
Examples of algebra-geometry correspondences

A geometric object, say a curve, a surface, a manifold, a variety, etc. can often be defined and studied in two equivalent, but *dual*, fashions: either as a set of points M endowed with some extra structure like a topology, or as a commutative algebra A possibly equipped with an extra structure. One recovers M from A as its *spectrum*, while A is an algebra of functions defined on M .

We give several examples of algebra-geometry correspondences. They all put into correspondence, or duality, certain categories of classical geometric objects with a corresponding category of commutative algebraic objects. In many cases, by relaxing the commutativity assumption we arrive at the corresponding noncommutative geometric object. Presumably, the more one knows about these duality relations the better one is prepared to pursue noncommutative geometry.

1.1 Locally compact spaces and commutative C^* -algebras

In functional analysis the celebrated *Gelfand–Naimark Theorem* [91] implies that the category of locally compact Hausdorff spaces and continuous proper maps is equivalent to the opposite of the category of commutative C^* -algebras and proper C^* -morphisms:

$$\{\text{locally compact Hausdorff spaces}\} \simeq \{\text{commutative } C^*\text{-algebras}\}^{\text{op}} \quad (1.1)$$

Under this correspondence, the category of compact Hausdorff spaces and continuous maps corresponds to the category of unital commutative C^* -algebras and unital C^* -morphisms:

$$\{\text{compact Hausdorff spaces}\} \simeq \{\text{commutative unital } C^*\text{-algebras}\}^{\text{op}} \quad (1.2)$$

By an *algebra* in this book we shall mean an associative algebra over the field of complex numbers \mathbb{C} . Algebras are not assumed to be commutative or unital, unless explicitly stated. An *involution* on an algebra A is a *conjugate linear* map $*$: $A \rightarrow A$, $a \mapsto a^*$, satisfying the extra relations

$$(ab)^* = b^*a^* \quad \text{and} \quad (a^*)^* = a$$

for all a and b in A . A \mathbb{C} -algebra endowed with an involution is called an *involution algebra*.

By a *normed algebra* we mean an algebra A such that A is a normed vector space and

$$\|ab\| \leq \|a\|\|b\|$$

for all a, b in A . If A is unital, with its unit denoted by 1 , we shall assume that $\|1\| = 1$. A *Banach algebra* is a normed algebra which is complete as a metric space in the sense that any Cauchy sequence in A is convergent.

Definition 1.1.1. A *C^* -algebra* is an involutive Banach algebra A such that for all $a \in A$ the C^* -identity

$$\|a^*a\| = \|a\|^2 \tag{1.3}$$

holds.

A *morphism* of C^* -algebras, or a *C^* -morphism*, is an algebra homomorphism $f: A \rightarrow B$ which preserves the $*$ -structure, namely

$$f(a^*) = f(a)^* \quad \text{for all } a \in A.$$

It can be shown that any C^* -morphism is automatically a *contraction* in the sense that $\|f(a)\| \leq \|a\|$ for all $a \in A$, and in particular is continuous. This ‘*automatic continuity*’ result, and its immediate consequence that the norm of a C^* -algebra is unique, is an example of ‘rigidity’ of C^* -algebras. Very often, purely algebraic conditions, thanks to the mighty C^* -identity (1.3), have topological consequences. This need not be true for general Banach algebras. See Appendix A for basics of C^* -algebra theory and Appendix D for the category theory language we use in this book. Let us explain the equivalences (1.1) and (1.2).

1.1.1 The spectrum. A *character* of an algebra A is a nonzero multiplicative linear map $\chi: A \rightarrow \mathbb{C}$. Notice that if A has a unit then necessarily $\chi(1) = 1$. Commutative algebras tend to have a large collection of characters. A noncommutative algebra on the other hand may have no character at all. For example a *simple algebra*, like $M_n(\mathbb{C})$, has no non-trivial two sided ideal and hence has no character. In fact the kernel of a character is easily seen to be a two-sided ideal.

