Operations on continuous bundles of C*-algebras

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1 Introduction

Recently several authors [2, 3, 8, 14] have considered bundles of C*-algebras obtained by applying to certain continuous fields of C*-algebras product operations such as taking a tensor product with a fixed C*-algebra or a crossed product by a group acting fibrewise. We consider the fundamental question, touched on in these articles, of the continuity of the resulting product bundles.

To describe the problem we will use the following idea of a continuous bundle of C*-algebras over a locally compact Hausdorff space.

Definition 1.1 A bundle of C*-algebras, or C*-bundle over a locally compact Hausdorff space X is a triple \( \mathcal{A} = (X, \pi_x : A \to A_x, A) \), where \( A \) is a C*-algebra, the bundle C*-algebra, and for each \( x \in X \), \( A_x \) is a C*-algebra and \( \pi_x \) a *-epimorphism of \( A \) onto \( A_x \) such that

(i) \( \{ \pi_x : x \in X \} \) is faithful, i.e. \( \| a \| = \sup_{x \in X} \| a_x \| \), where \( a_x = \pi_x(a) \) for each \( x \);

(ii) for \( f \in C_0(X) \) and \( a \in A \), there is an element \( fa \in A \) such that \( (fa)_x = f(x)a_x \) for \( x \in X \). When it is clear what the maps \( \pi_x \) are, \( \mathcal{A} \) will be written simply \( (X, A_x, A) \). A continuous bundle of C*-algebras is a C*-bundle \( \mathcal{A} = (X, A_x, A) \) which also satisfies

(iii) for \( a \in A \), the function \( N(a) : x \to \| a_x \| \) is in \( C_0(X) \).

This definition is equivalent to the classical definition of a continuous field of C*-algebras [5, 10.3.1] when \( X \) is compact, but has the conceptual and notational advantage that the algebra of sections is a C*-algebra even when \( X \) is not compact. We can identify \( A \) with the *-algebra of elements \( \gamma \) in the cartesian product \( \prod_{x \in X} A_x \) for which there is an \( a \in A \) with \( \gamma_x = \pi_x(a) \) for \( x \in X \). If \( \Gamma \) is the *-algebra of elements of \( \prod_{x \in X} A_x \) which coincide on compact subsets of \( X \) with elements of \( A \), the triple \((X, A_x, \Gamma)\) is a continuous field of C*-algebras in the sense of [5], and \( C_0(\Gamma) = A \). Conversely, if \((X, A_x, \Gamma)\)
is a continuous field of $\text{C}^*$-algebras on $X$ and $A$ is the $\ast$-algebra of $\gamma \in \Gamma$ such that the function $x \rightarrow \|\gamma_x\|$ is in $\text{C}_0(X)$, then $A$ is a $\text{C}^*$-algebra and $(X, \pi_x : A \rightarrow A_x, A)$, where the $\pi_x$ are the obvious coordinate morphisms, is a continuous bundle in the sense of (1.1), with $A = \text{C}_0(\Gamma)$. Thus the concepts of continuous bundle and continuous field of $\text{C}^*$-algebras are essentially equivalent for locally compact base spaces. The term bundle seems to us more appropriate than field in view of the increasing interplay between $\text{C}^*$-algebra theory and topology.

If $\mathcal{A} = (X, A_x, A)$ is a continuous bundle of $\text{C}^*$-algebras, and $B$ is another $\text{C}^*$-algebra, there are natural bundles $\mathcal{A} \otimes B$ and $\mathcal{A} \otimes_{\max} B$ over $X$ with bundle algebras $A \otimes_{\max} B$ and $A \otimes_{\max} B$, respectively, defined as follows.

1. Since $\{\pi_x : x \in X\}$ is a faithful family of morphisms on $A$, the family $\{\pi_x \otimes \text{id} : A \otimes B \rightarrow A_x \otimes B ; x \in X\}$ is faithful on $A \otimes B$ (where, from now on, $\otimes$ denotes $\otimes_{\min}$). Thus $A \otimes B$ can be identified with a $\ast$-algebra of bounded elements of $\prod_{x \in X} (A_x \otimes B)$. If $f_c$ is defined for $c \in \text{C}_0(X)$ and $c \in A \otimes B$ so that when $c = a \otimes b$ with $a \in A$ and $b \in B$, $f(a \otimes b) = (f(a) \otimes b)$, then the triple $\mathcal{A} \otimes B = (X, A_x \otimes B, A \otimes B)$ satisfies (i) and (iii) of Definition 1.1, but it is not clear that (ii) always holds. It is, in fact, always true that for $c \in A \otimes B$, $x \rightarrow \|((\pi_x \otimes \text{id})(c))\|$ is lower semicontinuous (see sect. 2), so that (ii) holds exactly when this function is also upper semicontinuous for all $c$.

