CLASSIFYING SPACES IN K-THEORY

Max Karoubi

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This article originates from an (unpublished) conjecture by Atiyah and Singer expressing the classifying spaces of the functors $K^n(X)$, $n = 0, 1, 2, ...$ by means of suitable Fredholm operators (cf. §1 for a precise statement). This generalizes the Jänich-Atiyah theorem for $n = 0$ [3], [8]. Relying on the techniques of [12] and [18], we prove this generalization here. This article is thus the natural continuation of [13], [15] where a stable version was proven. In fact, we propose two proofs of the conjecture here. The first, which relies essentially on the results of [12] and [13], is described in §3. The second, outlined in §4 and in the appendix, consists of finding a simple map between the ordinary classifying spaces and those of Atiyah-Singer that proves to be a homotopy equivalence. This map, expressed using "Jacobi matrices" [25], allows for a simultaneous proof of the conjecture and the classic Bott periodicity theorems.

We have also taken this opportunity to systematically develop the K-theory of paracompact spaces. On the whole, these results are not new (cf. for example [7]) and are part of what is conventionally called "representable K-theory". The advantage of the viewpoint adopted here is that it provides, thanks to "Banach categories," a common framework for "classic" and "representable" K-theories.

The essential content of this article is obviously contained in §§3 and 4, with the first two sections being simply a preparation for them. The Atiyah-Singer conjecture and its immediate implications are described in §1. In §2, we have, on one hand, transposed the techniques of [12] to the paracompact setting and, on the other hand, made explicit the classifying spaces naturally associated with a Banach algebra. We thus obtain a unified framework for the ordinary classifying spaces and those of Atiyah-Singer.

The essential results contained in this article have been summarized in two Notes to the Comptes-Rendus de l'Académie des Sciences.

Finally, one last remark: although we have strived to be complete in the definitions and proofs, we could not resist the temptation to use the Banach categories from [12][1]. We leave it to the reader to convince themselves of the utility of this notion for simply proving the theorems proposed here.

Note added in proof. I have just learned that the conjecture has been resolved (by a completely different method, it seems) by Atiyah and Singer. To avoid any ambiguity, let us specify that the essential part of this article (with the exception of §4) was communicated to Atiyah (letter of August 4, 1967) in response to a question he had asked me. Let us also specify that this work is the conclusion of articles that have already appeared ([13], [14], [15], [16]).

[1] Or, what amounts to the same thing here, Banach algebras (cf. [12] Proposition 1.3.4).

I The Atiyah-Singer Conjecture

Let k be the field of real numbers or complex numbers and let $C^{p,q+2}$ be the Clifford algebra of k^{p+q+2} equipped with the quadratic form $-x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_{p+q+2}^2$ (cf. [12, §1.1] for details). This algebra is written as $F(n)$ or $F(n) \oplus F(n)$, where F is one of the three fields R, C, or H ($F(n)$ representing the algebra of $n \times n$ matrices with coefficients in F). Let H be a Hilbert space (over the field k) on which the Clifford algebra acts. If $e_1, ..., e_p, \epsilon_1, ..., \epsilon_{q+2}$ are the generators of $C^{p,q+2}$, these can be interpreted as automorphisms of H satisfying the relations

$$(e_i)^2 = -1, \quad i = 1, ..., p$$

$$(\epsilon_j)^2 = +I, \quad j = 1, ..., q+2,$$

and anticommuting pairwise. Conversely, the given of such automorphisms is of course equivalent to that of a $C^{p,q+2}$-module structure on H. Let $G^{p,q+2}$ be the finite subgroup of $(C^{p,q+2})^*$ multiplicatively generated by the e_i and ϵ_j. There then exists a metric on H invariant under the action of $G^{p,q+2}$. For this metric, we have $e_i^* = -e_i$ and $\epsilon_j^* = \epsilon_j$. Furthermore, according to the general theorems on semi-simple algebras, the Hilbert space H can be written in the form H'^n or $H'^n \oplus H''^n$ depending on whether $C^{p,q+2}$ is a simple algebra or not. In the latter case, H'^n (resp. H''^n) is a module over the first (resp. the second) factor $F(n)$ of the sum $F(n) \oplus F(n) \approx C^{p,q+2}$.

DEFINITION (1.1). *The $C^{p,q+2}$-module H is said to be "infinite" if the metric of H is invariant under the action of $G^{p,q+2}$ and if H'^n and $H'^{\in}$ (if applicable) are vector spaces of infinite dimension.*

REMARK. *Two infinite $C^{p,q+2}$-modules are, of course, isometric. Without loss of generality, we can therefore assume that H is written as $C^{p,q+2} \otimes H'$ where H' is an (ordinary) Hilbert space of infinite dimension. We also deduce from this that H can be equipped with an infinite $C^{p',q'}$-module structure ($p' \geq p, q' \geq q+2$) which, by restriction of scalars, gives the initial $C^{p,q+2}$-module structure.*

To such a Hilbert space H, we will associate the set $\mathfrak{F}^{p,q}(H)$ of (bounded) Fredholm operators [3] in H satisfying the following relations:

$$D^* = D$$

$$De_i = -e_i D, \quad i = 1, ..., p$$

$$D\epsilon_j = -\epsilon_j D, \quad j = 1, ..., q+1$$

(note that no algebraic relation is imposed between D and ϵ_{q+2}). If we equip $\mathfrak{F}^{p,q}(H)$ with the norm topology induced by End H, we can consider the connected component $\mathfrak{F}_{p,q}$ of ϵ_{q+2} in $\mathfrak{F}^{p,q}(H)$. Let $k^{p,q}$ be the discrete group $K^{p-q}(\text{Point})$. We will prove the following theorem in §3:

THEOREM (1.2) (ATIYAH-SINGER CONJECTURE[2]). *The space $k^{p,q} \times \mathfrak{F}_{p,q}$ is a classifying space for the functor $K^{p-q}(X)$.*

REMARK 1. The theorem holds in the real case as well as in the complex case (H being a real or complex Hilbert space, respectively). $K^n(X)$ likewise denotes either real or complex K-theory. When there is a risk of confusion, we will denote $KU^n(X)$ (resp. $KO^n(X)$) for the K^n groups of complex (resp. real) K-theory. This type of convention is valid for this entire article.

[2] We modify its statement slightly to adapt it to the notation of [12].

REMARK 2. One can define a Hopf space structure on $\mathfrak{F}_{p,q}$ in the following way: H being infinite, there exists an isometry $f : H \oplus H \to H$ of $C^{p,q+2}$-modules. This allows for the definition of a continuous map

$$\mathbb{F} : \mathfrak{F}_{p,q} \times \mathfrak{F}_{p,q} \to \mathfrak{F}_{p,q}$$

by the formula $D \oplus D' = f \cdot (D \oplus D') \cdot f^{-1}$. According to Kuiper's theorem [18], [6], applied to the real, complex, or quaternionic case[3], the isotopy class of f, and thus that of $\mathbb{F}$, is independent of the particular choice of f. We will then have a group isomorphism $K^{p-q}(X) \approx [X, k^{p,q} \times \mathfrak{F}_{p,q}]$, for the obvious group structure on the second factor.

REMARK 3. It is false that $\mathfrak{F}^{p,q}(H)$ is a classifying space for $K^{p-q}(X)$ in general (for example $\mathfrak{F}^{1,0}(H)$ has three connected components; cf. Lemma 1.4). However, we will show later that this is indeed the case if $p - q \not\equiv 1 \pmod 4$ in the real case and if $p - q \not\equiv 1 \pmod 2$ in the complex case.

REMARK 4. In the statement of Theorem 1.2, the space X is assumed to be compact to avoid any ambiguity in the definition of $K^{p-q}(X)$. If X is not compact, we will see in §§2 and 3 how to extend the definition of $K^{p-q}(X)$ so that the theorem remains true.

To apply Theorem 1.2 with different choices of the pair (p, q), we must describe the spaces $\mathfrak{F}^{p,q}(H)$ and $\mathfrak{F}_{p,q}$ more "concretely". This description is not essential for the proof of Theorem 1.2, and the reader may omit the end of this section without inconvenience. Unless explicitly stated otherwise, all Hilbert spaces considered are of infinite dimension.