It can be shown that any character of a Banach algebra is automatically continuous and has norm one. It can also be shown that if A is a C^* -algebra then any character of A preserves the $*$ -structure.

Let \hat{A} denote the set of characters of the Banach algebra A . It is called the (*maximal ideal*) *spectrum* of A . We can endow \hat{A} with the weak* topology, i.e., the topology of pointwise convergence, on the continuous dual, A^* , of A . By the Banach–Alaoglu theorem, the unit ball of A^* is compact in the weak* topology and, since $\hat{A} \cup \{0\}$ is a closed subset of this unit ball, we can conclude that \hat{A} is a locally compact Hausdorff space. It is compact if and only if A is unital. When A is unital there is a one-to-one correspondence between characters of A and the set of maximal ideals of A : to a

character χ we associate its kernel which is a maximal ideal and to a maximal ideal I we associate the character $\chi: A \rightarrow A/I = \mathbb{C}$. Note that, by the Gelfand–Mazur theorem, for any maximal ideal I , the skew field A/I is isomorphic to \mathbb{C} .

Example 1.1.1. Given a locally compact Hausdorff space X , let $C_0(X)$ denote the algebra of complex-valued continuous functions on X , vanishing at infinity. Under pointwise addition and multiplication $C_0(X)$ is obviously a commutative algebra over the field of complex numbers. It is unital if and only if X is compact, in which case it will be denoted by $C(X)$. Endowed with the *sup-norm*

$$\|f\| := \|f\|_\infty = \sup\{|f(x)|; x \in X\},$$

and the $*$ -operation induced by complex conjugation

$$f \mapsto f^*, \quad f^*(x) = \overline{f(x)},$$

$C_0(X)$ can be easily shown to satisfy all the axioms of a C^* -algebra, including the all-important C^* -identity (1.3). Thus to any locally compact Hausdorff space we have associated a commutative C^* -algebra, and this C^* -algebra is unital if and only if the space is compact. The characters of $C_0(X)$ are easy to describe, as we show next.

For any $x \in X$ we have the *evaluation character*

$$\chi = \chi_x: C_0(X) \rightarrow \mathbb{C}, \quad \chi_x(f) = f(x).$$

It can be shown that all characters of $C_0(X)$ are of this form and that the map

$$X \rightarrow \widehat{C_0(X)}, \quad x \mapsto \chi_x,$$

is a homeomorphism. Thus, we can recover X , including its topology, as the space of characters of $C_0(X)$.

1.1.2 The Gelfand transform. By a fundamental theorem of Gelfand and Naimark [91] (see below), any commutative C^* -algebra A is isomorphic to $C_0(X)$ where X is the space of characters of A . The isomorphism is implemented by the Gelfand transform, to be recalled next.

For any commutative Banach algebra A , the *Gelfand transform*

$$\Gamma: A \rightarrow C_0(\widehat{A})$$

is defined by $\Gamma(a) = \hat{a}$, where

$$\hat{a}(\chi) = \chi(a).$$

It is a norm contractive algebra homomorphism, as can be easily seen. In general Γ need not be injective or surjective, though its image separates the points of the spectrum. The kernel of Γ is the *nilradical* of A consisting of quasi-nilpotent elements of A , that is, elements whose spectrum consists only of zero. In [91] Gelfand and Naimark, building on earlier work of Gelfand, show that under some extra assumptions on A , which is equivalent to A being a C^* -algebra, Γ is a $*$ -algebra isometric isomorphism between A and $C_0(\widehat{A})$.

Theorem 1.1.1 (Gelfand–Naimark Theorem [91]). *For any commutative C^* -algebra A with spectrum \widehat{A} , the Gelfand transform*

$$\Gamma: A \rightarrow C_0(\widehat{A}), \quad a \mapsto \widehat{a},$$

is an isomorphism of C^ -algebras.*

Theorem 1.1.1 is the main technical result needed to establish the equivalence of categories in (1.1) and (1.2). The description of these correspondences is a bit easier in the compact case (1.2) and that is what we shall do first.

