computing $\tilde{a}(\tilde{g})$, $\tilde{c}(\tilde{g})$ and $\tilde{d}(\tilde{g})$ accordingly, we find

$$\tilde{a}(\tilde{g}) = \begin{pmatrix} 1 & e^{-\sigma}\rho \\ 0 & e^{-\sigma} \end{pmatrix}, \quad \tilde{c}(\tilde{g}) = \begin{pmatrix} 0 & -e^{-\sigma}\rho \\ \rho & e^{-\sigma}\rho^2 \end{pmatrix}, \quad \tilde{d}(\tilde{g}) = \begin{pmatrix} 1 & 0 \\ -\rho & e^{\sigma} \end{pmatrix}, \tag{3.3.63}$$

where the inverse dual σ-model matrix $\hat{E}(e)$ at the unit element is

$$\hat{E}(\tilde{e}) = \begin{pmatrix} x & y \\ u & v \end{pmatrix}^{-1}. \tag{3.3.64}$$

Then the dual string background $\hat{E}(\tilde{g})$ can be obtained by substituting (3.3.63) and (3.3.64) into (3.3.38):

$$\hat{E}(\tilde{g}) = \frac{1}{\rho^2 + vx + \rho(u-y) - uy} \begin{pmatrix} v + \rho(u+y) + \rho^2 & -e^{\sigma}(y + \rho(x-1)) \\ -e^{\sigma}(u + \rho + \rho x) & e^{2\sigma} x \end{pmatrix}. \tag{3.3.65}$$

Chapter 4

Generalized geometry

4.1 Introduction and outline

Generalized geometry was first introduced by Hitchin [38] as a form of constructing differential geometry with a background B-field. Then the notion of generalized geometry was further developed by Cavalcanti and Gualtieri [16, 32] in their thesis.

Generalized geometry is a geometry on $TM \oplus T^*M$, the direct sum of tangent and cotangent bundles of a manifold M. Generally speaking, one would like to view $TM \oplus T^*M$ as a Courant algebroid over M.

Generalized geometry was first introduced as a generalized structure which unifies symplectic and complex structures. It was soon noticed by physicists that generalized geometry applies naturally to mirror symmetry [25, 30, 34].

As suggested by Gualtieri and Cavalcanti [16, 33], generalized geometry provides a natural geometry to study T-duality. They showed that global T-duality introduced by Bouwknegt, Evslin and Mathai [6] behaves naturally in the context of generalized geometry for the case of principal circle bundles. And since generalized geometry doubles the original geometry, it can also be related to the the double geometry of T-folds [40].

This chapter is organized as follows. In Section 4.2 we introduce natural operations on the generalized tangent space $TM \oplus T^*M$. Section 4.3 reviews the definition of Clifford algebra on the generalized tangent space, followed by an introduction to a generalized Cartan system on $TM \oplus T^*M$ in Section 4.4. In Section 4.5, the Courant bracket is viewed as an extension of the Lie bracket. Section 4.6 introduces the concept of a Dirac structure and maximal isotropic subspaces. In the last section, we introduce the generalized metric, which is a generalized version of a Riemannian metrics on $TM \oplus T^*M$.

4.2 Natural operations

Let M be a smooth manifold of dimension n. $TM \oplus T^*M$ has a natural symmetric non-degenerate bilinear form defined by

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\imath_Y \xi + \imath_X \eta)\,, \tag{4.2.1}$$

where $X, Y \in \Gamma(TM)$, and $\xi, \eta \in \Gamma(T^*M)$.

This symmetric form has signature (n,n) and is invariant under the orthogonal group $O(n,n)$.

There are a certain special symmetries of $TM \oplus T^*M$. To explore the various orthogonal symmetries of $TM \oplus T^*M$, let us consider the special orthogonal group $SO(TM \oplus T^*M) \cong SO(n,n)$ which preserves the non-degenerate, symmetric, bilinear form $\langle\ ,\ \rangle$. The Lie algebra of $SO(TM \oplus T^*M)$ is defined by

$$\mathfrak{so}(TM \oplus T^*M) = \{T | \langle T\mathfrak{X}_1, \mathfrak{X}_2\rangle + \langle \mathfrak{X}_1, T\mathfrak{X}_2\rangle = 0, \forall \mathfrak{X}_1, \mathfrak{X}_2 \in TM \oplus T^*M\}. \tag{4.2.2}$$

T can be decomposed as

$$T = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}, \tag{4.2.3}$$

where $A \in \mathrm{End}(TM)$, $\beta : TM \to T^*M$ and $B : T^*M \to TM$, and B and β are both skew-symmetric. Thus B is a 2-form and acts on $X \in \Gamma(TM)$ via the interior product $B(X) = \imath_X B$. Similarly, β is a bivector and acts on a form $\xi \in \Gamma(T^*M)$ via $\beta(\xi) = \imath_\xi \beta$.

A special case of T called the B**-field transform** can be obtained by exponentiation; this is an orthogonal symmetry of $TM \oplus T^*M$ given by

$$exp(B) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \tag{4.2.4}$$

the B-field transform sends $X + \xi \mapsto X + \xi + \imath_X B$, here $X \in \Gamma(TM)$ and $\xi \in \Gamma(T^*M)$.

β**-transform** is another important symmetry given by $\beta \in \wedge^2(TM)$, it is given by the element

$$exp(\beta) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \tag{4.2.5}$$

the β-transform sends $X + \xi \mapsto X + \xi + \imath_\xi \beta$.

Besides the non-degenerate bilinear form, there is a natural bracket operation on smooth sections of $TM \oplus T^*M$, called the Courant bracket. The Courant bracket was first introduced by Courant [18] to define a geometric structure called a Dirac structure, which is used by Courant and Weinstein [19] to unify Poisson geometry and presymplectic geometry by expressing each structure as a maximal isotropic subbundle of $TM \oplus T^*M$.

Definition 4.1. The **Courant bracket** is defined on pairs $(X, \xi) = X + \xi$ of a vector field X and a one-form ξ on a manifold M. Here $X + \xi, Y + \eta \in \Gamma(TM \oplus T^*M)$:

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(\imath_X\eta - \imath_Y\xi), \tag{4.2.6}$$

Note that the Courant bracket reduces to a Lie bracket on vector fields, i.e. let $\rho : TM \oplus T^*M \to TM$ be the projection on TM, then

$$\rho([\![\mathfrak{X}_1, \mathfrak{X}_2]\!]) = [\rho(\mathfrak{X}_1), \rho(\mathfrak{X}_2)], \tag{4.2.7}$$

where $\mathfrak{X}_i \in \Gamma(TM \oplus T^*M)$.

The Courant bracket has the following properties [32]:

1. It does not in general satisfy the Jacobi-identity, but its Jacobiator

   $$\mathrm{Jac}(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3) = [\![[\![\mathfrak{X}_1, \mathfrak{X}_2]\!], \mathfrak{X}_3]\!] + [\![[\![\mathfrak{X}_2, \mathfrak{X}_3]\!], \mathfrak{X}_1]\!] + [\![[\![\mathfrak{X}_3, \mathfrak{X}_1]\!], \mathfrak{X}_2]\!],$$

   can be expressed as the derivative of the **Nijenhuis operator**:

   $$\mathrm{Nij}(A, B, C) = \frac{1}{3}(\langle[\![\mathfrak{X}_1, \mathfrak{X}_2]\!], \mathfrak{X}_3\rangle + \langle[\![\mathfrak{X}_2, \mathfrak{X}_3]\!], \mathfrak{X}_1\rangle + \langle[\![\mathfrak{X}_3, \mathfrak{X}_1]\!], \mathfrak{X}_2\rangle)$$

   through

   $$\mathrm{Jac}(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3) = d(\mathrm{Nij}(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3)). \tag{4.2.8}$$

2. The Courant bracket satisfies a certain Leibnitz identity:

   $$[\![\mathfrak{X}_1, f\mathfrak{X}_2]\!] = f[\![\mathfrak{X}_1, \mathfrak{X}_2]\!] + (\rho(\mathfrak{X}_1)f)\mathfrak{X}_2 - \langle\mathfrak{X}_1, \mathfrak{X}_2\rangle df, \tag{4.2.9}$$

   where $f \in C^\infty(M)$.

3. Let B be a smooth two-form which maps $TM \to T^*M$ via the interior product $X \mapsto \imath_X B$. Then for $X + \xi, Y + \eta \in \Gamma(TM \oplus T^*M)$, the B-field transform

   $$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : X + \xi \mapsto X + \xi + \imath_X B$$

satisfies

$$[\![e^B(X+\xi), e^B(Y+\eta)]\!] = e^B([\![X+\xi, Y+\eta]\!]) + \imath_X \imath_Y dB.$$

i.e. the map e^B is an automorphism of the Courant bracket if and only if B is closed, i.e. $dB = 0$.
As the result, the Courant bracket has a non-trivial automorphism defined by a closed 2-form B.

4. The Courant bracket can be "twisted" by a real, closed, 3-form H. That is, define another bracket $[\![\ ,\ ]\!]_H$ on sections of $TM \oplus T^*M$:

$$[\![X+\xi, Y+\eta]\!]_H = [\![X+\xi, Y+\eta]\!] + \imath_X \imath_Y H. \tag{4.2.10}$$

$[\![\ ,\ ]\!]_H$ defines a Courant algebroid structure on $T \oplus T^*$ if and only if $dH = 0$. The notion of Courant algebroid has been discussed in Section 2.3.2.

The (twisted) Courant bracket is the anti-symmetrization of a bracket called the (twisted) Dorfmann bracket defined as follows:

Definition 4.2. The (twisted) **Dorfmann bracket** is a bracket on $\Gamma(TM \oplus T^*M)$ defined by

$$(X+\xi) \circ_H (Y+\eta) = ([X,Y] + \mathcal{L}_X \eta - \imath_Y d\xi + \imath_X \imath_Y H). \tag{4.2.11}$$

It is related to the (twisted) Courant bracket by

$$[\![(X+\xi), (Y+\eta)]\!]_H = (X+\xi) \circ_H (Y+\eta) - d\langle X+\xi, Y+\eta\rangle. \tag{4.2.12}$$

Properties of the (twisted) Dorfmann bracket are:

- The (twisted) Dorfmann bracket on $TM \oplus T^*M$ is not skew-symmetric, but its skew-symmetrization gives the Courant bracket, i.e.

$$[\![\mathfrak{X}_1, \mathfrak{X}_2]\!]_H = \frac{1}{2}(\mathfrak{X}_1 \circ_H \mathfrak{X}_2 - \mathfrak{X}_2 \circ_H \mathfrak{X}_1), \quad \mathfrak{X}_i \in \Gamma(TM \oplus T^*M). \tag{4.2.13}$$

- The (twisted) Dorfmann bracket satisfies the Jacobi-identity

$$\mathfrak{X}_1 \circ_H (\mathfrak{X}_2 \circ_H \mathfrak{X}_3) = (\mathfrak{X}_1 \circ_H \mathfrak{X}_2) \circ_H \mathfrak{X}_3 + \mathfrak{X}_2 \circ_H (\mathfrak{X}_1 \circ_H \mathfrak{X}_3). \tag{4.2.14}$$

4.3 Clifford algebra on $TM \oplus T^*M$

Definition 4.3. The **Clifford algebra** $CL(TM \oplus T^*M)$ is defined by the assignment $\mathfrak{X} \mapsto \gamma_{\mathfrak{X}}$, with the relation

$$\{\gamma_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}\} = 2\langle \mathfrak{X}_1, \mathfrak{X}_2 \rangle, \quad \forall \mathfrak{X}_1, \mathfrak{X}_2 \in \Gamma(TM \oplus T^*M). \tag{4.3.1}$$

Proposition 4.4. *The Clifford algebra has a natural representation on the exterior algebra $\wedge^\bullet(T^*M)$ given by the action of $\mathfrak{X} = X + \xi \in \Gamma(TM \oplus T^*M)$ defined by*

$$\gamma_{\mathfrak{X}}\Omega \equiv \imath_X \Omega + \xi \wedge \Omega, \tag{4.3.2}$$

where $\Omega \in \wedge^\bullet(T^*M)$.

Proof. We can verify this by showing that

$$\begin{aligned} \frac{1}{2}\{\gamma_{X+\xi}, \gamma_{X+\xi}\}\Omega &= \imath_X(\imath_X \Omega + \xi \wedge \Omega) + \xi \wedge (\imath_X \Omega + \xi \wedge \Omega) \\ &= (\imath_X \xi)\Omega \\ &= \langle X + \xi, X + \xi \rangle \Omega, \end{aligned} \tag{4.3.3}$$

as required. □

4.4 Generalized Cartan system

First recall Cartan's formulas with ingredients $(\imath_X, \mathcal{L}_X, d, [\ ,\ ])$ as follows

$$[d, d] = 0, \quad [\imath_X, \imath_Y] = 0 \quad \mathcal{L}_X = [\imath_X, d], \quad [\mathcal{L}_X, \imath_Y] = \imath_{[X,Y]}, \quad [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}. \tag{4.4.1}$$

Here X, Y are vector fields on a manifold, $\imath_X$ is the interior product with respect to X, $\mathcal{L}_X$ is the Lie derivative with respect to X, d is the de Rham differential and $[\ ,\ ]$ is the graded commutator of differential forms except on the right hand side of the last two formulas, which denote the Lie brackets.

On $TM \oplus T^*M$, there is a differential system with ingredients $(\gamma_{\mathfrak{X}}, \mathcal{L}_{\mathfrak{X}}, d_H, \circ_H)$ in analogy with the Cartan system we have just recalled. These ingredients are

- $\mathfrak{X} = (X, \xi) \in \Gamma(TM \oplus T^*M)$, in analogy with vector fields in Cartan's system.

- As defined in the previous section, $\gamma_{\mathfrak{X}} = \gamma_{(X,\Xi)}$ is the Clifford algebra on $TM \oplus T^*M$, in analogy with $\imath_X$ in Cartan's system. Recall that when acting on differential forms Ω,

$$\gamma_{(X,\xi)} \cdot \Omega = \imath_X \Omega + \xi \wedge \Omega.$$

- $\mathcal{L}_{\mathfrak{X}}$ in analogy with the Lie derivative $\mathcal{L}_X$ in Cartan's system. $\mathcal{L}_{\mathfrak{X}}$ acts on differential forms Ω via,

$$\mathcal{L}_{(X,\xi)} \cdot \Omega = \mathcal{L}_X \Omega + (d\xi + \imath_X H) \wedge \Omega. \tag{4.4.2}$$

- d_H is the twisted differential on differential forms, in analogy with the de Rham differential d in Cartan's system. $d_H = d + H$ acts on differential forms by

$$d_H \Omega = d\Omega + H \wedge \Omega. \tag{4.4.3}$$

- $\circ_H$ in analogy with the Lie bracket in the Cartan's system is the (twisted) Dorfmann bracket defined previously by (4.2.11) in Section 4.2.

We then claim that

Proposition 4.5. *In analogy with Cartan's formulas* (4.4.1)*, the algebraic structure of the differential graded algebra on $TM \oplus T^*M$ can be stated as follows*

$$\begin{aligned}
&(1)\ [d_H, d_H] = 0, && (4.4.4)\\
&(2)\ [\gamma_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}] = 2\langle \mathfrak{X}_1, \mathfrak{X}_2 \rangle, &&\\
&(3)\ [d_H, \gamma_{\mathfrak{X}}] = \mathcal{L}_{\mathfrak{X}}, &&\\
&(4)\ [\mathcal{L}_{\mathfrak{X}}, \gamma_{\mathfrak{Y}}] = \gamma_{\mathfrak{X} \circ_H \mathfrak{Y}}, &&\\
&(5)\ [\mathcal{L}_{\mathfrak{X}_1}, \mathcal{L}_{\mathfrak{X}_2}] = \mathcal{L}_{\mathfrak{X}_1 \circ_H \mathfrak{X}_2} = \mathcal{L}_{[\![\mathfrak{X}_1, \mathfrak{X}_2]\!]_H}, &&\\
&(6)\ [d_H, \mathcal{L}_{\mathfrak{X}}] = 0, && (4.4.5)
\end{aligned}$$

where $\mathfrak{X}, \mathfrak{X}_1, \mathfrak{X}_2 \in \Gamma(TM \oplus T^*M)$, $[\ ,\ ]$ is the graded commutator of the graded algebra on $TM \oplus T^*M$.

Proof. (1) and (2) follow from the definitions.

(3)

$$\begin{aligned}
[d_H, \gamma_{\mathfrak{X}}]\Omega &= (d + H\wedge)(\imath_X \Omega + \xi \wedge \Omega) + (\imath_X + \xi\wedge)(d + H\wedge)\Omega \\
&= \mathcal{L}_X \Omega + (d\xi + \imath_X H) \wedge \Omega = \mathcal{L}_{\mathfrak{X}} \Omega.
\end{aligned}$$

(4) Starting with LHS:

$$\begin{aligned}
&[\mathcal{L}_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}] \cdot \Omega \\
&= \mathcal{L}_{X_1}(\imath_{X_2} + \xi_2\wedge)\Omega + (d\xi_1 + \imath_{X_1}H) \wedge (\imath_{X_2} + \xi_2\wedge)\Omega \\
&\quad -\imath_{X_2}(\mathcal{L}_{X_1} + (d\xi_1 + \imath_{X_1}H) \wedge \Omega) - \xi_2 \wedge (\mathcal{L}_{X_1} + (d\xi_1 + \imath_{X_1}H)\wedge)\Omega \\
&= [\mathcal{L}_{X_1}, \imath_{X_2}]\Omega + (\mathcal{L}_{X_1}\xi_2 - \imath_{X_2}d\xi_1 - \imath_{X_2}\imath_{X_1}H) \wedge \Omega \\
&= \imath_{[X_1,X_2]}\Omega + (\mathcal{L}_{X_1}\xi_2 - \imath_{X_2}d\xi_1 + \imath_{X_1}\imath_{X_2}H) \wedge \Omega = \gamma_{\mathfrak{X}_1 \circ_H \mathfrak{X}_2}\Omega.
\end{aligned}$$

(6)

$$[d_H, \mathcal{L}_{\mathfrak{X}}] = d_H(d_H\gamma_{\mathfrak{X}} + \gamma_{\mathfrak{X}}d_H) - (d_H\gamma_{\mathfrak{X}} + \gamma_{\mathfrak{X}}d_H)d_H = 0.$$

(5)

$$\begin{aligned}
&[\mathcal{L}_{\mathfrak{X}_1}, \mathcal{L}_{\mathfrak{X}_2}] = \mathcal{L}_{\mathfrak{X}_1}[d_H, \gamma_{\mathfrak{X}_2}] - [d_H, \gamma_{X_2}]\mathcal{L}_{\mathfrak{X}_1} \quad \text{(via (3))} \\
&= \mathcal{L}_{\mathfrak{X}_1}(d_H\gamma_{\mathfrak{X}_2} + \gamma_{\mathfrak{X}_2}d_H) - (d_H\gamma_{\mathfrak{X}_2} + \gamma_{\mathfrak{X}_2}d_H)\mathcal{L}_{\mathfrak{X}_1} \\
&= d_H\mathcal{L}_{\mathfrak{X}_1}\gamma_{\mathfrak{X}_2} + \mathcal{L}_{\mathfrak{X}_1}\gamma_{\mathfrak{X}_2}d_H - d_H\gamma_{\mathfrak{X}_2}\mathcal{L}_{\mathfrak{X}_1} - \gamma_{\mathfrak{X}_2}\mathcal{L}_{\mathfrak{X}_1}d_H \quad \text{(via (6))} \\
&= \gamma_{\mathfrak{X}_1\circ_H\mathfrak{X}_2}d_H + d_H\gamma_{\mathfrak{X}_1\circ_H\mathfrak{X}_2} \quad \text{(via (4))} \\
&= \mathcal{L}_{\mathfrak{X}_1\circ_H\mathfrak{X}_2} \quad \text{(via (3))}.
\end{aligned} \tag{4.4.6}$$

Also by (5) we have

$$\mathcal{L}_{[\![\mathfrak{X}_1,\mathfrak{X}_2]\!]_H} = \frac{1}{2}(\mathcal{L}_{\mathfrak{X}_1\circ_H\mathfrak{X}_2} - \mathcal{L}_{\mathfrak{X}_2\circ_H\mathfrak{X}_1}) = \frac{1}{2}([\mathcal{L}_{\mathfrak{X}_1}, \mathcal{L}_{\mathfrak{X}_2}] - [\mathcal{L}_{\mathfrak{X}_2}, \mathcal{L}_{\mathfrak{X}_1}]) = \mathcal{L}_{\mathfrak{X}_1\circ_H\mathfrak{X}_2}.$$

□

4.5 Courant bracket - extension of the Lie bracket

The Courant bracket can be viewed as a natural extension of the Lie bracket on $TM \oplus T^*M$ in the following sense.

The Lie bracket satisfies the following identity when acting on a form Ω [52]:

$$\imath_{[X_1,X_2]}\Omega = [\imath_{X_1}, \imath_{X_2}]d\Omega + d[\imath_{X_1}, \imath_{X_2}]d\Omega + 2\imath_{X_1}d(\imath_{X_2}\Omega) - 2\imath_{X_2}d(\imath_{X_1}\Omega). \tag{4.5.1}$$

As observed by Gualtieri [32] (Lemma 4.24), the identity (4.5.1) for Lie bracket can be generalized to the Courant bracket on $TM \oplus T^*M$, acting on forms via the Clifford action:

$$\gamma_{[\![\mathfrak{X}_1,\mathfrak{X}_2]\!]}\Omega = \frac{1}{2}[\gamma_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}]d\Omega + \frac{1}{2}d[\imath_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}] + \imath_{\mathfrak{X}_1}d(\imath_{\mathfrak{X}_2}\Omega) - \imath_{\mathfrak{X}_2}d(\imath_{\mathfrak{X}_1}\Omega), \tag{4.5.2}$$

here $\mathfrak{X}_1, \mathfrak{X}_2 \in \Gamma(TM \oplus T^*M)$, and $[\ ,\ ]$ is the graded commutator on the graded algebra on $TM \oplus T^*M$. This extension of the Lie bracket can be generalized to the twisted Courant bracket as follows:

Proposition 4.6. *In terms of the generalized Cartan system* $(\gamma_{(X,\Xi)}, \mathcal{L}_{(X,\Xi)}, d_H, [\![\ ,\ ]\!]_H)$, *the twisted Courant bracket gives a natural extension of the Lie bracket on* $TM \oplus T^*M$ *acting on forms via the Clifford action:*

$$\begin{aligned}\gamma_{[\![\mathfrak{X}_1,\mathfrak{X}_2]\!]_H} \cdot \Omega &= \frac{1}{2}[\gamma_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}] \cdot d_H\Omega + \frac{1}{2} d_H([\gamma_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}] \cdot \Omega) \\ &+ \gamma_{\mathfrak{X}_1} \cdot d_H(\gamma_{\mathfrak{X}_2} \cdot \Omega) - \gamma_{\mathfrak{X}_2} \cdot d_H(\gamma_{\mathfrak{X}_1} \cdot \Omega). \end{aligned} \tag{4.5.3}$$

Proof. RHS. of (4.5.3) can first be rearranged as follows

$$\begin{aligned} RHS &= \frac{1}{2}[\gamma_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}] \cdot d_H\Omega + \frac{1}{2} d_H([\gamma_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}] \cdot \Omega) + \gamma_{\mathfrak{X}_1} \cdot d_H(\gamma_{\mathfrak{X}_2} \cdot \Omega) \\ &- \gamma_{\mathfrak{X}_2} \cdot d_H(\gamma_{\mathfrak{X}_1} \cdot \Omega) \\ &= \frac{1}{2}(\gamma_{\mathfrak{X}_1} \cdot \gamma_{\mathfrak{X}_2} - \gamma_{\mathfrak{X}_2} \cdot \gamma_{\mathfrak{X}_1}) \cdot d_H\Omega + \frac{1}{2} d_H(\gamma_{\mathfrak{X}_1} \cdot \gamma_{\mathfrak{X}_2} \cdot \Omega - \gamma_{\mathfrak{X}_2} \cdot \gamma_{\mathfrak{X}_1} \cdot \Omega) \\ &+ \gamma_{\mathfrak{X}_1} \cdot d_H(\gamma_{\mathfrak{X}_2} \cdot \Omega) - \gamma_{\mathfrak{X}_2} \cdot d_H(\gamma_{\mathfrak{X}_1} \cdot \Omega) \\ &= \frac{1}{2}(\gamma_{\mathfrak{X}_1} \cdot \mathcal{L}_{\mathfrak{X}_2}\Omega - \gamma_{\mathfrak{X}_2} \cdot \mathcal{L}_{\mathfrak{X}_1}\Omega + \mathcal{L}_{\mathfrak{X}_1} \cdot \gamma_{\mathfrak{X}_2}\Omega - \mathcal{L}_{\mathfrak{X}_2} \cdot \gamma_{\mathfrak{X}_1}\Omega) \\ &= \frac{1}{2}[\mathcal{L}_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}]\Omega - \frac{1}{2}[\mathcal{L}_{\mathfrak{X}_2}, \gamma_{\mathfrak{X}_1}]\Omega, \end{aligned}$$

using Property (4) in Proposition 4.5 along with the definition (4.2.13) of the Dorfmann bracket, it follows that

$$\begin{aligned} RHS. &= \frac{1}{2}\gamma_{\mathfrak{X}_1 \circ_H \mathfrak{X}_2}\Omega - \frac{1}{2}\gamma_{\mathfrak{X}_2 \circ_H \mathfrak{X}_1}\Omega \\ &= \gamma_{[\![\mathfrak{X}_1,\mathfrak{X}_2]\!]_H}\Omega \\ &= LHS. \end{aligned}$$

□

Imposing $d_H\Omega = d\Omega + H \wedge \Omega$, the above relation can be expand and be rewritten as

$$[\![\mathfrak{X}_1, \mathfrak{X}_2]\!]_H = [X_1, X_2] + \mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1 - \frac{1}{2} d(\imath_{X_1}\xi_2 - \imath_{X_2}\xi_1) + \imath_{X_1}\imath_{X_2}H,$$

where $\mathfrak{X}_i = (X_i, \xi_i)$ and the above is simply the twisted Courant bracket (4.2.10).

4.6 Dirac structure

The vector space $TM \oplus T^*M$ can often be decomposed into a direct sum of other spaces, for instance, maximally isotropic subspaces:

Definition 4.7. A subspace $L \in TM \oplus T^*M$ is **isotropic** if $\langle X, Y \rangle = 0$ for all $X, Y \in \Gamma(L)$. If the dimension of L is maximal, i.e. $dim(L) = dim(TM)$ then L is called **maximally isotropic**.

If L and L' are two maximally isotropic subspace of $TM \oplus T^*M$ such that $L \cap L' = 0$, then the inner product defines an isomorphism $L' \cong L^*$, and one can alternatively split $TM \oplus T^*M \cong L \oplus L'$.

Definition 4.8. A subspace $L \in \Gamma(TM \oplus T^*M)$ is called **involutive** if it is closed with respect to the Courant bracket, i.e.

$$[\![\mathfrak{X}_1, \mathfrak{X}_2]\!] \in \Gamma(L), \quad \forall \mathfrak{X}_1, \mathfrak{X}_2 \in \Gamma(L). \tag{4.6.1}$$

Definition 4.9. If a maximally isotropic subspace $L \in TM \oplus T^*M$ is involutive, which implies that L is integrable and in this case L is called a **Dirac structure**.

Proposition 4.10 ([32] Proposition 2.37)**.** *Let L be a maximally isotropic subbundle of $TM \oplus T^*M$, then L being involutive is equivalent to*

$$J(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3) = 0, \quad \forall \mathfrak{X}_i \in \Gamma(L). \tag{4.6.2}$$

Thus as a consequence:

Proposition 4.11. *Let L be a Dirac structure, together with the usual Leibnitz identity on L implies that L is a Lie algebroid.*

4.7 Generalized metric

Recall that a Courant algebroid is endowed with a natural non-degenerate pairing $\langle\ ,\ \rangle$. One can generalize the concept of a Riemannian metric g on manifold M to a *generalized metric* G on $TM \oplus T^*M$. First introduced by Gualtieri [32] and Witt [78], a generalized metric is defined by

Definition 4.12. Let $\mathcal{E} = TM \oplus T^*M$ be the generalized tangent space of M. A **generalized metric** $G : \mathcal{E} \to \mathcal{E}$ is an orthogonal and self adjoint operator such that

$$\langle Ge, e \rangle > 0, \quad \forall e \in \Gamma(\mathcal{E}) \setminus \{0\}. \tag{4.7.1}$$

Since G is symmetric and orthogonal, one notices that G is an involution

$$G^2 = GG^t = GG^{-1} = \mathrm{Id}. \tag{4.7.2}$$

Therefore G splits $\mathcal{E}$ into its ± 1-eigenspaces, $C_\pm$, i.e.

$$TM \oplus T^*M = C_+ \oplus C_-. \tag{4.7.3}$$

One can describe $C_\pm$ as a graph over TM, in terms of a metric $g \in \mathrm{Sym}^2 T^*M$ and a 2-form $b \in \wedge^2 T^*M$ [16]:

$$\begin{aligned} C_+ &= \{X + (b+g)(X) | X \in TM\} \\ C_- &= \{X + (b-g)(X) | X \in TM\}. \end{aligned} \tag{4.7.4}$$

One can express G in terms of the metric g and the B-fields b. Consider the case when $b = 0$, one finds $G_0 = G(g, b = 0)$ in matric form is given by:

$$G_0 = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \tag{4.7.5}$$

with the corresponding subbundles $C_{0\pm}$.

