PRELIMINARY DRAFT

Graded Jordan Algebras I

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To the memory of Adrian Albert (Nov. 9, 1905 - June 6, 1972) on the 70th anniversary of his birth

As this is being written, the theory of graded Lie algebras is developing rapidly. As is to be expected, there is a parallel theory of graded Jordan algebras. It seems to have independent interest, and may in due course shed some light on the Lie case. This paper is intended to lay the foundations for the study of the Jordan case.

It is appropriate in a paper dedicated to Adrian Albert that the setting matches that occurring in his pioneering paper [1]. The only simple algebras that arise are the expected ones. However, at the end of the paper examples are given to show that there will be extra simple algebras in future more general studies.

assume 2

Basic definitions will now be given briefly. The grading in this paper is by Z_2, the integers mod 2. Grading by Z, the integers, also merits study but is left to the future. We prefer to take our graded objects as set-theorestic unions rather than direct sums. So a graded vector space V is a union $V_0 \cup V_1$ of vector spaces, disjoint except for a common 0. (All vector spaces in this paper are finite-dimensional.) Elements of V_0 (V_1) are even (odd). A graded algebra $A = A_0 \cup A_1$ is a graded vector space with a multiplication satisfying

$A_iA_j \subset A_{i+j}$ (subscripts taken mod 2). The bracket $[xy]$ is $xy - yx$ except when x and y are both odd in which case it is $xy + yx$. The brace $\{xy\}$ is $xy + yx$ except when x and y are both odd in which case it is $xy - yx$. The bracket motivates graded Lie algebras, the brace graded Jordan algebras.

not comm. in usual sense

$V = V_0 \oplus V_1$

$\text{End}\, V = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$

$A_{00} + A_{11} = A_0$ even

$A_{10} + A_{01} = A_1$ odd

$Tr(T) = tr\begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix} = tr\, T_{00} - tr\, T_{11}$

The linear transformations on a graded vector space acquire a grading in a natural way. There is a trace which we call the <u>graded</u> <u>trace</u> and denote by Tr. (We keep the adjective "graded" since the usual ungraded trace will also play a role.) The graded trace of every odd linear transformation is 0; for an even T we define $Tr(T) = Tr(T_0) - Tr(T_1)$ where T_0 and T_1 are the restrictions of T to V_0 and V_1. A graded Jordan algebra of linear transformations (GJALT) is a subspace closed under $\{\ \}$. Ideals and simplicity are defined in the obvious way.

(1) $J = A^+$ for $A = \text{End}\, V$ graded assoc

Here are two key examples. The algebra of all linear transformations on a graded vector space is a simple GJALT under $\{\ \}$; call it <u>full</u> <u>linear</u>. Next assume a nondegenerate bilinear form which is symmetric on V_0, skew on V_1, and makes V_0 and V_1 orthogonal. Call a linear transformation T on V self-adjoint if, as usual, $(Tx, y) = (x, Ty)$ for all x and y, except that when T and x are both odd we insert a minus sign. The self-adjoint linear transformations form a simple GJALT. If V_0 and V_1 have dimensions s and 2r, its dimension is

$$\tfrac{1}{2}s(s + 1) + 2r^2 - r + 2rs.$$

$(Tx, y) = (-1)^{\varepsilon(T)\varepsilon(x)}(x, Ty)$

(1) $x, y \in V_0$ $\quad (T_{00}x, y) = (x, T_{00}y)$ $\quad T_{00}$ sym

(2) $x, y \in V_1$ $\quad \langle T_{11}x, y\rangle = \langle x, T_{11}y\rangle$ $\quad T_{11}$ sym

(3) $x \in V_0$, $y \in V_1$ $\quad \langle T_{10}x, y\rangle = (x, T_{01}y)$ $\quad T_{10}, T_{01}$ "adjoint" $\quad T_{01} = T_{10}^*$