From the general considerations of [12, §2.1], it is clear that the spaces $\mathfrak{F}^{p,q} = \mathfrak{F}^n$ depend, up to homeomorphism type, only on the difference $n = p - q \pmod 8$ (mod 2 in the complex case). To have an exhaustive list of the spaces $\mathfrak{F}_n$, 8 particular cases must therefore be considered (only 2 in the complex case). Note also that since two infinite $C^{p,q+2}$ Hilbert spaces are isometric, we have great latitude in the choice of the automorphisms e_i and ϵ_j.

Case $p = q = 0$. The algebra $C^{0,2}$ is then simple and we can choose $H = H' \oplus H'$

$$\epsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \epsilon_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

An element D of $\mathfrak{F}^{0,0}(H)$ is thus written

$$D = \begin{pmatrix} 0 & D' \\ D'^* & 0 \end{pmatrix}$$

where D' is a Fredholm operator in H'. If we denote $\mathfrak{F}(H')$ as the set of such operators, $\mathfrak{F}^{0,0}(H)$ is thus identified with $\mathfrak{F}(H')$ and $\mathfrak{F}_{0,0}$ with the subset of $\mathfrak{F}(H')$ formed by operators with index zero. Since the connected components of $\mathfrak{F}(H')$ are homeomorphic, we have $k^{0,0} \times \mathfrak{F}^{0,0} \approx \mathfrak{F}(H')$. Theorem 1.2 therefore implies the following well-known result[4]:

PROPOSITION (1.3) (ATIYAH-JÄNICH). *A classifying space for the group $K(X) \approx K^0(X)$ is the set $\mathfrak{F}^0$ of Fredholm operators in the Hilbert space H'.*

[3] We implicitly use the fact here that the category of $F(n)$-modules is equivalent to that of F-modules.

[4] In fact, we will use this result during the proof of Theorem 1.2.

Case $p-1=q=0$**.** The algebra $C^{1,2} \approx C^{0,1}(2)$ is no longer simple, but we can still choose $H = H' \oplus H'$,

$$\epsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \epsilon_2 = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$$

where $\alpha : H' \to H'$ is a self-adjoint endomorphism of H', with square one, such that Ker$(1-\alpha)$ and Ker$(1+\alpha)$ are of infinite dimension. A simple calculation on 2×2 matrices shows that $\mathfrak{F}^{1,0}(H)$ consists of endomorphisms of $H = H' \oplus H'$ that are written in the form

$$D = \begin{pmatrix} 0 & D' \\ D' & 0 \end{pmatrix};$$

where D' is a self-adjoint Fredholm operator. Let $\mathscr{F}$ denote the set of such operators D' and $D^{\vee\prime}$ the image of D' in the Banach algebra $A = \mathrm{End}H'/\mathfrak{K}$, where $\mathfrak{K}$ is the two-sided ideal of compact operators in H'[5].

LEMMA (1.4). *The set $\mathscr{F}$ has three connected components $\mathscr{F}^+$, $\mathscr{F}^-$ and $\mathfrak{F}_1$; $\mathscr{F}^+$ (resp. $\mathscr{F}^-$) is the set of Fredholm operators D' such that the spectrum of $D^{\vee\prime}$ is contained in R^+ (resp. R^-) and $\mathfrak{F}_1 = \mathscr{F} - \mathscr{F}^+ - \mathscr{F}^-$.*

Proof. By reasoning as in [3] or [15], we easily see that $\mathscr{F}$ has the same homotopy type as $\mathscr{F}^\vee$, $\mathscr{F}^\vee$ denoting the set of invertible elements Δ of the algebra A such that $\Delta^* = \Delta$. Let $\mathscr{F}^\circ$ be the subset of $\mathscr{F}^\vee$ formed by elements Δ of A that satisfy the additional condition $\Delta^2 = 1$. Classic theorems on the holomorphic functional calculus in a Banach algebra (cf. [27, p. 384]) imply that $\mathscr{F}^\circ$ is a deformation retract of $\mathscr{F}^\vee$. Furthermore, $\mathscr{F}^{\vee+}$, $\mathscr{F}^{\vee-}$ and $\mathfrak{F}_1^\vee$, images of $\mathscr{F}^+$, $\mathscr{F}^-$ and $\mathfrak{F}_1$ in A and having the same homotopy type as them, are invariant under this retraction. Consequently $\mathscr{F}^{\vee+}$ and $\mathscr{F}^{\vee-}$ (thus $\mathscr{F}^+$ and $\mathscr{F}^-$) are contractible. It remains to show that $\mathfrak{F}_1$ is connected. Consider two elements Δ and Δ' of $\mathfrak{F}_1$. Then Δ (resp. Δ') is homotopic in $\mathfrak{F}_1$ to $\Delta + \mathrm{Id}_{\mathrm{Ker}\Delta}$ (resp. to $\Delta' + \mathrm{Id}_{\mathrm{Ker}\Delta'}$). Without loss of generality, we can therefore assume that Δ and Δ' are invertible. In this case, the holomorphic functional calculus in the algebra End H also allows us to assume $\Delta^2 = \Delta'^2 = 1$. Let $K_1 = \mathrm{Ker}(\Delta - 1)$, $K_2 = \mathrm{Ker}(\Delta + 1)$, $K_1' = \mathrm{Ker}(\Delta' - 1)$, $K_2' = \mathrm{Ker}(\Delta' + 1)$. Since Δ and Δ' belong to $\mathfrak{F}_1$, the vector spaces K_1, K_2, K_1' and K_2' are of infinite dimension and we can find an isometry $\phi : H' = K_1 \oplus K_2 \to K_1' \oplus K_2' = H'$ such that $\Delta' = \phi^{-1}\Delta\phi$. The isometry ϕ being isotopic to the identity by Kuiper's theorem [6], [18], we deduce that Δ' is homotopic to Δ in $\mathfrak{F}_1$. □

Whence the proposition:

PROPOSITION (1.5). *A classifying space for the functor $K^1(X)$ is the set $\mathfrak{F}_1$ of self-adjoint Fredholm operators D such that the imaginary axis divides the spectrum of $D^\vee$ into two non-empty parts.*

The description of $\mathfrak{F}_{0,0}$ and $\mathfrak{F}_{1,0}$ being sufficient in the complex case, we will only consider the real case in what follows.

Case $p = q-1 = 0$**.** We have $C^{0,3} \approx C(2)$ and we can choose $H = H' \oplus H'$

$$\epsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \epsilon_3 = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix},$$

where α is a skew-symmetric automorphism of H' ($\alpha^* = -\alpha$) such that $\alpha^2 = -1$ (in other words, α is a complex structure on H' compatible with the metric). An element D of $\mathfrak{F}^{0,1}(H)$ is thus written in the form

$$D = \begin{pmatrix} 0 & -D' \\ D' & 0 \end{pmatrix};$$

where D' is Fredholm and skew-symmetric. Let $\mathfrak{F}^{-1}$ be the set of such operators.

[5] That is, limits of finite-rank operators.