Now turn on a B-field b. Then $C_\pm$ can be obtained by B-transform $C_\pm = e^b C_{0\pm}$, and the generalized metric G transforms as $G = e^b G_0 e^{-b}$. In matrix form G is thus given by

$$G = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}. \tag{4.7.6}$$

Chapter 5

T-duality and generalized geometry

5.1 Introduction and outline

T-duality originally arises as a symmetry of string theory which relates string theory compactified on large circles with string theory compactified on small circles. T-duality in string theory plays an important role as it relates string theory on different backgrounds and may be realized as a transformation between two-dimensional σ-models [29]. A two-dimensional σ-model describes the world-sheet theory of a string propagating on a target manifold M equipped with a Riemannian metric g_{ij} and an antisymmetric B-field b_{ij}, with string background defined by $E_{ij} \equiv g_{ij} + b_{ij}$. The transformation rules of the low energy effective fields under T-duality are given by the well-known Buscher rules. However, B-fields are only defined on local patches of the underlying manifold, while globally we have a well-defined 3-form $H = dB$. Thus it is tempting to interpret the geometry of the underlying manifold in terms of H-flux instead of B-fields.

Through examples in the literature [4, 34], it is argued that T-duality leads to a topology change of the underlying manifold. To understand the global properties of T-duality in the presence of $NS-NS$ 3-form H-flux, a systematic method has been developed by Bouwknegt, Evslin and Mathai [6, 7].

In this construction of T-duality, a principal torus bundle E over M with a curvature 2-form F and T-dualizable H-flux is topologically determined by the pair (H, F). In the most general case, the flux $[H] \in H^3(E)$ invariant with respect to the isometry can be decomposed via the Chern-Weil homomorphism into a four-tuple $(H_{(3)}, H_{(2)}, H_{(1)}, H_{(0)})$ while the curvature class $[F] \in H^2(M)$

can be characterized by a three-tuple $(F_{(2)}, F_{(1)}, F_{(0)})$. Here $H_{(i)}$ and $F_{(i)}$ are vector-valued i-forms on M. The T-dual object is found to be classified by $\hat{H} = (H_{(3)}, F_{(2)}, 0, 0)$ together with the triple $\hat{F} = (\hat{F}_{(2)}, \hat{F}_{(1)}, \hat{F}_{(0)}) = (H_{(2)}, H_{(1)}, H_{(0)})$ [9]. It turns out that when $H_{(1)} \neq 0$ and $H_{(0)} \neq 0$, the T-dual object characterized by the triple $(\hat{F}_{(2)}, \hat{F}_{(1)}, \hat{F}_{(0)})$ is no longer a principal torus bundle [9, 62]. There are some proposals offered to interpret such an object [40, 62].

It was recently discovered by Gualtieri and Cavalcanti [16, 33] that generalized geometry provides a natural setting to study global T-duality. They showed that for a principal circle bundle E, the space of invariant sections of $TE \oplus T^*E$ together with a non-degenerate bilinear form $\langle\ ,\ \rangle$ and the twisted Courant bracket $[\![\ ,\ ]\!]_H$ can be identified with a Courant algebroid. T-duality is then realized as an isomorphism of Courant algebroids.

In particular, we extend the results of Gualtieri and Cavalcanti from principal circle bundles to general principal torus bundles with a generalization of the Courant bracket which is invariant under T-duality in the presence of non-trivial background H-flux.

We show that for a general principal torus bundle E over M, on the invariant sections of the generalized tangent space $TM \oplus T^*E$, $TE \oplus T^*E$ together with a nondegenerate natural pairing $\langle\ ,\ \rangle$, a generalized Courant bracket $[\![\ ,\ ]\!]_{H,F}$ and an anchor map ρ defines a Courant algebroid $\mathcal{E} = (E, M, [\![\ ,\ ]\!]_{H,F}, \rho)$. Thus a principal torus bundle E and its T-dual space $\hat{E}$ can be described in terms of an isomorphism of Courant algebroids related by T-duality. We also show that using the language of Courant algebroids, the T-duality transformation rules for the fluxes (H, F) agree with the global T-duality due to Bouwknegt, Hannabuss and Mathai [9].

This chapter is organized as follows.

In section 5.2, we introduce some basic concepts of the global properties of T-duality first developed by Bouwknegt, Evslin and Mathai [6, 7] by considering two cases - principal circle bundles and principal torus bundles. In section 5.3 we begin by reviewing Gualtieri and Cavalcanti's [16, 32] formalism of T-duality using the framework of generalized geometry for the case of principal circle bundle, then we generalize this construction to the case of general principal torus bundles, and define a generalized Courant bracket on the generalized tangent space. In order to show that on the space of invariant sections, the generalized Courant bracket together with the natural pairing make the generalized tangent space into a Courant algebroid, we show in section 5.4 that one can consider the generalized Courant bracket as the bracket on the double of a proto-bialgebroid, thus a generalized tangent space together with the generalized Courant bracket can be interpreted as the double of a proto-bialgebroid, or generally speaking, a Courant algebroid.

Sections 5.3.2, 5.4.2 and 5.4 are collaborative works with Bouwknegt and Garretson [10, 11].

5.2 Global T-duality

T-duality arises as the generalization of the $R \to 1/R$ invariance of string theory compactified on a circle of radius R. Recall in Section 3.2, T-duality from a local perspective is derived by gauging the isometries of a two-dimensional σ-model action, followed by coupling to Lagrangian multipliers which provide the extra coordinates of the dual σ-model. Integrating out the Lagrangian multipliers leads to the original action, whereas integrating out the gauge fields gives the dual σ-model action. The resulting Buscher rules give the local transformation rules for the metric and B-field. Globally, one would then like to derive the transformation rules for the globally defined H-flux from the local transformation rules of the B-field.

In Section 5.2.1 we first consider T-dualizing on a circle, i.e. we view the spacetime E as a principal circle bundle. Following the construction developed by Bouwknegt, Evslin and Mathai [6], we start with the Buscher rules and derive the T-duality transformation rules of the globally defined H-flux. Section 5.2.2 generalizes the principal circle bundle case to general principal torus bundles, following the construction developed in [8].

5.2.1 Principal circle bundles

Let $\pi : E \to M$ be a principal S^1-bundle with H-flux $[H] \in H^3(E)$. When we choose a connection one-form A on E, and a metric $\bar{g}$ on the base M, the canonical metric on E is given by $g = \bar{g} + A \otimes A$.

Locally the coordinates on E can be chosen to be $x^i = (x^\mu, x^0)$ or denoted (x^μ, θ) on E such that the Killing vector of the S^1 isometry is given by $\kappa = \partial/\partial\theta$. And again locally $H = db$ with b a two form required to be invariant, i.e. $\mathcal{L}_\kappa b = 0$. The invariance condition $\mathcal{L}_\kappa H = \mathcal{L}_\kappa b = 0$ imply that the components H_{ijk} and b_{ij} are independent of θ.

The connection can be chosen locally as $A = A_i dx^i = d\theta + A_\mu dx^\mu$, where

$A_\mu dx^\mu \in \Omega^1(M)$. The metric, g, and B-field, b, can then be written as

$$\begin{aligned} g &= \bar{g} + A \otimes A = \bar{g}_{\mu\nu} dx^\mu \otimes dx^\nu + (d\theta + A_\mu dx^\mu)^2, \\ b &= \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu + B_\mu dx^\mu \wedge (d\theta + A_\nu dx^\nu), \end{aligned} \tag{5.2.1}$$

Or, in matrix form, the components of metric and B-field b are given by

$$g_{ij} = \begin{pmatrix} \bar{g}_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}, \quad b_{ij} = \begin{pmatrix} b_{\mu\nu} + (b_\mu A_\nu - A_\mu b_\nu) & b_\mu \\ -b_\nu & 0 \end{pmatrix}. \tag{5.2.2}$$

Applying the Buscher rules (3.2.9), the metric and the B-field components after the T-duality transformation become

$$\hat{g}_{ij} = \begin{pmatrix} \bar{g}_{\mu\nu} + b_\mu b_\nu & b_\mu \\ b_\nu & 1 \end{pmatrix}, \quad \hat{b}_{ij} = \begin{pmatrix} b_{\mu\nu} & A_\mu \\ -A_\nu & 0 \end{pmatrix}. \tag{5.2.3}$$

From Eqn. (5.2.2) and Eqn. (5.2.3), we see that T-duality locally corresponds to the interchange between A_μ and b_μ.

Denoting the coordinate of the dual circle by $\hat{\theta}$, we can define $\hat{A} = d\hat{\theta} + b_\mu dx^\mu$ as a connection on a dual circle bundle $\hat{\pi} : \hat{E} \to M$.

In local coordinates $(x^\mu, \theta, \hat{\theta})$, $\hat{b}$ becomes

$$\hat{b} = b + A \wedge \hat{A} - d\theta \wedge d\hat{\theta}, \tag{5.2.4}$$

so that

$$\hat{H} - H = d(A \wedge \hat{A}) = F \wedge \hat{A} - A \wedge \hat{F}, \tag{5.2.5}$$

where $F = dA$ and $\hat{F} = d\hat{A}$ are the curvatures of A and $\hat{A}$, respectively.

Eqn. (5.2.5) can be rewritten as

$$H - \hat{F} \wedge A = \hat{H} - F \wedge \hat{A}. \tag{5.2.6}$$

The left hand side of (5.2.6) is a form on E, while the right hand side is a form on $\hat{E}$. Therefore both sides need to equal a form $H_{(3)}$ defined on M, i.e.

$$\begin{aligned} H &= H_{(3)} + A \wedge \hat{F}, \\ \hat{H} &= H_{(3)} + \hat{A} \wedge F. \end{aligned} \tag{5.2.7}$$

Now let us denote $H = H_{(3)} + A \wedge H_{(2)}$ by $(H_{(3)}, H_{(2)})$ and $\hat{H} = H_{(3)} + \hat{A} \wedge F$ by $(H_{(3)}, F)$, with $H_{(2)} = \hat{F} = \pi_* H$ and $\hat{H}_{(2)} = F = \hat{\pi}_* \hat{H}$. Here π_* and $\hat{\pi}_*$ denote the pushforward maps of the bundle projections π and $\hat{\pi}$ on E and $\hat{E}$, respectively. Then a T-duality transformation corresponds to the interchange of the pairs $(H_{(2)}, F) \leftrightarrow (F, H_{(2)})$.

Theorem 5.1 (BHM [8]). *A principal torus bundle $E \to M$ with T-dualizable H-flux is determined topologically by (H, F) while its T-dual $\hat{E} \to M$ is determined by $(\hat{H}, \hat{F})$. The H-flux and its dual are given by*

$$\begin{aligned} H &= H_{(3)} + A \wedge \hat{F}, \\ \hat{H} &= H_{(3)} + \hat{A} \wedge F, \end{aligned} \tag{5.2.8}$$

where $H_{(3)} \in \Omega^3(M)$, $F \in \Omega^2(M, \mathfrak{t})$ (resp. $\hat{F} \in \Omega^2(M, \mathfrak{t}^*)$), and A (resp $\hat{A}$) is a connection one form on E (resp. $\hat{E}$) taking value in the Lie algebra $\mathfrak{t}$. Here $\mathfrak{t}$ denote the Lie algebra of S^1 and $\mathfrak{t}^*$ denote the dual Lie algebra.

As a result, H-flux and the first Chern class of the bundle are exchanged under T-duality:

$$\boxed{F = \hat{\pi}_* \hat{H}, \quad \hat{F} = \pi_* H.} \tag{5.2.9}$$

5.2.2 Principal torus bundles

To generalize the previous construction to principal torus bundles, we need to define the following notions.

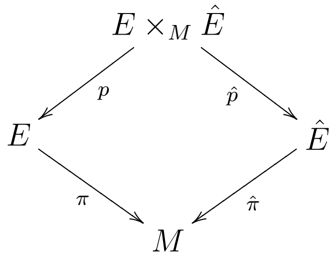
Let $\pi : E \to M$ be a principal $\mathbb{T}^n$-bundle, and $\mathfrak{t}$ and $\mathfrak{t}^*$ the Lie algebra of $\mathbb{T}^n$ and its dual, respectively. Let us choose a basis $\{t^a\}(a = 1, \ldots, n)$ of $\mathfrak{t}$ and a corresponding dual basis $\{t_a\}$ of $\mathfrak{t}^*$. The connection A and curvature $F = dA$ are $\mathfrak{t}$-valued, expressed as

$$A = A_a t^a, \quad F = F_a t^a = dA_a t^a. \tag{5.2.10}$$

Denoting the T-dual of E by $\hat{E}$, one defines

Definition 5.2. The **correspondence space** of E and $\hat{E}$ is the fibered product $E \times_M \hat{E} = \{(x, \hat{x}) \in E \times \hat{E} | \pi(x) = \hat{\pi}(\hat{x})\}$, with the following commutative diagram:

(5.2.11)



That is to say, the correspondence space is the higher dimensional space containing both the original space and its dual, such that there exist two independent projections which give the original space E and the dual geometry $\hat{E}$.

The action of $\mathbb{T}^n$ on E associates to each element $X \in \mathfrak{t}$ a vector field which we will also denote by X.

Definition 5.3. For all $X \in \mathfrak{t}$, a form $\Omega \in \Omega^k(E)$ is called **invariant** if $\mathcal{L}_X \Omega = 0$.

In the construction of global T-duality, we only consider principal torus bundles E with the "T-dualizable" H-fluxes which admit a T-dual.

Definition 5.4. An H-flux on E is a closed integral 3-form $H \in \Omega^3(E)$. H is called **T-dualizable** if there exists a closed $\mathfrak{t}^*$-valued 2-form $\hat{F}$ on M such that

$$\begin{aligned} dH &= 0 \\ \imath_X H &= \pi^* \hat{F}(X), \end{aligned} \tag{5.2.12}$$

for all $X \in \mathfrak{t}$, and $\hat{F}(X) \in \Omega^2(M)$ is the dual pairing of $\hat{F} \in \Omega^2(M, \mathfrak{t}^*)$ with $X \in \mathfrak{t}$.

Pairs $(H, \hat{F})$ satisfying (5.2.12) are called **T-dualizable fluxes**.

It follows from Definition (5.4) that all T-dualizable fluxes $(H, \hat{F})$ satisfy

$$\mathcal{L}_X H = \mathcal{L}_X \hat{F} = 0, \quad \forall X \in \mathfrak{t}. \tag{5.2.13}$$

Proposition 5.5 ([31])**.** *Let $\Omega(E)_{inv}$ be forms invariant under the $U(1)^n$ isometry. There is an isomorphism $H^\bullet(E) \cong H^\bullet(\Omega(E)_{inv}, d)$, i.e. every cohomology class in $H(E)$ contains an invariant representative.*

One can then decompose an invariant form $\Omega^k(E)_{\text{inv}}$ by

$$\Omega^k(E)_{\text{inv}} \cong \bigoplus_{p+q=k} (\Omega^p(M) \otimes \wedge^q \mathfrak{t}^*). \tag{5.2.14}$$

For instance, one can decompose H-flux as:

$$H = H_{(3)} + A_a \wedge H^a_{(2)} + \frac{1}{2} A_a \wedge A_b \wedge H^{ab}_{(1)} + \frac{1}{6} A_a \wedge A_b \wedge A_c \wedge H^{abc}_{(0)}, \tag{5.2.15}$$

where $H_{(i)} \in H^i(M, \wedge^{3-i}\mathfrak{t}^*)$.

In a short-hand notation, let us denote the above decomposition by the 4-tuple $H = (H_{(3)}, H_{(2)}, H_{(1)}, H_{(0)})$.

This way of decomposing the fluxes $[H] \in H^3(E)$ by the $H_{(a)}$'s is referred to as **dimensional reduction** or **Chern-Weil homomorphism**.

Suppose we are given a principal $\mathbb{T}^n$-bundle over M with H-flux $H \in H^3(E)$ and curvature $F \in H^2(M)$ satisfying (5.2.12). Upon dimensional reduction,

(H, F) is characterized by the tuple $((H_{(3)}, H_{(2)}, H_{(1)}, H_{(0)}), (F_{(2)}, 0, 0))$, where $H_{(i)} \in \Omega^i(M) \otimes \wedge^{3-i}\mathfrak{t}^*$ and $F_{(i)} \in \Omega^i(M) \otimes \wedge^{3-i}\mathfrak{t}$. The T-dual fluxes $\hat{H}$ and $\hat{F}$ characterizing the T-dual object can be deduced from the Gysin sequence of principal torus bundles [9], and the resulting $(\hat{H}, \hat{F})$ on $\hat{E}$, upon dimensional reduction, is found to be characterized by the tuple $((\hat{H}_{(3)}, \hat{H}_{(2)}, \hat{H}_{(1)}, \hat{H}_{(0)}), (\hat{F}_{(2)}, \hat{F}_{(1)}, \hat{F}_{(0)})) = ((H_{(3)}, F_{(2)}, 0, 0), (H_{(2)}, H_{(1)}, H_{(0)}))$. I.e. T-duality exchanges the role of $H_{(i)}$ and $F_{(i)}$. In the case when E is a principal circle bundle, $H_{(1)}$ and $H_{(0)}$ vanish and T-duality exchanges $H_{(2)}$ and $F_{(2)}$. This is in agreement with the T-duality transformation rule given by (5.2.9).

A question we would like to pose at this point is: What is the topology of the dual space characterized by such an $\hat{H}$ and $\hat{F}$, i.e. when $\hat{F}_{(1)}$ and (or) $\hat{F}_{(0)}$ are nonzero? What type of topology change of the underlying manifold is a result of T-duality transformation?

Possible interpretations:

Here are two possible interpretations for the T-dual of a principal torus bundle with non-trivial H-flux:

(1) *The T-dual manifold as a field of non-commutative / non-associative tori:*

For a principal torus bundle E with non-vanishing $H_{(1)}$, E is characterized by $H = (H_{(3)}, H_{(2)}, H_{(1)}, 0)$ and $F = (F_{(2)}, 0, 0)$, while the T-dual object is characterized by $\hat{H} \equiv (H_{(3)}, F_{(2)}, 0, 0)$ and $\hat{F} \equiv (H_{(2)}, H_{(1)}, 0)$. It was proposed by Mathai and Rosenberg [62, 63, 64] that the T-dual space in this case turns out to be a continuous field C of noncommutative tori. I.e. the fibre over a point in the base M is a noncommutative torus. Let $\theta \in [0, 1]$ be the non-commutativity parameter, the non-commutative torus A_θ can be realized as taking the cross product $C(\mathbb{T}) \rtimes \mathbb{Z}$, where the generator of $\mathbb{Z}$ acts on $\mathbb{T}$ by rotation through an angle of $2\pi\theta$. It turns out that when θ is rational, A_θ is Morita equivalent to $C(\mathbb{T}^2)$ [62]. Thus in this approach, the action of T-duality is considered as taking the crossed-product algebra.

Next consider a principal $\mathbb{T}^n$-bundle E with H-flux and curvature F such that upon dimensional reduction, (H, F) is characterized by $((H_{(3)}, H_{(2)}, H_{(1)}, H_{(0)}), (F_{(2)}, 0, 0))$. In this case when $H_{(0)}$ is also nonzero, $(\hat{H}, \hat{F})$ on the dual bundle $\hat{E}$ upon dimensional reduction, is classified by $((H_{(3)}, F_{(2)}, 0, 0), (H_{(2)}, H_{(1)}, H_{(0)}))$. In this case the T-dual object carries an integral class $[H_{(0)}] \in H^0(M, \wedge^3\mathfrak{t}^*)$. It is well known that such a class often corresponds to nonassociativity [15, 42]. Thus in this case the T-dual bundle is proposed to be a continuous field of noncommutative, nonassociative tori [9].

(2) *The T-dual manifold as a non-geometric object called a* ***T-fold***:

Consider a string theory on a spacetime which is locally viewed as a $\mathbb{T}^n$-bundle E over M. In this construction, the fibre of E is doubled from a $\mathbb{T}^n$ to a $\mathbb{T}^{2n}$, so that globally there is an extended space $\tilde{E}$ which is a $\mathbb{T}^{2n}$-bundle over N. String theory on E – a $\mathbb{T}^n$-bundle over M – becomes a string theory on the extended space $\tilde{E}$. A choice of a subspace $\mathbb{T}^n \in \mathbb{T}^{2n}$ gives rise to a physical subspace. For a geometric background, the local choice of $\mathbb{T}^n$ fit together to give a space which is a $\mathbb{T}^n$-bundle. When considering a non-geometric string background, the local choice of $\mathbb{T}^n$ do not fit together to form a manifold but rather a non-geometric object called the T-fold [39, 40, 41].

T-folds, originally constructed by Hull [39], are spaces where T-dualities can be transition functions between local patches. T-folds locally look like a conventional patch of a spacetime with a torus fibration, where the transition functions also include T-duality transformations.

5.3 T-duality and generalized geometry

Let E be a principal $\mathbb{T}^n$-bundle. The generalized tangent space, $TE \oplus T^*E$, is the natural object in generalized geoemtry. It was first realized by Cavalcanti and Gualtieri [16, 33] that one can build a framework on generalized tangent space to study T-duality, in particular, T-duality can be viewed as an isomorphism of Courant algebroids.

This section is organized as follows. Section 5.3.1 starts with the simple case, T-duality for principal circle bundles with non-trivial H-flux. To built up a framework on generalized geometry for T-duality, we follow the constructions in [16] and firstly define a T-duality map between the complexes of invariant differential forms and a T-duality map between invariant sections of the generalized tangent space. A twisted Courant bracket is then defined as the natural bracket on the generalized tangent space. At the end of this section we review an important theorem due to Cavalcanti and Gualtieri [16] – Theorem 5.9 concludes that on the invariant sections of the generalized tangent spaces, a principal circle bundle and its T-dual space can be related as a pair of isomorphic Courant algebroids. In section 5.3.2, we generalize the previous construction to the case when the underlying manifold is a principal torus bundle. We then generalized Theorem 5.9 to general principal torus bundles.

5.3.1 Principal circle bundles

Let us consider a principal S^1-bundle E over M, with $\mathfrak{t}$ $(=\mathbb{R})$ the Lie algebra of S^1 and $\mathfrak{t}^*$ the dual algebra. Let A be a $\mathfrak{t}$-valued connection one-form on E.

An invariant section of $TE\oplus T^*E$ is denoted by $\mathfrak{X}=(X,\Xi)\in\Gamma(TE\oplus T^*E)_{S^1}$. Any invariant section can further be written in dimensionally reduced form as $(x,f;\xi,g)\in(\Gamma(TM)\times C^\infty(M,\mathfrak{t}))\times(\Gamma(T^*M)\times C^\infty(M,\mathfrak{t}^*))$. This is equivalent to $X=x+f\partial_A$ and $\Xi=\xi+gA$, where A is a connection on E and ∂_A is the vector field dual to A.

An invariant k-form Ω on E can be decomposed as $\Omega=\Omega_{(k)}+A\wedge\Omega_{(k-1)}$ where $\Omega_{(i)}\in\Omega^i(M)$, and we will often use the short-hand notation $\Omega=(\Omega_{(k)},\Omega_{(k-1)})$.

In this section, we define T-duality maps τ and Ω between invariant differential forms on E and $\hat{E}$ and invariant sections of the generalized tangent spaces $TE\oplus T^*E$ and $T\hat{E}\oplus T^*\hat{E}$, respectively. Following the introduction of the twisted Courant bracket on the generalized tangent space, we review the important theorem 5.9 linking generalized geometry and T-duality due to Cavalcanti and Gualtieri [16, 33].

T-duality maps

Let us first introduce a T-duality map τ between the complexes of invariant differential forms, $\tau:\Omega^\bullet(E)_{S^1}\to\Omega^\bullet(\hat{E})_{\hat{S}^1}$ by

$$\tau(\Omega_{(k)}+A\wedge\Omega_{(k-1)})=-\Omega_{(k-1)}+\hat{A}\wedge\Omega_{(K)} \tag{5.3.1}$$

which can be written in the short handed form:

$$\tau(\Omega_{(k)},\Omega_{(k-1)})=(-\Omega_{(k-1)},\Omega_{(k)}). \tag{5.3.2}$$

Theorem 5.6 ([6]). *The map $\tau:(\Omega^\bullet(E)_{S^1},d_H)\to(\Omega^\bullet(\hat{E})_{\hat{S}^1},-d_{\hat{H}})$ is an isomorphism of differential complexes, where d_H is the twisted differential $d_H=d+H$ and acts on invariant forms by*

$$d_H\Omega=d\Omega+H\wedge\Omega. \tag{5.3.3}$$

The T-duality map τ has the following property. If we T-dualize twice and choose $\hat{\hat{A}}=A$, then on invariant forms it is clear that $\tau^2=-\mathrm{Id}$.

We also have a T-duality map of invariant sections $\varphi:\Gamma((TE\oplus T^*E)_{S^1})\to\Gamma((T\hat{E}\oplus T^*\hat{E})_{S^1})$ by

$$\varphi(x,f;\xi,g)=(x,g;\xi,f). \tag{5.3.4}$$

It is obvious that $\varphi^2 = \mathrm{Id}$ and φ is an isomorphism between invariant sections of $TE \oplus T^*E$ and $T\hat{E} \oplus T^*\hat{E}$.

Remark 5.7. This map φ along with the fibre orientation reversing automorphism $\varphi'(x, f; \xi, g) = (x, -f; \xi, -g)$ generate the four element T-duality group $O(1, 1, \mathbb{Z})$. As we will see in Section 5.5, the generalized metric defined in Section 4.7 can be transported using the T-duality map φ and as a result reclaim the Buscher rules.

τ also induces an isomorphism of Clifford modules $\wedge^\bullet(T^*E)_{S^1} \cong \wedge^\bullet(T^*\hat{E})_{\hat{S}^1}$, i.e.

Theorem 5.8. *Let Ω be an invariant form and recall that the Clifford action of $\gamma_{(X,\Xi)}$ on Ω is given by*

$$\gamma_{(X,\Xi)} \cdot \Omega = \imath_X\Omega + \Xi \wedge \Omega.$$

Then for $\mathfrak{X} = (X, \Xi) \in \Gamma(TE) \times \Gamma(T^*E)$

$$\tau(\gamma_{\mathfrak{X}} \cdot \Omega) = -\gamma_{\varphi(\mathfrak{X})} \cdot \tau(\Omega). \tag{5.3.5}$$

Proof. First, let us compute the Clifford action using the short-hand notation:

$$\begin{aligned} \gamma_{(x,f;\xi,g)}(\Omega_{(k)}, \Omega_{(k-1)}) &= ((\imath_x\Omega_{(k)} + \xi \wedge \Omega_{(k)} + f\Omega_{(k-1)}), (-\imath_x\Omega_{(k-1)} - \xi \wedge \Omega_{(k-1)} \\ &\quad +g\Omega_{(k)})) \\ &= (\gamma_{(x,\xi)} \cdot \Omega_{(k)} + f\Omega_{(k-1)}, -\gamma_{(x,\xi)} \cdot \Omega_{(k-1)} + g\Omega_{(k)}). \end{aligned} \tag{5.3.6}$$

Then apply τ to the above equation we find

$$\tau(\gamma_{(x,f;\xi,g)} \cdot \Omega) = (\gamma_{(x,\xi)} \cdot \Omega_{(k-1)} - g\Omega_{(k)}, \gamma_{(x,\xi)} \cdot \Omega_{(k)} + f\Omega_{(k-1)}), \tag{5.3.7}$$

while computing the right-hand side of equation (5.3.5) we find

$$\begin{aligned} \gamma_{(x,g;\xi,f)} \cdot (\tau\Omega) &= \gamma_{(x,g;\xi,f)} \cdot (-\Omega_{(k-1)}, \Omega_{(k)}) \\ &= (-\gamma_{(x,\xi)} \cdot \Omega_{(k-1)} + g\Omega_{(k)}, -\gamma_{(x,\xi)} \cdot \Omega_{(k)} - f\Omega_{(k-1)}). \end{aligned} \tag{5.3.8}$$

From relations (5.3.7) and (5.3.8), we have arrived at the proof. □

Twisted Courant bracket

Recall from Section 4.2 that the twisted Courant bracket for $\mathfrak{X}_i = (X_i, \Xi_i) \in \Gamma(TE \oplus T^*E)$ is defined by

$$[\![(X_1, \Xi_1), (X_2, \Xi_2)]\!]_H = ([X_1, X_2], \mathcal{L}_{X_1}\Xi_2 - \mathcal{L}_{X_2}\Xi_1 - \frac{1}{2}d(\imath_{X_1}\Xi_2 - \imath_{X_2}\Xi_1) + \imath_{X_1}\imath_{X_2}H), \tag{5.3.9}$$

where $H = H_{(3)} + A \wedge H_{(2)}$.