$\therefore y \in V_1, x \in V_0 \Rightarrow (T_{01}y, x) = -\langle y, T_{10}x\rangle$ $\quad T_{10} = -T_{01}^* = -T_{10}^{**}$ natural!

self adj $T^* = T$

$\begin{pmatrix} H_{00} & T_{01} \\ -T_{01}^* & H_{11} \end{pmatrix} = \begin{pmatrix} H_{00} & T_{10}^* \\ T_{10} & H_{11} \end{pmatrix}$

Orthosymplectic invol

$T^* = \begin{pmatrix} T_{00}^t & T_{10}^t \\ -T_{01}^t & T_{11}^t \end{pmatrix}$

graded invol $(TS)^* = -S^*T^*$ if S, T odd

② $J = H(A,*)$ for $(A,*)$ graded $*$-alg orthosymplectic

We call this algebra orthosymplectic.

Theorem. Let J be a simple GJALT over an algebraically closed field of characteristic 0. Assume that the odd part of J is nonzero and that Tr is not identically 0 on J. Then J is isomorphic to a full linear algebra or an orthosymplectic algebra.

not always tr T_{11} + tr T_{00} will force $K = K_{11} \oplus K_{00}$

The proof will be carried out in a number of steps.

$Tr\{xy\} = (-1)^{\epsilon x \epsilon y} Tr\{yx\}$ since $\{xy\} = (-1)^{\epsilon x \epsilon y}\{yx\}$. If $x, y \in J_1$ then $Tr[x,y] = Tr[x,y]$ $= tr(X_{01}Y_{10} - Y_{01}X_{10}) - tr(X_{10}Y_{01} - Y_{10}X_{01}) = 2tr(X_{01}Y_{10} - Y_{01}X_{10})$

(1) $Tr\{\{xy\}z\} = Tr\{x\{yz\}\}$. This routine verification is left to the reader.

$Tr = 0$ if $x \in J_i$, $y \in J_j$, $z \in J_k$, $i+j+k \equiv 1$; $i+j+k = 0$: $(x \circ y) \circ z - x \circ (y \circ z) = [y[xz]]$; $i+j+k=2$: $Tr[a_1 \circ b_0, c_1] = Tr\, b_0 \circ [a_1, c_1] - Tr[a_1, b_0 c_1]$ from $Tr\, a_1 \circ [b_0, c_1]$ $[tr\, A_{01}(B_{11}C_{10}) + (B_{00}C_{01})A_{10} - A_{01}(C_{00}B_{01}) - (C_{01}B_{11})A_{10}$ vs $tr\, A_{10}(B_{00}C_{01}) + B_{11}C_{10})A_{01} - B_{10}(C_{01}\ldots) - (C_{10}B_{00})A_{01}$

(2) If $Tr\{xy\} = 0$ for all y then $x = 0$. That the set of all such x's is an ideal follows from (1). If this ideal is all of J, then Tr vanishes on $\{JJ\}$, which equals J by simplicity. This contradicts our hypothesis. Hence the ideal is 0.

We write K and L for the even and odd parts of J, reserving subscripts for components of Peirce decompositions.

(3) K is semisimple. Let x be in the radical of K. Then $Tr\{xK\} = 0$ since Tr vanishes on nilpotent elements and $Tr\{xL\} = 0$ since Tr vanishes on odd elements. Hence $Tr\{xJ\} = 0$, whence $x = 0$ by (2).

from Jordan theory

general [no $x = 0$, no alg. closed]: If J simple, K unital, then J unital

✓(4) J has a unit element. Let u be the unit element of K. We shall prove that u is the unit element of J. Given x and y in J, our task will be accomplished if we verify $Tr\{(\{ux\} - x)y\} = 0$. Only the case where x and y are both odd needs attention. Then

x, y different: $Tr = 0$; x, y even: $\{ux\} = x$

$$Tr\{\{ux\}y\} = Tr\{u\{xy\}\} = Tr\{xy\}$$

since $\{xy\}$ is even.