Let \mathcal{S} denote the category whose objects are compact Hausdorff spaces and whose morphisms are continuous maps between such spaces. Let \mathcal{C} denote another category whose objects are commutative unital C^* -algebras and whose morphisms are unital C^* -algebra morphisms between such algebras.

We define contravariant functors

$$C: \mathcal{S} \rightsquigarrow \mathcal{C} \quad \text{and} \quad \widehat{}: \mathcal{C} \rightsquigarrow \mathcal{S}$$

as follows. We send a compact Hausdorff space X , to $C(X)$, the algebra of complex-valued continuous functions on X . If $f: X \rightarrow Y$ is a continuous map, we let

$$C(f) = f^*: C(Y) \rightarrow C(X), \quad f^*(g) = g \circ f,$$

be the *pullback* of f . It is clearly a C^* -algebra homomorphism which preserves the units. We have thus defined the functor C .

The functor $\widehat{}$, called the *functor of points* or the *maximal ideal spectrum* functor, sends a commutative unital C^* -algebra A to its space of characters, or equivalently maximal ideals, \widehat{A} , and sends a C^* -morphism $f: A \rightarrow B$ to the continuous map $\widehat{f}: \widehat{B} \rightarrow \widehat{A}$ defined by

$$\widehat{f}(\chi) = \chi \circ f$$

for any character $\chi \in \widehat{B}$.

To show that C and $\widehat{}$ are equivalences of categories, quasi-inverse to each other, we must show that the functor $C \circ \widehat{}$ is isomorphic to the identity functor of \mathcal{C} and similarly the functor $\widehat{} \circ C$ is isomorphic to the identity functor of \mathcal{S} . That is, we have to show that for any compact Hausdorff space X and any commutative unital C^* -algebra A , there are natural isomorphisms $\widehat{C(X)} \simeq X$ and $C(\widehat{A}) \simeq A$. But in fact we have already done this. Consider the maps

$$\begin{aligned} X &\xrightarrow{\simeq} \widehat{C(X)}, & x &\mapsto \chi_x, \\ A &\xrightarrow{\simeq} C(\widehat{A}), & a &\mapsto \widehat{a}. \end{aligned}$$

Here χ_x is the *evaluation at x* map defined by $\chi_x(f) = f(x)$, and $a \mapsto \widehat{a}$ is the *Gelfand transform* Γ defined above by $\widehat{a}(\chi) = \chi(a)$. The first isomorphism is elementary and

does not require the theory of Banach algebras. The second isomorphism is the content of Gelfand–Naimark’s Theorem 1.1.1 whose proof is based on Gelfand’s theory of commutative Banach algebras.

Care must be applied in the more general non-compact/non-unital case. The main issue is to get the morphisms right. Let us define a category \mathcal{S}_0 consisting of locally compact Hausdorff spaces and *proper* continuous maps between them. Recall that a continuous map $f: X \rightarrow Y$ between locally compact spaces is called *proper* if for any compact $K \subset Y$, $f^{-1}(K)$ is compact. Let \mathcal{C}_0 be another category whose objects are commutative not necessarily unital C^* -algebras. For morphisms between A and B we take *proper* C^* -algebra homomorphisms $f: A \rightarrow B$. Here proper means that for any approximate identity $e_i, i \in I$, of A , $f(e_i), i \in I$ is an approximate identity for B . Equivalently, for any nonzero character χ on B , $\chi \circ f$ should be nonzero. Alternatively, f is proper if and only if $f(A)B$ is dense in B . Recall that an *approximate identity* for a C^* -algebra A is a net $e_i, i \in I$ of elements of A such that for all $a \in A$, $ae_i \rightarrow a$ and $e_i a \rightarrow a$. Thus the properness of a morphism can be seen as a replacement for being unital. In particular the zero map, while it is a C^* -map, is not proper.