Let $F = dA$ be the curvature two-form on E. The Lie bracket on $(TE)_{S^1}$ is given by [32]

$$[(x_1, f_1), (x_2, f_2)] = ([x_1, x_2], x_1(f_2) - x_2(f_1) + \imath_{x_1}\imath_{x_2}F), \tag{5.3.10}$$

where $(x_i, f_i) \in \Gamma(TM) \times C^\infty(M, \mathfrak{t})$.

Thus in dimensionally reduced form, the twisted Courant bracket is given by

$$\begin{aligned}&[\![(x_1, f_1; \xi_1, g_1), (x_2, f_2; \xi_2, g_2)]\!]_H = ([x_1, x_2], x_1(f_2) - x_2(f_1) + \imath_{x_1}\imath_{x_2}F;\\ &(\mathcal{L}_{x_1}\xi_2 - \mathcal{L}_{x_2}\xi_1) + (g_2\imath_{x_1}F - g_1\imath_{x_2}F) - \frac{1}{2}d(\imath_{x_1}\xi_2 - \imath_{x_2}\xi_1) + \frac{1}{2}(df_1 g_2 + f_2 dg_1\\ &-f_1 dg_2 - df_2 g_1) + \imath_{x_1}\imath_{x_2}H_{(3)} + (f_2\imath_{x_1}H_{(2)} - f_1\imath_{x_2}H_{(2)}), x_1(g_2) - x_2(g_1)\\ &+\imath_{x_1}\imath_{x_2}H_{(2)}).\end{aligned} \tag{5.3.11}$$

The above bracket by construction has the properties of a Courant bracket, i.e. it is antisymmetric and it satisfies a Jacobiator condition (4.2.8).

Together with Theorem 5.8, the properties of the T-duality maps φ and τ are encoded in the following theorem, proved in [16]:

Theorem 5.9 ([16])**.** *The following hold:*

1. *φ is orthogonal with respect to the non-degenerate bilinear form $\langle\ ,\ \rangle$ on invariant sections $(TE \oplus T^*E)_{S^1}$ and $(T\hat{E} \oplus T^*\hat{E})_{\hat{S}^1}$, hence it induces an isomorphism of Clifford algebras, $CL(TE \oplus T^*E)_{S^1} \cong CL(T\hat{E} \oplus T^*\hat{E})_{\hat{S}^1}$;*

2. *φ defines an automorphism of the Courant bracket, $(\Gamma(TE \oplus T^*E)_{S^1}, [\![\ ,\ ]\!]_H) \cong (\Gamma(T\hat{E} \oplus T^*\hat{E})_{\hat{S}^1}, [\![\ ,\ ]\!]_{\hat{H}})$:*

$$\varphi([\![(x_1, f_1; \xi_1, g_1), (x_2, f_2; \xi_2, g_2)]\!]_H) = [\![\varphi(x_1, f_1; \xi_1, g_1), \varphi(x_2, f_2; \xi_2, g_2)]\!]_{\hat{H}}. \tag{5.3.12}$$

3. *As a result of (5.3.5), τ induces an isomorphism of Clifford modules, i.e. $\wedge^\bullet T^*_{S^1}E \cong \wedge^\bullet T^*_{\hat{S}^1}\hat{E}$.*

An important conclusion of the above theorem can be stated as follows:

Any structure on E in terms of the natural pairing, the twisted Courant bracket and invariant forms (closed forms) has a corresponding one on the dual space $\hat{E}$.

Since $(TE \oplus T^*E)_{S^1}$ together with the natural pairing $\langle\ ,\ \rangle$ and the twisted Courant bracket $[\![\ ,\ ]\!]_H$ defines a Courant algebroid, Theorem 5.9 states that the Courant algebroids defined by invariant sections of a pair of T-dual principal circle bundles are isomorphic.

5.3.2 Principal torus bundles

In this section we extend the previous construction of principal S^1-bundles to general principal $\mathbb{T}^n$-bundles.

T-duality map

The T-duality maps defined in section (5.3.1) can be generalized for principal torus bundles as follows.

For a principal $\mathbb{T}^n$-bundle, the T-duality map between invariant sections of $TE \oplus T^*E$ and $T\hat{E} \oplus T^*\hat{E}$ is defined by $\varphi = \varphi_1\varphi_2 \cdots \varphi_i \cdots \varphi_n$, here

$$\begin{aligned}&\varphi_a(x, f_1, \ldots, f_a, \ldots, f_n; \xi, g^1, \ldots, g^a, \ldots, g^n)\\&= (x, f_1, \ldots, g^a, \ldots, f_n; \xi, g^1, \ldots, f_a, \ldots, g^n),\end{aligned} \tag{5.3.13}$$

i.e. φ_a is the T-duality map of invariant sections in the $a-th$ circle of $\mathbb{T}^n$.

The T-duality map between invariant forms τ is simply defined by $\tau = \tau_1 \ldots \tau_n$ where τ_a is T-duality with respect to the a-th circle.

Properties of τ_a:

(1) Due to the antisymmetry of $A_a \wedge A_b$, the τ_a's anti-commute:

$$\tau_a\tau_b = -\tau_b\tau_a. \tag{5.3.14}$$

(2) The map τ_a acts on invariant forms as the Clifford action of $(-\partial_{A_a}, A_a)$ followed by replacing the remaining A_a with $\hat{A}_a$. As an example, consider the case when E is a principal $\mathbb{T}^2$-bundle. The map τ_1 acts on an invariant form $\Omega = A_1 \wedge \Omega^1$ via

$$\tau_1(\Omega) = (-\partial_{A_{(1)}}, A_1)(A_1 \wedge \Omega^1) = -\Omega^1.$$

Similarly, the map τ_2 acts on Ω as

$$\begin{aligned}\tau_2(\Omega) &= (-\partial_{A_2}, A_2)(A_1 \wedge \Omega^1)\\&= A_2 \wedge A_1 \wedge \Omega^1 \quad (A_2 \to \hat{A}_2)\\&= \hat{A}_2 \wedge A_1 \wedge \Omega^1.\end{aligned}$$

Since $\tau = \tau_1 \ldots \tau_n$ and each τ_a satisfies Eqn. (5.3.5), we immediately have

Theorem 5.10.

$$\tau(\gamma_{\mathfrak{X}}\Omega) = (-1)^n \gamma_{\varphi(\mathfrak{X})} \cdot \tau(\Omega), \tag{5.3.15}$$

where $\mathfrak{X} \in \Gamma(TE \oplus T^*E)_{\mathbb{T}^n}$.

Example 5.11. Consider a principal $\mathbb{T}^2$-torus bundle E. Any invariant k-form on E can be written in the following dimensionally reduced form

$$\begin{aligned}\Omega &= \Omega_{(k)} + A_a \wedge \Omega^a_{(k-1)} + \frac{1}{2} A_a \wedge A_b \wedge \Omega^{ab}_{(k-2)} \\ &= \Omega_{(k)} + A_1 \wedge \Omega^1_{(k-1)} + A_2 \wedge \Omega^2_{(k-1)} + A_1 \wedge A_2 \wedge \Omega^{12}_{(k-2)}. \end{aligned} \tag{5.3.16}$$

In shorthand, we denote this by

$$\Omega \equiv (\Omega_{(k)}, \begin{pmatrix} \Omega^1_{(k-1)} \\ \Omega^2_{(k-1)} \end{pmatrix}, \Omega^{12}_{(k-2)}).$$

Applying T-duality map τ

$$\begin{aligned}\tau\Omega &= \tau_1\tau_2(\Omega_{(k)} + A_1 \wedge \Omega^1_{(k-1)} + A_2 \wedge \Omega^2_{(k-1)} + A_1 \wedge A_2 \wedge \Omega^{12}_{(k-2)}) \\ &= (-\partial_{A_1}, A_1) \cdot (-\partial_{A_2}, A_2) \cdot (\Omega_{(k)} + A_1 \wedge \Omega^1_{(k-1)} + A_2 \wedge \Omega^2_{(k-1)} + A_1 \wedge A_2 \wedge \Omega^{12}_{(k-2)}) \\ &\quad \text{with} \quad (A_2 \to \hat{A}_2) \\ &= (-\partial_{A_1}, A_1) \cdot (\hat{A}_2 \wedge \Omega_{(k)} + \hat{A}_2 \wedge A_1 \wedge \Omega^1_{(k-1)} - \Omega^2_{(k-1)} + A_1 \wedge \Omega^{12}_{(k-2)}) \\ &\quad \text{with} \quad (A_1 \to \hat{A}_1)) \\ &= \hat{A}_1 \wedge \hat{A}_2 \wedge \Omega_{(k)} + \hat{A}_2 \wedge \Omega^1_{(k-1)} - \hat{A}_1 \wedge \Omega^2_{(k-1)} - \Omega^{12}_{(k-2)} \\ &\equiv (-\Omega^{12}_{(k-2)}, \begin{pmatrix} -\Omega^2_{(k-1)} \\ \Omega^1_{(k-1)} \end{pmatrix}, \Omega_{(k)}).\end{aligned}$$

Thus

$$\tau(\Omega_{(k)}, \begin{pmatrix} \Omega^1_{(k-1)} \\ \Omega^2_{(k-1)} \end{pmatrix}, \Omega^{12}_{(k-2)}) = (-\Omega^{12}_{(k-2)}, \begin{pmatrix} -\Omega^2_{(k-1)} \\ \Omega^1_{(k-1)} \end{pmatrix}, \Omega_{(k)}) \tag{5.3.17}$$

And it is easy to verify that $\tau^2 = -1$.

Generalized Courant bracket

To generalize the Courant bracket on a principal $\mathbb{T}^n$-bundle, let us first make the following observation.

Consider the $\imath_{(x_1,f_{1,a})}\imath_{(x_2,f_{2,a})}H$ term in the twisted Courant bracket (5.3.11), expanding this twisting term gives us

$$\begin{aligned}\imath_{(x_1,f_{1,a})}\imath_{(x_2,f_{2,b})}H &= \imath_{x_1}\imath_{x_2}H_{(3)} - f_{2,b}\imath_{x_1}H^b_{(2)} + f_{1,a}\imath_{x_2}H^a_{(2)} \\ &\quad + f_{1,a}f_{2,b}H^{ab}_{(1)} + A_a \wedge (\imath_{x_1}\imath_{x_2}H^a_{(2)} - f_{2,b}\imath_{x_1}H^{ab}_{(1)} + f_{1,b}\imath_{x_2}H^{ab}_{(1)} \\ &\quad + f_{1,b}f_{2,c}H^{abc}_{(0)}).\end{aligned} \tag{5.3.18}$$

Applying the T-duality map φ to the Courant bracket (5.3.11), we find the corresponding $\imath_{(x_1,f_{1,a})}\imath_{(x_2,f_{2,b})}\hat{H}$ term

$$\imath_{(x_1,f_{1,a})}\imath_{(x_2,f_{2,b})}\hat{H} = \imath_{x_1}\imath_{x_2}H_{(3)} + \hat{A}_a \wedge \imath_{x_1}\imath_{x_2}F^a_{(2)} - f_{2,a}\imath_{x_1}F^a_{(2)} + f_{1,a}\imath_{x_2}F^a_{(2)}, \quad (5.3.19)$$

here $F \in \Omega^2(M) \otimes \mathfrak{t}$ and we denote $F = (F_{(2)}, 0, 0)$.

Recall that from Theorem 5.9, the twisted Courant bracket is preserved under the map φ in the case of principal circle bundles. We expect the same rule for principal torus bundles. However, it is obvious that for non-zero $H_{(1)}$ and $H_{(0)}$ the map φ does not preserve the twisted Courant bracket by comparing (5.3.18) and (5.3.19). To resolve this, we need to generalize the twisted Courant bracket (5.3.11) by adding terms involving $g's$ and contractions of x with new variables $F_{(1)} \in \Omega^1(M) \otimes \wedge^2\mathfrak{t}$ and $F_{(0)} \in C^\infty(M) \otimes \wedge^3\mathfrak{t}$, i.e. we arrive at the "**generalized Courant bracket**" $[\![\ ,\ ]\!]_{H,F}$ given in terms of dimensional reduced form

$$\begin{aligned}
&[\![(x_1, f_{1,a}; \xi_1, g_1^a), (x_2, f_{2,a}; \xi_2, g_2^a)]\!]_{H,F} = ([x_1, x_2], && (5.3.20)\\
&(x_1(f_{2,a}) - x_2(f_{1,a})) + \imath_{x_1}\imath_{x_2}F_{(2)a} + g_2^b\imath_{x_1}F_{(1)ab} - g_1^b\imath_{x_2}F_{(1)ab} - g_1^b g_2^c F_{(0)abc};\\
&(\mathcal{L}_{x_1}\xi_2 - \mathcal{L}_{x_2}\xi_1) - \frac{1}{2}d(\imath_{x_1}\xi_2 - \imath_{x_2}\xi_1) + \imath_{x_1}\imath_{x_2}H_{(3)} +\\
&(g_2^a\imath_{x_1}F_{(2)a} - g_1^a\imath_{x_2}F_{(2)a}) + (f_{2,a}\imath_{x_1}H^a_{(2)} - f_{1,a}\imath_{x_2}H^a_{(2)}) - f_{1,a}f_{2,b}H^{ab}_{(1)}\\
&-g_1^a g_2^b F_{(1)ab} + \frac{1}{2}(df_{1,a}g_2^a + f_{2,a}dg_1^a - f_{1,a}dg_2^a - df_{2,a}g_1^a), x_1(g_2^a)\\
&-x_2(g_1^a) + \imath_{x_1}\imath_{x_2}H^a_{(2)} + (f_{2,a}\imath_{x_1}H^{ab}_{(1)} - f_{1,a}\imath_{x_2}H^{ab}_{(1)}) - f_{1,b}f_{2,c}H^{abc}_{(0)}).
\end{aligned}$$

To write (5.3.20) in a more invariant notation, let us introduce $2d$-dimensional vectors and forms

$$h = \begin{pmatrix} f \\ g \end{pmatrix} \in C^\infty(M, \mathfrak{t} \oplus \mathfrak{t}^*), \quad \mathcal{F}_i = \begin{pmatrix} F_{(i)} \\ H_{(i)} \end{pmatrix} \in \Omega^i(M, \wedge^{3-i}\mathfrak{t} \oplus \wedge^{3-i}\mathfrak{t}^*), \quad (5.3.21)$$

as well as

$$\mathcal{J} = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}, \quad (5.3.22)$$

and write $\mathcal{X} = (x, \xi)$.

Then the generalized Courant bracket can be rewritten as

$$\begin{aligned}
&[\![(\mathcal{X}_1, h_1), (\mathcal{X}_2, h_2)]\!]_{\mathcal{F}} = ([\![\mathcal{X}_1, \mathcal{X}_2]\!]_{H_{(3)}} - (h_1\mathcal{J}\imath_{x_2}\mathcal{F}_2 - h_2\mathcal{J}\imath_{x_1}\mathcal{F}_2) + \frac{1}{2}(dh_1\mathcal{J}h_2\\
&-h_1\mathcal{J}dh_2) - (h_1\mathcal{J})(h_2\mathcal{J})\mathcal{F}_1, x_1(h_2) - x_2(h_1) + \imath_{x_1}\imath_{x_2}\mathcal{F}_2 + (h_1\mathcal{J})\mathcal{F}_1\\
&-(h_1\mathcal{J})(h_2\mathcal{J})\mathcal{F}_0), && (5.3.23)
\end{aligned}$$

from which is is obvious that the $O(n,n)$ T-duality group provides automorphisms of this generalized Courant bracket.

With the T-duality maps τ and φ and the newly defined generalized Courant bracket, we have thus generalized Theorem 5.9 to general principal torus bundles:

Theorem 5.12. *The following hold*

1. *φ is orthogonal with respect to the non-degenerate bilinear form $\langle\ ,\ \rangle$ on the invariant sections of $(TE \oplus T^*E)$ and $(T\hat{E} \oplus T^*\hat{E})$, hence it induces an isomorphism of Clifford algebras on the invariant sections of $(TE \oplus T^*E)$ and $(T\hat{E} \oplus T^*\hat{E})$.*

2. *φ defines an automorphism of the generalized Courant bracket $[\![\ ,\ ]\!]_{H,F}$ on the invariant sections of $(TE \oplus T^*E)$ and $(T\hat{E} \oplus T^*\hat{E})$, i.e.*

$$\begin{aligned}&\varphi([\![(x_1, f_{1,a}; \xi_1, g_1^a), (x_2, f_{2,a}; \xi_2, g_2^a)]\!]_{H,F})\\ &= [\![\varphi(x_1, f_{1,a}; \xi_1, g_1^a), \varphi(x_2, f_{2,a}; \xi_2, g_2^a)]\!]_{\hat{H},\hat{F}}.\end{aligned} \tag{5.3.24}$$

3. *τ induces an isomorphism of Clifford modules on the invariant sections of $(TE \oplus T^*E)$ and $(T\hat{E} \oplus T^*\hat{E})$ via* (5.3.15).

A question which arises naturally at this point is, does the space of invariant sections of $(TE \oplus T^*E)$ together with the natural non-degenerate pairing $\langle\ ,\ \rangle$ and the generalized Courant bracket $[\![\ ,\ ]\!]_{H,F}$ define a Courant algebroid, which is introduced previously in Section 2.3.2? Can we still interpret T-duality between a Principal $\mathbb{T}^n$-bundle and its dual in terms of an isomorphism between a pair of Courant algebroids?

Originally we expect that the generalized Courant bracket can be interpreted as the bracket of a Courant algebroid. However, upon taking the natural anchor map $\rho : (x, f; \xi, g) \to (x, f)$, one finds

$$\begin{aligned}\rho([\![(x_1, f_{1,a}; \xi_1, g_1^a), (x_2, f_{2,a}; \xi_2, g_2^a)]\!]_{H,F}) \quad &= \quad [\rho(x_1, f_{1,a}; \xi_1, g_1^a), \rho(x_2, f_{2,a}; \xi_2, g_2^a)]\\ &\quad +(g_2^b \imath_{x_1} F_{(1)ab} - g_1^b g_2^c F_{(0)abc}),\end{aligned} \tag{5.3.25}$$

i.e.

$$\rho([\![(x_1, f_{1,a}; \xi_1, g_1^a), (x_2, f_{2,a}; \xi_2, g_2^a)]\!]_{H,F}) \neq [\rho(x_1, f_{1,a}; \xi_1, g_1^a), \rho(x_2, f_{2,a}; \xi_2, g_2^a)] \tag{5.3.26}$$

Thus the space of invariant sections of $(TE \oplus T^*E)$ together with the non-degenerate pairing $\langle\ ,\ \rangle$ and the generalized Courant bracket fail to be a Courant algebroid with the natural anchor map.

We can, however, choose the anchor map as a projection onto the base manifold M, $\rho_0 : (x, f; \xi, g) \to (x, 0)$. The Courant algebroid with this anchor map is only defined over the base manifold, i.e. one can take $(TM \oplus T^*M, \rho_0, \langle\ ,\ \rangle, [\![\ ,\ ]\!]_{H,F})$ and this is a Courant algebroid.

To resolve this problem of $(TE \oplus T^*E)_{\mathbb{T}^n}$ failing to have an interpretation of a Courant algebroid, we show in the next section that one can construct a Courant algebroid on the invariant sections of $(TE \oplus T^*E)$ with the generalized Courant bracket from the double of a proto-bialgebroid (TE, T^*E).

5.4 Generalized Courant bracket of a Courant algebroid

Recall that for E a principal S^1-bundle, the (twisted) Courant bracket on invariant sections of $TE \oplus T^*E$ makes it into a Courant algebroid, it is natural to ask if this is true for general principal $\mathbb{T}^n$-bundles with the generalized Courant bracket (5.3.2).

In this section, we will redefine the generalized Courant bracket as a derived bracket on a symplectic manifold (cf. Section 2.3.4). Thus when E is a principal $\mathbb{T}^n$-bundle, on the invariant sections of $(TE \oplus T^*E)$, $(TE \oplus T^*E)$ can be realized as the double of a proto-bialgebroid defined previously in Section 2.3.4, i.e. a Courant algebroid. As a result, for a principal $\mathbb{T}^n$-bundle E, the space of invariant sections of $TE \oplus T^*E$ together with a non-degenerate bilinear form $\langle\ ,\ \rangle$ and the generalized Courant bracket $[\![\ ,]\!]_{H,F}$ defines a Courant algebroid. As a result, T-duality can be realized as a map relating isomorphic Courant algebroids.

5.4.1 $TE \oplus T^*E$ as a Courant algebroid

Let E be a principal $\mathbb{T}^n$-bundle over M ($dim(M) = d$). Let us choose coordinates $\{x_i\} = \{x_\mu, x_a\}(\mu = 1, \ldots, d; a = 1, \ldots, n)$ on E, and coordinates $(x^i, \xi_i^*, x_i^*, \xi^i)$ on $T^*(\Pi T^*E)$.

Let (TE, T^*E) be a proto-bialgebroid defined previously in Section 2.3.4, with proto-bialgebroid structures $(\mu, \gamma, \varphi, \psi)$ which are degree 3-functions on $T^*(\Pi T^*E)$. The structures $(\mu, \gamma, \varphi, \psi)$ are required to satisfy the condition (2.3.42).

We choose the proto-algebroid structures $(\mu, \gamma, \varphi, \psi)$ on (TE, T^*E) as follows

$$\begin{aligned}
\mu &= \xi^i x_i^* + \frac{1}{2} f_{\mu\nu}{}^a \xi^\mu \xi^\nu \xi_a^*, \\
\gamma &= \frac{1}{2} q_\mu{}^{ab} \xi_a^* \xi_b^* \xi^\mu, \\
\varphi &= \frac{1}{6} \varphi^{abc} \xi_a^* \xi_b^* \xi_c^*, \\
\psi &= \frac{1}{6} \psi_{\mu\nu\gamma} \xi^\mu \xi^\nu \xi^\gamma + \frac{1}{2} \psi_{a\mu\nu} \xi^a \xi^\mu \xi^\nu + \frac{1}{2} \psi_{ab\mu} \xi^a \xi^b \xi^\mu + \frac{1}{6} \psi_{abc} \xi^a \xi^b \xi^c.
\end{aligned} \tag{5.4.1}$$

Here f, q, φ and ψ can be identified with H's and F's via

$$\begin{aligned}
&F_{(2)a} \equiv f_{\mu\nu}{}^a \xi^\mu \xi^\nu \xi_a^*, \quad F_{(1)ab} \equiv q^{ab}{}_\mu \xi_a^* \xi_b^* \xi^\mu, \quad F_{(0)abc} \equiv \varphi^{abc} \xi_a^* \xi_b^* \xi_c^*, \\
&H_{(3)} \equiv \psi_{\mu\nu\gamma} \xi^\mu \xi^\nu \xi^\gamma, \quad H_{(2)}^a \equiv \psi_{a\mu\nu} \xi^a \xi^\mu \xi^\nu, \quad H_{(1)}^{ab} \equiv \psi_{ab\mu} \xi^a \xi^b \xi^\mu, \\
&H_{(0)}^{abc} \equiv \psi_{abc} \xi^a \xi^b \xi^c.
\end{aligned} \tag{5.4.2}$$

Next we will construct the anchor map, the quasi-differential and the Courant bracket correspond to the above defined proto-bialgebroid structures.

Anchor map

The anchor map ρ on $TE \oplus T^*E$ is defined by

$$\rho(X + \xi) = \rho_{TE}(X) + \rho_{T^*E}(\Xi), \tag{5.4.3}$$

with ρ_{TE} and ρ_{T^*E} given, respectively by

$$\begin{aligned}
\rho_{TE}(X)(f) &= \{\{X, \theta\}, f\}, \\
\rho_{T^*E}(\Xi)(f) &= \{\{\Xi, \theta\}, f\},
\end{aligned} \tag{5.4.4}$$

where $X \in \Gamma(TE), \Xi \in \Gamma(T^*E)$, $f \in C^\infty(M)$ and $\theta = \mu + \gamma + \varphi + \psi$.

Consider the anchor map on invariant sections of $TE \oplus T^*E$. For $(x, f^a; \xi, g_a)$ an invariant section of $TE \oplus T^*E$, the anchor map gives

$$\rho(x, f^a; \xi, g_a)(f) = x^\mu x_\mu^*(f) + f^a x_a^* f = x^\mu \partial_\mu f + f^a \partial_a f. \tag{5.4.5}$$

In this case the anchor becomes

$$\rho(x, f^a; \xi, g_a) = (x, f^a \partial_a). \tag{5.4.6}$$

Quasi-differentials vs twisted differential

The associated quasi-differentials, d_μ and d_γ on $\Gamma(\wedge^\bullet T^*E)$ and $\Gamma(\wedge^\bullet TE)$ are given by

$$d_\mu \Xi = \{\mu, \Xi\}, \quad d_\gamma X = \{\gamma, X\}. \tag{5.4.7}$$

The quasi-differentials d_μ and d_γ are "quasi" in the sense that they do not square to zero but satisfy the following relations:

$$\begin{cases} (d_\mu)^2 \cdot + \{d_\gamma \psi, \cdot\} = 0 \\ (d_\gamma)^2 \cdot + \{d_\mu \varphi, \cdot\} = 0. \end{cases} \tag{5.4.8}$$

The quasi-differential relations (5.4.8) are equivalent to the first two of the five constraints placed on the proto-bialgebroid structures (2.3.42):

$$\begin{cases} \frac{1}{2}\{\mu, \mu\} + \{\gamma, \psi\} = 0, \\ \frac{1}{2}\{\gamma, \gamma\} + \{\mu, \varphi\} = 0, \\ \{\mu, \gamma\} + \{\varphi, \psi\} = 0, \\ \{\mu, \psi\} = 0, \\ \{\gamma, \varphi\} = 0. \end{cases} \tag{5.4.9}$$

Substituting (5.4.1) into (5.4.9) and making the identification given by (5.4.2), we find the following constraints on H's and F's

$$\begin{cases} dH_{(3)} + F_{(2)a} \wedge H^a_{(2)} = 0, \\ dH^a_{(2)} + F_{(2)b} \wedge H^{ab}_{(1)} = 0, \\ dH^{ab}_{(1)} + F_{(2)c} \wedge H^{cab}_{(0)} = 0, \\ dH^{abc}_{(0)} = 0, \\ dF_{(2)a} + F_{(1)ab} \wedge H^b_{(2)} = 0, \\ dF_{(1)ab} + F_{(0)abc} \wedge H^c_{(2)} = 0, \\ dF_{(0)abc} = 0. \end{cases} \tag{5.4.10}$$

Courant bracket

Recall in Section 2.3.4 that the double of a proto-bialgebroid (TE, T^*E), given by $TE \oplus T^*E$ is a Courant algebroid, and is equipped with the Dorfmann bracket on $TE \oplus T^*E$ defined by the derived bracket

$$(X_1 + \Xi_1) \circ (X_2 + \Xi_2) = \{\{X_1 + \Xi_1, \theta\}, X_2 + \Xi_2\}, \tag{5.4.11}$$

for $X_i \in \Gamma(TE)$ and $\Xi_i \in \Gamma(T^*E)$.