$Tr((u \cdot x - x) \cdot y) = Tr(u \cdot xy - xy) = Tr(x \cdot [yu - y])$

vanishes if x or y even or xy even

At this point we can harmlessly assume that J contains the identity linear transformation, since the rest of the graded vector space on which J is acting is irrelevant. Note that the unit element of J equals ½ the identity linear transformation, because of our use of xy + yx rather than (xy + yx)/2 for the Jordan operation.

$U = $ proj on $U_0 \oplus U_1 \subset V_0 \oplus \ldots$ all other $x \in J$ supported on $U_0 \oplus U_1$

$U_e = 2L_e^2 - L_e$

(5) <u>If</u> e <u>is a primitive idempotent of</u> K <u>then</u> eJe <u>is one-dimensional</u>. We know that eKe is one-dimensional, and so our problem is to prove that eLe = 0. Write J_{11} (= eJe), J_{12}, J_{22} for the subspaces in the Peirce decomposition relative to e. With x ε eLe we wish to show that Tr $\{xy\}$ = 0 for any y; we can suppose y to be odd. The vanishing of Tr $\{xy\}$ is clear for y in J_{12} or J_{22}. For y in J_{11}, the alternative requires that $\{xy\}$ = xy - yx be a nonzero scalar multiple of e. This makes the ungraded trace of e equal to 0, an impossibility. (Note the use of characteristic 0 here. A similar argument appears in the next paragraph.)

(6) <u>Let</u> e_1 <u>and</u> e_2 <u>be orthogonal primitive idempotents belonging to the same simple summand of</u> K. <u>Then</u> $(e_1 + e_2)J(e_1 + e_2)$ <u>contains no nonzero odd elements</u>. With the usual subscripts for Peirce subspaces, our only problem is to prove $L_{12} = 0$, since $L_{11} = L_{22} = 0$ is known from (5). By ordinary Jordan theory there exists $z \in K_{12}$ with $z^2 = e_1 + e_2$. Take x in L_{12}. In showing that Tr $\{xJ\}$ vanishes our only problem arises with Tr $\{xy\}$ for y in L_{12}. The elements $\{xz\}$ and $\{yz\}$ are odd elements in $J_{11} + J_{22}$ and hence are 0, again by (5).

say $eJe = eKe \subset K$ for capacity 2 e's

Better: $[x,y] = \alpha e_1 + \beta e_2$ in $K_{11} + K_{22}$

Ungraded trace $\alpha \cdot \dim(e_1 V) + \beta \dim(e_2 V)$

If $\alpha = \beta$ then we see $\alpha(n_1 + n_2) = 0 \Rightarrow \alpha = 0$ and $[xy] = 0$

But Z anticommutes $x, y \Rightarrow$ commutes $[xy]$ and $\alpha(e_1 + e_2)$, hence $(\beta - \alpha)e_2$

If $\alpha \neq \beta$ get $z \leftrightarrow e_1, e_2$ impossible for $z \in K_{12}$

We conclude the argument using a block matrix notation for the elements of $(e_1 + e_2)J(e_1 + e_2)$. For a suitable choice of basis we have

$$z = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

The equations $\{zx\} = 0$, $\{zy\} = 0$ show that x and y have the form

$$x = \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix}.$$

we have

$$\{xy\} = xy - yx = \begin{pmatrix} YX - XY & 0 \\ 0 & YX - XY \end{pmatrix}.$$

If this is nonzero then $YX - XY$ is a nonzero scalar multiple of the identity matrix, an impossibility in characteristic 0.

(7) K <u>is</u> <u>not</u> <u>simple</u>. Take a maximal set of orthogonal primitive idempotents in K and apply (6). If K is simple we find that $L = 0$, contrary to hypothesis.

Now consider orthogonal primitive idempotents e_1 and e_2 residing in different simple summands of K. We shall in due course prove that (in the same Peirce notation as above) L_{12} is nonzero. As a temporary expedient, let us call e_1 and e_2 <u>related</u> if $L_{12} \neq 0$.