Define two contravariant functors, similar to what we had before,

$$C_0: \mathcal{S}_0 \rightsquigarrow \mathcal{C}_0 \quad \text{and} \quad \hat{\cdot}: \mathcal{C}_0 \rightsquigarrow \mathcal{S}_0$$

as follows. We send a locally compact Hausdorff space X to $C_0(X)$ and send a proper continuous map $f: X \rightarrow Y$, to its pullback $C_0(f) = f^*: C_0(Y) \rightarrow C_0(X)$. It is easily seen that, thanks to properness of f , $C_0(f)$ is well defined and is a proper morphism of C^* -algebras. The definition of the spectrum functor is the same as in the unital case. Notice that under a proper morphism $f: A \rightarrow B$, the map $f^*: \hat{B} \rightarrow \hat{A}$ sends a nonzero character to a nonzero character and hence is well defined. Using unitization and 1-point compactification, one can deduce the equivalence (1.2) from (1.1) as follows.

If A is a commutative C^* -algebra, its *unitization* $A^+ = A \oplus \mathbb{C}$ is obtained by adjoining a unit to A . Thus its multiplication and $*$ -structure are defined by $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$, $(a, \lambda)^* = (a^*, \bar{\lambda})$. The definition of its norm is less obvious and one should take the left action of A^+ on A by multiplication and the corresponding operator norm. Thus $\|(a, \lambda)\| := \sup\{\|ab + \lambda b\|; \|b\| \leq 1\}$. Then it can be shown that A^+ is a (unital) C^* -algebra. Let $\chi_0: A^+ \rightarrow \mathbb{C}$ be the character defined by $\chi_0(a, \lambda) = \lambda$. Now we have $\hat{A} = \hat{A}^+ \setminus \{\chi_0\}$, which shows that, being the complement of a closed set in a compact Hausdorff space, \hat{A} is a locally compact Hausdorff space. Now if $f: A \rightarrow B$ is a proper morphism, then one checks that $\hat{f}(\chi) := \chi \circ f$ is a non-trivial character. This defines the functor of spectrum in the non-unital case. One can now deduce the equivalence (1.1) from (1.2).

Example 1.1.2. The spectrum of a commutative C^* -algebra may not be ‘visible’ at first sight, unless the algebra is already in the form $C_0(X)$. For example, the algebra $C_b(\mathbb{R})$ of *bounded* continuous functions on \mathbb{R} is a unital commutative C^* -algebra in a natural way. So, by Theorem 1.1.1, we know that $C_b(\mathbb{R}) = C(\beta\mathbb{R})$, where $\beta\mathbb{R}$ denotes the

spectrum of $C_b(\mathbb{R})$. It is easy to see that $\beta\mathbb{R}$ is in fact the Stone–Čech compactification of \mathbb{R} . More generally, for a locally compact Hausdorff space X , the spectrum of $C_b(X)$ can be shown to be homeomorphic to βX , the Stone–Čech compactification of X (cf. also the next example).

For an example of a different flavor, let X be a topological space which is manifestly non-Hausdorff and let $A = C_0(X)$. Then the spectrum of A has the effect of turning X into a Hausdorff space and is in some sense the ‘Hausdorff-ization’ of X . For yet a different type of example, the reader should try to describe the spectrum of $L^\infty[0, 1]$, the algebra of essentially bounded measurable functions on $[0, 1]$.