The Lie bracket on TE and T^*E are given by

$$[X_1, X_2]_\mu = \{\{X_1, \mu\}, X_2\},$$
$$[\Xi_1, \Xi_2]_\gamma = \{\{\Xi_1, \gamma\}, \Xi_2\}. \tag{5.4.12}$$

Using an analogous notation for the interior product and the Lie derivations, let us define

$$\mathcal{L}^\mu_X = d_\mu \imath_X + \imath_X d_\mu, \quad \mathcal{L}^\gamma_\Xi = d_\gamma \imath_\Xi + \imath_\Xi d_\gamma.$$

Thus the components of (5.4.11) can be written as

$$\begin{aligned} X_1 \circ X_2 &= [X_1, X_2]_\mu + \imath_{X_1}\imath_{X_2}\psi, \\ X_1 \circ \Xi_2 &= -\imath_{\Xi_2} d_\gamma X_1 + \mathcal{L}^\mu_{X_1}\Xi_2, \\ \Xi_1 \circ X_2 &= \mathcal{L}^\gamma_{\Xi_1} X_2 - \imath_{X_2} d_\mu \Xi_1, \\ \Xi_1 \circ \Xi_2 &= [\Xi_1, \Xi_2]_\gamma + \imath_{\Xi_1}\imath_{\Xi_2}\varphi. \end{aligned} \tag{5.4.13}$$

On invariant sections of $TE \oplus T^*E$ the bracket becomes

$$\begin{aligned} &(x_1, f_{1,a}; \xi_1, g_1^a) \circ (x_2, f_{2,a}; \xi_2, g_2^a) \\ &= \{\{x_1^\mu \xi^*_\mu + f_1^a \xi^*_a + \xi_{1,\mu}\xi^\mu + g_{1,a}\xi^a, \theta\}, x_2^\mu \xi^*_\mu + f_2^a \xi^*_a + \xi_{2,\mu}\xi^\mu + g_{2,a}\xi^a\} \\ &= [x_1, x_2] + x_1(f_2) - x_2(f_1) + f_{\mu\nu}{}^a x_1^\mu x_2^\nu \xi^*_a + \psi_{\mu\nu A} x_1^\mu x_2^\nu \xi^i + \psi_{a\mu i} f_1^a x_2^\mu \xi^i \\ &+\psi_{\mu a i} x_1^\mu f_2^a \xi^i + \psi_{abi} f_1^a f_2^b \xi^i + q^{ab}{}_\mu x_1^\mu g_{2,a}\xi^*_b + \mathcal{L}_{x_1}\xi_2 + \imath_{x_1} dg_2 + f_{\mu\nu}{}^a x_1^\mu g_{2,a}\xi^\nu \\ &+q^{ab}{}_\mu x_2^\mu g_{1,a}\xi^*_b + f_{\mu\nu}{}^a x_2^\mu g_{1,a}\xi^\nu - \imath_{x_2} d\xi_1 - \imath_{x_2} dg_1 + df_1^a g_{2,a} + f_2^a dg_{1,a} \\ &+\varphi^{abc} g_{1,a} g_{2,b}\xi_c + q^{ab}{}_\mu g_{1,a} g_{2,b}\xi^\mu, \end{aligned} \tag{5.4.14}$$

the indices $\mu, \nu, \gamma = 1, \ldots, dim(M)$, $a, b, c = 1, \ldots, n$, while $i, j = 1, \ldots, rank(E)$.

In terms of H's and F's, the above Dorfmann bracket can be rewritten as

$$\begin{aligned} &(x_1, f_{1,a}; \xi_1, g_1^a) \circ (x_2, f_{2,a}; \xi_2, g_2^a) = ([x_1, x_2], x_1(f_2) - x_2(f_1) + \imath_{x_1}\imath_{x_2} F_{(2)a} \\ &+g_2^a \imath_{x_1} F_{(1)ab} - g_1^a \imath_{x_2} F_{(1)ab} + \imath_{x_1}\imath_{x_2} H_{(3)} - g_1^a g_2^b F_{(0)abc}; \mathcal{L}_{x_1}\xi_2 - \imath_{x_2} d\xi_1 \\ &+(f_{2,a}\imath_{x_1} H^a_{(2)} - f_{1,a}\imath_{x_2} H^a_{(2)}) - f_{1,a} f_{2,b} H^{ab}_{(1)} + (g_2^a \imath_{x_1} F_{(2)a} - g_1^a \imath_{x_2} F_{(2)a}) \\ &-g_1^a g_2^b F_{(1)ab} + df_{1,a} g_2^a + f_{2,a} dg_1^a, x_1(g_2^a) - x_2(g_1^a) + \imath_{x_1}\imath_{x_2} H^a_{(2)} \\ &+(f_{2,a}\imath_{x_1} H^{ab}_{(1)} - f_{1,a}\imath_{x_2} H^{ab}_{(1)}) - f_{1,a} f_{2,b} H^{abc}_{(0)}). \end{aligned} \tag{5.4.15}$$

The bracket (5.4.15) is exactly the Dorfmann bracket corresponding to the generalized Courant bracket (5.3.20) defined in Section 5.3.2. Therefore for a principal torus bundle, the generalized Courant bracket (5.3.20) on invariant sections of $TE \oplus T^*E$ can be alternatively defined via the derived bracket (5.4.11) with structures $(\mu, \gamma, \varphi, \psi)$ given by (5.4.1).

Therefore we conclude that the space of invariant sections of $TE \oplus T^*E$ with the generalized Courant bracket defines a Courant algebroid, with H's and F's required to satisfy (5.4.10).

At this point we have not specified $[\ ,\ ]_\gamma$, the Lie bracket on T^*M given in terms of the proto-bialgebroid structure $q^{ab}{}_\mu$. Later in Sections 5.4.3 and 5.4.4, we will study the special cases when $[\ ,\ ]_\gamma$ is defined via a Poisson structure [56].

The first example in Section 5.4.3 is my own work, while the second example in Section 5.4.4 is due to Garretson [26].

5.4.2 Generalized Cartan system

Recall in Section 4.5 that the Courant bracket can be viewed as an extension of the Lie bracket, acting on forms via the Clifford action on the generalized tangent space. The twisted Courant bracket $[\![\ ,\ ]\!]_H$ and the twisted differential d_H are compatible in the sense of Proposition 4.6.

In this section, we generalize Proposition 4.6 to general principal $\mathbb{T}^n$-bundles, showing that the generalized Cartan system in Proposition 4.5 can be formulated similarly for principal torus bundles, with the generalized Courant bracket $[\![\ ,\ ]\!]_{H,F}$ and the differential d_θ.

Let E ba a principal torus bundle, with d_θ the differential defined in terms of the structure $\theta = \mu + \gamma + \varphi + \psi$ and the canonical Poisson bracket on $T^*(\Pi T^*E)$:

$$d_\theta \cdot = \{\theta, \cdot\}. \tag{5.4.16}$$

The differential d_θ acts on $\mathfrak{X} = (X + \Xi)$ and gives

$$d_\theta(X + \Xi) = d_\mu \Xi + d_\gamma X + \imath_X \psi + \imath_\Xi \varphi, \tag{5.4.17}$$

where d_μ and d_γ are the quasi-differential defined by (5.4.7).

Also recall from the previous section, the (generalized) Dorfmann bracket is given by

$$\mathfrak{X}_1 \circ_{H,F} \mathfrak{X}_2 = \{\{\theta, \mathfrak{X}_1\}, \mathfrak{X}_2\}. \tag{5.4.18}$$

Let us define the generalized Cartan system on $TE \oplus T^*E$ with the ingredients $(\gamma_\mathfrak{X}, \mathcal{L}_\mathfrak{X}, d_\theta, \circ_{H,F})$ given by:

- $\mathfrak{X} = (X, \xi) \in \Gamma(TE \oplus T^*E)$, in analogy with vector fields in Cartan's system.
- $\gamma_\mathfrak{X}$ in analogy with $\imath_X$ in Cartan's system, acts on differential forms via $\gamma_\mathfrak{X} = \{\mathfrak{X}, \cdot\}$.

- $\mathcal{L}_{\mathfrak{X}}$ in analogy with the Lie derivative in Cartan's system, is defined by

$$\begin{aligned}\mathcal{L}_{\mathfrak{X}}\cdot &= [d_\theta, \gamma_{\mathfrak{X}}]\cdot \\ &= \{\{\theta, \mathfrak{X}\}, \cdot\} + \{\mathfrak{X}, \{\theta, \cdot\}\}.\end{aligned} \tag{5.4.19}$$

- A differential d_θ defined by (5.4.16), in analogy with the de Rham differential d in Cartan's system.
- The Dorfmann bracket $\circ_{H,F}$ defined by (5.4.18), in analogy with the Lie bracket in Cartan's system.

Thus we claim:

Proposition 5.13. *The generalized Cartan formulaes defined in Section 4.5 on $TE \oplus T^*E$ can be generalized as follows*

$$\begin{aligned}&(1)[d_\theta, d_\theta] = 0,\\ &(2)[\gamma_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}] = 2\langle \mathfrak{X}_1, \mathfrak{X}_2\rangle,\\ &(3)[d_\theta, \gamma_{\mathfrak{X}}] = \mathcal{L}_{\mathfrak{X}},\\ &(4)[\mathcal{L}_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}] = \gamma_{\mathfrak{X}_1 \circ_{H,F} \mathfrak{X}_2},\\ &(5)[\mathcal{L}_{\mathfrak{X}_1}, \mathcal{L}_{\mathfrak{X}_2}] = \mathcal{L}_{\mathfrak{X}_1 \circ_{H,F} \mathfrak{X}_2},\\ &(6)[d_\theta, \mathcal{L}_{\mathfrak{X}}] = 0,\end{aligned}$$

where $\mathfrak{X}_i \in \Gamma(TE \oplus T^*E)$ and $[\ ,\ ]$ is the graded commutator of the graded algebra on $TE \oplus T^*E$.

Proof. (1)

$$[d_\theta, d_\theta] = 2\{\theta, \{\theta, \cdot\}\} = 0. \tag{5.4.20}$$

(2) and (3) by definition.
(4)

$$\begin{aligned}[\mathcal{L}_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}] &= \mathcal{L}_{\mathfrak{X}_1}\gamma_{\mathfrak{X}_2} - \gamma_{\mathfrak{X}_2}\mathcal{L}_{\mathfrak{X}_1}\\ &= d_\theta\gamma_{\mathfrak{X}_1}\gamma_{\mathfrak{X}_2} + \gamma_{\mathfrak{X}_1}d_\theta\gamma_{\mathfrak{X}_2} - \gamma_{\mathfrak{X}_2}d_\theta\gamma_{\mathfrak{X}_1} - \gamma_{\mathfrak{X}_2}\gamma_{\mathfrak{X}_1}d_\theta\\ &= \{\theta, \{\mathfrak{X}_1, \{\mathfrak{X}_2, \cdot\}\}\} + \{\mathfrak{X}_1, \{\theta, \{\mathfrak{X}_2, \cdot\}\}\} - \{\mathfrak{X}_2, \{\theta, \{\mathfrak{X}_1, \cdot\}\}\}\\ &\quad -\{\mathfrak{X}_2, \{\mathfrak{X}_1, \{\theta, \cdot\}\}\}\\ &= \{\{\theta, \mathfrak{X}_1\}, \{\mathfrak{X}_2, \cdot\}\} - \{\mathfrak{X}_2, \{\{\mathfrak{X}_1, \theta\}, \cdot\}\}\\ &= \{\{\{\theta, \mathfrak{X}_1\}, \mathfrak{X}_2\}, \cdot\}\\ &= \gamma_{\mathfrak{X}_1 \circ_{H,F} \mathfrak{X}_2}\cdot .\end{aligned} \tag{5.4.21}$$

(6)

$$\begin{aligned}[d_\theta, \mathcal{L}_{\mathfrak{X}}] &= d_\theta d_\theta \gamma_{\mathfrak{X}} + d_\theta \gamma_{\mathfrak{X}} d_\theta - d_\theta \gamma_{\mathfrak{X}} d_\theta - d_\theta d_\theta \gamma_{\mathfrak{X}} \\ &= 0. \end{aligned} \tag{5.4.22}$$

(5)

$$\begin{aligned}[\mathcal{L}_{\mathfrak{X}_1}, \mathcal{L}_{\mathfrak{X}_2}] &= \mathcal{L}_{\mathfrak{X}_1}\mathcal{L}_{\mathfrak{X}_2} - \mathcal{L}_{\mathfrak{X}_2}\mathcal{L}_{\mathfrak{X}_1} \\ &= (d_\theta\gamma_{\mathfrak{X}_1} + \gamma_{\mathfrak{X}_1} d_\theta)(d_\theta\gamma_{\mathfrak{X}_2} + \gamma_{\mathfrak{X}_2} d_\theta) - (d_\theta\gamma_{\mathfrak{X}_2} + \gamma_{\mathfrak{X}_2} d_\theta)(d_\theta\gamma_{\mathfrak{X}_1} + \gamma_{\mathfrak{X}_1} d_\theta) \\ &= \{\theta, \{\mathfrak{X}_1, \{\theta, \{\mathfrak{X}_2, \cdot\}\}\}\} + \{\theta, \{\mathfrak{X}_1, \{\mathfrak{X}_2, \{\theta, \cdot\}\}\}\} \\ &\quad + \{\mathfrak{X}_1, \{\theta, , \{\theta, \{\mathfrak{X}_2, \cdot\}\}\}\} + \{\mathfrak{X}_1, \{\theta, \{\mathfrak{X}_2, \{\theta, \cdot\}\}\}\} \\ &\quad - \{\theta, \{\mathfrak{X}_2, \{\theta, \{\mathfrak{X}_1, \cdot\}\}\}\} - \{\theta, \{\mathfrak{X}_2, \{\mathfrak{X}_1, \{\theta, \cdot\}\}\}\} \\ &\quad - \{\mathfrak{X}_2, \{\theta, \{\theta, \{\mathfrak{X}_1, \cdot\}\}\}\} - \{\mathfrak{X}_2, \{\theta, \{\mathfrak{X}_1, \{\theta, \cdot\}\}\}\} \\ &= \{\theta, \{\mathfrak{X}_1, \{\{\mathfrak{X}_2, \theta\}, \cdot\}\}\} + \{\mathfrak{X}_1, \{\theta, \{\{\theta, \mathfrak{X}_2\}, \cdot\}\}\} \\ &\quad - \{\theta, \{\mathfrak{X}_2, \{\{\mathfrak{X}_1, \theta\}, \cdot\}\}\} - \{\mathfrak{X}_2, \{\theta, \{\{\theta, \mathfrak{X}_1\}, \cdot\}\}\} \\ &= \{\theta, \{\{\{\theta, \mathfrak{X}_1\}, \mathfrak{X}_2\}, \cdot\}\} \\ &= \{\{\theta, \{\{\theta, \mathfrak{X}_1\}, \mathfrak{X}_2\}\}, \cdot\} + \{\{\{\theta, \mathfrak{X}_1\}, \mathfrak{X}_2\}, \{\theta, \cdot\}\} \\ &= \mathcal{L}_{\mathfrak{X}_1 \circ_{H,F} \mathfrak{X}_2}. \end{aligned} \tag{5.4.23}$$

□

The following theorem follows immediately

Theorem 5.14. *The generalized Courant bracket is related to the Lie bracket and twisted differential d_θ through the relation:*

$$\begin{aligned}\gamma_{[\![\mathfrak{X}_1, \mathfrak{X}_2]\!]_{H,F}}\Omega &= \frac{1}{2}[\gamma_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}] \cdot d_\theta\Omega + \frac{1}{2}d_\theta([\gamma_{\mathfrak{X}_1}, \gamma_{\mathfrak{X}_2}] \cdot \Omega) + \gamma_{\mathfrak{X}_1} \cdot d_\theta(\gamma_{\mathfrak{X}_2} \cdot \Omega) \\ &\quad - \gamma_{\mathfrak{X}_2} \cdot d_\theta(\gamma_{\mathfrak{X}_1} \cdot \Omega). \end{aligned} \tag{5.4.24}$$

5.4.3 Example: Principal torus bundles with Poisson structures

Consider a principal $\mathbb{T}^n$-bundle E over M with a Poisson structure Π. Let us choose coordinates $\{x_i\} = \{x_\mu, x_a\}(\mu = 1, \ldots, dim(M); a = 1, \ldots, n)$ on E. Suppose Π has components only on the fibre of the bundle, i.e. locally on $T^*(\Pi T^* E)$ the Poisson structure is given by $\Pi = \frac{1}{2}\Pi^{ab}\xi^*_a\xi^*_b$.

(TE, T^*E) is a proto-bialgebroid with structures $(\mu, \gamma, \varphi, \psi)$ given by (5.4.1), except in this case the structure γ is replaced by $\gamma = \{\mu, \Pi\}$, i.e. the derived bracket corresponding to γ is the Koszul bracket on one-forms [53]:

$$[\xi, \eta]_\gamma = \{\{\gamma, \xi\}, \eta\} = \mathcal{L}_{\pi^\sharp \xi}\eta - \mathcal{L}_{\pi^\sharp \eta}\xi - d(\Pi(\xi, \eta)). \tag{5.4.25}$$

We can re-define $\varphi = \wedge^3 \pi^\sharp \psi$. Thus in local coordinates on $T^*\Pi A$, the proto-algebroid structures are given by

$$\begin{aligned}
\mu &= \xi^i x_i^* - \frac{1}{2} f_{\mu\nu}{}^a \xi^\mu \xi^\nu \xi_a^*, \\
\gamma &= \{\mu, \Pi\} = \partial_\mu(\Pi^{ab}(x))\xi^\mu \xi_a^* \xi_b^*, \\
\varphi &= \frac{1}{6}\{\Pi, \{\Pi, \{\Pi, \psi\}\}\} = \frac{1}{6}\Pi^{ad}\Pi^{be}\Pi^{cf}\psi_{abc}\xi_d^*\xi_e^*\xi_f^* = \frac{1}{6}\varphi^{def}\xi_d^*\xi_e^*\xi_f^*, \\
\psi &= \frac{1}{6}\psi_{\mu\nu\gamma}\xi^\mu\xi^\nu\xi^\gamma + \frac{1}{2}\psi_{\mu ab}\xi^\mu\xi^a\xi^b + \frac{1}{2}\psi_{\mu\nu a}\xi^\mu\xi^\nu\xi^a + \frac{1}{6}\psi_{abc}\xi^a\xi^b\xi^c,
\end{aligned} \tag{5.4.26}$$

for implicity, we relabel $\Pi^{ad}\Pi^{be}\Pi^{cf}\psi_{abc} \equiv \varphi^{def}$ and $q^{ab}{}_\mu \equiv \partial_\mu \Pi^{ab}$.

Recall in Section 2.3.4 that in order for $((TE, T^*E), \mu, \gamma, \varphi, \psi)$ to define a proto-bialgebroid, we need the consistency relations given by (5.4.9).

It follows from (5.4.26) that $\{\mu, \mu\} = \{\gamma, \gamma\} = 0$. Thus the consistency relations (5.4.9) breakdown to the following sets of relations:

$$\begin{aligned}
&\{\mu, \psi\} = \{\mu, \varphi\} = 0, \\
&\{\mu, \gamma\} + \{\varphi, \psi\} = 0, \\
&\{\gamma, \psi\} = \{\gamma, \mu\} = \{\gamma, \gamma\} = \{\mu, \mu\} = 0,
\end{aligned} \tag{5.4.27}$$

Substituting (5.4.26) into (5.4.27) gives us the following set of conditions:

$$\begin{aligned}
&\partial_\mu(\psi_{\nu\gamma\kappa}) - \frac{3}{2} f_{\mu\nu}{}^a \psi_{a\gamma\kappa} = 0, \\
&\partial_\mu(\psi_{a\nu\gamma}) + f_{\mu\nu}{}^b \psi_{ba\gamma} = 0, \\
&\partial_\mu(\psi_{ab\nu}) - \frac{3}{2} f_{\mu\nu}{}^c \psi_{abc} = 0, \\
&\partial_\mu(\psi_{abc}) = 0. \\
&\varphi^{abc}\psi_{aij} = 0, \quad (\partial_\mu \Pi^{ab})\psi_{bij} = 0, \quad \partial_\mu \varphi^{abc} = 0,
\end{aligned} \tag{5.4.28}$$

On invariant sections of $TE \oplus T^*E$, the Dorfmann bracket defined by these structures is given by

$$
\begin{aligned}
&(x_1, f_{1,a}; \xi_1, g_1^a) \circ (x_2, f_{2,a}; \xi_2, g_2^a) \\
&= \{\{x_1^\mu \xi_\mu^* + f_1^a \xi_a^* + \xi_{1,\mu}\xi^\mu + g_{1,a}\xi^a, \theta\}, x_2^\mu \xi_\mu^* + f_2^a \xi_a^* + \xi_{2,\mu}\xi^\mu + g_{2,a}\xi^a\} \\
&= [x_1, x_2] + x_1(f_2) - x_2(f_1) + f_{\mu\nu}{}^a x_1^\mu x_2^\nu \xi_a^* + \psi_{\mu\nu i} x_1^\mu x_2^\nu \xi^i + \psi_{a\mu i} f_1^a x_2^\mu \xi^i \\
&+\psi_{\mu a i} x_1^\mu f_2^a \xi^i + \psi_{abi} f_1^a f_2^b \xi^i + \partial_\mu(\Pi^{ab}(x)) x_1^\mu g_{2,a}\xi_b^* + \mathcal{L}_{x_1}\xi_2 + \imath_{x_1} dg_2 \\
&+f_{\mu\nu}{}^a x_1^\mu g_{2,a}\xi^\nu + \partial_\mu(\Pi^{ab}(x)) x_2^\mu g_{1,a}\xi_b^* + f_{\mu\nu}{}^a x_2^\mu g_{1,a}\xi^\nu - \imath_{x_2} d\xi_1 - \imath_{x_2} dg_1 \\
&+df_1^a g_{2,a} + f_2^a dg_{1,a} + \varphi^{abc} g_{1,a} g_{2,b} \xi_c + \partial_\mu(\Pi^{ab}(x)) g_{1,a} g_{2,b}\xi^\mu, \qquad (5.4.29)
\end{aligned}
$$

is invariant under T-duality transformation φ defined in Section 5.3.2, with the corresponding terms exchange under T-duality:

$$ f_{\mu\nu}{}^a \leftrightarrow \psi_{\mu\nu a}, \qquad (5.4.30) $$

$$ q^{ab}{}_\mu \leftrightarrow \psi_{\mu ab}, \qquad (5.4.31) $$

$$ \varphi^{abc} \leftrightarrow \psi_{abc}. \qquad (5.4.32) $$

If one makes the identification

$$
\begin{aligned}
&H_{(3)} \equiv \psi_{\mu\nu\gamma}\xi^\mu\xi^\nu\xi^\gamma, \quad H_{(2)}^a \equiv \psi_{a\mu\nu}\xi^a\xi^\mu\xi^\nu, \quad H_{(1)}^{ab} \equiv \psi_{ab\mu}\xi^a\xi^b\xi^\mu, \\
&H_0^{abc} \equiv \psi_{abc}\xi^a\xi^b\xi^c, \quad F_{(2)a} \equiv f_{\mu\nu}{}^a \xi^\mu\xi^\nu\xi_a^*, \qquad (5.4.33) \\
&F_{(1)ab} \equiv \partial_\mu(\Pi^{ab}(x))\xi^\mu\xi_a^*\xi_b^* = d\Pi, \quad F_{(0)abc} \equiv \varphi^{abc}\xi_a^*\xi_b^*\xi_c^* = (\wedge^3\pi^\sharp)H,
\end{aligned}
$$

we see that T-duality exchanges $(H_{(2)}, H_{(1)}, H_{(0)})$ with $(F_{(2)}, F_{(1)}, F_{(0)})$ as one would expect.

In terms of the H and F, the set of consistency relations (5.4.27) become

$$
\begin{cases}
dH_{(3)} - \frac{3}{2}F_{(2)a}H_{(2)}^a = 0, \\
dH_{(2)}^a + F_{(2)b}H_{(1)}^{ab} = 0, \\
dH_1^{ab} - \frac{3}{2}F_{(2)c}H_{(0)}^{cab} = 0, \\
dH_{(0)}^{abc} = 0, \\
dF_{(2)a} = 0, \\
dF_{(1)ab} = 0,\ dF_{(1)ab}H_{(2)}^a = 0,\ dF_{(1)ab}H_{(1)}^{ac} = 0,\ dF_{(1)ab}H_{(0)}^{acd} = 0, \\
dF_{(0)abc} = 0,\ dF_{(0)abc}H_2^a = 0,\ dF_{(0)abc}H_1^{ad} = 0,\ dF_{(0)abc}H_{(0)}^{ade} = 0.
\end{cases} \qquad (5.4.34)
$$

By specifying the structure γ so that the Lie bracket on T^*E is replaced by the Koszul bracket on one forms, the set of constraints (5.4.28) is a set of stricter constraints on the H and F than (5.4.10).

In this example, the principal $\mathbb{T}^n$-bundle with Poisson structure Π is characterized by the H-flux $H = (H_{(3)}, H_{(2)}, H_{(1)}, H_{(0)})$ and the three tuple $(F_{(2)}, F_{(1)}, F_{(0)})$. Here $F_{(2)}$ is the curvature two form, $F_{(1)} = d\Pi$ and $F_{(0)} = (\wedge^3\pi^\sharp)H$.

5.4.4 Example: Principal torus bundles twisted by a Poisson structure

In this example, let us consider a principal torus bundle E with Π a Poisson structure, satisfying

$$[\Pi, \Pi]_\mu = \{\{\mu, \Pi\}, \Pi\} = 0. \tag{5.4.35}$$

Consider a Poisson structure Π such that its components only live on the fibre, i.e., locally on $T^*(\Pi T^* E)$ the Poisson structure is given by $\Pi = \frac{1}{2}\Pi^{ab}(x)\xi^*_a\xi^*_b$. We will now consider twisting a proto-bialgebroid by Π as introduced by Roytenberg [66] and Kosmann-Schwarzbach [56].

Let us start with a proto-bialgebroid (TE, T^*E) with structures $(\mu, 0, 0, \psi)$, where μ and ψ are the structures defined in (5.4.1). One can twist this proto-bialgebroid by a bivector Π to construct a proto-bialgebroid with twisted structures $(\mu_\Pi, \gamma_\Pi, \varphi_\Pi, \psi_\Pi)$ given by ([56] section (4.1.2)).

$$\begin{aligned}
\mu_\Pi &= \mu + \pi^\sharp\psi = \mu + \{\Pi, \psi\} \\
&= \xi^i x^*_i - \frac{1}{2} f_{\mu\nu}{}^a \xi^\mu \xi^\nu \xi^*_a + f_{\mu a}{}^b \xi^\mu \xi^a \xi^*_b + f_{ab}{}^c \xi^a \xi^b \xi^*_c, \\
\gamma_\Pi &= \{\mu, \Pi\} + (\wedge^2\pi^\sharp)\psi = \{\mu, \Pi\} + \frac{1}{2}\{\Pi, \{\Pi, \psi\}\} \\
&= \frac{1}{2} q_\mu{}^{ab} \xi^\mu \xi^*_a \xi^*_b + \frac{1}{2} q_a{}^{bc} \xi^a \xi^*_b \xi^*_c, \\
\varphi_\Pi &= (\wedge^3\pi^\sharp)\psi = \frac{1}{6}\{\Pi, \{\Pi, \{\Pi, \psi\}\}\} \\
&= \varphi^{abc} \xi^*_a \xi^*_b \xi^*_c, \\
\psi_\Pi &= \psi \\
&= \frac{1}{6}\psi_{\mu\nu\gamma}\xi^\mu\xi^\nu\xi^\gamma + \frac{1}{2}\psi_{a\mu\nu}\xi^a\xi^\mu\xi^\nu + \frac{1}{2}\psi_{ab\mu}\xi^a\xi^b\xi^\mu + \frac{1}{6}\psi_{abc}\xi^a\xi^b\xi^c,
\end{aligned} \tag{5.4.36}$$

where

$$\begin{aligned}
&f_{\mu c}{}^a = \Pi^{ab}\psi_{b\mu c}, \quad f_{cd}{}^a = \Pi^{ab}\psi_{bcd}, \quad q^{ab}{}_\mu = \partial_\mu \Pi^{ab}, \\
&q^{cd}{}_a = \Pi^{ce}\Pi^{df}\psi_{aef}, \quad \varphi^{abc} = (\wedge^3\pi^\sharp)\psi.
\end{aligned} \tag{5.4.37}$$

This twisted proto-bialgebroid has consistency relations given by the original (untwisted) proto-bialgebroid, which is

$$\{\mu, \psi\} = d_\mu \psi = 0. \tag{5.4.38}$$

If as usual we identify $\psi \equiv H$, (5.4.38) simply gives the consistency relation

$$dH = 0, \tag{5.4.39}$$

which is equivalent to saying $d_H^2 = 0$.