LEMMA (1.6). *The space $\mathfrak{F}^{-1}$ consists of two isomorphic connected components. If $D' \in \mathfrak{F}^{-1}$, the parity of the dimension of the kernel of D' characterizes the connected component of D'.*

Proof. Let $(\mathfrak{F}^{-1})^+$ (resp. $(\mathfrak{F}^{-1})^-$) be the subset of $\mathfrak{F}^{-1}$ formed by operators D such that Ker D is of even (resp. odd) dimension. Reasoning as in the previous case and noting that there always exists an invertible skew-symmetric operator in an even-dimensional vector space, we see that $(\mathfrak{F}^{-1})^+$ and $(\mathfrak{F}^{-1})^-$ are connected. It remains to show that a point in $(\mathfrak{F}^{-1})^+$ and a point in $(\mathfrak{F}^{-1})^-$ cannot be joined by a path in $\mathfrak{F}^{-1}$. Recall for this the existence of a bijection (cf. [13] or [15]):

$$j : \overline{K}^{p,q}(X) \to K^{p,q}(X)$$

for any compact space X. On the other hand, we have an obvious surjection (thanks to Kuiper's theorem)

$$[X, \mathfrak{F}^{p,q}(H)] \to \overline{K}^{p,q}(X)$$

(cf. §4 for more details). In particular, if X is reduced to a point and if $p = q - 1 = 0$, we deduce a surjection

$$\pi_0(\mathfrak{F}^{-1}) \to Z_2.$$

The announced result follows immediately. □

REMARK. *To prove that $\pi_0(\mathfrak{F}^{-1}) = Z_2$, one could also have considered a family of real operators on the circle S^1. The author does not know of an "elementary" proof.*

COROLLARY (1.7). *A classifying space for the group $KO^{-1}(X)$ is the set $\mathfrak{F}^{-1} = \mathfrak{F}_{0,1}$ of skew-symmetric Fredholm operators in an infinite-dimensional (real) Hilbert space.*

Case $p - 2 = q = 0$**.** We have $C^{2,2} \approx R(4)$ and $C^{2,1} \approx C^{1,0}(2) \approx C(2)$. We can therefore write $H = H' \oplus H'$, H' being a complex Hilbert space, with

$$e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \epsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \epsilon_2 = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix};$$

where c is complex conjugation. Consequently, an element D of $\mathfrak{F}^{2,0}(H)$ is written in the form

$$D = \begin{pmatrix} 0 & D' \\ D' & 0 \end{pmatrix};$$

where $D' : H' \to H'$ is an anti-linear and self-adjoint (for the underlying real metric) Fredholm operator.

LEMMA (1.8). *The space $\mathfrak{F}^{2,0}(H)$ is connected.*

Proof. If $D' : H' \to H'$ is an anti-linear and self-adjoint Fredholm operator, KerD' is a (complex) vector subspace of H' of finite dimension. Furthermore, H' splits into the sum Ker$D' \oplus ($Ker$D')^\perp$ of subspaces invariant under the action of D'. Using the fact that one can find such an operator D' on any complex vector space, we see that we can assume D' is injective (thus bijective). By the holomorphic functional calculus in End$_R H'$, we are reduced by homotopy to the case $D'^2 = 1$. We can then split H' into the sum $H'' \oplus H''$ of real vector subspaces with

$$D' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

If c is complex conjugation, we can similarly split H' into the sum $H_1'' \oplus H_1''$ relative to c and find a (complex) isometry

$$\phi : H' = H'' \oplus H'' \to H_1'' \oplus H_1'' = H'$$

such that $D' = \phi^{-1} \cdot c \cdot \phi$. The isometry ϕ being homotopic to the identity, D' and c are homotopic, whence the result. □

The following proposition follows immediately:

PROPOSITION (1.9). *A classifying space for the group $KO^2(X)$ is the set $\mathfrak{F}_{2,0}$ of anti-linear and self-adjoint Fredholm operators in an infinite-dimensional complex Hilbert space.*

Case $p = q - 2 = 0$. Since $C^{0,4} \approx H(2)$, we can write $H = H' \oplus H'$ where H' is a quaternionic Hilbert space and

$$\epsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \epsilon_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \epsilon_4 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}.$$

An element D of $\mathfrak{F}^{0,2}(H)$ is thus written

$$D = \begin{pmatrix} 0 & -D' \\ D' & 0 \end{pmatrix},$$

where D' is anti-linear (for the C-structure of H') and skew-symmetric (for the underlying real metric).

LEMMA (1.10). *The space $\mathfrak{F}^{0,2}(H)$ has two isomorphic connected components. If $D \in \mathfrak{F}^{0,2}(H)$, the parity of the (complex) dimension of Ker$D$ characterizes the connected component of D.*

Proof. The proof of this lemma is analogous to that of Lemma 1.6. □

COROLLARY (1.11). *A classifying space for the group $KO^{-2}(X)$ is the set $\mathfrak{F}_{0,2}$ of anti-linear skew-symmetric Fredholm operators in an infinite-dimensional complex Hilbert space.*

Case $p = q - 4 = 0$. We have in general $C^{p,q+4} \approx C^{p,q} \otimes C^{0,4}$. Since $C^{0,4} \approx H(2)$, it follows from the preceding arguments that $\mathfrak{F}^{0,4}(H)$ is identified with the set of quaternionic Fredholm operators on a Hilbert space H' which is equipped with a module structure over the field of quaternions. From this we deduce the following well-known result:

PROPOSITION (1.12). *A classifying space for the functor $KO^4(X)$ is the set of H-Fredholm operators on a quaternionic Hilbert space.*

Case $p - 5 = q = 0$ and $p - 3 = q = 0$. In these two cases, we have the following propositions:

PROPOSITION (1.13). *A classifying space for the functor $KO^5(X) \approx KO^{-3}(X)$ is the set of self-adjoint H-Fredholm operators D on a quaternionic Hilbert space such that the imaginary axis divides the spectrum of $D^\vee$ into two non-empty parts.*

PROPOSITION (1.14). *A classifying space for $KO^3(X)$ is the set of skew-symmetric H-Fredholm operators on a quaternionic Hilbert space.*

Proof. The proof of these two propositions is analogous to that of Proposition 1.5 and Corollary 1.7. □

The various special cases thus considered in this section can be summarized in the following two theorems:

THEOREM (1.15). *We have a natural isomorphism*

$$KU^n(X) \approx [X, \mathfrak{F}U^n],$$

where $\mathfrak{F}U^n$ denotes the subset of $\mathfrak{F}(H)$ formed by C-linear operators D satisfying the following condition:

- $n \equiv 0 \pmod 2$*: no additional condition.*
- $n \equiv 1 \pmod 2$*: D is self-adjoint* $(D^* = D)$ *and the spectrum of* $D^\vee$ *is not located on only one side of the imaginary axis.*

THEOREM (1.16). *We have a natural isomorphism*

$$KO^n(X) \approx [X, \mathfrak{F}O^n],$$

where $\mathfrak{F}O^n$ *denotes the subset of* $\mathfrak{F}(H)$ *formed by real operators D satisfying the following condition:*

- $n \equiv 0 \pmod 8$*: no additional condition.*
- $n \equiv 1 \pmod 8$*: D is self-adjoint, the spectrum of* $D^\vee$ *not being located on only one side of the imaginary axis.*
- $n \equiv 2 \pmod 8$*: H is equipped with a complex structure for which D is self-adjoint and anti-linear.*
- $n \equiv 3 \pmod 8$*: H is equipped with a quaternionic structure; D is skew-symmetric* $(D^* = -D)$ *and is compatible with this structure.*
- $n \equiv 4 \pmod 8$*: H is equipped with a quaternionic structure and D is compatible with this structure.*
- $n \equiv -3 \pmod 8$*: H is equipped with a quaternionic structure; D is self-adjoint and is compatible with this structure. In addition, the spectrum of* $D^\vee$ *is not located on only one side of the imaginary axis.*
- $n \equiv -2 \pmod 8$*: H is equipped with a complex structure; D is skew-symmetric and anti-linear.*
- $n \equiv -1 \pmod 8$*: D is skew-symmetric.*

REMARK. *The adjoint* D^* *of an operator D is understood in the sense of the metric of the underlying real vector space. In the case where the Hilbert space H is complex (resp. quaternionic), we will write* $H = H' \otimes C \approx H' \oplus H'$ *(resp.* $H = H' \otimes H \approx H' \oplus H' \oplus H' \oplus H'$*), the metric of H being the sum of the metrics of each copy of* H'*.*

II Review and complements of K-theory

Let $\mathscr{C}$ be a Banach category in the sense of [10][6] and let X be a paracompact space. We want to define the notion of a "$\mathscr{C}$-bundle" over X. To do this, we start by considering the category $\mathscr{C}_T(X)$ of "trivial $\mathscr{C}$-bundles" over X: this category has the objects of $\mathscr{C}$ as its objects, and the morphisms from source E to target F are simply the continuous maps from X into $\mathrm{Hom}_{\mathscr{C}}(E, F)$. The composition of morphisms is immediate. The category of trivial $\mathscr{C}$-bundles is obviously additive, and the associated pseudo-abelian category (cf. [11]) will be denoted $\mathfrak{P}(X)$: an object of $\mathfrak{P}(X)$ is thus a pair (E, p) where E is a trivial $\mathscr{C}$-bundle and p is a projector of E (think "Im p"). A morphism from source (E, p) to target (F, q) is a morphism $f : E \to F$ of trivial $\mathscr{C}$-bundles such that $f \cdot p = q \cdot f = f$. We easily see that a $\mathscr{C}$-bundle (i.e., an object of $\mathfrak{P}(X)$) is "locally trivial" (cf. [12], Proposition 1.2.8).