✓(8) <u>Let</u> e_1, e_2, <u>and</u> e_3 <u>be</u> <u>orthogonal</u> <u>primitive</u> <u>idempotents</u> <u>in</u> K, <u>with</u> e_1, e_2 <u>in</u> <u>the</u> <u>same</u> <u>simple</u> <u>summand</u> <u>of</u> K <u>and</u> e_3 <u>in</u> <u>a</u> <u>different</u> <u>one</u>. <u>If</u> e_1 <u>and</u> e_3 <u>are</u> <u>related</u>

general: e_1 related to e_3
e_1 intercomm to e_2
then e_2 related to e_3

so are e_2 *and* e_3. Here and in the rest of the discussion we continue to use the standard notation for a Peirce decomposition. Pick z in K_{12} with $z^2 = e_1 + e_2$; pick y nonzero in L_{13}. Then $\{zy\}$ lies in L_{23} and a matrix computation shows that it is nonzero.

know $L_{23} = L_{13} \circ K_{12}$ by interconnectivity

z invertible in $J_0(e_3)$, so V_z invertible on $J_{1/2}(e_3) = J_{13} + J_{23}$

(9) *Let* e_1, e_2, e_3 *be orthogonal primitive idempotents in three different simple summands of* K. *Then* e_1 *cannot be related to both* e_2 *and* e_3. Assume the contrary. Pick x and y nonzero in L_{12} and L_{13}. There must exist z in L_{13} with $\{yz\} \neq 0$; otherwise $\mathrm{Tr}\{yJ\} = 0$. The vanishing of the ungraded trace of $\{yz\} = yz - zy$ $= \alpha_1 e_1 + \alpha_3 e_3$ shows that $\{yz\}$ is a nonzero scalar multiple of $e_1 - e_3$.

CAREFUL! $\alpha_1 n_1 + \alpha_3 n_3 = 0$ but needn't have $n_1 = n_3$ so $\alpha_1, \alpha_3 \neq 0$

NO! just deduce $\alpha_1 \neq 0$?

Next we note that $\{xy\}$ and $\{xz\}$ are 0, for they lie in $K_{23} = 0$. So x commutes with y and z, hence with $e_1 - e_3$, a contradiction.

$0 = [x, [yz]] = [x, \alpha_1 e_1 + \alpha_3 e_3] = \alpha_1 [x, e_1]$ for $\alpha_1 \neq 0$, impossible

We choose, and hold fixed for the rest of the discussion, a maximal set of orthogonal primitive idempotents in K.

✓(10) *Let* A *and* B *be two simple summands of* K. *Suppose that some primitive idempotent in* A *is related to some primitive idempotent in* B. *Then any primitive idempotent in* A *is related to any primitive idempotent in* B. This is quickly seen by iterated use of (8).

$e \in A$ rel $f \in B$

$e \sim f \Rightarrow e \sim$

$e \sim f' \Rightarrow e' \sim$

General fact: e_1 related to e_3, e_1 interconn to e_2, then e_2 related to e_3

(11) K *has precisely two simple summands*. K has at least two by (7). As in (7), some primitive idempotents from different simple summands of K (say A and B) must be related, for otherwise L = 0. Let u be the unit element

of $A \oplus B$. It follows from (9) and (10) that the
$L_{12} \neq 0 \Rightarrow L_{12} = L$
off-diagonal Peirce space for the decomposition given by u is 0. Thus if $u \neq 1$, u yields a nontrivial direct sum decomposition of J, contradicting simplicity. Hence $K = A \oplus B$.