The Courant bracket defined by

$$(x+\xi)\circ(y+\eta) = \{\{\theta_\Pi, x+\xi\}, y+\eta\}, \quad \theta_\Pi = \mu_\Pi + \gamma_\Pi + \varphi_\Pi + \psi_\Pi \tag{5.4.40}$$

is still invariant under T-duality, with the corresponding terms exchange under T-duality:

$$\begin{aligned} &f_{\mu\nu}{}^a \leftrightarrow \psi_{\mu\nu a}, \quad q^{ab}{}_\mu \leftrightarrow \psi_{\mu ab}, \quad \varphi^{abc} \leftrightarrow \psi_{abc}, \\ &f_{\mu a}{}^b \leftrightarrow f_{\mu b}{}^a \quad f_{bc}{}^a \leftrightarrow q^{bc}{}_a. \end{aligned} \tag{5.4.41}$$

If we make the following identification

$$\begin{aligned} &H_{(3)} \equiv \psi_{\mu\nu\gamma}\xi^\mu\xi^\nu\xi^\gamma, \quad H_{(2)}^a \equiv \psi_{a\mu\nu}\xi^a\xi^\mu\xi^\nu, \quad H_{(1)}^{ab} \equiv \psi_{ab\mu}\xi^a\xi^b\xi^\mu, \\ &H_{(0)}^{abc} \equiv \psi_{abc}\xi^a\xi^b\xi^c, \quad F_{(2)a} \equiv f_{\mu\nu}{}^a\xi^\mu\xi^\nu\xi_a^*, \\ &F_{(1)ab} \equiv (\partial_\mu\Pi^{ab}(x))\xi^\mu\xi_a^*\xi_b^* = d\Pi, \quad F_{(0)abc} \equiv \varphi^{abc}\xi_a^*\xi_b^*\xi_c^* = (\wedge^3\pi^\sharp)H, \end{aligned} \tag{5.4.42}$$

it it obvious that T-duality exchanges $(H_{(2)}, H_{(1)}, H_{(0)})$ with $(F_{(2)}, F_{(1)}, F_{(0)})$ as one expects.

However, we end up with additional $f_{\mu a}{}^b$, $f_{cd}{}^a$ and the $q^{bc}{}_a$ components coming from (5.4.36), as a result of this twisting. These extra terms are either dual to each other or self-dual under T-duality.

5.5 T-duality and the Generalized metric

As pointed out by Cavalcanti [17], the generalized metric $G : TE \oplus T^*E \to TE \oplus T^*E$ defined in Section 4.7 is another geometric structure that can be transported via T-duality.

A generalized metric G on $TE \oplus T^*E$ is introduced by Hull in terms of a symmetric matrix g (the metric) and an antisymmetric matrix b (the B-field) on E [40]

$$G = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix}. \tag{5.5.1}$$

The dual generalized metric $\hat{G}$ on $T\hat{E} \oplus T^*\hat{E}$ can be obtained by

$$\hat{G} = \varphi G \varphi^{-1}. \tag{5.5.2}$$

Recall from Section 4.7 that G is a self-adjoint, orthogonal metric which splits $TE \oplus T^*E$ into $\pm$-eigenspaces $C_\pm \in TE \oplus T^*E$. $C_\pm$ are given as the graph of

$g + b$ and $g - b$, respectively. I.e.

$$\begin{aligned} C_+ &= \text{Span}\{x + (g+b)(x) | x \in TE\}, \\ C_- &= \text{Span}\{x + (g-b)(x) | x \in TE\}. \end{aligned} \tag{5.5.3}$$

One can transfer $C_\pm$ to its dual counterpart $\hat{C}_\pm$ using the T-duality map φ, i.e.

$$\begin{aligned} \varphi(C_\pm) &= \varphi(\text{Span}\{X + (g+b)(X)\}) \\ &= \text{Span}\{\hat{x} + (\hat{g} \pm \hat{b})(\hat{x}) | \hat{x} \in T\hat{E}\} \\ &= \hat{C}_\pm. \end{aligned} \tag{5.5.4}$$

Since the dual +-eigenspace $\hat{C}_+$ is given by the graph of $\hat{g} + \hat{b}$ on the dual space $\hat{E}$, the corresponding $\hat{g}$ and $\hat{b}$ can be determined from $\hat{C}_+$.

We will start with the simplest case – when E is a principal circle bundle – and show that the corresponding $\hat{g}$ and $\hat{b}$ on $\hat{E}$ are related to the original one by the Buscher rules, followed by a generalization to the case of a general principal torus bundle.

5.5.1 Principal circle bundle case

E is a principal circle bundle over M, with local coordinates $\{x^i\} = \{x^\mu, \theta\}$, where $\{x_\mu\}$ are the coordinates on M and θ is the coordinate on the circle.

For simplicity, let us denote $E = g + b$. In this case E is given by

$$E = E_{00} d\theta \otimes d\theta + E_{\mu 0} dx^\mu \otimes d\theta + E_{0\nu} d\theta \otimes dx^\nu + E_{\mu\nu} dx^\mu \otimes dx^\nu. \tag{5.5.5}$$

To determine the corresponding $\hat{E}$, recall that $\hat{C}_+ = \varphi(C_+)$:

$$\begin{aligned} \varphi(C_+) &= \varphi(Span\{\partial_\theta + \partial_{x^\mu} + (E_{00} + E_{\mu 0}) d\theta + (E_{0\nu} + E_{\mu\nu}) dx^\nu\}) \\ &= \text{Span}\{(E_{00} + E_{\mu 0})\partial_{\hat{\theta}} + \partial_{x^\mu} + d\hat{\theta} + E_{\mu\nu} dx^\nu\} \\ &= \text{Span}\left\{ \begin{pmatrix} E_{00} & 0 \\ E_{\mu 0} & \mathbb{I} \end{pmatrix} \begin{pmatrix} \partial_{\hat{\theta}} \\ \partial_{x^\mu} \end{pmatrix} ; \begin{pmatrix} 1 & E_{0\nu} \\ 0 & E_{\mu\nu} \end{pmatrix} \begin{pmatrix} d\hat{\theta} \\ dx^\nu \end{pmatrix} \right\} \\ &= \text{Span}\left\{ \partial_{\hat{\theta}} + \partial_{x^\mu} + \begin{pmatrix} E_{00} & 0 \\ E_{\mu 0} & \mathbb{I} \end{pmatrix}^{-1} \begin{pmatrix} 1 & E_{0\nu} \\ 0 & E_{\mu\nu} \end{pmatrix} \begin{pmatrix} d\hat{\theta} \\ dx^\nu \end{pmatrix} \right\} \\ &= \text{Span}\{\partial_{\hat{\theta}} + \partial_{x^\mu} + (E_{00}^{-1} - (E_{00}^{-1} E_{\mu 0})) d\hat{\theta} + (E_{00}^{-1} E_{0\nu} + E_{\mu\nu} \\ &\quad - E_{00}^{-1} E_{\mu 0} E_{0\nu}) dx^\nu\} \\ &= \text{Span}\{\partial_{\hat{\theta}} + \partial_{x^\mu} + (\hat{E}_{00} + \hat{E}_{\mu 0}) d\hat{\theta} + (\hat{E}_{0\nu} + \hat{E}_{\mu\nu}) dx^\nu\}. \end{aligned}$$

Therefore we have the following correspondence between $\hat{E}$ and E

$$\hat{E}_{00} = E_{00}^{-1}, \quad \hat{E}_{\mu 0} = -E_{00}^{-1} E_{\mu 0}, \quad \hat{E}_{0\nu} = E_{00}^{-1} E_{0\nu}, \quad \hat{E}_{\mu\nu} = E_{\mu\nu} - E_{00}^{-1} E_{\mu 0} E_{0\nu}.$$

If we insert $E = g + b$ and $\hat{E} = \hat{g} + \hat{b}$ into the previous line, we find

$$\begin{aligned}
&\hat{g}_{00} = g_{00}^{-1}, \quad \hat{g}_{\mu 0} = \frac{g_{\mu 0}}{g_{00}}, \quad \hat{g}_{0\nu} = \frac{g_{0\nu}}{g_{00}}, \\
&\hat{b}_{\mu 0} = \frac{b_{\mu 0}}{g_{00}}, \quad \hat{b}_{0\nu} = -\frac{b_{0\nu}}{g_{00}}, \\
&\hat{g}_{\mu\nu} = g_{\mu\nu} - \frac{1}{g_{00}}(g_{\mu 0} g_{0\nu} + b_{\mu 0} b_{0\nu}), \quad \hat{b}_{\mu\nu} = b_{\mu\nu} - \frac{1}{g_{00}}(b_{\mu 0} g_{0\nu} + g_{\mu 0} b_{0\nu}),
\end{aligned}$$

i.e. we retrieve the Buscher rules given by (3.2.9).

5.5.2 Principal torus bundle case

Now let us consider a general principal $\mathbb{T}^n$-bundle E over M. Let $\{x^\mu\}$ be coordinates on M and $\{\theta^a\}$ be coordinates on the torus.

The string background $E = b + g$ decomposes as

$$E = E_{\mu\nu} dx^\mu \otimes dx^\nu + E_{\mu b} dx^\mu \otimes d\theta^b + E_{a\nu} d\theta^a \otimes dx^\nu + E_{ab} d\theta^a \otimes d\theta^b. \tag{5.5.6}$$

The T-duality map φ transforms C_+ to $\tilde{C}_+$ via

$$\begin{aligned}
\varphi(C_+) &= \varphi(\mathrm{Span}\{\partial_{x^\mu} + \partial_{\theta^a} + (E_{\mu\nu} + E_{a\nu}) dx^\nu + (E_{ab} + E_{\mu b}) d\theta^b\}) \\
&= \mathrm{Span}\{\partial_{x^\mu} + (E_{ab} + E_{\mu b}) \partial_{\hat{\theta}^b} + (E_{\mu\nu} + E_{a\nu}) dx^\nu + d\hat{\theta^a}\} \\
&= Span\left\{ \begin{pmatrix} \mathbb{I} & E_{\mu b} \\ 0 & E_{ab} \end{pmatrix} \begin{pmatrix} \partial_{x^\mu} \\ \partial_{\hat{\theta}^b} \end{pmatrix} + \begin{pmatrix} E_{\mu\nu} & 0 \\ E_{a\nu} & \mathbb{I} \end{pmatrix} \begin{pmatrix} dx^\nu \\ d\hat{\theta}^b \end{pmatrix} \right\} \\
&= \mathrm{Span}\left\{ \partial_{x^\mu} + \partial_{\hat{\theta}^a} + \begin{pmatrix} \mathbb{I} & E_{\mu b} \\ 0 & E_{ab} \end{pmatrix}^{-1} \begin{pmatrix} E_{\mu\nu} & 0 \\ E_{a\nu} & \mathbb{I} \end{pmatrix} \begin{pmatrix} dx^\nu \\ d\hat{\theta}^b \end{pmatrix} \right\} \\
&= \mathrm{Span}\left\{ \partial_{x^\mu} + \partial_{\hat{\theta}^a} + \begin{pmatrix} E_{\mu\nu} - E_{\mu a}(E^{-1})^{ab} E_{b\nu} & -E_{\mu a}(E^{-1})^a{}_b \\ (E^{-1})_a{}^b E_{b\nu} & (E^{-1})_{ab} \end{pmatrix} \begin{pmatrix} dx^\nu \\ d\hat{\theta}^b \end{pmatrix} \right\} \\
&= \mathrm{Span}\{\partial_{x^\mu} + \partial_{\hat{\theta}^a} + (\hat{E}_{\mu\nu} + \hat{E}_{a\nu}) dx^\nu + (\hat{E}_{ab} + \hat{E}_{\mu b}) d\hat{\theta}^b\}.
\end{aligned}$$

Thus the dual string background is given by

$$\hat{E} = \begin{pmatrix} E_{\mu\nu} - E_{\mu a}(E^{-1})^{ab} E_{b\nu} & -E_{\mu a}(E^{-1})^a{}_b \\ (E^{-1})_a{}^b E_{b\nu} & (E^{-1})_{ab} \end{pmatrix}, \tag{5.5.7}$$

which agrees with the Buscher rules (3.2.8).

Chapter 6

Poisson-Lie T-duality and generalized geometry

6.1 Introduction and outline

The Poisson-Lie T-duality is a generalization of (Abelian) T-duality proposed by Klimčík and Ševera [49] in 1995. This construction of T-duality does not need the requirement of an isometry, instead, the backgrounds of a dual pair of σ-models are required to obey the Poisson-Lie condition, which is the necessary condition for the existence of the dual worldsheet. The dual pair of σ-models with targets G and $\tilde{G}$ are defined on a Drinfel'd double $\mathcal{D}$ as a generalized space in the following sense.

The Lie group D with Lie algebra $\mathcal{D}$ has subgroups G and $\tilde{G}$, which is a pair of Poisson-Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively. The algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ form the maximally isotropic subspaces of $\mathcal{D}$ with respect to the ad-invariant non-degenerate bilinear form on $\mathcal{D}$. Starting with the tangent space $T_eD \cong \mathcal{D} = \mathfrak{g} \oplus \mathfrak{g}^*$ at the unit element of $\mathcal{D}$, we take a $dim(G)$-dimensional subspace R_+ of $\mathcal{D}$ which is the graph of a non-degenerate linear mapping $E(e) : \mathfrak{g} \to \mathfrak{g}^*$. The subspace R_+ can be transferred to every point $g \in G$ and the resulting subspace R_+^g is the graph of the string background $E(g)$ of the σ-model on G. Similarly transferring R_+ to $\tilde{g} \in \tilde{G}$ gives rise to the dual string background $\hat{E}(\tilde{g})$ of the dual σ-model on $\tilde{G}$. Thus we transfer the subspace R_+ of $\mathcal{D} = \mathfrak{g} \oplus \mathfrak{g}^*$ onto G and $\tilde{G}$.

The above can be extended to Courant algebroids. Let E and $\hat{E}$ be principal G and $\tilde{G}$-bundles over M. Similar to the Abelian T-duality case, the spaces of invariant sections of $TE \oplus T^*E$ and $T\hat{E} \oplus T^*\hat{E}$ can be viewed as isomorphic Courant algebroids related by Poisson-Lie T-duality.

This chapter is organized as follows.

Section 6.2 introduces the Semenov-Tian-Shansky Poisson structure on a Drinfel'd double D and is used to define a Poisson structure Π on G. In this section, we showed that the structure Π which appeared previously in Proposition 3.14 is indeed a Poisson structure on G.

In Section 6.3 we revisit the Poisson-Lie T-duality first introduced in Chapter 3. In particular we explicitly show that the dual pair of string backgrounds expressed in terms of the Poisson structures on G and $\tilde{G}$ satisfy the Poisson-Lie condition.

Recall that our general argument of Chapter 5 is that the Abelian T-duality can be viewed as a duality between isomorphic Courant algebroids. This motivates us to consider Poisson-Lie T-duality on a Drinfel'd double as a generalized space, which is discussed in Section 6.4. In Section 6.4.3, we generalize the (Abelian) T-duality case and establish an isomorphism of Courant algebroids related by Poisson-Lie T-duality.

This chapter is a collaborative work with Bouwknegt.

6.2 The Poisson structure on G

In this section we introduce the Semenov-Tian-Shansky Poisson structure [70] on a Drinfel'd double D. When restricted on G, this Poisson structure turns out to be the structure Π appeared previously in Proposition 3.14.

Let D be a Drinfel'd double containing both groups G and $\tilde{G}$ with Lie algebra $\mathcal{D}$. $\mathcal{D}$ can be decomposed as $\mathfrak{g} \oplus \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}^*$ is the Lie algebra of $\tilde{G}$.

Let us choose a basis $\{T_a, \tilde{T}^a\}$ of $\mathcal{D}$ such that $\{T_a\}$ is a basis of $\mathfrak{g}$ while $\{\tilde{T}^a\}$ is a basis of $\mathfrak{g}^*$. T_a and $\tilde{T}^a$ are orthogonal with respect to the non-degenerate bilinear form on $\mathcal{D}$, i.e.

$$\langle T_a, \tilde{T}^b \rangle = \delta_a^b. \tag{6.2.1}$$

According to Lemma 3.10, there is an adjoint representation of the group G on $\mathcal{D}$.

Definition 6.1. The adjoint representations of G on $\mathcal{D}$ can be defined in terms of the matrices a, c and d as the coefficients in the expansion

$$g^{-1} T_a g \equiv a(g)_a{}^b T_b, \quad g^{-1} \tilde{T}^a g \equiv c(g)^{ab} T_b + d(g)^a{}_b \tilde{T}^b, \tag{6.2.2}$$

where $g \in G$.

There exists a natural Poisson structure introduced by Semenov-Tian-Shansky [70] on D, however, before introducing such a Poisson structure the following must be defined.

Left and right gradients

For any group G there are left and right gradients ∇_L and ∇_R taking values in $\mathfrak{g}^*$:

$$\begin{aligned} \langle \nabla_L f(g), \xi \rangle &= \frac{d}{dt} f(e^{t\xi} g)|_{t=0}, \\ \langle \nabla_R f(g), \xi \rangle &= \frac{d}{dt} f(g e^{t\xi})|_{t=0}, \end{aligned} \tag{6.2.3}$$

where $\xi \in \mathfrak{g}$ and f is a function of $g \in G$.

In terms of $\{T_a\}$ and $\{\tilde{T}^a\}$, we have

$$df = (\nabla_L f)_a w_L^a = (\nabla_R f)_a w_R^a, \tag{6.2.4}$$

where w_L^a and w_R^a are the (Maurer-Cartan) one-form given by

$$g^{-1} dg = w_L^a T_a, \quad dg\ g^{-1} = w_R^a T_a. \tag{6.2.5}$$

and

$$\begin{aligned} (\nabla_L f)_a(g) &= \langle df, v_a^L \rangle(g) = \frac{d}{dt} f(e^{tT_a} g)|_{t=0}, \\ (\nabla_R f)_a(g) &= \langle df, v_a^R \rangle(g) = \frac{d}{dt} f(g e^{tT_a})|_{t=0}, \end{aligned} \tag{6.2.6}$$

where v_a^L and v_a^R are the left and right-invariant vector fields corresponding to T_a, respectively.

Apply the previous construction on D, it gives

$$\begin{aligned} (\nabla_L f)^a(g) &= \frac{d}{dt} f(e^{t\tilde{T}^a} g)|_{t=0}, \\ (\nabla_R f)^a(g) &= \frac{d}{dt} f(g e^{t\tilde{T}^a})|_{t=0}, \end{aligned} \tag{6.2.7}$$

where $g \in D$.

Next, we introduce the Semenov-Tian-Shansky Poisson structure on D, and in particular we show that $\Pi = c(g)a(g)^{-1}$ is a Poisson structure on G.

Poisson structure on G

Definition 6.2. Let $\mathfrak{g}$ a Lie algebra, $r \in \mathrm{End}\ \mathfrak{g}$ a linear operator and $\langle\ ,\ \rangle$ a nondegenerate invariant scalar product on $\mathfrak{g}$, then $(\mathfrak{g}, r)$ is called a **Boxter Lie algebra** if r is skew-symmetric and satisfies the Yang-Baxter equation

$$[rX, rY] = r([rX, Y] + [X, rY]) - [X, Y], \quad \forall X, Y \in \mathfrak{g}. \tag{6.2.8}$$

Such an operator r is called a **classical r-matrix**.

Proposition 6.3. *The Yang-Baxter equation* (6.2.8) *implies*

$$[X, Y]_r = \frac{1}{2}([rX, Y] + [X, rY]) \tag{6.2.9}$$

is a Lie bracket.

Consider a Drinfel'd double D with Lie algebra $\mathcal{D} = \mathfrak{g} \oplus \mathfrak{g}^*$. The left and right gradients give rise to $\nabla_L f$ and $\nabla_R f \in C^\infty(D, \mathcal{D})$. Let $P_{\mathfrak{g}}$ and $P_{\mathfrak{g}^*}$ be projection operators onto $\mathfrak{g}$ and $\mathfrak{g}^*$ parallel to $\mathfrak{g}^*$ and $\mathfrak{g}$, respectively.

Proposition 6.4 ([70])**.**

$$P_{\mathcal{D}} = P_{\mathfrak{g}} - P_{\mathfrak{g}^*} \in \mathrm{End}\ \mathcal{D} \tag{6.2.10}$$

is skewsymmetric with respect to the natural pairing on $\mathcal{D}$ and satisfies the Yang-Baxter equation (6.2.8)*.*

The right and left gradients defined in the previous section generalize to $\nabla_L f, \nabla_R f \in C^\infty(D, \mathcal{D})$ via (6.2.3).

Proposition 6.5 ([70])**.** *The Semenov-Tian-Shansky Poisson structure on $C^\infty(D)$ is given by*

$$\{f, f'\}_D = -\frac{1}{2}(\langle P_{\mathcal{D}} \nabla_L f, \nabla_L f' \rangle - \langle P_{\mathcal{D}} \nabla_R f, \nabla_R f' \rangle). \tag{6.2.11}$$

In terms of a basis $\{T_a\}$ of $\mathfrak{g}$ and a dual basis $\{\tilde{T}^a\}$ of $\mathfrak{g}^*$, (6.2.11) *can be expressed as*

$$\begin{aligned} \{f, f'\}_D &= \frac{1}{2}((\nabla_L f)_a (\nabla_L f')^a - (\nabla_L f)^a (\nabla_L f')_a) \\ &- \frac{1}{2}((\nabla_R f)_a (\nabla_R f')^a - (\nabla_R f)^a (\nabla_R f')_a). \end{aligned} \tag{6.2.12}$$

The manifold D equipped with the above Poisson bracket is called a **Heisenberg double**.

Proposition 6.6. *Consider functions f, f' on D, which are invariant under the right action of $\tilde{G}$. Then these functions can be interpreted as functions on G. The Poisson bracket in equation* (6.2.12) *on such functions defines a Poisson bracket on G and can be written as*

$$\{f, f'\}_G = \Pi^{ab}(g)(\nabla_L f)_a(\nabla_L f')_b, \tag{6.2.13}$$

such that $\Pi(g)$ is given by

$$\Pi(g) = c(g)a(g)^{-1}, \tag{6.2.14}$$

where $a(g)$ and $c(g)$ are given by (6.2.2).

Proof. Firstly, for functions on D, invariant under the right action of $\tilde{G}$, we have the following relations

$$\begin{aligned}
(\nabla_L f)^a(g) &= \frac{d}{dt} f(e^{t\tilde{T}^a} g)|_{t=0} = \frac{d}{dt} f(g g^{-1} e^{t\tilde{T}^a} g)|_{t=0} = c(g)^{ab}(\nabla_R f)_b(g), \\
(\nabla_R f)_a(g) &= \frac{d}{dt} f(g e^{tT_a})|_{t=0} = \frac{d}{dt} f(g e^{tT_a} g^{-1} g)|_{t=0} = a(g^{-1})_a{}^b(\nabla_L f)_b(g), \\
(\nabla_R f)^a(g) &= 0.
\end{aligned} \tag{6.2.15}$$

Therefore the Poisson bracket (6.2.12) restricted on G becomes

$$\{f, f'\}_G = c(g)^{ac}(a(g)^{-1})_c{}^b(\nabla_L f)_a(\nabla_L f')_b = \Pi^{ab}(\nabla_L f)_a(\nabla_L f')_b. \tag{6.2.16}$$

□

Before showing that $\Pi = c(g)a(g)^{-1}$ defines a Poisson structure on G, we need the following results.

Lemma 6.7. *The left gradient on Π is given by*

$$(\nabla_L \Pi^{ab})_c = f_{cd}{}^a \Pi^{db} + f_{cd}{}^b \Pi^{ad} - \tilde{f}^{ab}{}_c. \tag{6.2.17}$$

Proof. Starting with $(a(g)^{-1})_a{}^b = \langle g T_a g^{-1}, \tilde{T}^b \rangle = \langle T_a, g^{-1} \tilde{T}^b g \rangle$ and $c(g)^{ab} = \langle g^{-1} \tilde{T}^a g, \tilde{T}^b \rangle$, $(\nabla_L (a^{-1})_a{}^b)_c$ and $(\nabla_L b^{ab})_c$ are found to be

$$\begin{aligned}
(\nabla_L (a(g)^{-1})_a{}^b)_c &= \frac{d}{dt} \langle T_a, g^{-1} e^{-T_c t} \tilde{T}^b e^{tT_c} g \rangle|_{t=0} = \langle T_a, g^{-1} [\tilde{T}^b, T_c]\; g \rangle \\
&= \langle T_a, g^{-1}\; (f_{cd}{}^b \tilde{T}^d + \tilde{f}^{db}{}_c T_d)\; g \rangle = f_{cd}{}^b d(g)_a{}^d = f^b_{cd}(a(g)^{-1})_a{}^d, \\
(\nabla_L c(g)^{ab})_c &= \frac{d}{dt} \langle g^{-1} e^{-T_c t} \tilde{T}^a e^{tT_c} g, \tilde{T}^b \rangle|_{t=0} = \langle g^{-1}\; [\tilde{T}^a, T_c]\; g, \tilde{T}^b \rangle \\
&= \langle g^{-1}(f_{cd}{}^a \tilde{T}^d - \tilde{f}^{ad}{}_c T_d) g, \tilde{T}^b \rangle = f_{cd}{}^a c(g)^{db} - \tilde{f}^{ad}{}_c a(g)_d{}^b.
\end{aligned}$$

Therefore

$$\begin{aligned}(\nabla_L\Pi^{ab})_c &= (\nabla_L c^{ad}(g))_c (a(g)^{-1})_d{}^b + c^{ad}(g)(\nabla_L (a(g)^{-1})_d{}^b)_c \\ &= f_{cd}{}^a c(g)^{de}(a(g)^{-1})_e{}^b - \tilde{f}^{ab}{}_c a(g)_b{}^d (a(g)^{-1})_d{}^b + f_{cd}{}^b c(g)^{ae}(a(g)^{-1})_e{}^d \\ &= f_{cd}{}^a \Pi^{db} - \tilde{f}^{ab}{}_c + f_{cd}{}^b \Pi^{ad}. \qquad (6.2.18)\end{aligned}$$

□

Lemma 6.8. *The left gradients* (∇_L) *do not commute but satisfy*

$$(\nabla_L(\nabla_L f)_a)_b - (\nabla_L(\nabla_L f)_b)_a = -f_{ab}{}^c (\nabla_L f)_c. \qquad (6.2.19)$$

Proof. From definition, $(\nabla_L(\nabla_L f)_a)_b$ is given by

$$\frac{d}{ds}\frac{d}{dt} f(e^{sT_b}e^{tT_a}g)|_{t=0,s=0}. \qquad (6.2.20)$$

Thus using Taylor expansion around $e \in G$, we have

$$\begin{aligned}(\nabla_L(\nabla_L f)_a)_b - (\nabla_L(\nabla_L f)_b)_a &= \frac{d}{ds}\frac{d}{dt}(f(e^{sT_b}e^{tT_a}g) - f(e^{sT_a}e^{tT_b}g))|_{t=0,s=0} \\ &= -\sum_{i=1}^{\infty} f^{(i)}(e)[T_a, T_b]g^i \\ &= -f_{ab}{}^c \sum_{i=0}^{\infty} \frac{d}{dt}(f(e) + \frac{1}{i!} f^{(i)}(e)(e^{T_c t}g)^i)|_{t=0} \\ &= -f_{ab}{}^c (\nabla_L f)_c. \qquad (6.2.21)\end{aligned}$$

□

Proposition 6.9. $\Pi = c(g)a(g)^{-1}$ *defines a Poisson structure on* G.

Proof. Let us first simplify our notation and denote $(\nabla_L f)_a \equiv \nabla_a f$.

To check that $\Pi(g) = c(g)a(g)^{-1}$ is a Poisson structure, recall in Section 3.3.2 that $a(g)$, $c(g)$ and $d(g)$ are constrained by

$$a(g)^T = d(g)^{-1}, \quad c(g)d(g)^T = -d(g)c(g)^T, \qquad (6.2.22)$$

which directly guarantee that $\Pi^{ab} = -\Pi^{ba}$. It is also obvious that Π satisfies the Leibnitz rule. Thus we only need to check if Π satisfies the Jacobi-identity.