2. Since $A$ is an ideal of its multiplier algebra $\text{M}(A)$, and $B$ is an ideal of its unitization $\hat{B}$, $A \otimes_{\max} B$ is an ideal of $\text{M}(A) \otimes_{\max} \hat{B}$, and any irreducible representation $\pi$ of $A \otimes_{\max} B$ extends to an irreducible representation $\tilde{\pi}$ of $\text{M}(A) \otimes_{\max} \hat{B}$. The algebra $\text{C}_0(X)$ is identified with a $\text{C}^*$-subalgebra of the centre $Z(\text{M}(A))$ of $\text{M}(A)$, and so $\text{C}_0(X) \otimes 1 \subseteq Z(\text{M}(A) \otimes_{\max} \hat{B})$. Thus $\tilde{\pi}|_{\text{C}_0(X) \otimes 1}$ is a character of $\text{C}_0(X) \otimes 1$, corresponding to evaluation at some point $x \in X$ which depends only on $\pi$. Let $J_x = \{a \in A : a_x = 0\}$. Then $J_x = \text{C}_{0,x}(X)A$, where $\text{C}_{0,x}(X) = \{f \in \text{C}_0(X) : f(x) = 0\}$, and $\tilde{\pi}(J_x \otimes 1) = 0$. Since $(A \otimes_{\max} B)/(J_x \otimes_{\max} B) \cong A_x \otimes_{\max} B$, it follows that $\pi$ factors through $A_x \otimes_{\max} B$ via the quotient map $\pi_x \otimes_{\max} \text{id} : A \otimes_{\max} B \rightarrow A_x \otimes_{\max} B$. If $c \in A \otimes_{\max} B$, then $\|c\| = \sup_{\pi} \|\pi(c)\|$, where the supremum is over all irreducible representations of $A \otimes_{\max} B$. Thus $\|c\| = \sup_{x \in X} \|((\pi_x \otimes_{\max} \text{id})(c))\|$, and $A \otimes_{\max} B$ can be identified with a $\ast$-subalgebra of $\prod_{x \in X} (A_x \otimes_{\max} B)$. If the module action of $\text{C}_0(X)$ on $A \otimes_{\max} B$ is defined so that $f(a \otimes b) = (f(a) \otimes b)$ for $f \in \text{C}_0(X), a \in A$ and $b \in B$, and is extended to arbitrary $c \in A \otimes_{\max} B$ by linearity and continuity, conditions (i) and (iii) of Definition (1.1) again hold, and it is always true that for $c \in A \otimes_{\max} B$, the function $x \rightarrow \|((\pi_x \otimes_{\max} \text{id})(c))\|$ is upper semicontinuous. Thus (ii) will hold if and only if this function is lower semicontinuous. The bundle $(X, \pi_x \otimes_{\max} \text{id} : A \otimes_{\max} B \rightarrow A_x \otimes_{\max} B, A \otimes_{\max} B)$ will be denoted by $\mathcal{A} \otimes_{\max} B$.

Before we embarked on this investigation it was widely believed that a product bundle of form $\mathcal{A} \otimes B$ would always satisfy (ii) for continuous $\mathcal{A}$. Surprisingly, we have found that there are instances where upper semicontinuity can fail. Similarly lower semicontinuity can fail for bundles of form $\mathcal{A} \otimes_{\max} B$. 
For a continuous bundle \( \mathcal{A} = (X, \pi_x : A \to A_x, A) \) with \( J_x = \ker \pi_x \) for \( x \in X \), the continuity of \( \mathcal{A} \otimes B \) is closely linked to exactness for the sequences

\[
0 \to J_x \otimes B \to A \otimes B \to A_x \otimes B \to 0 \quad (\ast)
\]

(Rieffel [14] had already observed this). In general exactness of a short exact sequence of C*-algebras need not be preserved on taking a minimal C*-tensor product with another C*-algebra [16], [18], and this fact turns out to be the key to producing examples of continuous C*-bundles \( \mathcal{A} \) and C*-algebras \( B \) such that \( \mathcal{A} \otimes B \) does not satisfy (ii). We actually go further, and give a new characterisation of exact C*-algebras (Theorem 4.5), which may be summarised as

**Theorem A** Let \( B \) be a C*-algebra. Then \( B \) is exact if and only if for any continuous bundle \( \mathcal{A} = (\tilde{N}, A_t, A) \) of C*-algebras on the one-point compactification \( \tilde{N} \) of \( N \), with \( A \) separable, \( \mathcal{A} \otimes B \) is continuous.

A sufficient condition for (\( \ast \)) to be exact for arbitrary \( B \) is that \( A \) be an exact C*-algebra. This implies [10] that each \( A_x \) is exact, and the question arises whether, conversely, exactness of each \( A_x \) implies that of \( A \). Again, this is not always true, and we have the following criterion (see Theorem 4.6) for exactness of \( A \):

**Theorem B** Let \( \mathcal{A} = (X, A_x, A) \) be a continuous bundle of C*-algebras with each \( A_x \) exact. Then the following conditions are equivalent:

(i) \( A \) is exact;
(ii) for any C*-algebra \( B \), the bundle \( \mathcal{A} \otimes B \) is continuous;
(iii) each of the quotient maps \( \pi_x : A \to A_x \) is locally liftable (in the sense of [11]).