(12) Any L_{ij} linking primitive idempotents in the two summands of K is 2-dimensional. It follows from (10) and (11) that L_{ij} is nonzero. For x and y

NO?

in L_{ij} we have that $\{xy\} = ce_i - c'e_j$ (NO!) for a scalar c. $cn_i = c'n_j$ but not nec $c = c'$

The map assigning c to x, y is an alternate form on L_{ij}. The usual appeal to Tr shows that this form is nondegenerate. $c = 0 \Rightarrow c' = 0$, so $f\{xL_{ij}\} = 0 \Rightarrow \{xL_{ij}\} = 0 \Rightarrow Tr\{xL_{ij}\} = 0$ forces $x = 0$

If the dimension of L_{ij} exceeds 2, we can arrange to have $\{xt\} = \{yt\} = 0$ with t nonzero and $\{xy\} = e_i - de_j$. Then t commutes with x and y, and with $d(e_i + e_j)$, hence with $(1+d)e_i$ $\therefore$ with e_i as $d \neq -1$; $n_i = dn_j$, $d = \frac{n_i}{n_j} > 0$

hence with $\{xy\}$, a contradiction.

several times uses $[xu] = [xv] = 0 \Rightarrow [x, ab \pm ba] = 0$ when $[ab \pm ba]$ is not the algebra product

From this point on the identification of J with a full linear or orthosymplectic GJALT is fairly straightforward and also tedious (involving case distinctions and matrix computations). The procedure will therefore only be sketched, with details left to the reader.

The crucial case - the first in which there is work to do - is that in which the simple summands A, B of K have degrees 2 and 1. The easy deduction of the general case from this will be totally omitted.

(NO) deg A = deg B = 1
$A = Fe_1$ $B = Fe_2$
$L = Fx + Fy$
$[xy] = e_1 - e_2$
FULL MATRIX

So B is 1-dimensional, and A is an (ordinary) Jordan algebra of degree 2. We shall build up the

But if $[xy] = n_2 e_1 - n_1 e_2$

$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $y = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x = e_{12} + e_{31}$ $y = e_{13} - e_{21}$

$[xy] = (-e_{11} + e_{33}) - (e_{11} - e_{22})$
$= (e_{22} + e_{33}) - 2e_{11} = e_2 - 2e_1$

GET ONLY ISOTOPES OF FULL MATRIX?

structure of J in a 3 by 3 block matrix notation, the blocks having the sizes indicated:

why $r' = r$?
because interconnected $z_{12}^2 = e_1 + e_2$

$$\begin{pmatrix} r \text{ by } r & r \text{ by } r' & r \text{ by } s \\ r \text{ by } r & r' \text{ by } r' & r' \text{ by } s \\ s \text{ by } r & s \text{ by } r' & s \text{ by } s \end{pmatrix} .$$

A

The algebra A occupies the four blocks in the upper left. The matrices down the main diagonal are scalar. We write

$$p = \begin{pmatrix} 0 & 0 & P \\ 0 & 0 & 0 \\ Q & 0 & 0 \end{pmatrix} , \quad t = \begin{pmatrix} 0 & 0 & R \\ 0 & 0 & 0 \\ S & 0 & 0 \end{pmatrix}$$

for a basis of L_{13}. As we noted above, A contains an element z with $z^2 = e_1 + e_2$ and it can be taken in the form

$$z = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

We have

$$\{pt\} = \begin{pmatrix} PS - RQ & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & QR - SP \end{pmatrix} .$$

So PS - RQ and QR - SP are nonzero scalar matrices. The elements $\{zp\}$, $\{zt\}$ form a basis of L_{23}, built in exactly the same fashion out of P, Q, R, S. Multiplying L_{13} and L_{23} we find that

$p' = \{zp\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & P \\ 0 & Q & 0 \end{pmatrix}$ $\quad P_{23} + Q_{32}$

$t' \; \{zt\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & R \\ 0 & S & 0 \end{pmatrix}$ $\quad R_{23} + S_{32}$

$q = [p, t'] = [P_{13} + Q_{31}, R_{23} + S_{32}] = PS_{12} - RQ_{21} = \begin{pmatrix} 0 & PS & 0 \\ -RQ & 0 & 0 \end{pmatrix}$

$$q = \begin{pmatrix} 0 & PS & 0 \\ -RQ & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[p, p'] = \begin{pmatrix} 0 & RQ & 0 \\ -PQ & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

lies in A. Now come the various cases.