Using Lemma 6.7 and Lemma 6.8, one finds

$$\begin{aligned}
&\{\{g,h\},f\} - \{\{f,h\},g\} + \{\{f,g\},h\} \\
&= \{\Pi^{bc}\nabla_b g\nabla_c h, f\} - \{\Pi^{ac}\nabla_a f\nabla_c h, g\} + \{\Pi^{ab}\nabla_b f\nabla_b g, g\} \\
&= (\Pi^{da}\nabla_d\Pi^{bc} - \Pi^{db}\nabla_d\Pi^{ac} + \Pi^{dc}\nabla_d)\nabla_a f\nabla_b g\nabla_c h \\
&\quad +\Pi^{dc}\Pi^{ab}(\nabla_d\nabla_a f)\nabla_b g\nabla_c h + \Pi^{dc}\Pi^{ab}(\nabla_b\nabla_d g)\nabla_a f\nabla_c h \\
&\quad -\Pi^{ad}\Pi^{bc}(\nabla_d\nabla_b g)\nabla_a f\nabla_c h - \Pi^{ad}\Pi^{bc}(\nabla_d\nabla_c h)\nabla_a f\nabla_b g \\
&\quad +\Pi^{bd}\Pi^{ab}(\nabla_d\nabla_b f)\nabla_b g\nabla_c h + \Pi^{bd}\Pi^{ac}(\nabla_d\nabla_c h)\nabla_a f\nabla_b g \\
&= (\Pi^{da}\nabla_d\Pi^{bc} - \Pi^{db}\nabla_d\Pi^{ac} + \Pi^{dc}\nabla_d)\nabla_a f\nabla_b g\nabla_c h \\
&\quad +\Pi^{ab}\Pi^{dc}f_{da}{}^{a} + \Pi^{ad}\Pi^{cb}f_{dc}{}^{c} + \Pi^{ab}\Pi^{cd}f_{bd}{}^{b})\nabla_a f\nabla_b g\nabla_c h \\
&= (\Pi^{da}\nabla_d\Pi^{bc} - \Pi^{db}\nabla_d\Pi^{ac} + \Pi^{dc}\nabla_d \quad + f_{ad}{}^{b}\Pi^{dc}\Pi^{ab} + f_{bd}{}^{b}\Pi^{dc}\Pi^{ab} \\
&\quad -f_{cd}{}^{c}\Pi^{ad}\Pi^{bc})\nabla_a f\nabla_b g\nabla_c h \\
&= (f_{bd}{}^{b}\Pi^{ad}\Pi^{bc} + f_{ad}{}^{a}\Pi^{ba}\Pi^{dc} + f_{dc}{}^{c}\Pi^{bd}\Pi^{ac} - \tilde{f}^{ab}{}_{d}\Pi^{dc} + \tilde{f}^{bc}{}_{d}\Pi^{ad} \\
&\quad -\tilde{f}^{ac}{}_{d}\Pi^{bd})\nabla_a f\nabla_b g\nabla_c h.
\end{aligned} \tag{6.2.23}$$

Next, consider the adjoint action of g on $[T_a, T_b]$

$$\begin{aligned}
&g^{-1}[T_a,T_b]g = g^{-1}(f_{ab}{}^{c}T_c)g = f_{ab}{}^{c}a_c{}^{e}T_e \\
&= [g^{-1}T_a g, g^{-1}T_b g] = [a_a{}^{c}T_c, a_b{}^{d}T_d] = a_a{}^{c}a_b{}^{d}f_{cd}{}^{e}T_e,
\end{aligned}$$

thus we have the following constraint

$$f_{ab}{}^{c}a_c{}^{e} = f_{cd}{}^{e}a_a{}^{c}a_b{}^{d}. \tag{6.2.24}$$

Similarly for $[\tilde{T}^a, \tilde{T}^b]$ we have

$$\begin{aligned}
&g^{-1}[\tilde{T}^a,\tilde{T}^b]g = g^{-1}(\tilde{f}^{ab}{}_{c}\tilde{T}^c)g = \tilde{f}^{ab}{}_{c}(c^{ce}T_e + d^{c}{}_{e}\tilde{T}^e) \\
&= [c^{ac}T_c + d^{a}{}_{c}\tilde{T}^c, c^{bd}T_d + d^{b}{}_{d}\tilde{T}^d] \\
&= (c^{ac}c^{bd}f_{cd}{}^{e} + c^{ac}d^{b}{}_{d}\tilde{f}^{de}{}_{c} - d^{a}{}_{c}c^{bd}\tilde{f}^{ce}{}_{d})T_e + (f_{ec}{}^{d}c^{ac}d^{b}{}_{d} - f_{ed}{}^{c}d^{a}{}_{c}c^{bd} \\
&\quad +d^{a}{}_{c}d^{b}{}_{d}\tilde{f}^{cd}{}_{e})\tilde{T}^e,
\end{aligned}$$

as a result, we obtain the following constraints

$$\tilde{f}^{ab}{}_{c}c^{ce} = c^{ac}c^{bd}f_{cd}{}^{e} + c^{ac}d^{b}{}_{d}\tilde{f}^{de}{}_{c} - d^{a}{}_{c}c^{bd}\tilde{f}^{ce}{}_{d} \tag{6.2.25}$$

$$\tilde{f}^{ab}{}_{c}d^{c}{}_{e} = f_{ec}{}^{d}c^{ac}d^{b}{}_{d} - f_{ed}{}^{c}d^{a}{}_{c}c^{bd} + d^{a}{}_{c}d^{b}{}_{d}\tilde{f}^{cd}{}_{e}. \tag{6.2.26}$$

Computing $\tilde{f}^{ab}{}_{d}\Pi^{dc}$ we find

$$\begin{aligned}
\tilde{f}^{ab}{}_{d}\Pi^{dc} &= \tilde{f}^{ab}{}_{d}c^{de}(a^{-1})_e{}^{c} \\
(6.2.25) \Rightarrow \quad &= (c^{ae}c^{bd}f_{ed}{}^{f} + c^{ae}d_d{}^{b}\tilde{f}^{df}{}_{e} - d_e{}^{a}c^{bd}\tilde{f}^{ef}{}_{d})(a^{-1})^{c}{}_{f} \\
(6.2.24) \Rightarrow \quad &= f_{dc}{}^{c}\Pi^{bd}\Pi^{ac} + \tilde{f}^{df}{}_{d}\Pi^{ac}d^{b}{}_{f} - \tilde{f}^{df}{}_{f}\Pi^{bc}d^{a}{}_{d}.
\end{aligned} \tag{6.2.27}$$

Permuting a, b, c in the above equation (6.2.27) and add up $\tilde{f}\Pi$'s, we find

$$\begin{aligned}
&\tilde{f}^{ab}{}_{d}\Pi^{dc} + \tilde{f}^{bc}{}_{d}\Pi^{da} + \tilde{f}^{ca}{}_{d}\Pi^{db} = f_{cd}{}^{c}\Pi^{bd} + \tilde{f}^{ed}{}_{e}d^{b}{}_{d} - \tilde{f}^{de}{}_{e}\Pi^{bc}d^{a}{}_{d} \\
&+f_{ad}{}^{a}\Pi^{cd}\Pi^{ba} + \tilde{f}^{ed}{}_{e}d^{c}{}_{d} - \tilde{f}^{de}{}_{e}\Pi^{ca}d^{b}{}_{d} + f_{bd}{}^{b}\Pi^{ad}\Pi^{cb} + \tilde{f}^{ed}{}_{e}\Pi^{cb}d^{a}{}_{d} - \tilde{f}^{de}{}_{e}d^{c}{}_{d} \\
&= f_{dc}{}^{c}\Pi^{bd}\Pi^{ac} + f_{da}{}^{a}\Pi^{cd}\Pi^{ba} + f_{db}{}^{b}\Pi^{ad}\Pi^{cb}.
\end{aligned} \tag{6.2.28}$$

Substituting (6.2.28) into (6.2.23) we obtain the Jacobi-identity. And therefore $\Pi = c(g)a(g)^{-1}$ defines a Poisson structure on G. □

Example 6.10 (Borelian double)**.** Let us consider the simplest non-Abelian double appeared previously in Section 3.3.3, the Borelian double $D = GL(2,\mathbb{R})$ consists of the Borel group G and the dual group $\tilde{G}$ such that their Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ have basis $\{T_a\}$ and $\{\tilde{T}^a\}$ given by

$$\begin{aligned}
T_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
\tilde{T}^1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{T}^2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.
\end{aligned} \tag{6.2.29}$$

An element $g \in G$ can be chosen by

$$g = \begin{pmatrix} e^{\varphi} & \theta \\ 0 & 1 \end{pmatrix}, \tag{6.2.30}$$

and it follows that the matrices $a(g)$, $c(g)$ and $d(g)$ from (6.2.2) are found to be

$$a(g) = \begin{pmatrix} 1 & e^{-\varphi}\theta \\ 0 & e^{-\varphi} \end{pmatrix} \quad c(g) = \begin{pmatrix} 0 & -e^{-\varphi}\theta \\ \theta & e^{-\varphi}\theta^2 \end{pmatrix}, \quad d(g) = \begin{pmatrix} 1 & 0 \\ -\theta & e^{\varphi} \end{pmatrix}. \tag{6.2.31}$$

Thus the Poisson structure $\Pi^{ab} = c^{ad}(g)(a(g)^{-1})^{b}{}_{d}$ on G is

$$\Pi = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}. \tag{6.2.32}$$

Now, let us compute $(\nabla_L \Pi^{ab})_c$. Since $(\nabla_L \Pi^{ab})_c$ is given by

$$(\nabla_L \Pi^{ab})_c = f_{cd}{}^{a}\Pi^{db} + f_{cd}{}^{b}\Pi^{ad} - \tilde{f}^{ab}{}_{c}, \tag{6.2.33}$$

the components of $(\nabla_L \Pi^{ab})_c$ are found to be

$$(\nabla_L \Pi^{12})_1 = -(\nabla_L \Pi^{21})_1 = -\theta, \quad (\nabla_L \Pi^{12})_2 = -(\nabla_L \Pi^{21})_2 = -1. \tag{6.2.34}$$

One can then easily check that the Poisson structure Π given by (6.2.32) satisfies the Jacobi-identity, therefore Π is a Poisson structure on the Borel group G. Similarly, if we choose an element $\tilde{g} \in \tilde{G}$ parameterized by

$$\tilde{g} = \begin{pmatrix} 1 & 0 \\ -\tilde{\theta} & e^{\tilde{\varphi}} \end{pmatrix}, \tag{6.2.35}$$

we obtain the Poisson structure $\tilde{\Pi}$ on the dual group $\tilde{G}$ as

$$\tilde{\Pi} = \tilde{c}(\tilde{g})(\tilde{a}(\tilde{g})^{-1}) = \begin{pmatrix} 0 & -\tilde{\theta} \\ \tilde{\theta} & 0 \end{pmatrix}. \tag{6.2.36}$$

6.3 Poisson-Lie T-duality revisited

In this section, we give a short revision of the Poisson-Lie T-duality introduced previously in Chapter 3. In particular, we focus on the solutions of the dual pair of σ-models defined in terms of the Poisson structures on the dual groups G and $\tilde{G}$.

Recall in Section 3.3.2, for a σ-model (3.2.1) to possess a Poisson-Lie dual σ-model, the string background E_{ij} is required to satisfy the Poisson-Lie condition:

$$\boxed{\mathcal{L}_{v_a}(E_{ij}) = \tilde{f}^{bc}{}_a v_b^m v_c^n E_{mj} E_{in},} \tag{6.3.1}$$

where $\tilde{f}^{bc}{}_a$ is the structure constants of $\tilde{G}$, and $v_a^L = v_a^i \partial_i$ are the left invariant vector fields on G.

A dual pair of σ-models with the targets being Lie groups G and $\tilde{G}$ can be constructed with the Lagrangian

$$\begin{aligned} L &= E(g)_{ab}(\partial g g^{-1})^a(\bar{\partial} g g^{-1})^b, \\ \hat{L} &= \hat{E}(\tilde{g})^{ab}(\partial \tilde{g}\tilde{g}^{-1})_a(\bar{\partial}\tilde{g}\tilde{g}^{-1})_b, \end{aligned} \tag{6.3.2}$$

where $g \in G$ and $\tilde{g} \in \tilde{G}$.

The string background E_{ij} is related to $E_{ab}(g)$ via

$$E_{ij} dx^i \bar{d}x^j = E_{ab} w_L^a w_L^b = E_{ab} w_i^a dx^i w_j^b \bar{d}x^j, \tag{6.3.3}$$

where $w_L^a = w_i^a dx^i$ are the (Maurer-Cartan) one forms on G.

Next, we show explicitly that the String background E defined in terms of the Poisson structure Π on G satisfies the Poisson-Lie condition.

Proposition 6.11. *A pair of σ-models with targets being G and $\tilde{G}$ have string backgrounds $E(g)$ and $\hat{E}(\tilde{g})$ given explicitly by*

$$\begin{aligned} E(g) &= (a(g) + E(e)c(g))^{-1}E(e)d(g), \\ \hat{E}(\tilde{g}) &= \tilde{d}(\tilde{g})\hat{E}(\tilde{e})(\tilde{a}(\tilde{g}) + \tilde{c}(\tilde{g})\hat{E}(\tilde{e}))^{-1}, \end{aligned} \tag{6.3.4}$$

or as have been shown in Section 3.3.2 that up to a similarity transformation, $E(g)$ can be conveniently written as

$$\begin{aligned} E^{-1}(g)^{ab} &= (E^{-1}(e) + \Pi(g))^{ab} \\ \hat{E}^{-1}(\tilde{g})_{ab} &= (E(e) + \tilde{\Pi}(\tilde{g}))_{ab}. \end{aligned} \tag{6.3.5}$$

Proof. We show that (6.3.5) satisfies the Poisson-Lie condition (6.3.3).

Let us start with $\mathcal{L}_{v_a}E_{bc}$ and use the Poisson-Lie condition (6.3.3), we find

$$\begin{aligned} \mathcal{L}_{v_a}E_{bc} &= \mathcal{L}_{v_a}(v_b^i v_c^j E_{ij}) \\ &= f_{ab}{}^d v_d^i v_c^j E_{ij} + f_{ac} v_b^i v_d^j E_{ij} + \tilde{f}^{de}{}_a v_b^i v_c^j v_d^m v_e^n E_{mj} E_{be} \\ &= f_{ab}{}^d E_{dc} + f_{ac}{}^d E_{bd} + \tilde{f}^{de}{}_a E_{dc} E_{be}. \end{aligned} \tag{6.3.6}$$

Substituting (6.3.6) into $\mathcal{L}_{v_a}(E^{-1})^{bc}$, we have

$$\begin{aligned} \mathcal{L}_{v_a}(E^{-1})^{bc} &= -(E^{-1})^{br}(\mathcal{L}_{v_a}E_{rs})(E^{-1})^{sc} \\ &= -(E^{-1})^{br}(f_{ar}{}^d E_{ds} + f_{as}{}^d E_{rd} + \tilde{f}^{de}{}_a E_{ds} E_{re})(E^{-1})^{sc} \\ &= -(E^{-1})^{br} f_{ar}{}^c - (E^{-1})^{sc} f_{as}{}^b - \tilde{f}^{cb}{}_a. \end{aligned} \tag{6.3.7}$$

We have seen in Lemma 6.7 that

$$(\nabla_L \Pi^{bc})_a = -\mathcal{L}_{v_a}\Pi^{bc} = f_{ad}{}^c \Pi^{dc} + f_{ad}{}^c \Pi^{bd} - \tilde{f}^{bc}{}_a$$

hence Π^{bc} satisfies (6.3.7), while $E^{-1}(e) + \Pi$ satisfies (6.3.7) provided

$$f_{ad}{}^b (E^{-1}(e))^{dc} + f_{ad}{}^c (E^{-1}(e))^{bd} = 0, \tag{6.3.8}$$

i.e. this condition is equivalent to the requirement $\mathcal{L}_{v_a}E(e)_{bc} = 0$. Thus $E(g) = [E^{-1}(e) + \Pi]^{-1}$ satisfies the Poisson-Lie condition requiring that (6.3.8) is satisfied.

□

6.4 Poisson-Lie T-duality and generalized geometry

Recall in Chapter 5 that using the framework of generalized geometry, (Abelian) T-duality between a pair of dual σ-models on E and $\hat{E}$ can be viewed as duality between invariant sections of $TE \oplus T^*E$ and $T\hat{E} \oplus T^*\hat{E}$. This result motivates us to consider Poisson-Lie T-duality as a duality on the Drinfel'd double $\mathcal{D} = \mathfrak{g} \oplus \mathfrak{g}^*$ in a similar way.

We begin in section 6.4.1 defining the natural operations on $\mathfrak{g} \oplus \mathfrak{g}^*$. In Section 6.4.2, we consider the Poisson-Lie T-duality between a pair of σ-models on G and $\tilde{G}$ as a duality on the orthogonal subspaces of the generalized space $\mathcal{D}$. In Section 6.4.3, We generalize the construction for (Abelian) T-duality in Chapter 5 and establish an isomorphism of Courant algebroids related by Poisson-Lie T-duality.

6.4.1 Natural operation on the Drinfel'd double $\mathcal{D}$

The double $\mathcal{D} = \mathfrak{g} \oplus \mathfrak{g}^*$ is equipped with an ad-invariant non-degenerate bilinear form

$$\langle x + \xi, y + \eta \rangle = \langle x, \eta \rangle + \langle y, \xi \rangle, \tag{6.4.1}$$

where $x, y \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$. $\langle \ , \ \rangle$ is the canonical orthogonal pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, i.e. $\langle T_a, \tilde{T}^b \rangle = \delta_a^b$, and otherwise 0.

$\mathcal{D}$ is equipped with the bracket

$$\begin{aligned} [x + \xi, y + \eta]_{\mathcal{D}} &= [x, y] + ad^*_\xi y - ad^*_\eta x \\ &\quad + [\xi, \eta] + ad^*_x \eta - ad^*_y \xi, \end{aligned} \tag{6.4.2}$$

where $x, y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$. ad^*_x is the ad^*-operator for $\mathfrak{g}$ acting on $\mathfrak{g}^*$ and ad^*_ξ corresponds to the coadjoint action of $\mathfrak{g}^*$ on $\mathfrak{g}$, i.e.

$$\langle ad^*_y \xi, x \rangle = \langle \xi, [x, y] \rangle, \quad \langle ad^*_\eta x, \xi \rangle = \langle x, [\xi, \eta] \rangle. \tag{6.4.3}$$

The Lie brackets on $\mathfrak{g}$ and $\mathfrak{g}^*$ are given respectively by

$$\begin{aligned} &[T_a, T_b]_{\mathcal{D}} = f_{ab}{}^c T_c, \\ &[\tilde{T}^a, \tilde{T}^b]_{\mathcal{D}} = \tilde{f}^{ab}{}_c \tilde{T}^c, \end{aligned} \tag{6.4.4}$$

where $f_{ab}{}^c$ and $\tilde{f}^{ab}{}_c$ are structure constants on $\mathfrak{g}$ and $\mathfrak{g}^*$. Then it follows from (6.4.2) that the brackets between $\mathfrak{g}$ and $\mathfrak{g}^*$ are given by

$$\begin{aligned} [T_a, \tilde{T}^b]_{\mathcal{D}} &= -ad^*_{\tilde{T}^b} T_a + ad^*_{T_a} \tilde{T}^b \\ &= \tilde{f}^{bc}{}_a T_c + f_{ca}{}^b \tilde{T}^c, \\ [\tilde{T}^a, T_b]_{\mathcal{D}} &= ad^*_{\tilde{T}^a} T_b - ad^*_{T_b} \tilde{T}^a \\ &= -\tilde{f}^{ac}{}_b T_c + f_{bc}{}^a \tilde{T}^c. \end{aligned} \tag{6.4.5}$$

These brackets are simply the brackets between $\mathfrak{g}$ and $\mathfrak{g}^*$ given previously in Section 2.2.3.

The bracket (6.4.2) on $\mathcal{D}$ has the following properties:

1. It is antisymmetric.

2. The bracket on the Drinfel'd double $\mathcal{D}$ satisfies the Jacobi-identity by placing the following condition on the structure constants:

$$\tilde{f}^{ed}{}_c f_{ab}{}^c = \tilde{f}^{cd}{}_b f_{ac}{}^e + \tilde{f}^{ef}{}_b f_{af}{}^d - \tilde{f}^{cd}{}_a f_{bc}{}^e - \tilde{f}^{ef}{}_a f_{bf}{}^d, \tag{6.4.6}$$

   we recognize this as the integrability condition of the Poisson-Lie condition (6.3.3) given previously in Proposition 3.9.

Here is an example of a Borelian double following Example 6.10.

Example 6.12. The Borelian double $D = GL(2,\mathbb{R})$ with Lie algebra $\mathcal{D} = \mathfrak{g} \oplus \mathfrak{g}^*$ has subgroups G and $\tilde{G}$, such that the Borel group G has Lie algebra $\mathfrak{g}$ and the dual group $\tilde{G}$ has Lie algebra $\mathfrak{g}^*$, respectively.

The Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ have basis $\{T_a\}$ and $\{\tilde{T}^a\}$ given by (6.2.29). There is a non-degenerate pairing on $\mathcal{D}$ satisfying $\langle T_a, \tilde{T}^b \rangle = \delta_a^b$:

$$\langle x, y \rangle = Det(x+y), \quad \forall x, y \in \mathcal{D}. \tag{6.4.7}$$

It follows from (6.2.29) that the Lie brackets on $\mathfrak{g}$ and $\mathfrak{g}^*$ are

$$[T_1, T_2] = T_2, \quad [\tilde{T}^1, \tilde{T}^2] = \tilde{T}^2, \tag{6.4.8}$$

thus according to (6.4.5), the brackets on $\mathcal{D}$ between $\mathfrak{g}$ and $\mathfrak{g}^*$ are found to be

$$[T_1, \tilde{T}^2] = -\tilde{T}^2, \quad [\tilde{T}^1, T_2] = -T_2, \tag{6.4.9}$$

and otherwise 0.

These are simply the commutation relations of the Lie algebra $\mathcal{D} = gl(2,\mathbb{R})$ with a basis $\{T_1, T_2, \tilde{T}^1, \tilde{T}^2\}$.

6.4.2 Orthogonal subspaces of the Drinfel'd double $\mathcal{D}$

Let us consider the tangent space $T_e D \cong \mathcal{D}$ at e the unit element of $\mathcal{D}$. I.e. e is the unit element of both G and $\tilde{G}$ at the same time (i.e. $e = \tilde{e}$).

The generalized space $\mathcal{D} = \mathfrak{g} \oplus \mathfrak{g}^*$ can be decomposed as linear orthogonal subspaces $R_\pm$ as follows:

$$\mathcal{D} = \mathfrak{g} \oplus \mathfrak{g}^* = R_+ \oplus R_-, \quad \langle R_+, R_- \rangle = 0. \tag{6.4.10}$$

Let $E(e)$ be a non-degenerate linear mapping $E(e) : \mathfrak{g} \to \mathfrak{g}^*$. $R_\pm$ can be defined as the graph of $E(e)$ in $\mathcal{D}$, i.e.

$$\begin{aligned} R_+ &= \mathrm{Span}\{t + E(e)(t,\cdot)|t \in \mathfrak{g}\}, \\ R_- &= \mathrm{Span}\{t - E(e)(\cdot,t)|t \in \mathfrak{g}\}, \end{aligned} \tag{6.4.11}$$

or $R_\pm$ can be expressed as

$$\begin{aligned} R_+ &= Span\{T_a + E(e)_{ab}\tilde{T}^b\} \\ R_- &= Span\{T_a - E(e)_{ba}\tilde{T}^b\}. \end{aligned} \tag{6.4.12}$$

Remark 6.13. The subspaces $R_\pm$ are self dual, i.e. $R_\pm = \tilde{R}_\pm$, where

$$\tilde{R}_+ = \mathrm{Span}\{\tilde{T}^a + \hat{E}(\tilde{e})^{ab}T_b\}, \quad \tilde{R}_- = \mathrm{Span}\{\tilde{T}^a - \hat{E}(\tilde{e})^{ba}T_b\}. \tag{6.4.13}$$

$E(e)$ and $\hat{E}(\tilde{e})$ are related by $E(e)\hat{E}(\tilde{e}) = \hat{E}(\tilde{e})E(e) = 1$.

There is a generalized metric G on $T_eD = \mathfrak{g} \oplus \mathfrak{g}^*$, i.e. $G : T_eD \to T_eD$ with the following properties:

1. G is an involution, i.e. $G^2 = 1$,
2. G is required to be a (linear) homomorphism of the Lie algebras, $G[x,y] = [Gx, Gy]$, for all $x, y \in \mathcal{D}$.
3. G is self-adjoint with respect to the non-degenerate bilinear form $\langle\ ,\ \rangle$ on $\mathcal{D}$, i.e. $\langle Gx, y\rangle = \langle x, Gy\rangle$.
4. $G : \mathcal{D} \to \mathcal{D}$ has eigenspaces $R_\pm$,

$$G(R_\pm) = \pm R_\pm. \tag{6.4.14}$$

Remark 6.14. Since G is a homomorphism of Lie algebras, we have the following properties:

$$G[R_+, R_+] = [R_+, R_+], \quad G[R_+, R_-] = -[R_+, R_-], \quad G[R_-, R_-] = [R_-, R_-], \tag{6.4.15}$$

and thus

$$[R_+, R_+] \in R_+, \quad [R_-, R_-] \in R_+, \quad [R_+, R_-] \in R_-. \tag{6.4.16}$$

Since R_+ and R_- are orthogonal subspaces with respect to $\langle\ ,\ \rangle$, we have the following relation:

$$\langle [R_+, R_+], R_-\rangle = \langle [R_+, R_-], R_+\rangle = \langle [R_-, R_-], R_-\rangle = 0. \tag{6.4.17}$$

In terms of the symmetric part $G(e)$ and the anti-symmetric part $B(e)$ of $E(e)$, the generalized metric G can be expressed as a matrix

$$G = \begin{pmatrix} -G(e)^{-1}B(e) & G(e)^{-1} \\ G(e) - B(e)G(e)^{-1}B(e) & B(e)G(e)^{-1} \end{pmatrix}. \tag{6.4.18}$$

Now, consider a pair of σ-models with targets being G and $\tilde{G}$. For each element $g \in G$, T_eD can be transported to T_gD via the action of g on $\mathcal{D}$ (i.e. (6.2.2)) such that the subspaces $R_\pm$ are transported to $R^g_\pm \in T_gD \cong \mathcal{D}$ according to

$$\begin{aligned} R^g_+ &= g^{-1}R_+g = \mathrm{Span}\{T_a + E(g)_{ab}\tilde{T}^b\}, \\ R^g_- &= g^{-1}R_-g = \mathrm{Span}\{T_a - E(g)_{ba}\tilde{T}^b\}, \end{aligned} \tag{6.4.19}$$

where $E(g)$ is defined explicitly in Proposition 6.11 in terms of the Poisson structure Π on G and satisfies the Poisson-Lie condition, i.e. $E(g)$ defines the string background on a σ-model.

Apparently $R^g_\pm$ span $T_gD = \mathcal{D}$ and since the non-degenerate bilinear form $\langle\ ,\ \rangle$ is ad-invariant, thus $R^g_\pm$ are orthogonal subspaces of $\mathcal{D}$, i.e.

$$\langle R^g_+, R^g_- \rangle = \langle g^{-1}R_+g, g^{-1}R_-g \rangle = \langle R_+, R_- \rangle = 0. \tag{6.4.20}$$

There is also a generalized metric $G_g : T_gD \to T_gD$ with $R^g_\pm$ its $\pm$-eigenspaces. I.e. we have the following commutative diagram

$$\begin{array}{ccc} T_eD & \xrightarrow{G} & T_eD \\ \downarrow{\scriptstyle g} & & {\scriptstyle g}\downarrow \\ T_gD & \xrightarrow{G_g} & T_gD \end{array} \tag{6.4.21}$$

According to [2], every element $f \in D$ can have two different decompositions

$$f = g\tilde{h} = \tilde{g}h, \quad g, h \in G,\ \tilde{g}, \tilde{h} \in \tilde{G}. \tag{6.4.22}$$

Thus, similarly for $\tilde{g} \in \tilde{G}$, T_eD can be transported to $T_{\tilde{g}}D$ via the action of $\tilde{g}$ on $\mathcal{D}$ such that the subspaces $R_\pm$ are transported to $R^{\tilde{g}}_\pm \in T_{\tilde{g}}D \cong \mathcal{D}$. The subspaces $R^{\tilde{g}}_\pm$ are given by

$$R^{\tilde{g}}_+ = \mathrm{Span}\{\tilde{T}^a + \hat{E}(\tilde{g})^{ab}T_b\}, \quad R^{\tilde{g}}_- = \mathrm{Span}\{\tilde{T}^a - \hat{E}(\tilde{g})^{ba}T_b\}, \tag{6.4.23}$$

where $\hat{E}(\tilde{g})$ defines the string background on the dual σ-model and is given in terms of the Poisson structure $\tilde{\Pi}$ on $\tilde{G}$ in Proposition 6.11.

Let $G_{\tilde{g}}$ be the generalized metric on $T_{\tilde{g}}D \cong \mathcal{D}$ with $R^{\tilde{g}}_{\pm}$ its $\pm$-eigenspaces, then we have the commutative diagram

$$\begin{array}{ccc} T_eD & \xrightarrow{G} & T_eD \\ \downarrow \tilde{g} & & \tilde{g} \downarrow \\ T_{\tilde{g}}D & \xrightarrow{G_{\tilde{g}}} & T_{\tilde{g}}D \end{array} \tag{6.4.24}$$

Example 6.15. Consider again the Borelian double $D = GL(2,\mathbb{R})$ following Examples 6.10 and 6.12. D has subgroups B_2 and $\tilde{G}$ with Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ generated by $\{T_1, T_2\}$ and $\{\tilde{T}^1, \tilde{T}^2\}$ defined previously in (6.2.29).