When $\mathscr{C}$ is the category of finite-dimensional vector spaces, $\mathfrak{P}(X)$ is equivalent to the category of vector bundles that are direct factors of trivial bundles; this coincides with the category

[6] In fact a "pre-Banach" category will suffice for our considerations. For the reader's convenience, recall that this means that $\mathrm{Hom}_{\mathscr{C}}(M, N)$ is, for any pair (M, N), equipped with a Banach space structure such that the composition of morphisms $\mathrm{Hom}_{\mathscr{C}}(M, N) \times \mathrm{Hom}_{\mathscr{C}}(N, P) \to \mathrm{Hom}_{\mathscr{C}}(M, P)$ is bilinear and continuous.

of all vector bundles if X is compact. Another interesting particular case is when $\mathscr{C}$ is the category $\mathfrak{P}(A)$ of finitely generated projective modules over a Banach algebra A (when $A = R, C$ or H, we recover the previous particular case). Then $\mathfrak{P}(X)$ is equivalent to the category of Banach bundles whose fiber is a finitely generated projective A-module and which can be expressed as a direct factor of a trivial A-module bundle $X \times A^n$, $n = 0, 1, 2, ...$

DEFINITION (2.1). *Let X be a paracompact space and $\mathscr{C}$ a Banach category. Then $K(X; \mathscr{C})$ is the Grothendieck group of the additive category $\mathfrak{P}(X)$. If $\mathscr{C} = \mathfrak{P}(A)$, we simply denote $K(X; A)$ for the group $K(X; \mathfrak{P}(A))$.*

REMARK. *With the usual notations, we thus have (for X compact) the identities $K(X; R) = KO(X)$, $K(X; C) = KU(X)$, $K(X; H) = KSp(X)$.*

Let $C^{p,q}$ be the Clifford algebra of R^{p+q} equipped with the quadratic form of type (p, q). We define a group $K^{p,q}(X; \mathscr{C})$ as follows: let I be the set of triples (E, η_1, η_2) where E is a $\mathscr{C}$-bundle equipped with a $C^{p,q}$-module structure and where η_1 and η_2 are two "gradings" of E (i.e., involutive automorphisms that anticommute with the generators e_i, ϵ_j of the algebra $C^{p,q}$). Two triples (E, η_1, η_2) and (F, ξ_1, ξ_2) are equivalent if there exists a $C^{p,q}$-module $\mathscr{C}$-bundle G and a grading ξ of G, such that $\eta_1 \oplus \xi_2 \oplus \xi$ is homotopic to $\eta_2 \oplus \xi_1 \oplus \xi$ among the gradings of $E \oplus F \oplus G$. The quotient of I by the equivalence relation thus defined is denoted $K^{p,q}(X; \mathscr{C})$: it is clearly a group under the sum of triples. If $\mathscr{C} = \mathfrak{P}(A)$, we also denote it $K^{p,q}(X; A)$.

According to Lemma 2.1.13 of [12], this definition coincides with that of [12] (in the case where X is compact). On the other hand, according to the general considerations of [12] (cf. for example Proposition 2.1.18), generality is not restricted by assuming that E is a bundle of the form $C^{p,q+1} \otimes F$, where F is a trivial $\mathscr{C}$-bundle, with the first grading η_1 being defined by the last generator ϵ_{q+1} of $C^{p,q+1}$. The second grading η_2 is then interpreted as a continuous map from X into the space Grad(E) of "gradings" of E. To fix ideas, suppose that $\mathscr{C} = \mathfrak{P}(A)$ and consider $F = A^n$. We then denote $\text{Grad}^{p,q}(n, A)$ as the set of "gradings" of $E = C^{p,q+1} \otimes F$, i.e., the set of A-endomorphisms Δ of this A-module such that $\Delta^2 = 1$, $\Delta e_i = -e_i \Delta$ and $\Delta \epsilon_j = -\epsilon_j \Delta$, $i = 1, ..., p$; $j = 1, ..., q$. If $n = 1$, we simply denote $\text{Grad}^{p,q}(A)$ as the set $\text{Grad}^{p,q}(1, A)$; we obviously have $\text{Grad}^{p,q}(n, A) = \text{Grad}^{p,q}(A(n))$. On the other hand, we have obvious maps $\text{Grad}^{p,q}(n, A) \to \text{Grad}^{p,q}(n+1, A)$ defined by $\Delta \mapsto \Delta \oplus \epsilon_{q+1}$. Finally, we have a commutative diagram (for any paracompact space X):

$$\begin{array}{ccc} [X, \text{Grad}^{p,q}(n, A)] & \longrightarrow & [X, \text{Grad}^{p,q}(n+1, A)] \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ K^{p,q}(X; A) & = & K^{p,q}(X; A) \end{array}$$

where f_n is defined by the formula $f_n(\Delta) = d(E, \epsilon_{q+1}, \Delta)$. With the techniques of [12, §2.1], the following proposition is obvious:

PROPOSITION (2.2). *The maps f_n induce an isomorphism from $K^{p,q}(X; A)$ onto $\varinjlim_n [X, \text{Grad}^{p,q}(n, A)]$ for any paracompact space X.*

REMARK 1. If X is compact, we have an identity of a well-known type $\varinjlim_n [X, \text{Grad}^{p,q}(n, A)] \approx [X, \varinjlim_n \text{Grad}^{p,q}(n, A)]$.

REMARK 2. If $p = q = 0$, we obviously recover the ordinary group $K(X; A)$ (cf. [12], Proposition 2.1.7 for example).

REMARK 3. To define $\text{Grad}^{p,q}(n,A)$, we started from the $C^{p,q+1}$-A-module $N = C^{p,q+1} \otimes A$ and considered the gradings of N^n (regarded as a $C^{p,q}$-module). Of course, if we are only interested in the inductive limit

$$\varinjlim_n [X, \text{Grad}^{p,q}(n,A)],$$

we can replace N with any finitely generated projective $C^{p,q+1}$-A-module M, provided that it "generates" the category of such modules. By this we mean that any finitely generated projective $C^{p,q+1}$-A-module is a direct factor of M^n for n large enough. We can then identify

$$\varinjlim_n [X, \text{Grad}^{p,q}(n,A)] \quad \text{and} \quad \varinjlim_n [X, \text{Grad}^{p,q}(M^n)]$$

with obvious notations.

The definitions we have just given for the groups $K^{p,q}(X;\mathscr{C})$ and $K^{p,q}(X;A)$ extend immediately to the relative case. For example $K^{p,q}(X,Y;\mathscr{C})$ (for Y closed in X) is constructed using triples $(E,\epsilon_1,\epsilon_2)$ where ϵ_1 and ϵ_2 are two gradings that coincide over Y (cf. [12, §2.1] for details). We then have $K^{p,q}(X,Y;\mathscr{C}) \approx K^{p,q}(X/Y,\{y\};\mathscr{C})$, $p:(X,Y)\to(X/Y,\{y\})$ being the canonical projection (excision theorem). We denote $k^{p,q}(X;\mathscr{C})$ (resp. $k^{p,q}(X;A)$) as the subgroup of $K^{p,q}(X;\mathscr{C})$ (resp. $K^{p,q}(X;A)$) whose elements vanish when restricted to any point of X. Denoting $k^{p,q}$ as the (discrete) groups $K^{p,q}(\text{Point};\mathscr{C}) = K^{p,q}(\mathscr{C})$, we have an obvious decomposition of $K^{p,q}(X;\mathscr{C})$ into the direct sum $k^{p,q}(X;\mathscr{C}) \oplus H^0(X;k^{p,q})$. If X is a based space with base point x_0, we similarly set

$$\tilde{K}^{p,q}(X;\mathscr{C}) = K^{p,q}(X,\{x_0\};\mathscr{C}) = \text{Ker}(K^{p,q}(X;\mathscr{C}) \to K^{p,q}(\{x_0\};\mathscr{C})).$$

If X is connected, we of course have $k^{p,q}(X;\mathscr{C}) \approx \tilde{K}^{p,q}(X;\mathscr{C})$. In the case where $\mathscr{C}$ is the Banach category $\mathfrak{P}(A)$, let us finally denote $\text{grad}^{p,q}(n,A)$ (resp. $\text{grad}^{p,q}(A)$) as the connected component of ϵ_{q+1} in $\text{Grad}^{p,q}(n,A)$ (resp. $\text{Grad}^{p,q}(A)$). Taking into account the preceding definitions and remarks, Proposition 2.2 can be reformulated as:

PROPOSITION (2.3). *The obvious map from*

$$\varinjlim_n [X, \text{grad}^{p,q}(n,A)] \quad \textit{into} \quad k^{p,q}(X;A)$$

is an isomorphism.