Case I. A is 3-dimensional. Then q must be a scalar multiple of z, so that PS = -QR = a scalar multiple of I; we can normalize to make them equal I. In the matrix for q we may replace B (S) by Q and R by P and the resulting element will also be a multiple of z. This tells us that $\boxed{PQ = 0}$ and likewise we have $\boxed{RS = 0}$. Suppose that QR - SP is the scalar c times the identity. Right-multiply by Q to get -Q = cQ, c = -1. Since the ungraded trace of {pt} is 0, we deduce that s = 2r. Bases may now be chosen so as to make P = (I 0), R = (0 -I),

$p' = (S, R)$

$\downarrow$

$p' = (Q, P)$

$PS = -QR = I_r$

always

$$\boxed{\begin{gathered} PS - RQ = d\, I_r \\ QR - SP = c\, I_s \\ r\cdot d + s\cdot c = 0 \end{gathered}}$$

$\underbrace{QR}_{-1}Q - \underbrace{\cancel{SP}Q}_{0} = cQ$

$-Q = cQ$ $c = -1$

$Q\underbrace{PS}_{1} - \underbrace{QR}_{1}Q = dQ$

$2Q = dQ$

$d = 2$

$2r - s = 0$

$s = 2r$

$$p = \begin{pmatrix} 0 & 0 & 1\,0 \\ 0 & 0 & 0\,0 \\ 1 & 0 & 0\,0 \\ 0 & 1 & 0\,0 \end{pmatrix} \quad t = \begin{pmatrix} 0 & 0 & 0\,{-1} \\ 0 & 0 & 0\,0 \\ 1 & 0 & 0\,0 \\ 0 & 0 & 0\,0 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 \\ I \end{pmatrix},\ S = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

On deflating the matrices by a factor of r we recognize ~~an~~ the symplectic algebra, acting on a four-dimensional vector space with two-dimensional symmetric and skew components.

From now on A is more than 3-dimensional. The remaining basis elements of A may be assumed to anticommute with z and therefore have the form

$$\begin{pmatrix} 0 & X & 0 \\ -X & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We may further assume $X^2 = I$.

Case II. A is 4-dimensional and X is a scalar. We may assume $X = I$. It is possible to take $Q = R = 0$. Then PS and SP are both scalar matrices, r must equal s, and P and S can be normalized to I. A deflation by r yields the full linear ~~algebra~~ GJALT on a 3-dimensional vector space.

Case III. A is 4-dimensional and X is not a scalar. Again we find $r = s$. It turns out that, J can be exhibited as the full linear GJALT on a 3-dimensional vector space repeated a certain number of times, and then repeated a certain number of times with each matrix transposed, the total number of replicas being r. In any event, J is isomorphic to that algebra.

Case IV. A is more than 4-dimensional. Since A admits a 4-dimensional module, Jordan module theory (see [5]) shows that the dimension of A is at most 6. A fifth basis element may be chosen with square the identity and anticommuting with both z and our 4th basis element. One then sees that our 4th and 5th basis elements can be put in the form

$$\begin{pmatrix} 0 & \begin{matrix} I & 0 \\ 0 & -I \end{matrix} & 0 \\ \begin{matrix} -I & 0 \\ 0 & I \end{matrix} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \begin{matrix} 0 & I \\ I & 0 \end{matrix} & 0 \\ \begin{matrix} 0 & -I \\ -I & 0 \end{matrix} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here we find that r must equal 2s and we can normalize so that $Q = (0\ I)$, $S = (I\ 0)$,

$$P = \begin{pmatrix} I \\ 0 \end{pmatrix} \quad , R = \begin{pmatrix} 0 \\ -I \end{pmatrix} .$$

After deflating the matrices by a factor of s we have the orthosymplectic algebra that acts on a 5-dimensional vector space, consisting of a 4-dimensional skew piece and a 1-dimensional symmetric piece. With this the proof of the theorem is complete.