Example 1.1.3 (Essential ideals and compactifications). Let X be a locally compact Hausdorff space. Recall that a *Hausdorff compactification* of X is a compact Hausdorff space Y , where X is homeomorphic to a dense subset of Y . We consider X as a subspace of Y . Then X is open in Y and its boundary $Y \setminus X$ is compact. We have an exact sequence

$$0 \rightarrow C_0(X) \rightarrow C(Y) \rightarrow C(Y \setminus X) \rightarrow 0,$$

where $C_0(X)$ is an *essential ideal* of $C(Y)$. (An ideal $I \subset A$ is called essential if whenever $aI = 0$, then $a = 0$.) Conversely, any extension

$$0 \rightarrow C_0(X) \rightarrow A \rightarrow B \rightarrow 0,$$

where A , and hence B , is a commutative unital C^* -algebra and $C_0(X)$ is an essential ideal of A , defines a Hausdorff compactification of X . Thus, we have a one-to-one correspondence between Hausdorff compactifications of X and (isomorphism classes of) essential extensions of $C_0(X)$. In particular, the 1-point compactification and the Stone–Čech compactification correspond to

$$0 \rightarrow C_0(X) \rightarrow C_0(X)^+ \rightarrow \mathbb{C} \rightarrow 0,$$

and

$$0 \rightarrow C_0(X) \rightarrow C_b(X) \rightarrow C(\beta X \setminus X) \rightarrow 0.$$

Example 1.1.4. Under the correspondence (1.1), constructions on spaces have their algebraic counterparts and vice versa. We list a few of these correspondences. The *disjoint union* of spaces $X \cup Y$ corresponds to the *direct sum* of algebras $A \oplus B$; the *Cartesian product* of spaces $X \times Y$ corresponds to a certain topological tensor product $A \hat{\otimes} B$ of algebras. Closed subspaces $Y \subset X$ of a space correspond to closed ideals; A compact Hausdorff space X is connected if and only if the algebra $C(X)$ has no non-trivial idempotent. Recall that an idempotent in an algebra is an element e such that $e^2 = e$. It is called non-trivial if $e \neq 0, 1$. The 1-point compactification of a locally compact Hausdorff space corresponds to unitization, i.e., the operation of adding a unit to the algebra. We record some of these correspondences in the following table.

Space	Algebra
compact	unital
1-point compactification	unitization
Stone–Čech compactification	multiplier algebra
closed subspace; inclusion	closed ideal; quotient algebra
surjection	injection
injection	surjection
homeomorphism	automorphism
Borel measure	positive functional
probability measure	state
disjoint union	direct sum
Cartesian product	minimal tensor product

1.1.3 Noncommutative spaces. Theorem 1.1.1 and the correspondence (1.1) form the foundation of the idea that the category of noncommutative C^* -algebras may be regarded as the dual of an, otherwise undefined, category of noncommutative (NC) spaces. Thus formally one can propose a category of noncommutative spaces as the dual of the category of C^* -algebras and C^* -morphisms:

$$\{\text{NC locally compact spaces}\} := \{\text{NC } C^*\text{-algebras and } C^*\text{-morphisms}\}^{\text{op}} \quad (1.4)$$

Notice that while (1.1) is a theorem, (1.4) is a proposal and at the moment there is no other way to define its left-hand side by any other means. Various operations and concepts for spaces can be paraphrased in terms of algebras of functions on spaces and then one can try to generalize them to noncommutative spaces. This is the rather easy part of noncommutative geometry. The more interesting and harder part is to find properties and phenomena that have no commutative counterpart.

While (1.4) is a useful definition, it is by far not enough and one should take a broader perspective on the nature of a noncommutative space. For example, (1.4) captures only the topological aspects, and issues like smooth or complex structures, metric and Riemannian structures, etc. are totally left out. One can probe a space with a hierarchy of classes of functions:

$$\text{polynomial} \subset \text{analytic} \subset \text{smooth} \subset \text{continuous} \subset \text{measurable}$$

There is a similar hierarchy in the noncommutative realm, though it is much less well understood at the moment.

Our working definition of a noncommutative space is a noncommutative algebra, possibly endowed with some extra structure. Operator algebras, i.e., algebras of bounded operators on a Hilbert space, provided the first really deep insights into this noncommutative realm.