COROLLARY (2.4). *For any paracompact space X, we have a natural isomorphism*

$$\varinjlim_n [X, K^{p,q}(A) \times \text{grad}^{p,q}(n,A)] \xrightarrow{\approx} K^{p,q}(X;A)^7.$$

It is quite easy (and classic, cf. [27] for example) to give an interpretation of the space $\text{grad}^{p,q}(A)$ (and consequently of $\text{grad}^{p,q}(n,A)$) in terms of a homogeneous space. Specifically, let $GL^{p,q}(A)$ be the group of A-$C^{p,q}$-automorphisms of $C^{p,q+1}\otimes A$ and let $gl^{p,q}(A)$ be the identity component of $GL^{p,q}(A)$. Let $GL'^{p,q+1}(A)$ (resp. $gl'^{p,q+1}(A)$) be the subgroup of $GL^{p,q}(A)$ (resp. $gl^{p,q}(A)$) formed by $C^{p,q+1}$-automorphisms. Let h be the map from $gl^{p,q}(A)$ to $\text{grad}^{p,q}(A)$ defined by $h(g) = g\epsilon_{q+1}g^{-1}$. Finally, let

$$\overline{h} : gl^{p,q}(A)/gl'^{p,q+1}(A) \to \text{grad}^{p,q}(A)$$

be the induced map.

LEMMA (2.5). *The map $\overline{h}$ is a homeomorphism.*

[7] We denote $K^{p,q}(\mathscr{C})$ and $K^{p,q}(A)$ as the groups $K^{p,q}(\text{Point};\mathscr{C})$, $K^{p,q}(\text{Point};A)$.

Proof. Let ϵ be an element of grad$^{p,q}(A)$ that belongs to the image of h, so $\epsilon = \alpha\epsilon_{q+1}\alpha^{-1}$. Consider an element η of grad$^{p,q}(A)$ close to ϵ. We then have the identity $\eta\beta = \beta\epsilon$ with $\beta = (1+\epsilon\eta)/2$, whence $\eta = (\beta\alpha)\epsilon_{q+1}(\beta\alpha)^{-1}$. This shows that h is an open map and that $gl^{p,q}(A)$ is a principal bundle over its image. Now let η be a point in the closure of Im h and let ϵ_n be a sequence of points in grad$^{p,q}(A)$ such that $\epsilon_n \to \eta$. For n sufficiently large, $\beta_n = (1+\epsilon_n\eta)/2$ will be invertible. Setting $\epsilon_n = \alpha_n\epsilon_{q+1}\alpha_n^{-1}$; we will thus have $\eta = h(\beta_n\alpha_n) \in \text{Im } h$. □

REMARK. *The spaces* Grad$^{p,q}(A)$ *and* $GL^{p,q}(A)/GL'^{p,q+1}(A)$ *are not homeomorphic in general (example:* $p = q = 0$ *and* $A = C$*).*

COROLLARY (2.6). *For X compact, the functor* $K^{p,q}(X; A)$ *is represented by the space*

$$K^{p,q}(A) \times \varinjlim_n gl^{p,q}(n, A)/gl'^{p,q+1}(n, A).$$

We will now clarify the fundamental theorem of [12, §2.2] in this context. This theorem states that we have an isomorphism

$$t : K^{p,q+1}(\text{Point}; \mathscr{C}) \to K^{p,q}(D^1, S^0; \mathscr{C}),$$

given by a formula of the following type

$$t(d(E, \eta_1, \eta_2)) = d(E', \eta_1(\theta), \eta_2(\theta)),$$

where E' is the underlying $C^{p,q}$-module of E and where $\eta_i(\theta) = \epsilon_{q+1}\cos\theta + \eta_i \sin\theta$, $i = 1, 2$, is the grading of E' over the point $e^{i\theta}$ of $D^1 \subset S^1 \subset C$, $\theta \in [0, \pi]$. In fact, the same formula allows us to define a homomorphism

$$t : K^{p,q+1}(X; \mathscr{C}) \to K^{p,q}(X \times D^1, X \times S^0; \mathscr{C})$$

for any paracompact space X. Let us give another interpretation of t in terms of reduced K-theory. We thus assume that X is a based space with base point x_0 and we want to define a homomorphism

$$\tilde{t} : \tilde{K}^{p,q+1}(X) \to \tilde{K}^{p,q}(SX) \approx K^{p,q}(X \times S^1, \{x_0\} \times S^1 \cup X \times \{1\}),$$

where SX is the suspension of X (the category $\mathscr{C}$ is understood, to lighten the notation). We set for this

$$\tilde{t}(d(E, \eta_1, \eta_2)) = d(E', \eta'_1(\theta), \eta'_2(\theta)).$$

where $\eta'_i(\theta)$, $i = 1, 2$ are the gradings of E' defined over the point $e^{i\theta}$ of S^1 (more correctly of $X \times S^1$) by the formulas:

$$\eta'_1(\theta) = \epsilon_{q+1}\cos\theta + \eta_1 \sin\theta,$$

$$\eta'_2(\theta) = \epsilon_{q+1}\cos\theta + \eta_2 \sin\theta \quad \text{for } 0 \le \theta \le \pi$$
$$\eta'_2(\theta) = \epsilon_{q+1}\cos\theta + \eta_1 \sin\theta \quad \text{for } \pi \le \theta \le 2\pi$$

We then have a commutative diagram

$$\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\tilde{K}^{p,q+1}(X) & \xrightarrow{\tilde{t}} & \tilde{K}^{p,q}(SX) \\
\downarrow & & \downarrow \\
K^{p,q+1}(X) & \xrightarrow{t} & K^{p,q}(X \times D^1, X \times S^0) \\
\downarrow & & \downarrow \\
K^{p,q+1}(\{x_0\}) & \xrightarrow{t} & K^{p,q}(\{x_0\} \times D^1, \{x_0\} \times S^0) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array}$$

From this diagram, $\tilde{t}$ is determined by t. Conversely, t is also determined by $\tilde{t}$ (consider X^+ the disjoint union of X and a point). This allows us to consider t or $\tilde{t}$ indifferently, according to the convenience of the presentation.

According to [12, §2.2], it is clear that the generalized homomorphism t (and thus $\tilde{t}$) is bijective when X is compact. In the case where X is paracompact, we cannot rely on the techniques of [12]. The essential reason why the arguments of [12] cannot be applied to the paracompact case is the impossibility of approximating a Fourier series, depending parametrically on X, by the sum of a finite number of its terms (more precisely, by Cesàro means). Suppose, however, that the functors $K^{p,q}(X)$ are representable (for X paracompact) by based H-spaces $B^{p,q}$ having the homotopy type of CW-complexes. We must therefore have a natural isomorphism $\tilde{K}^{p,q}(X) \approx [X, B^{p,q}]'$ where $[,]'$ denotes the set of homotopy classes of continuous maps respecting the base points. Suppose further that the homomorphism $\tilde{t}$ is induced by a continuous map

$$t' : B^{p,q+1} \to \Omega B^{p,q}$$ [8]

It will then follow from Theorem 2.2.2 of [12] that

$$\pi_i(t') : \pi_i(B^{p,q+1}) \to \pi_i(\Omega B^{p,q})$$

is an isomorphism. According to Milnor [21], the space $\Omega B^{p,q}$ also has the homotopy type of a CW-complex. A well-known theorem by J. H. C. Whitehead [26] shows that t' is a homotopy equivalence[9], which implies that $\tilde{t}$ (and thus t) is an isomorphism in the paracompact case. We can thus summarize the preceding discussion:

DEFINITION (2.7). *A Banach category $\mathscr{C}$ is said to be "CW-representable" if, for X paracompact, the functors $\tilde{K}^{p,q}(X; \mathscr{C})$ are representable by H-spaces having the homotopy type of CW-complexes and if the homomorphism $\tilde{t}$ is induced by a continuous map between representing spaces.*

THEOREM (2.8). *Suppose the Banach category $\mathscr{C}$ is CW-representable. Then the homomorphism*

$$t : K^{p,q+1}(X; \mathscr{C}) \to K^{p,q}(X \times D^1, X \times S^0; \mathscr{C})$$

is an isomorphism for any paracompact space X.