We now present a number of examples which hopefully will point the way to future investigations.

Example 1. The characteristic is 3, the algebra is 3-dimensional, and it acts on a 3-dimensional graded vector space in which the first two basis vectors are even and the third odd. A basis for the algebra consists of the identity matrix and

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} .$$

The commutator of these two matrices is the identity, so the algebra closes and it is simple. The graded trace of the identity is 1. This shows the theorem failing for characteristic $\neq 0$. From now on all examples are in characteristic 0.

Example 2. Take the $2n^2$-dimensional graded associative algebra where the even and odd elements are of the form

Isomorphic? to $J_0 = M_n(k) = J_1$

$A = M_n(k) \otimes_k k[u]$ graded assoc $u^2=1$

$J = A^+$

$$\begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}, \quad \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix},$$

P and Q ranging ~~over all n~~ by ~~n~~ matrices. Under { } this is a simple GJALT. The graded trace vanishes identically. If we use [ ] instead, and trim by one dimension at top and bottom, we obtain the graded Lie algebras of Gell-Mann and Radicati (see page 567 of [3]).

Example 3. Take 2n by 2n matrices divided into blocks of size n. The map

$J = H(A, *)$

* symplectic involution on graded $A = M_{2n}(k)$

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \rightarrow \begin{pmatrix} S' & -Q' \\ R' & P' \end{pmatrix} \qquad (' = \text{transpose})$$

is an involution in the graded sense (one requires $(xy)^* = -y^*x^*$ when x and y are both odd). Give the algebra a Z-grading as follows:

(?)

$$\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} \text{degree } 0, \quad \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} \text{degree } 1, \quad \begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix} \text{degree } -1$$

We can convert this to a Z_2-grading by lumping together the portions of degrees 1, -1. The self-adjoint elements have the form

$$\begin{pmatrix} P & \text{skew} \\ \text{symm.} & P' \end{pmatrix}$$

and form a simple GJALT. Again the graded trace vanishes identically. The Lie counterpart (consisting of all elements skew with respect to the involution) appears in [6] under the designation P(m) and also at the end of [4].

Consider the following GJALT's: full linear, orthosymplectic, and those of Examples 2 and 3. It is a fact that they exhaust the simple GJALT's obtainable by taking all of a simple graded associative algebra (these are known - see [7]), or the self-adjoint elements under an involution. The analogous remark applies to the Lie case.

Garbled. Will be fixed when rewritten

What identity (in addition to commutativity, graded style) should one postulate for general graded Jordan algebras? One notes that $a^2b.a = a^2.ba$ is inadequate since it yields nothing when a is odd. The linearization is fine, however. Assume

$$ab.cd + ac.db + ad.bc = (bc.a)d + (cd.a)b + (db.a)c$$

except when two elements are odd and two even. If a,b are even and c, d are odd the assumption is changed to

$$ab.cd + ac.db - ad.bc = (bc.a)d + (cd.a)b - (db.a)c.$$

Example 4. With x and y odd and $xy = -yx = 1$ we have a 3-dimensional simple graded Jordan algebra. For characteristic 0 it is not "special", i. e. it cannot be represented by linear transformations, at least on a finite-dimensional vector space; but in Example 1 it is so exhibited for characteristic 3. Anderson [2, p. 1200] has encountered this algebra in a different context.

To conclude the paper we make a remark on the connection with triple systems. Consider the odd elements of a GJALT. They of course close under $\{\{ab\}c\}$. From the ungraded point of view the operation is actually $\{[ab]c\}$. Likewise the odd elements of a graded Lie algebra lead to $[\{ab\}c]$. These "mixed triple systems" have occurred in physics and may merit the attention of mathematicians.

Bibliography

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