Let us choose the matrix $E(e)^{-1}$ as

$$E(e)^{-1} = \begin{pmatrix} x & y \\ u & v \end{pmatrix}, \tag{6.4.25}$$

i.e.

$$E(e) = \frac{1}{uy - vx} \begin{pmatrix} -v & y \\ u & -x \end{pmatrix}. \tag{6.4.26}$$

Then the subspaces $R_\pm$ are found to be

$$\begin{aligned} R_+ &= \mathrm{Span}\{T_a + E(e)_{ab}\tilde{T}^b\} = \mathrm{Span}\left\{ \begin{pmatrix} 1 & 0 \\ \frac{-y}{uy-vx} & \frac{-v}{uy-vx} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \frac{x}{uy-vx} & \frac{u}{uy-vx} \end{pmatrix} \right\} \\ R_- &= \mathrm{Span}\{T_a - E(e)_{ba}\tilde{T}^b\} = \mathrm{Span}\left\{ \begin{pmatrix} 1 & 0 \\ \frac{-u}{uy-vx} & \frac{v}{uy-vx} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \frac{-x}{uy-vx} & \frac{-y}{uy-vx} \end{pmatrix} \right\}. \end{aligned} \tag{6.4.27}$$

An element $g \in B_2$ can be parameterized by

$$g = \begin{pmatrix} e^{\varphi} & \theta \\ 0 & 1 \end{pmatrix}. \tag{6.4.28}$$

Thus R_+ can be transported to R_+^g via

$$\begin{aligned}
R_+^g &= g^{-1}R_+g \\
&= \text{Span}\left\{\frac{1}{uy-vx}\begin{pmatrix} (uy-vx)r^{\varphi}+\theta y & \theta(uy-vx)+\theta^2 ye^{-\varphi}+\theta e^{-\varphi}v \\ -ye^{\varphi} & -y\theta - v\end{pmatrix},\right. \\
&\qquad \left.\frac{1}{uy-vx}\begin{pmatrix} -\theta x & -\theta^2 e^{-\varphi}x - \theta u e^{-\varphi} + uy - vx \\ xe^{\varphi} & x\theta + u\end{pmatrix}\right\} \\
&= \text{Span}\left\{\begin{pmatrix} 0 & 1 \\ \frac{e^{2\varphi}x}{\theta(y-u)+e^{\varphi}(uy-vx)+e^{-\varphi}\theta^2} & \frac{\theta+(\theta x+u)e^{\varphi}}{\theta(y-u)+e^{\varphi}(uy-vx)+e^{-\varphi}\theta^2}\end{pmatrix},\right. \\
&\qquad \left.\begin{pmatrix} 1 & 0 \\ \frac{-ye^{\varphi}-\theta}{\theta(y-u)+e^{\varphi}(uy-vx)+e^{-\varphi}\theta^2} & \frac{-x\theta^2-\theta(u+y)-v}{\theta(y-u)+e^{\varphi}(uy-vx)+e^{-\varphi}\theta^2}\end{pmatrix}\right\} \\
&= \text{Span}\{T_a + E(g)_{ab}\tilde{T}^b\}. \qquad (6.4.29)
\end{aligned}$$

Thus comparing terms we find that the string background $E(g)$ is given by

$$E(g) = \frac{1}{\theta(y-u)+e^{\varphi}(uy-vx)+e^{-\varphi}\theta^2}\begin{pmatrix} -(x\theta^2+\theta(u+y)+v) & ye^{\varphi}+\theta \\ \theta+(\theta x+u)e^{\varphi} & -e^{2\varphi}x\end{pmatrix}. \qquad (6.4.30)$$

Let us choose a parametrization of the dual group $\tilde{G}$

$$\tilde{g} = \begin{pmatrix} 1 & 0 \\ -\tilde{\theta} & e^{\tilde{\varphi}}\end{pmatrix}. \qquad (6.4.31)$$

One can check that both g and $\tilde{g}$ give a different decomposition of $l \in D$ via (6.4.22).

Again transporting R_+ to $R_+^{\tilde{g}}$ gives us

$$\begin{aligned}
R_+^{\tilde{g}} &= \tilde{g}^{-1}R_+\tilde{g} \\
&= \text{Span}\left\{\frac{1}{uy-vx}\begin{pmatrix} uy-vx & 0 \\ e^{-\tilde{\varphi}}(v\tilde{\theta}-y)+\tilde{\theta}e^{-\tilde{\varphi}}(uy-vx) & -v\end{pmatrix},\right. \\
&\qquad \left.\begin{pmatrix} -\tilde{\theta}(uy-vx) & e^{\tilde{\theta}}(uy-vx) \\ e^{-\theta\varphi}(x-y\tilde{\theta}-\tilde{\theta}^2(uy-vx)) & \tilde{\theta}(uy-vx)+y\end{pmatrix}\right\} \\
&= \text{Span}\left\{\begin{pmatrix} \frac{e^{-\tilde{\varphi}}(uy-vx)(x-\tilde{\theta}^2 v)}{-e^{-\tilde{\varphi}}(vx+y^2)+\tilde{\theta}^2(uy-vx)} & \frac{-(uy-vx)(v\tilde{\theta}-y+\tilde{\theta}(uy-vx))}{-e^{-\tilde{\varphi}}(vx+y^2)+\tilde{\theta}^2(uy-vx)} \\ 0 & 1\end{pmatrix},\right. \\
&\qquad \left.\begin{pmatrix} \frac{(uy-vx)(\tilde{\theta}(uy-vx)+y-v\tilde{\theta})}{e^{-\tilde{\varphi}}(\tilde{\theta}^2(uy-vx)^2+vx)} & \frac{e^{\tilde{\varphi}}v(uy-vx)}{e^{-\tilde{\varphi}}(\tilde{\theta}^2(uy-vx)^2+vx)} \\ 1 & 0\end{pmatrix}\right\} \\
&= \text{Span}\{\tilde{T}^a + \hat{E}(\tilde{g})^{ab}T_b\}. \qquad (6.4.32)
\end{aligned}$$

Comparing terms we find that the string background $\hat{E}(\tilde{g})$ on the dual σ-model is given by

$$\hat{E}(\tilde{g}) = \begin{pmatrix} \frac{e^{-\tilde{\varphi}}(uy-vx)(x-\tilde{\theta}^2 v)}{-e^{-\tilde{\varphi}}(vx+y^2)+\tilde{\theta}^2(uy-vx)} & \frac{-(uy-vx)(v\tilde{\theta}-y+\tilde{\theta}(uy-vx))}{-e^{-\tilde{\varphi}}(vx+y^2)+\tilde{\theta}^2(uy-vx)} \\ -\frac{(uy-vx)(\tilde{\theta}(uy-vx)+y-v\tilde{\theta})}{e^{-\tilde{\varphi}}(\tilde{\theta}^2(uy-vx)^2+vx)} & -\frac{e^{\tilde{\varphi}}v(uy-vx)}{e^{-\tilde{\varphi}}(\tilde{\theta}^2(uy-vx)^2+vx)} \end{pmatrix}. \tag{6.4.33}$$

We can compare the above dual pair of string backgrounds $E(g)$ and $\hat{E}(\tilde{g})$ with the string backgrounds computed previously in (3.19) in Section 3.3.3 using (6.3.5). These solutions agree up to re-parameterizations.

6.4.3 Poisson-Lie T-duality on Courant algebroids

In this section we establish an isomorphism of Courant algebroids related by Poisson-Lie T-duality.

Consider a Poisson-Lie group G and its dual group $\tilde{G}$. Let $\{v_a\}$ be a basis of left-invariant vector fields on G and $\{w^a\}$ be the dual basis of left-invariant one forms on G.

The Lie bracket on the left-invariant vector fields is given by

$$[v_a, v_b] = f_{ab}{}^c v_c, \tag{6.4.34}$$

while the dual one forms are defined by

$$w^a(v_b) \equiv \imath_{v_b} w^a = \delta^a_b, \tag{6.4.35}$$

satisfy

$$dw^a = \frac{1}{2} f_{bc}{}^a w^b \wedge w^c. \tag{6.4.36}$$

Lemma 6.16. *In terms of left action of g, Lemma* (6.7) *becomes*

$$(\nabla_L \Pi^{ab})_c = f_{dc}{}^a \Pi^{db} + f_{dc}{}^b \Pi^{ad} + \tilde{f}^{ab}{}_c. \tag{6.4.37}$$

Proof. With g a left action, the adjoint action of g on $\mathcal{D}$ is given by

$$gT_a g^{-1} = a(g)_a{}^b T_b, \quad g\tilde{T}^a g^{-1} = c(g)^{ab} T_b + d(g)^a{}_b \tilde{T}^b. \tag{6.4.38}$$

Computing $(\nabla_L (a(g)^{-1})_a{}^b)_c$ and $(\nabla_L c(g)^{ab})_c$, we find

$$\begin{aligned} (\nabla_L (a(g)^{-1})_a{}^b)_c &= f_{dc}{}^b (a(g)^{-1})_a{}^d, \\ (\nabla_L c(g)^{ab})_c &= f_{dc}{}^a c(g)^{db} + \tilde{f}^{ad}{}_c a(g)_d{}^b. \end{aligned} \tag{6.4.39}$$

Thus

$$\begin{aligned}(\nabla_L \Pi^{ab})_c &= (\nabla_L c(g)^{ad})_c (a(g)^{-1})_d{}^b + c(g)^{ad} (\nabla_L (a(g)^{-1})_d{}^b)_c \\ &= f_{dc}{}^a \Pi^{db} + f_{dc}{}^b \Pi^{ad} + \tilde{f}^{ab}{}_c. \end{aligned} \tag{6.4.40}$$

□

Lemma 6.17. *The Koszul bracket* $[\ ,\ ]_\Pi$ *defined by* (2.3.1) *on invariant sections of* T^*G *is given by*

$$[w^a, w^b]_\Pi = \tilde{f}^{ab}{}_c w^c, \tag{6.4.41}$$

where $\tilde{f}^{ab}{}_c$ *is the structure constant of* $\tilde{G}$.

Proof. The bracket $[\ ,\ ]_\Pi$ on invariant forms is given by

$$\begin{aligned}[w^a, w^b]_\Pi &= \mathcal{L}_{\pi^\sharp(w^a)} w^b - \mathcal{L}_{\pi^\sharp(w^b)} w^a - d\Pi(w^a, w^b) \\ &= d\Pi(w^a, w^b) + \imath_{\pi^\sharp(w^a)} dw^b - d\Pi(w^b, w^a) - \imath_{\pi^\sharp(w^b)} dw^a - d\Pi(w^a, w^b) \\ &= f_{cd}{}^b \Pi(w^c, w^a) w^d - f_{cd}{}^a \Pi(e^c, e^b) w^d - d\Pi(w^b, w^a) \\ &= f_{cd}{}^b \Pi^{ca} w^d - f_{cd}{}^a \Pi^{cb} w^d + d\Pi^{ab} \\ &= f_{cd}{}^b \Pi^{ca} w^d - f_{cd}{}^a \Pi^{cb} w^d + (\nabla_L \Pi^{ab})_c w^c. \end{aligned} \tag{6.4.42}$$

Substituting (6.4.37) into the above equation, we obtain

$$[w^a, w^b]_\Pi = \tilde{f}^{ab}{}_c w^c. \tag{6.4.43}$$

□

Lemma 6.18. *The differential* $d_\Pi = [\Pi, \cdot]_{SN}$ *on left-invariant vector fields is given by*

$$d_\Pi v_a = \frac{1}{2} \tilde{f}^{bc}{}_a v_b \wedge v_c. \tag{6.4.44}$$

where $[\ ,\ ]_{SN}$ *is the Schouten bracket on multi-vector fields.*

Proof.

$$\begin{aligned} d_\Pi v_c &= [\frac{1}{2} \Pi^{ab} v_a v_b, v_c]_{SN} \\ &= \frac{1}{2} (\Pi^{ab} [v_a, v_c] v_b - \Pi^{ab} [v_b, v_c] v_a + v_c(\Pi^{ab}) v_a \wedge v_b) \\ &= \frac{1}{2} (-\Pi^{db} f_{dc}{}^a - \Pi^{ad} f_{dc}{}^b + v_c(\Pi^{ab})) v_a \wedge v_b \\ &= \frac{1}{2} (-\Pi^{db} f_{dc}{}^a - \Pi^{ad} f_{dc}{}^b + (\nabla_L \Pi^{ab})_c) v_a \wedge v_b \\ &= \frac{1}{2} \tilde{f}^{ab}{}_c v_a \wedge v_b, \end{aligned} \tag{6.4.45}$$

where we have used (6.4.37). □

Recall in Example 2.32 in Section 2.3.1 that when G is a Poisson manifold with a Poisson structure Π, then $(T^*G, [\ ,\ ]_\Pi, \rho_* = \pi^\sharp, d_\Pi)$ is a Lie algebroid over G, with the associated bracket on $\Gamma(T^*G)$ being the Koszul bracket $[\ ,\ ]_\Pi$, anchor $\pi^\sharp$ and a differential d_Π on $\Gamma(\wedge^\bullet TG)$ defined by $d_\Pi = [\Pi, \cdot]_{SN}$. $(T^*G, [\ ,\ ]_\Pi, \rho_* = \pi^\sharp, d_\Pi)$ is often called a cotangent Lie algebroid.

As we have seen in Example 2.36 that (TG, T^*G) defines a Lie bialgebroid, where the Lie algebroid $(TG, [\ ,\ ], \rho, d)$ is equipped with the usual Lie bracket on $\Gamma(TG)$, identity anchor map ρ and de Rham differential d. T^*G is a cotangent Lie algebroid with the above mentioned structures. Since (TG, T^*G) defines a Lie bialgebroid, according to Theorem 2.40, its double $TG \oplus T^*G$ form a Courant algebroid equipped with a non-degenerate bilinear form $\langle\ ,\ \rangle$ and a Courant bracket $[\![\ ,]\!]$ on $\Gamma(TG \oplus T^*G)$. On invariant sections, the non-degenerate bilinear form is given by

$$\langle v_a, w^b \rangle = \delta_a^b. \tag{6.4.46}$$

The Courant bracket on invariant sections of $TG \oplus T^*G$ is found to be

$$\begin{aligned}
&[\![f_1^a v_a + g_{1,a} w^a, f_2^a v_a + g_{2,a} w^a]\!] = [f_1^a v_a, f_2^b v_b] + \mathcal{L}^\Pi_{g_{1,a} w^a} f_2^b v_b - \mathcal{L}^\Pi_{g_{2,a} w^a} f_1^b v_b \\
&+[g_{1,a} w^a, g_{2,b} w^b] + \mathcal{L}_{f_1^a v_b}(g_{2,a} w^b) - \mathcal{L}_{f_2^a v_a}(g_{1,b} w^b) + \frac{1}{2} d(g_{1,a} f_2^a - g_{2,a} f_1^a) \\
&= (f_1^a f_2^b f_{ab}{}^c + f_2^a g_{1,b} \tilde{f}^{bc}{}_a - f_1^a g_{2,b} \tilde{f}^{bc}{}_a) v_c + (g_{1,a} g_{2,b} \tilde{f}^{ab}{}_c + f_1^b g_{2,a} f_{bc}{}^a \\
&- f_2^b g_{1,a} f_{bc}{}^a) w^c,
\end{aligned} \tag{6.4.47}$$

where $f^a, g_a \in C^\infty(G)$ are constant functions and $\mathcal{L}^\Pi = [\imath, d_\Pi]$.

Since $(TG \oplus T^*G)_G \cong \mathfrak{g} \oplus \mathfrak{g}^*$, it is obvious that the above Courant bracket on $\mathfrak{g} \oplus \mathfrak{g}^*$ coincides with (2.2.26), i.e. the Lie brackets on the Drinfel'd double $\mathcal{D}$.

Similarly for the dual group $\tilde{G}$ with a Poisson structure $\tilde{\Pi}$, let $\{\tilde{v}^a\}$ be a basis of left-invariant vector fields on $\tilde{G}$ and $\{\tilde{w}_a\}$ be a basis of left-invariant one forms on $\tilde{G}$. The Lie bracket on the left-invariant vector fields on $\tilde{G}$ is given by

$$[\tilde{v}^a, \tilde{v}^b] = \tilde{f}^{ab}{}_c \tilde{v}^c, \tag{6.4.48}$$

while the dual basis of one forms on $\tilde{G}$ are defined by

$$\tilde{w}_a(\tilde{v}^b) \equiv \imath_{\tilde{v}^b} \tilde{w}_a = \delta_a^b, \tag{6.4.49}$$

satisfy

$$d\tilde{w}_a = \frac{1}{2} \tilde{f}^{bc}{}_a \tilde{w}_b \wedge \tilde{w}_c. \tag{6.4.50}$$

According to Lemma 6.17 and Lemma 6.18, on left invariant forms of $\tilde{G}$, the Koszul bracket is given by

$$[\tilde{w}_a, \tilde{w}_b]_{\tilde{\Pi}} = f_{ab}{}^c \tilde{w}_c, \tag{6.4.51}$$

and the associated differential $d_{\tilde{\Pi}}$ is given by

$$d_{\Pi}\tilde{v}^a = \frac{1}{2} f_{bc}{}^a \tilde{v}^b \wedge \tilde{v}^c. \tag{6.4.52}$$

It follows that $(T\tilde{G}, T^*\tilde{G})$ is a Lie bialgebroid, where the Lie algebroid $(T\tilde{G}, [\ ,\ ], \tilde{\rho}, d)$ is equipped with the usual Lie bracket on $\Gamma(T\tilde{G})$, identity anchor map $\tilde{\rho}$ and de Rham differential d. And $(T^*\tilde{G}, [\ ,\ ]_{\tilde{\Pi}}, \tilde{\rho} = \tilde{\pi}^\sharp, d_{\tilde{\Pi}})$ is a cotangent Lie algebroid with the Koszul bracket $[\ ,\ ]_{\tilde{\Pi}}$ on $\Gamma(T^*\tilde{G})$, anchor map $\tilde{\rho} = \tilde{\pi}^\sharp$ and a differential $d_{\tilde{\Pi}} = [\tilde{\Pi}, \cdot]_{SN}$ on $\Gamma(\wedge^\bullet T\tilde{G})$. Since $(T\tilde{G}, T^*\tilde{G})$ defines a Lie bialgebroid, its double $T\tilde{G} \oplus T^*\tilde{G}$ is a Courant algebroid.

Let φ be a map relating invariant sections of $TG \oplus T^*G$ and $T\tilde{G} \oplus T^*\tilde{G}$ and is defined by

$$\varphi : f^a v_a + g_a w^a \to g_a \tilde{v}^a + f^a \tilde{w}_a. \tag{6.4.53}$$

φ is an isomorphism of invariant sections and $\varphi^2 = 1$. It is obvious that the map φ simply exchanges the role of $\mathfrak{g}$ with $\mathfrak{g}^*$. Thus, let us refer to φ as a "Poisson-Lie T-duality map" relating invariant sections of $TG \oplus T^*G$ and $T\tilde{G} \oplus T^*\tilde{G}$.

Then we have the following result:

Theorem 6.19. *(1) φ is orthogonal with respect to the the non-degenerate bilinear form $\langle\ ,\ \rangle$ on the invariant sections of $TG \oplus T^*G$ and $T\tilde{G} \oplus T^*\tilde{G}$.*

(2) φ preserves the Courant brackets on the invariant sections of $TG \oplus T^*G$ and $T\tilde{G} \oplus T^*\tilde{G}$

$$\varphi([\![f_1^a v_a + g_{1,a} w^a, f_2^a v_a + g_{2,a} w^a]\!]) = [\![\varphi(f_1^a v_a + g_{1,a} w^a), \varphi(f_2^a v_a + g_{2,a} w^a)]\!]. \tag{6.4.54}$$

Proof. (1) This is trivial from the definition.

(2) It is obvious that φ preserves the Lie brackets on $\mathcal{D}$ given by (2.2.26). $\square$

Therefore on the space of invariant sections, $TG \oplus T^*G$ and $T\tilde{G} \oplus T^*\tilde{G}$ are isomorphic Courant algebroids related by the Poisson-Lie T-duality map φ.

This can be generalized to principal G and $\tilde{G}$-bundles over a common base manifold M.

Let E be a principal G-bundle over M, and $\hat{E}$ be a principal $\tilde{G}$-bundle over M. The invariant sections of $TE \oplus T^*E$ decompose as $(TM \times \mathfrak{g}) \oplus (T^*M \times \mathfrak{g}^*)$ while the invariant sections of $T\hat{E} \oplus T^*\hat{E}$ decompose as $(TM \times \tilde{\mathfrak{g}}) \oplus (T^*M \times \tilde{\mathfrak{g}}^*) \equiv (TM \times \mathfrak{g}^*) \oplus (T^*M \times \mathfrak{g})$. Thus the previous construction applies to principal G and $\tilde{G}$-bundles in the obvious way, such that the Poisson-Lie T-duality map φ simply exchanges the role of $\mathfrak{g}$ and $\mathfrak{g}^*$ on the invariant sections of $TE \oplus T^*E$ and $T\hat{E} \oplus T^*\hat{E}$.

In the Abelian case when E is a principal $\mathbb{T}^n$-bundle and $\hat{E}$ its dual space, the Koszul brackets on T^*E and $T^*\hat{E}$ vanish, and the map φ coincides with the T-duality map between invariant sections of $TE \oplus T^*E$ and $T\hat{E} \oplus T^*\hat{E}$ defined previously in Section 5.3.2.

Chapter 7

Non-geometric flux compactification vs global T-duality

7.1 Introduction and outline

(Super)string theory requires 10-dimensional spacetime, and one of the most challenging aspects of string theory is to reduce the 10-dimensional spacetime to our 4-dimensional world consistently. There is a systematic method developed by Scherk and Schwarz [69] for reducing supergravities as a generalization of the Kaluza-Klein reduction. Such a reduction of supergravities gives a lower dimensional supergravity, in particular when a non-Abelian Yang-Mills group is involved.

The Scherk-Schwarz reduction on a $D+d$ dimensional field theory with target E on a d-dimensional internal manifold T gives rise to a D-dimensional field theory with gauge symmetry, mass terms and a scalar potential. Kaloper and Myers [44] showed that the Scherk-Schwarz reduction can be constructed on an internal space which is a twisted torus, thus the Scherk-Schwarz reduction can be generalized by introducing a flux labeled ${f_{ab}}^c$ corresponding to the twisting. In string theory, this construction can be used to compactify a $D+d$-dimensional string theory on d-dimensional (twisted) tori with the presence of background H-flux. In the literature, the H-flux and the fluxes corresponding to the twisting of the internal space are referred to as the **geometric fluxes** [41, 44, 74].

Shelton, Taylor and Wecht [74] then take the Scherk-Schwarz reduction of string theory on twisted tori one step further by including compactifications on spaces which cannot be described geometrically. In their constructions additional algebraic structures are introduced to be included on a given string background which results in the string background no longer describing a manifold. These algebraic structures are referred to as the **non-geometric fluxes**. The non-geometric fluxes, labeled ${q^{ab}}_c$ and r^{abc}, appeared naturally by T-dualizing the original internal manifold. As a result, Shelton, Taylor and Wecht proposed the following T-duality rule

$$h_{abc} \xleftrightarrow{T_a} {f_{bc}}^a \xleftrightarrow{T_b} {q^{ab}}_c \xleftrightarrow{T_c} r^{abc},$$

where a, b, c denote the indices of the coordinates on the internal space.

It turns out that the brackets on the gauge algebras of the reduced theory simply correspond to the Courant bracket on the invariant sections of the generalized tangent space $TE \oplus T^*E$ when restricted on T. And the non-geometric fluxes ${q^{ab}}_c$ and r^{abc} simply corresponds to the fluxes $F_{(1)ab}$ and $F_{(0)abc}$ introduced by Bouwknegt, Evslin and Mathai [6] for the global T-duality.

This Chapter is organized as follows: In Section 7.2 we review the basic idea of Scherk-Schwarz reduction and the gauge algebra of the reduced theory when the internal space is a flat torus. In Section 7.3, we follow Kaloper and Myers' generalization of Scherk-Schwarz reduction to the case when the internal space is a twisted torus. Section 7.4 reviews the idea of non-geometric compactification proposed by Shelton, Taylor and Wecht [74]. In Section 7.5 we show how to obtain the full gauge algebra which is invariant under T-duality, while in Section 7.6 we show that the non-geometric fluxes introduced by Shelton, Taylor and Wecht are related to the fluxes $F_{(1)}$ and $F_{(0)}$ introduced previously in Chapter 5.

This Chapter is collaborative work with Bouwknegt and Garretson [11].

7.2 Scherk-Schwarz reduction and gauge algebra

Let us consider a $D + d = 10$-dimensional string theory compactified on a d-dimensional internal manifold T which gives rise to a D-dimensional string theory on the manifold M. Let y^a ($a = 1, \ldots, d$) be coordinates on T and x^μ ($\mu = 1, \ldots, D$) be coordinates on M.

The Scherk-Schwarz reduction [69] of a string theory on a n-torus $\mathbb{T}^n$ gives a decomposition of the metric g, the B-field b and the gauge fields as follows

$$\begin{aligned} g &= g_{\mu\nu}(x)dx^\mu \otimes dx^\nu + g_{ab}(x,y)A^a \otimes A^b \\ &= (g_{\mu\nu}(x) + g_{ab}A^a_\mu A^b_\nu)dx^\mu \otimes dx^\nu + g_{ab}(x)A^b_\nu dy^a \otimes dx^\nu + g_{ab}(x)A^a_\mu dx^\mu \otimes dy^b \\ &\quad + g_{ab}(x)dy^a \otimes dy^b, \\ b &= \frac{1}{2}b_{\mu\nu}(x)dx^\mu \wedge dx^\nu + b_{\mu a}(x)dx^\mu \wedge A^a + \frac{1}{2}b_{ab}(x,y)A^a \wedge A^b \\ &= (\frac{1}{2}b_{\mu\nu}(x) + b_{\mu a}(x)A^a_\nu + \frac{1}{2}b_{ab}(x,y)A^a_\mu A^b_\nu)dx^\mu \wedge dx^\nu \\ &\quad + (b_{\mu a}(x) + \frac{1}{2}b_{ab}A^b_\mu - \frac{1}{2}b_{ba}A^b_\mu)dx^\mu \wedge dy^a + \frac{1}{2}b_{ab}(x)dy^a \wedge dy^b, \end{aligned} \tag{7.2.1}$$

where $A^a = dy^a + A^a_\mu dx^\mu$ and $A^a_\mu dx^\mu$ are Kaluza-Klein fields.

Similarly, a p-form gauge potential V decomposes as

$$V = V_{(p)} + V_{(p-1)a} \wedge A^a + \frac{1}{2}V_{(p-2)ab} \wedge A^a \wedge A^b + \dots . \tag{7.2.2}$$

Let us organize the B-field b in (7.2.1) as

$$\begin{aligned} b_{\mu\nu} &= b_{\mu\nu}(x) + 2b_{\mu a}(x)A^a_\nu + b_{ab}(x,y)A^a_\mu A^b_\nu, \\ b_{\mu a} &= b_{\mu a}(x) + \frac{1}{2}b_{ab}(x,y)A^b_\mu - \frac{1}{2}b_{ba}(x,y)A^b_\mu, \\ b_{ab} &= b_{ab}(x,y). \end{aligned} \tag{7.2.3}$$

Similarly, we organize the metric g terms in (7.2.1) as

$$\begin{aligned} g_{\mu\nu} &= g_{\mu\nu}(x) + g_{ab}A^a_\mu A^b_\nu, \\ g_{a\nu} &= g_{ab}(x)A^b_\nu, \\ g_{ab} &= g_{ab}(x). \end{aligned} \tag{7.2.4}$$

One can introduce the flux h_{abc} and decompose $b_{ab}(x,y)$ by including an explicit dependence on the coordinate y as follows

$$b_{ab}(x,y) = b_{ab}(x) + h_{abc}y^c. \tag{7.2.5}$$

By introducing the flux h_{abc}, the fields A^a_i arise from the metric and the B-field transform under two types of reduced gauge transformations [44]:

(1) Kalb-Ramond gauge transformations with gauge algebra λ_a:

Let ξ^a be a basis of non-vanishing one-forms on the internal space. The B-field b has a one-form gauge symmetry in the original 10-dimensional theory given by a one-form Λ:

$$b \to b' = b + d\Lambda, \quad \Lambda = \lambda_a \xi^a + \lambda_i dx^i. \tag{7.2.6}$$

If we set $\lambda_i = 0$, an explicit computation shows that the reduced two form B-field transforms according to

$$b'_{\mu a} = b_{\mu a} + \partial_\mu \lambda_a,$$
$$b'_{\mu\nu} = b_{\mu\nu} + F_{\mu\nu}{}^a \lambda_a, \qquad (7.2.7)$$

where $F_{\mu\nu}{}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ is the field strength of A^a_i.