[8] The base point of $\Omega B^{p,q}$ is not necessarily the "constant" loop at the base point of $B^{p,q}$.

[9] $B^{p,q}$ being an H-space, all connected components have the same homotopy type.

Let us now consider the important particular case where $\mathscr{C} = \mathfrak{P}(A)$. According to Corollary 2.4, we have an isomorphism

$$\tilde{K}^{p,q}(X;A) \approx \varinjlim_n [X, K^{p,q}(A) \times \text{grad}^{p,q}(n,A)]'.$$

Moreover, on the connected components of the classifying spaces, $\tilde{t}$ is obviously induced by a map

$$s' : \text{grad}^{p,q+1}(n,A) \to \Omega^0 \text{grad}^{p,q}(2n,A)^{10}$$

where $s'(\epsilon)$ for a grading ϵ, is defined by the formula:

$$s'(\epsilon)(\theta) = \epsilon_{q+1} \cos\theta + \epsilon \sin\theta \quad \theta \in [0,\pi],$$
$$s'(\epsilon)(\theta) = \epsilon_{q+1} \cos\theta + \epsilon_{q+2} \sin\theta \quad \theta \in [\pi, 2\pi].$$

If we assume that the map $\text{grad}^{p,q}(n,A) \to \text{grad}^{p,q}(n+1,A)$ is a homotopy equivalence and that $\text{grad}^{p,q}(A)$ has the homotopy type of a CW-complex[11], we will have from the above a homotopy equivalence

$$s' : \text{grad}^{p,q+1}(A) \to \Omega^0 \text{grad}^{p,q}(A).$$

By setting $B^{p,q} = K^{p,q}(A) \times \text{grad}^{p,q}(A)$, we then find that the hypotheses of Theorem 2.8 are satisfied (the H-space structure of $\text{grad}^{p,q}(A)$ being induced by the direct sum of the gradings). Whence the proposition:

PROPOSITION (2.9). *Let A be a Banach algebra such that, for all integers p, q, and n, the map*

$$\text{grad}^{p,q}(n,A) \to \text{grad}^{p,q}(n+1,A)$$

is a homotopy equivalence. Then the category $\mathfrak{P}(A)$ is CW-representable. In particular, the homomorphism

$$t : K^{p,q+1}(X;A) \to K^{p,q}(X \times D^1, X \times S^0; A)$$

is bijective for any paracompact space X.

Consider again a general Banach category $\mathscr{C}$ and a pair (X, Y), where X is paracompact and Y is closed in X (thus paracompact). We can then define a connecting operator

$$\partial : K^{p,q}(Y \times D^1, Y \times S^0; \mathscr{C}) \to K^{p,q}(X, Y; \mathscr{C}).$$

Let us briefly indicate how ∂ is defined in this context (we refer the reader to [15] for more details[12]). Let $d(E, \epsilon_1(\theta), \epsilon_2(\theta))$ be an element of $K^{p,q}(Y \times D^1, Y \times S^0; \mathscr{C})$, where E is a trivial $C^{p,q}$-bundle and where $\epsilon_1(\theta)$ and $\epsilon_2(\theta)$ are two gradings of E (considered as a bundle over Y), depending continuously on a parameter θ varying between 0 and π. By adding an elementary triple if necessary (i.e., equal to 0 in the corresponding $K^{p,q}$ group), we can assume that $\epsilon_1(0)$ comes from a grading ϵ of E considered as a bundle over X. A "homotopy extension theorem" for gradings then allows us to extend $\epsilon_1(\theta)$ and $\epsilon_2(\theta)$ to gradings $\tilde{\epsilon}_1(\theta)$ and $\tilde{\epsilon}_2(\theta)$ of E, considered as a $C^{p,q}$-bundle over X, such that $\tilde{\epsilon}_1(0) = \tilde{\epsilon}_2(0) = \epsilon$. We then set

$$\partial(d(E, \epsilon_1(\theta), \epsilon_2(\theta))) = d(E, \tilde{\epsilon}_1(\pi), \tilde{\epsilon}_2(\pi)),$$

a definition independent of the choice of extensions.

[10] Ω^0 denotes the connected component of the loop $\epsilon_{q+1} \cos\theta + \epsilon_{q+2} \sin\theta$, $\theta \in [0, 2\pi]$, in $\text{grad}^{p,q}(2n,A)$, it being understood that $(C^{p,q+2} \otimes A)^n$ is identified with $(C^{p,q+1} \otimes A^2)^n$ as a $C^{p,q+1}$-module.

[11] In fact, $\text{grad}^{p,q}(A)$ always has the homotopy type of a CW-complex because it has the homotopy type of operators D such that Spec $D \cap iR = \emptyset$, which is open in a suitable Banach space (cf. [6, Lemma 1] or [24]).

[12] Strictly speaking, in [15] we are only interested in the compact case, but the generalization to the paracompact case presents no difficulty.

PROPOSITION (2.10) (cf. [15]. *] For any Banach category $\mathscr{C}$, we have an exact sequence*

$$K^{p,q}(X\times D^1, X\times S^0;\mathscr{C}) \longrightarrow K^{p,q}(Y\times D^1, Y\times S^0;\mathscr{C}) \xrightarrow{\partial} K^{p,q}(X,Y;\mathscr{C}) \longrightarrow K^{p,q}(X;\mathscr{C}) \longrightarrow K^{p,q}(Y;\mathscr{C}).$$

Finally, note that the $K^{p,q}$ groups depend only on the congruence of $p-q \pmod 8$ in the real case and $p-q \pmod 2$ in the complex case (cf. [12, §2.1]). Thus we will often write K^n instead of $K^{p,q}$ when $p-q=n$. Theorem 2.8 and the preceding proposition therefore imply the following result:

THEOREM (2.11). *Suppose the Banach category $\mathscr{C}$ is CW-representable. We then have a "cohomology exact sequence"*

$$K^{n-1}(X;\mathscr{C}) \longrightarrow K^{n-1}(Y;\mathscr{C}) \longrightarrow K^{n}(X,Y;\mathscr{C}) \longrightarrow K^{n}(X;\mathscr{C}) \longrightarrow K^{n}(Y;\mathscr{C})$$

for any paracompact space X and any closed subspace Y, ∂^{n-1} being the composite of the homomorphisms t and ∂.

COROLLARY (2.12). *Suppose the Banach category $\mathscr{C}$ is CW-representable. Then the functors $K^n(X,Y;\mathscr{C})$, equipped with the coboundary operators ∂ defined above, constitute a cohomology theory on the category of pairs (X, Y) where X is paracompact and Y is closed in X.*

REMARK. *It is not at all obvious a priori that non-trivial CW-representable Banach categories exist. Indeed, the usual Banach categories in K-theory $\mathfrak{P}(R)$, $\mathfrak{P}(C)$ and $\mathfrak{P}(H)$ are not. One of the goals of the next section is to construct one, which will also allow us to prove Theorem 1.2 at the same time.*

III Proof of Theorem 1.2

We will apply the general considerations of the previous section to the following situation: let H' be an infinite-dimensional Hilbert space[13] and let B be the Banach algebra of continuous endomorphisms of H'. The compact operators form a two-sided ideal $\mathfrak{K}$ in B, and we will denote A as the quotient Banach algebra $B/\mathfrak{K}$. We will see that this Banach algebra and the associated Banach category $\mathfrak{P}(A)$ play an essential role in the proof of Theorem 1.2. Beforehand, it is good to note that the category $\mathfrak{P}(A)$ is equivalent (as a Banach category) to the category $\mathscr{H}^\vee$ considered in [13]. Indeed, $\mathscr{H}^\vee$ is by definition the Banach category associated with the following pre-Banach category $\mathscr{H}'$: the objects of $\mathscr{H}'$ are infinite-dimensional Hilbert spaces, and the morphisms are (bounded) homomorphisms between Hilbert spaces considered modulo compact operators. Up to replacing $\mathscr{H}'$ with an equivalent category, we can assume that the objects of $\mathscr{H}'$ are H'^n, $n=0,1,...$ We then define a functor ψ from $\mathscr{H}'$ to $\mathfrak{P}(A)$ (category of free modules over A) by the obvious formulas $\psi(H'^n)=A^n$, $\psi(\alpha)=\alpha^\vee$ (matrix notation for homomorphisms). It is trivial that ψ is an equivalence of pre-Banach categories and that consequently $\psi^\vee:\mathscr{H}^\vee\to\mathfrak{P}(A)$ is an equivalence of the associated Banach categories. Note that, similarly, the category $\mathfrak{P}(B)$ is equivalent to the category $\mathscr{H}$ of Hilbert spaces (infinite-dimensional or not). In this context, the obvious functor $\phi:\mathscr{H}\to\mathscr{H}^\vee$ is interpreted as the extension of scalars functor $\mathfrak{P}(B)\to\mathfrak{P}(A)$ defined by the usual formula $M\mapsto A\otimes_B M$.

Now let H be an infinite $C^{p,q+2}$-module in the sense of §1. Without loss of generality, we can assume that $H=C^{p,q'+1}\otimes H'$[14], the $C^{p,q'+1}$-module structure being induced by the first factor of the tensor product. Let D be an element of $\mathfrak{F}^{p,q}(H)$. Since $D^*=D$ and $D^\vee=\phi(D)$ is invertible,

[13] Real or complex depending on the K-theory under consideration.

[14] We set $q'=q+1$ to simplify the notation throughout this section.

the spectrum of $D^\vee$ is contained in $R - \{0\}$ (cf. [19]). Let γ^+ (resp. γ^-) be a closed differentiable curve containing the spectrum of D located to the right (resp. to the left) of the imaginary axis, and let Δ be the element of Grad$^{p,q'}(A)$ defined by the integral (which depends only on $D^\vee$):

$$\Delta = I(D^\vee) = \frac{1}{2i\pi}\int_{\gamma^+}\frac{dz}{z - D^\vee} - \frac{1}{2i\pi}\int_{\gamma^-}\frac{dz}{z - D^\vee}.$$

In fact, Δ satisfies the additional condition $\Delta^* = \Delta$ (we can define an adjoint in the category $\mathscr{H}^\vee$ since the adjoint of a compact operator is compact). Let Grad$^{p,q'}(A)^*$ be the subset of Grad$^{p,q'}(A)$ formed by gradings that satisfy this additional condition. Similarly, let GRAD$^{p,q'}(A)$ (resp. GRAD$^{p,q'}(A)^*$) be the set of endomorphisms D of H such that $D^\vee \in$ Grad$^{p,q'}(A)$ (resp. Grad$^{p,q'}(A)^*$) and which anticommute with the generators e_i, ϵ_j of $C^{p,q'} \subset C^{p,q'+1}$.

LEMMA (3.1). *The maps*

$$\mathfrak{F}^{p,q}(H) \longrightarrow \text{Grad}^{p,q'}(A)^*$$

and

$$\text{GRAD}^{p,q'}(A) \xrightarrow{\phi|\text{GRAD}^{p,q'}(A)} \text{Grad}^{p,q'}(A)$$
$$\text{GRAD}^{p,q'}(A)^* \xrightarrow{\phi|\text{GRAD}^{p,q'}(A)^*} \text{Grad}^{p,q'}(A)^*$$

are homotopy equivalences.

Proof. The last two maps are induced on a suitable subset by the projection

$$Q : \text{End}_{\mathscr{H}} H \to \text{End}_{\mathscr{H}^\vee} H.$$

According to a well-known theorem [20], Q admits a global section s (perhaps non-linear). If we restrict ourselves to endomorphisms that anticommute with the generators of $C^{p,q'}$, say $\text{End}^{\sim}_{\mathscr{H}} H$ and $\text{End}^{\sim}_{\mathscr{H}^\vee} H$, we find a section $\tilde{s}$ of the corresponding projection by setting

$$\tilde{s}(\Delta) = \frac{1}{2^{p+q'+1}} \sum_{g \in G^{p,q'}} (-1)^{\omega(g)} g^{-1} s(g\Delta),$$

where $\omega(g)$ denotes the degree of g. The same remark applying for the adjoint, we deduce that the last two maps of the lemma are fibrations with contractible fiber. Indeed, this fiber is the vector space of compact operators anticommuting with the generators of $C^{p,q'}$. It follows that these maps are homotopy equivalences. Now let $\mathfrak{F}^{\vee p,q}(H)$ be the subset of $\text{End}^{\sim}_{\mathscr{H}^\vee} H$ image of $\mathfrak{F}^{p,q}(H)$ by the canonical functor ϕ. By an analogous argument, the projection

$$\mathfrak{F}^{p,q}(H) \xrightarrow{\phi|\mathfrak{F}^{p,q}(H)} \mathfrak{F}^{\vee p,q}(H)$$

is also a homotopy equivalence. To show that the same holds for the first map, it suffices to show that Grad$^{p,q'}(A)^*$ is a strong deformation retract of $\mathfrak{F}^{\vee p,q}(H)$. To do this, we set

$$r_t(D) = tD + (1-t)I(D), \quad D \in \mathfrak{F}^{\vee p,q}(H), \quad t \in [0,1].$$

Then r_t is the desired retraction. □

LEMMA (3.2). *The inclusion map*

$$i : \text{GRAD}^{p,q'}(A)^* \to \text{GRAD}^{p,q'}(A)$$

is a homotopy equivalence.

Proof. The space GRAD$^{p,q'}(A)^*$ has the homotopy type of $\mathfrak{F}^{p,q}(H)$, which is open in an obvious Banach space. According to [21] (see also [6, Lemma 1]), GRAD$^{p,q'}(A)^*$ therefore has the homotopy type of a CW-complex. Let $\mathfrak{F}'^{p,q}(H)$ be the set of endomorphisms D of H that anticommute with the generators of $C^{p,q'}$ and are such that the spectrum of D does not meet the imaginary axis. An analogous argument to the preceding one shows that GRAD$^{p,q'}(A)$ has the homotopy type of $\mathfrak{F}'^{p,q}(H)$. It follows that GRAD$^{p,q'}(A)$ also has the homotopy type of a CW-complex. We are thus led to show that the inclusion map induces a bijection

$$\Phi_X : [X, \text{GRAD}^{p,q'}(A)^*]' \to [X, \text{GRAD}^{p,q'}(A)]'$$

for any paracompact space X (in fact, spheres suffice, cf. [26]). In this case $[X, \text{GRAD}^{p,q'}(A)]'$ is identified with the set of homotopy classes of Hilbert bundles as infinite $C^{p,q'+1}$-modules over X, trivializable and equipped with an endomorphism D satisfying certain properties (in fact, any bundle of this type is trivializable according to Kuiper's theorem [6], [18], but the application of this theorem at this precise spot is not essential). We have an analogous description of the set $[X, \text{GRAD}^{p,q'}(A)^*]'$. If D is an endomorphism of the bundle E that does not necessarily satisfy the condition $D^{\vee *} = D^{\vee}$, it is easy to change the metric of E so that this is the case. Indeed, if $<,>'$ is the initial metric of E, it suffices to set

$$< x, y >=< x, y >' + < Dx, Dy >'$$

Then $D^{\vee *} = D^{\vee}$ for the new metric because $D^{\vee 2} = 1$. Since any two metrics are homotopic and X is paracompact, we can trivialize E equipped with the new metric. In other words, there exists an automorphism α of the bundle E such that $\Delta^* = \Delta$ for the old metric, with $\Delta = \alpha D \alpha^{-1}$ and α homotopic to the identity. This reasoning shows that Φ_X is surjective. One would similarly show that Φ_X is injective by considering $X \times [0, 1]$. □

The two preceding lemmas imply that the map

$$\mathfrak{F}^{p,q}(H) \to \text{Grad}^{p,q'}(A)$$

is a homotopy equivalence. By replacing H' with H'^n and H with H^n, one would show that the same holds for the map

$$\mathfrak{F}^{p,q}(H^n) \to \text{Grad}^{p,q'}(n, A).$$

From this, we immediately deduce the following proposition:

PROPOSITION (3.3). *The maps*

$$\mathfrak{F}_{p,q}(H^n) \to \text{grad}^{p,q'}(n, A)$$

are homotopy equivalences.