(2) Kaluza-Klein gauge transformations with gauge algebra ω^a:

The forms dy^a transform under spacetime diffeomorphism according to $dy^a \to dy'^a = dy^a + d\omega^a$. In order to ensure the invariance of the internal space d-bein, i.e $dy^a + A^a_\mu dx^\mu \to dy'^a + A'^a_\mu dx^\mu$ is invariant, A^a_μ transforms under Kaluza-Klein transformation according to $A^a_\mu \to A'^a_\mu = A^a_\mu - \partial_\mu \omega^a$. While the B-field b transforms according to

$$b'_{\mu a} = b_{\mu a} + h_{abc}\omega^c A^b_\mu,$$
$$b'_{ab} = b_{ab} + h_{abc}\omega^c,$$
$$b'_{\mu\nu} = b_{\mu\nu} + h_{\mu\nu a}\omega^a + 3h_{abc}\omega^a A^b_\mu A^c_\nu, \qquad (7.2.8)$$

where $h_{\mu\nu a} = \partial_\mu b_{\nu a} - \partial_\nu b_{\mu a}$ is the field strength of $b_{\mu a}$.

Those fields that are not listed above are invariant under the corresponding transformations.

Let X^a be generators of the Kalb-Ramond gauge transformation and Z_a be generators of the Kaluza-Klein gauge transformation.

The generators Z_a and X^a satisfy

$$[Z_a, Z_b] = A_{ab}{}^c Z_c + B_{abc} X^c,$$
$$[Z_a, X^b] = C_{ac}{}^b X^c + D_a{}^{bc} Z_c,$$
$$[X^a, X^b] = E^{abc} Z_c + F^{ab}{}_c X^c, \qquad (7.2.9)$$

where A, B, C, D, E and F are some structure constants to be determined.

To find these structure constants, let us consider the successive application of the Kalb-Ramond and the Kaluza-Klein gauge transformations. Let $g_i = e^{i\alpha^a_i Z_a}$ and $h_i = e^{i\beta_{i,a} X^a}$ be gauge transformations corresponding to the Kalb-Ramond and the Kaluza-Klein gauge transformations with α^a and β_a the gauge transformation parameters, respectively.

Now computing the forms $g_1 g_2 - g_2 g_1$, $g_1 h_2 - h_2 g_1$ and $h_1 h_2 - h_2 h_1$, and comparing the products with the corresponding successive gauge transformations given by (7.2.7) and (7.2.8), we obtain the structure constants in (7.2.9)

$$B_{abc} = h_{abc}, \quad \text{otherwise } 0. \qquad (7.2.10)$$

Thus the gauge algebra becomes

$$[Z_a, Z_b] = h_{abc}X^c,$$
$$[Z_a, X^b] = [X^a, X^b] = 0. \tag{7.2.11}$$

7.3 Gauge algebra on a twisted torus

In the previous section, the 10-dimensional string theory is compactified on the flat n-torus with coordinates y^a. In this section, we will consider the case when the internal space T is not restricted to be homeomorphic to fibred torus $\mathbb{T}^n$, but a space called the twisted torus.

To illustrate the idea of a twisted torus, let us first consider an example of a twisted three-torus T provided in [43].

Consider string theory compatified on $\mathbb{T}^3$ with non-trivial H-flux. Let (y^1, y^2, y^3) be coordinates on $\mathbb{T}^3$, each with period 1. We start with a metric and H-flux (which corresponds to $h_{123} = k$ in the previous notation) given by

$$g = (dy^1)^2 + (dy^2)^2 + (dy^3)^2, \quad H = kdy^1 \wedge dy^2 \wedge dy^3. \tag{7.3.1}$$

This metric has symmetry $(y^1, y^2, y^3) \backsim (y^1 + 1, y^2, y^3) \backsim (y^1, y^2 + 1, y^3) \backsim (y^1, y^2, y^3 + 1)$. If we then choose a local gauge $H = db$, with $b = ky^3dy^1 \wedge dy^2$, we can do a T-duality transformation on the y^1 or y^2 coordinate. Performing T-duality transformation on the y^1-direction, the Buscher rules (3.2.9) give us

$$\hat{g} = (d\hat{y^1} - ky^3dy^2)^2 + (dy^2)^2 + (dy^3)^2, \quad \hat{b} = 0 \ (\equiv \hat{H} = 0). \tag{7.3.2}$$

The dual metric $\hat{g}$ has symmetry $(\tilde{y}^1, y^2, y^3) \backsim (\tilde{y}^1 + ky^2, y^2, y^3 + 1) \backsim (\tilde{y}^1 + 1, y^2, y^3) \backsim (\tilde{y}^1, y^2 + 1, y^3)$.

The shift of dy^1 by ky^3zdy in the metric turns $\mathbb{T}^3$ into a **twisted three-torus**.

The twisted three-torus is a manifold. The T-duality transformation mixes the metric and the B-field b by an $SL(2, \mathbb{Z}) \in O(2, 2; \mathbb{Z})$ transformation which is characterized by $k \in \mathbb{Z}$.

To generalize the above construction to twisted d-torus T, let ξ^a be a basis of one-forms on T defined by a vielbein ${\sigma^a}_b(y)$ which is related to coordinate twisting of y^a

$$\xi^a = {\sigma^a}_b(y)dy^b. \tag{7.3.3}$$

The inverse of the above map is

$$dy^a = {\sigma_b}^a(y)\xi^b, \tag{7.3.4}$$

where ${\sigma_b}^a(y)$ is the inverse of ${\sigma^a}_b$.

The generators Z_a of the Kaluza-Klein gauge transformation are the Killing vector fields generating the spacetime isometry of the internal space, and are dual to the basis one-forms ξ^a. The Killing vector fields Z_a can be expressed as

$$Z_a = {\sigma_a}^b \partial_b, \tag{7.3.5}$$

where $\partial_b = \frac{\partial}{\partial y^b}$.

The Killing vector fields Z_a must satisfy

$$[Z_a, Z_b] = {f_{ab}}^c Z_c, \tag{7.3.6}$$

where ${f_{ab}}^c$ are some structure constants.

Let us include the flux ${f_{ab}}^c$ by a shifting of dy^a via

$$dy^a \to dy^a - {f_{bc}}^a y^b dy^c. \tag{7.3.7}$$

Thus the components of the metric g and B-field b in (7.2.1) now decompose as:

$$\begin{aligned} g_{\mu\nu} &= g_{\mu\nu}(x) + g_{ab} A^a_\mu A^b_\nu, \\ g_{\mu b} &= g_{ab}(x) A^a_\mu - g_{ad}(x) {f_{cb}}^d A^a_\mu y^c, \\ g_{ab} &= g_{ab}(x,y) - g_{ad}(x) {f_{cb}}^d y^c. \end{aligned} \tag{7.3.8}$$

and

$$\begin{aligned} b_{\mu\nu} &= b_{\mu\nu}(x) + b_{\mu a}(x) A^a_\nu + \frac{1}{2} b_{ab}(x,y) A^a_\mu A^b_\nu, \\ b_{\mu a} &= b_{\mu a}(x) + \frac{1}{2} b_{ab}(x,y) A^b_\mu - \frac{1}{2} b_{ba}(x,y) A^b_\mu - b_{\mu b}(x) {f_{ca}}^b y^c - b_{cb}(x,y) A^c_\mu {f_{ea}}^b y^e, \\ b_{ab} &= b_{ab}(x,y) - \frac{1}{2} b_{ad}(x,y) {f_{be}}^d y^e - \frac{1}{2} b_{db}(x,y) {f_{ea}}^d y^e. \end{aligned} \tag{7.3.9}$$

Therefore by including the flux ${f_{ab}}^c$ which is related to the twisting, the gauge transformations (7.2.7) and (7.2.8) that appeared in the previous section are now generalized as follows [44]:

(1) Kalb-Ramond gauge transformations with gauge algebra λ_a

$$\begin{aligned} b'_{ab}(x) &= b_{ab} - {f_{ab}}^c \lambda_c, \\ b'_{\mu a}(x) &= b_{\mu a}(x) + \partial_\mu \lambda_a - {f_{ab}}^c A^b_\mu \lambda_c, \\ b'_{\mu\nu}(x) &= b_{\mu\nu}(x) + {F_{\mu\nu}}^a \lambda_a + \frac{1}{2} {f_{ab}}^c A^a_\mu A^b_\nu \lambda_c, \end{aligned} \tag{7.3.10}$$

(2) Kaluza-Klein gauge transformations with gauge algebra ω^a:

$$\begin{aligned}
A'^a_\mu &= A^a_\mu - \partial_\mu \omega^a - f_{bc}{}^a \omega^b A^c_\mu, \\
b'_{\mu a}(x) &= b_{\mu a}(x) + h_{abc}\omega^c A^b_\mu + f_{ab}{}^c \omega^b b_{\mu c} + O(\omega^2), \\
b'_{ab}(x) &= b_{ab}(x) + h_{abc}\omega^c + O(\omega^2), \\
b'_{\mu\nu}(x) &= b_{\mu\nu}(x) + h_{\mu\nu a}\omega^a - 3h_{abc}\omega^a A^b_\mu A^c_\nu - f_{ab}{}^c \omega^a b_{[\mu c} A^b{}_{\mu]} + O(\omega^2).
\end{aligned} \tag{7.3.11}$$

Those fields that are not listed above are invariant under the corresponding transformations.

Following the procedure in the previous section by applying successive gauge transformations, the structure constants in (7.2.9) are found to be

$$A_{ab}{}^c = -C_{ab}{}^c = f_{ab}{}^c, \quad B_{abc} = h_{abc}, \quad \text{otherwise } 0. \tag{7.3.12}$$

Thus the gauge algebras Z_a and X^a satisfy

$$\begin{aligned}
[Z_a, Z_b] &= f_{ab}{}^c Z_c + h_{abc} X^c, \\
[X^a, Z_b] &= -f_{bc}{}^a X^c, \\
[X^a, X^b] &= 0.
\end{aligned} \tag{7.3.13}$$

These brackets define a Lie algebra if we further require the following conditions on h's and f's:

$$\begin{aligned}
&f_{ab}{}^d f_{dc}{}^e + f_{bc}{}^d f_{da}{}^e + f_{ca}{}^d f_{db}{}^e = 0, \\
&f_{ab}{}^d h_{dce} + f_{ce}{}^d h_{adb} + f_{bc}{}^d h_{dae} + f_{ae}{}^d h_{cbd} + f_{ca}{}^d h_{dbe} + f_{be}{}^d h_{acd} = 0.
\end{aligned}$$

7.4 Non-geometric flux compactification

First, let us recall the twisted three-torus example in Section 7.3. Denoting $f_{23}{}^1 = k$, the shift of dy^1 by $f_{23}{}^1 y^3 dy^2$ in the metric (7.3.2) turns $\mathbb{T}^3$ into a twisted three-torus. These fluxes $f_{23}{}^1 \in \mathbb{Z}$ are referred to as **geometric fluxes** since they characterize twisting of the coordinates. As a result, T-dualizing on the y^1 direction takes:

$$h_{123} \xrightarrow{T_1} f_{23}{}^1. \tag{7.4.1}$$

If we further perform another T-duality on the y^2 direction, the Buscher rules give us

$$\hat{g} = \frac{1}{1+k^2(y^3)^2}((d\tilde{y}^1)^2 + (d\tilde{y}^2)^2) + (dy^3)^2, \quad \hat{b} = \frac{ky^3}{1+k^2(y^3)^2} d\tilde{y}^1 \wedge d\tilde{y}^2. \tag{7.4.2}$$

Although the metric $\hat{g}$ and B-field $\hat{b}$ are defined locally, the string background with such $\hat{g}$ and $\hat{b}$ is globally not a manifold. The T-duality transformation mixes the metric and the B-field by a $SL(2,\mathbb{Z}) \in O(2,2;\mathbb{Z})$ transformation which is characterized by ${q^{12}}_3 = k \in \mathbb{Z}$, and q is called a **non-geometric flux**. Thus T-duality takes

$$h_{123} \overset{T_1}{\longleftrightarrow} {f_{23}}^1 \overset{T_2}{\longleftrightarrow} {q^{12}}_3. \tag{7.4.3}$$

For general twisted d-torus, Shelton, Taylor and Wecht [74] proposed the following T-duality transformation rules for non-geometric fluxes:

$$h_{abc} \overset{T_a}{\longleftrightarrow} {f_{bc}}^a \overset{T_b}{\longleftrightarrow} {q^{ab}}_c \overset{T_c}{\longleftrightarrow} r^{abc}, \tag{7.4.4}$$

where h_{abc} corresponds to the NS-NS H-flux with three legs in the internal space, ${f_{bc}}^a$ is the geometric flux corresponding to coordinate twisting, and ${q^{ab}}_c$ and r^{abc} are both non-geometric fluxes.

7.5 Full gauge algebra under T-duality

Recall that in the previous sections 7.2 and 7.3, the gauge algebra Z_a associated with the Kaluza-Klein gauge transformations (7.3.11) is the generator of the spacetime isometry on the internal space, which corresponds to Killing vector fields on the internal space T. Since the gauge algebra X^a associated with the Kalb-Ramond gauge transformations (7.3.10) is the generator corresponding to the one-form gauge symmetry of the B-field, thus X^a correspond to the one-form $\Lambda = \lambda_a \xi^a$ and can be associated with the one-form basis ξ^a on the internal space T.

Thus Z_a and X^a can be viewed as gauge algebras corresponding to invariant vector fields and invariant one-form on the internal space T.

Let E be a T-bundle over M. Recall in Chapter 5, the T-duality map φ interchanges the invariant vector fields and the invariant one-forms on the invariant section of $TE \oplus T^*E$. Thus we expect in the a-th coordinate on the internal space T, $Z_a \leftrightarrow X^a$ under the T-duality transformation.

For T a twisted d-torus, the gauge algebras Z_a and X^a satisfy

$$\begin{aligned}
[Z_a, Z_b] &= {f_{ab}}^c Z_c + h_{abc} X^c, \\
[X^a, Z_b] &= -{f_{bc}}^a X^c, \\
[X^a, X^b] &= 0.
\end{aligned}$$

where h_{abc} is the component of the H-flux on T, while $f_{ab}{}^c$ comes from the twisting of the coordinates y^a on T.

Now, applying the T-duality transformation to the above set of brackets, we obtain

$$[X^a, X^b] = q^{ab}{}_c X^c + r^{abc} Z_c,$$
$$[X^a, Z_b] = q^{ac}{}_b Z_c,$$
$$[Z_a, Z_b] = 0.$$

where $q^{ab}{}_c$ is the flux T-dual to $f_{ab}{}^c$ and r^{abc} is the flux T-dual to h_{abc}.

Thus for the algebras to be closed under T-duality, we need to modify the brackets (7.3.13) by including the T-dual fluxes $q^{ab}{}_c$ and r^{abc}, i.e.

$$\begin{aligned} [Z_a, Z_b] &= f_{ab}{}^c Z_c + h_{abc} X^c, \\ [X^a, Z_b] &= -f_{bc}{}^a X^c + q^{ac}{}_b Z_c, \\ [X^a, X^b] &= q^{ab}{}_c X^c + r^{abc} Z_c, \end{aligned} \tag{7.5.1}$$

and the fluxes are related via T-duality as

$$f_{ab}{}^c \stackrel{T}{\longleftrightarrow} q^{ab}{}_c, \quad h_{abc} \stackrel{T}{\longleftrightarrow} r^{abc}. \tag{7.5.2}$$

Next, we show that the brackets (7.5.1) on Z_a and X^a correspond to a Courant bracket on the generalized space of T.

Let us first introduce a basis $\{v_a\}$ of vector field on T and a dual basis $\{w^a\}$ of one-forms on T. Here we have introduced the notation $v_a(f) = \partial_a(f)$, while the dual basis of one-forms are defined by

$$w^a(v_b) \equiv \imath_{v_b} w^a = \delta^a_b \tag{7.5.3}$$

then w^a satisfies

$$dw^a = \frac{1}{2} f_{bc}{}^a w^b \wedge w^c. \tag{7.5.4}$$

The Lie bracket between vector fields in this case is given by

$$[v_a, v_b] = f_{ab}{}^c v_c, \tag{7.5.5}$$

while the bracket on w^a's is given by

$$[w^a, w^b]_* = [w^a, w^b]_\gamma = q^{ab}{}_c w^c, \tag{7.5.6}$$

where $[\ ,\ ]_\gamma$ is defined by (5.4.12).

Recall in Section 5.4, the Dorfmann bracket on the generalized space can be defined in terms of proto-bialgebroid structures $\theta = \mu+\gamma+\varphi+\psi$ given by (5.4.1). The Dorfmann brackets are given as follows:

$$\begin{aligned} X_1 \circ X_2 &= [X_1, X_2]_\mu + \imath_{X_1}\imath_{X_2}\psi, \\ X_1 \circ \Xi_2 &= -\imath_{\Xi_2} d_\gamma X_1 + \mathcal{L}^\mu_{X_1}\Xi_2, \\ \Xi_1 \circ X_2 &= \mathcal{L}^\gamma_{\Xi_1} X_2 - \imath_{X_2} d_\mu \Xi_1, \\ \Xi_1 \circ \Xi_2 &= [\Xi_1, \Xi_2]_\gamma + \imath_{\Xi_1}\imath_{\Xi_2}\varphi, \end{aligned} \tag{7.5.7}$$

where $X \in \Gamma(T\mathrm{T})$ and $\Xi \in \Gamma(T^*\mathrm{T})$. d_μ and d_γ are the quasi-differentials defined by (5.4.7) while $\mathcal{L}^\mu_X = d_\mu \imath_X + \imath_X d_\mu$ and $\mathcal{L}^\gamma_\Xi = d_\gamma \imath_\Xi + \imath_\Xi d_\gamma$.

On the basis $\{v_a\}$ and $\{w^a\}$, the Dorfmann bracket becomes

$$\begin{aligned} v_a \circ v_b &= [v_a, v_b] + \imath_{v_a}\imath_{v_b}\varphi = f_{ab}{}^c v_c + \varphi_{abc} w^c, \\ v_a \circ w^b &= -\imath_{w^b} d_\gamma v_a + \mathcal{L}^\mu_{v_a} w^b = \frac{1}{2} f_{ac}{}^b w^c - \frac{1}{2} q^{bc}{}_a v_c, \\ w^a \circ v_b &= \mathcal{L}^\gamma_{w^a} v_b - \imath_{v_b} d_\mu w^a = \frac{1}{2} q^{ac}{}_b v_c - \frac{1}{2} f_{bc}{}^a w^c, \\ [w^a, w^b] &= [w^a, w^b]_* + \imath_{w^a}\imath_{w^b}\varphi = q^{ab}{}_c w^c + \varphi^{abc} v_c. \end{aligned} \tag{7.5.8}$$

Anti-symmetrizing the above Dorfmann bracket, we obtain the Courant brackets on v_a and w^a as

$$\begin{aligned} &[\![v_a, v_b]\!] = f_{ab}{}^c v_c + h_{abc} w^c, \\ &[\![v_a, w^b]\!] = f_{ac}{}^b w^c - q^{bc}{}_a v_c, \\ &[\![w^a, w^b]\!] = q^{ab}{}_c w^c + \varphi^{abc} v_c. \end{aligned} \tag{7.5.9}$$

Thus comparing the charges of algebras in (7.5.1) with (7.5.9), we identify the charges $(f_{ab}{}^c, q^{ab}{}_c, h_{abc}, r^{abc})$ of (7.5.1) with $(f_{ab}{}^c, q^{ab}{}_c, h_{abc}, \varphi^{abc})$ of (7.5.9).

7.6 Global T-duality and non-geometric flux compactification

In this section we relate the non-geometric flux compactification with the global T-duality discussed previously in Chapter 5.

Let E be the target manifold of a 10-dimensional σ-model compactified on a d-dimensional internal space T over a manifold M. Let us re-define coordinates on the internal manifold T. Let y^i $(i = 1, \ldots, d)$ be the coordinates on T and we choose from T n-coordinates y^a $(1,\dots,n)$ to carry out T-duality transformations and denote the rest as y^μ $(\mu = n+1,\dots,d)$.

Recall in Chapter 5 that T-duality exchanges the role of invariant vector fields with invariant one-forms on the generalized tangent space $TE \oplus T^*E$ via the T-duality map φ defined by (5.3.4). As remarked in the previous section, this is equivalent to the exchange between Z_a and X^a.

Now, recall that the generalized Courant bracket on invariant sections of $TE\oplus T^*E$ given in Section 5.3.2 is

$$\begin{aligned}
&[\![(x_1, f_{1,a}; \xi_1, g_1^a), (x_2, f_{2,a}; \xi_2, g_2^a)]\!]_{H,F} = ([x_1, x_2], \qquad (7.6.1)\\
&(x_1 f_{2,a} - x_2 f_{1,a}) + \imath_{x_1 x_2} F_{(2)a} + g_2^b \imath_{x_1} F_{(1)ab} - g_1^b \imath_{x_2} F_{(1)ab} - g_1^b g_2^c F_{(0)abc};\\
&(\mathcal{L}_{x_1}\xi_2 - \mathcal{L}_{x_2}\xi_1) - \frac{1}{2} d(\imath_{x_1}\xi_2 - \imath_{x_2}\xi_1) + \imath_{x_1}\imath_{x_2} H_{(3)} +\\
&(g_2^a \imath_{x_1} F_{(2)a} - g_1^a \imath_{x_2} F_{(2)a}) + (f_{2,a}\imath_{x_1} H_{(2)}^a - f_{1,a}\imath_{x_2} H_{(2)}^a) - f_{1,a} f_{2,b} H_{(1)}^{ab}\\
&-g_1^a g_2^b F_{(1)ab} + \frac{1}{2}(df_{1,a} g_2^a + f_{2,a} dg_1^a - f_{1,a} dg_2^a - df_{2,a} g_1^a), x_1(g_2^a)\\
&-x_2(g_1^a) + \imath_{x_1}\imath_{x_2} H_2^a + (f_{2,a}\imath_{x_1} H_{(1)}^{ab} - f_{1,a}\imath_{x_2} H_{(1)}^{ab}) - f_{1,b} f_{2,c} H_{(0)}^{abc}),
\end{aligned}$$

where x and ξ are invariant vector fields and one-forms on M, and f_a and g^a correspond to invariant vector fields and one-form, taking the value Z_a and X^a respectively.

The generalized Courant bracket (7.6.1) on a basis of vector fields v_a and one-forms w^a decomposes as

$$\begin{aligned}
[\![v_\mu, v_\nu]\!]_{H,F} &= f_{\mu\nu}{}^\gamma v_\gamma + F_{(2)a}(\cdot, v_\mu, v_\nu) + H_{(3)}(v_\mu, v_\nu, \cdot) + H_{(2)}^a(\cdot, v_\mu, v_\nu),\\
[\![v_a, v_\mu]\!]_{H,F} &= H_{(2)}^a(v_a, v_\mu, \cdot) + H_{(1)}^{ab}(v_a, \cdot, v_\mu)\\
[\![v_a, v_b]\!]_{H,F} &= -H_{(1)}^{ab}(v_a, v_b, \cdot) - H_{(0)}^{abc}(v_a, v_b, \cdot),\\
[\![v_\mu, w^a]\!]_{H,F} &= F_{(1)ab}(v_\mu, w^a, \cdot) + F_{(2)a}(v_\mu, \cdot, w^a),\\
[\![w^a, w^b]\!]_{H,F} &= -F_{(1)ab}(\cdot, w^a, w^b) - F_{(0)abc}(w^a, w^b, \cdot),\\
[\![v_a, w^b]\!]_{H,F} &= 0, \qquad (7.6.2)
\end{aligned}$$

where $H_{(i)} \in \Gamma(\wedge^3 T^*\mathrm{T})$ has $(3-i)$-legs in the T-duality coordinates y^a, and $F_{(i)} \in \Gamma(\wedge^i T^*\mathrm{T} \oplus \wedge^{3-i} T\mathrm{T})$ has all three legs in the T-duality coordinates.

Then let us decompose the brackets (7.5.1) in terms of coordinates of y^i and

y^a as follows:

$$
\begin{aligned}
[Z_\mu, Z_\mu] &= f_{\mu\nu}{}^a Z_a + f_{\mu\nu}{}^\gamma Z_\gamma + h_{\mu\nu a} X^a + h_{\mu\nu\gamma} X^\gamma, \\
[Z_\mu, Z_a] &= f_{\mu a}{}^b Z_b + f_{\mu a}{}^\nu Z_\nu + h_{\mu ab} X^b + h_{\mu a\nu} X^\nu, \\
[Z_a, Z_b] &= f_{ab}{}^c Z_c + f_{ab}{}^\mu Z_\mu + h_{ab\mu} X^\mu + h_{abc} X^c, \\
[X^a, X^b] &= q^{ab}{}_c X^c + q^{ab}{}_\mu Z^\mu + r^{abc} Z_c + r^{ab\mu} Z_\mu, \\
[Z_a, X^b] &= f_{ac}{}^b X^c + f_{a\mu}{}^b X^\mu - q^{bc}{}_a Z_c - q^{b\mu}{}_a Z_\mu.
\end{aligned} \tag{7.6.3}
$$

Comparing (7.6.1) and (7.6.3) gives us the following correspondence between the charges of the algebra (7.6.3) and the fluxes:

$$
\begin{aligned}
&h_{\mu\nu\gamma} \equiv H_{(3)}(v_\mu, v_\nu, v_\gamma) \quad h_{a\mu\nu} \equiv H^a_{(2)}(v_a, v_\mu, v_\nu), \\
&h_{ab\mu} \equiv -H^{ab}_{(1)}(v_a, v_b, v_\mu), \quad h_{abc} \equiv -H^{abc}_{(0)}(v_a, v_b, v_c), \\
&f_{\mu\nu}{}^a \equiv F_{(2)a}(v_\mu, v_\nu, w^a), \quad q^{ab}{}_\mu \equiv -F_{(1)ab}(v_\mu, w^a, w^b) \quad r^{abc} \equiv -F_{(0)abc}(w^a, w^b, w^c), \\
&\text{otherwise } 0.
\end{aligned} \tag{7.6.4}
$$

Thus the gauge algebras (7.6.3) become

$$
\begin{aligned}
[Z_\mu, Z_\nu] &= h_{\mu\nu a} X^a + h_{\mu\nu\gamma} X^\gamma, \\
[Z_\mu, Z_a] &= h_{\mu ab} X^b + h_{\mu a\nu} X^\nu, \\
[Z_a, Z_b] &= h_{ab\mu} X^\mu + h_{abc} X^c, \\
[Z_\mu, X^a] &= f_{\mu\nu}{}^a X^\nu - q^{ab}{}_\mu Z_b, \\
[X^a, X^b] &= q^{ab}{}_\mu X^\mu + r^{abc} Z_c, \\
[Z_a, X^b] &= 0.
\end{aligned} \tag{7.6.5}
$$

Therefore the charges $(f_{ab}{}^c, q^{ab}{}_c, r^{abc}, h_{abc})$ appear in (7.4.4) are in correspondence with the fluxes $(F_{(2)a}, F_{(1)ab}, F_{(0)abc}, H^{abc}_{(0)})$ coming from the global T-duality.

These brackets are invariant under T-duality transformation. Let $\varphi : X^a \leftrightarrow Z_a$ be a T-duality map, and consider the T-duality map φ as a homomorphism of the gauge algebras, i.e. $\varphi[Z_i, Z_j] = [\varphi(Z_i), \varphi(Z_j)]$, $\varphi[Z_i, X^j] = [\varphi(Z_i), \varphi(X^j)]$ and $\varphi[X^i, X^j] = [\varphi(X^i), \varphi(X^j)]$ implies the following T-duality transformation between the charges:

$$
h_{\mu\nu\gamma} \leftrightarrow h_{\mu\nu\gamma}, \quad h_{a\mu\nu} \leftrightarrow f_{\mu\nu}{}^a \quad h_{ab\mu} \leftrightarrow q^{ab}{}_\mu, \quad h_{abc} \leftrightarrow r^{abc}. \tag{7.6.6}
$$

This transformation rule can be rewritten in terms of H's and F's using the correspondence (7.6.4), as a result, T-duality changes the fluxes according to

$$
(H_{(3)}, H^a_{(2)}, H^{ab}_{(1)}, H^{abc}_{(0)}) \overset{T}{\longleftrightarrow} (H_{(3)}, F_{(2)a}, F_{(1)ab}, F_{(0)abc}). \tag{7.6.7}
$$

This agrees with the global T-duality discussed previously in Chapter 5.

As a conclusion, the non-geometric flux transformation rule (7.4.4) proposed by Shelton, Taylor and Wecht [74] is a particular example of the global T-duality transformation rules introduced by Bouwknegt, Evslin and Mathai [6] when considering T-duality transformation in one coordinate at a time.

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