Consider now a real, complex, or quaternionic Hilbert space E and let A_E be the corresponding Banach algebra End$E/\mathfrak{K}$.

LEMMA (3.4). *The natural map from $GL(n, A_E)$ to $GL(n + 1, A_E)$ is a homotopy equivalence.*

Proof. We do not restrict generality by considering only the case $n = 1$. We also have a commutative diagram

$$\begin{array}{ccc} \mathfrak{F}(E) & \longrightarrow & \mathfrak{F}(E \oplus E) \\ \downarrow & & \downarrow \\ GL(1, A_E) & \longrightarrow & GL(2, A_E) \end{array}$$

where the vertical arrows are trivial fibrations with contractible fiber by an already-tested argument ($\mathfrak{F}(H)$ denoting the set of real, complex, or quaternionic Fredholm operators, as the case may be, in the space H). We are thus led to prove the lemma's conclusion for the first horizontal arrow of the diagram. Since the spaces considered have the homotopy type of CW-complexes, it suffices to show that

$$[X, \mathfrak{F}(E)] \to [X, \mathfrak{F}(E \oplus E)]$$

is bijective for a compact space X (in fact, a sphere suffices here too). But this is a well-known consequence of the Jänich-Atiyah theorem (cf. [3], [8] or [18] and [23]). □

PROPOSITION (3.5). *The natural map*

$$\mathsf{grad}^{p,q'}(n, A) \to \mathsf{grad}^{p,q'}(n+1, A)$$

defined by $\Delta \mapsto \Delta \oplus \epsilon_{q'+1}$ *is a homotopy equivalence.*

Proof. The spaces considered having the homotopy type of CW-complexes, it suffices to show that this map induces an isomorphism on the homotopy groups π_i, $i = 1, 2, ...$ According to Lemma 2.5, we have two locally trivial fibrations

$$gl'^{p,q'+1}(n, A) \longrightarrow gl^{p,q'}(n, A) \longrightarrow \mathsf{grad}^{p,q'}(n, A)$$
$$gl'^{p,q'+1}(n+1, A) \longrightarrow gl^{p,q'}(n+1, A) \longrightarrow \mathsf{grad}^{p,q'}(n+1, A)$$

We then apply the preceding lemma and the five lemma to the two corresponding exact homotopy sequences, taking into account the classification of Clifford algebras in terms of matrix algebras over the real, complex, or quaternionic numbers [12, §1.1][15]. □

COROLLARY (3.6). *The natural map* $\mathfrak{F}_{p,q}(H) \to \mathfrak{F}_{p,q}(H \oplus H)$ *is a homotopy equivalence.*

COROLLARY (3.7). *The category* $\mathfrak{P}(A)$ *is CW-representable in the sense of Definition 2.7. In particular, Corollary 2.12 applies to this category. The representing spaces for the functors* $K^{p,q'}(X; A) = K^{p,q+1}(X; A)$ *are the spaces*

$$K^{p,q'}(A) \times \mathsf{grad}^{p,q'}(A) \sim K^{p,q+1}(A) \times \mathfrak{F}_{p,q}(H)^{16}.$$

Proof. This is indeed a direct consequence of the preceding proposition and Propositions 2.9 and 3.3. □

End of the proof of Theorem 1.2. Taking into account that the groups $K^{p,q}$ only depend on the difference $p-q \pmod 8$, it remains for us to show that, for X compact, we have an isomorphism $K^n(X) \approx K^{n-1}(X; A)$ with $n \geq 1$ for example. This isomorphism is described in great detail in [13] and [15][17]; moreover, its general scope was emphasized in [14] and [16]. We will therefore limit ourselves to the following few remarks: First, we have $K^{n-1}(X; A) \approx K^{n-1}(X; \mathscr{H}^\vee)$ as a consequence of the equivalence of Banach categories $\mathscr{H}^\vee$ and $\mathfrak{P}(A)$ noted at the beginning of this section. Second, we have the exact sequence of pre-Banach categories

$$0 \to \mathscr{C}_T(X) \to \mathscr{H}_T(X) \to \mathscr{H}_T^\vee(X) \to 0$$

where $\mathscr{C}_T(X)$ generally denotes the category of trivial $\mathscr{C}$-bundles (cf. §2). Since the category $\mathscr{H}_T(X)$ is flabby [14, Definition 3], the connecting operator $\partial^{n-1} : K^n(\mathscr{H}_T^\vee(X)) \to K^{n-1}(\mathscr{C}_T(X))$ is an isomorphism [14, Theorem 7]; a "concrete" description of ∂^{n-1} in this particular case is given in [13]. This completes the proof of Theorem 1.2.

[15]For the calculation of the π_1 groups, note that $\pi_0(gl'^{p,q'+1}(n, A))$ is the number of connected components of $GL'^{p,q'+1}(n, A)$ contained in $gl^{p,q'}(n, A)$ and that a homotopy equivalence respects connected components.

[16]The preceding reasoning shows that Proposition 3.5 (thus Corollary 2.12 for $\mathscr{C} = \mathfrak{P}(A)$) is true more generally for Banach algebras A such that $GL(n, A') \to GL(n+1, A')$ is a homotopy equivalence, $A' = A, A \otimes R, C$ and $A \otimes R, H$, for any n. Such an algebra is called stable.

[17]See also the end of §4 for an "elementary" proof in the spirit of this article.

REMARK (3.8). *The homotopy equivalence*

$$s' : \mathsf{grad}^{p,q+1}(A) \to \Omega^0 \mathsf{grad}^{p,q}(A)$$

obviously transfers to the spaces $\mathfrak{F}_{p,q}$. *We thus have a homeomorphism*

$$S : \mathfrak{F}_{p,q+1} \to \Omega^0 \mathfrak{F}_{p,q}$$

which is defined by the formula

$$S(D)(\theta) = \epsilon_{q+1} \cos\theta + D \sin\theta \quad \textit{for } 0 \leq \theta \leq \pi$$
$$S(D)(\theta) = \epsilon_{q+1} \cos\theta + \epsilon_{q+2} \sin\theta \quad \textit{for } \pi \leq \theta \leq 2\pi.$$

As was noted in §2, this homeomorphism corresponds on the Grothendieck groups to the "Thom isomorphism":

$$\tilde{K}^{p,q+2}(X; \mathscr{H}^\vee) \to \tilde{K}^{p,q+1}(SX; \mathscr{H}^\vee).$$

For X compact, this again gives the ordinary isomorphism $\tilde{K}^n(X) \to \tilde{K}^{n+1}(SX)$ *when one identifies* $K^n(Y)$ *with* $K^{n-1}(X; \mathscr{H}^\vee)$ *(cf. [13], [15]).*

The above arguments show that there exists a natural candidate for the functor $K^n(X)$, X paracompact, if one wants Theorem 1.2 to remain valid. It suffices to choose $K^{n-1}(X; A) \approx K^{n-1}(X; \mathscr{H}^\vee)$[18]. The advantage of this definition over the classical definition is obviously that the category $\mathscr{H}^\vee$ is CW-representable. Another advantage is that the considerations of [12] relative to Banach categories, etc. allow for the immediate extension to the paracompact setting of K-theory theorems ordinarily proven in the compact setting. For example, one can prove a "Thom isomorphism" for spinor bundles (in the formulation of [12, Proposition 3.1.5] or [13, Theorem 8] with families of support), using the techniques of [12] and [22].

IV Relations with classic classifying spaces

[Section IV omitted from translation.]

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