Using results from [11], we construct examples of bundles \( (X, A_x, A) \), with each \( A_x \) exact, for which one of the \( \pi_x \)'s has no local lifting. It then follows by Theorem B that \( A \) is not exact.

Turning to an analogous question for the maximal C*-tensor product, we obtain

**Theorem C** Let \( B \) be a C*-algebra. Then \( B \) is nuclear if and only if for any continuous bundle \( \mathcal{A} = (\tilde{N}, A_t, A) \) of C*-algebras, with \( A \) separable, the bundle \( \mathcal{A} \otimes_{\text{max}} B \) is continuous.

The plan of the paper is as follows. Section 2 is devoted to preliminaries on bundles of C*-algebras, in particular, criteria for upper and lower semicontinuity. In Sect. 3 we consider the continuity question for product bundles of form \( \mathcal{A} \otimes_{\text{max}} B \), where \( \mathcal{A} \) is a continuous C*-bundle. As well as proving Theorem C, we give an explicit example of a product bundle of this type which is not continuous. In Sect. 4 we examine analogous questions for product bundles of form \( \mathcal{A} \otimes B \). We give several examples of discontinuous bundles of this type and establish the characterisations summarised above as Theorems A and B.
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2 Continuity criteria for bundles of C*-algebras

2.1 Upper and lower semicontinuity for a bundle of C*-algebras

Let \((X,A_x,A)\) be a continuous bundle of C*-algebras on a locally compact Hausdorff space \(X\). Condition (iii) of Definition (1.1) says that \(A\) is a \(C_0(X)\)-module. The following nondegeneracy property holds.

Lemma 2.1 \(A\) is the closure of \(C_0(X)A\).

Proof. (i) \(C_0(X)A \subseteq A\).

(ii) Let \(0 < a \in A, \epsilon > 0\), and let \(Y = \{x \in X : \|a_x\| \geq \epsilon\}\) and \(Y' = \{x \in X : \|a_x\| > \epsilon/2\}\). Then \(Y \subseteq Y'\), and \(Y\) and \(Y'\) are compact. Let \(f : X \to [0,1]\) be a continuous function such that \(f|_Y = 1, f|_{X-Y'} = 0\). Then for \(x \not\in Y\),

\[
\|(f-a)x\| \leq (f(x) + 1)\|a_x\| < 2\epsilon.
\]

Thus \(\|f-a\| < 2\epsilon\), and, since \(fa \in C_0(X)A\) and \(\epsilon\) is arbitrary, the result follows. \(\square\)

If \(A\) is a C*-algebra which is a \(C_0(X)\)-module, for a locally compact Hausdorff space \(X\), then \(C_0(X) \subseteq Z(M(A))\). More generally, let \(A\) be a C*-algebra, suppose \(C_0(X) \subseteq Z(M(A))\) for such an \(X\), with \(C_0(X)A = A\), and let \(\pi_x : A \to A_x\) be a faithful family of *-epimorphisms such that for each \(x \in X\),

\[
\pi_x(fa) = f(x)\pi_x(a) \quad (a \in A).
\]

Then \((X,\pi_x : A \to A_x, A)\) satisfies conditions (i) and (iii) of Definition 1.1, but in general the function \(x \to \|\pi_x(a)\|\) will not be continuous for \(a \in A\). The next lemma gives a criterion for the lower semicontinuity of this function.

Lemma 2.2 The function \(x \to \|\pi_x(a)\|\) is lower semicontinuous for \(a \in A\) if and only if, for any closed subset \(X' \subseteq X\) and dense subset \(D\) of \(X'\), the morphisms \(\pi_D = \bigoplus_{x \in D} \pi_x\) and \(\pi_{X'} = \bigoplus_{x \in X'} \pi_x\) have the same kernel.

Proof. \(\Rightarrow\): Let \(X'\) be a closed subset of \(X\), let \(D\) be dense in \(X'\), and let \(a \in \ker\pi_D\). For \(x \in X'\) and \(\epsilon > 0\), there is a neighbourhood \(U\) of \(x\) in \(X\) such that \(\|\pi_y(a)\| \geq \|\pi_x(a)\| - \epsilon\) for \(y \in U\). Now \(D \cap U \neq \emptyset\), and for \(y \in D \cap U\)

\[
\|\pi_x(a)\| \leq \|\pi_y(a)\| + \epsilon = \epsilon.
\]

Since \(\epsilon\) is arbitrary, \(\|\pi_x(a)\| = 0\) and \(a \in \ker\pi_{X'}\).

\(\Leftarrow\): Suppose that \(x \to \|\pi_x(a)\|\) is not lower semicontinuous for some \(a\) at some \(x \in X\). Then for some \(\epsilon > 0\), there is a net \(\{x_i\}\) in \(X\) such that
x = \lim_{\lambda} x_{\lambda} and \|\pi_{x_{\lambda}}(a)\| < \|\pi_{x}(a)\| - \varepsilon for all \lambda. Replacing a with |a| and using functional calculus, we can assume that \pi_{x_{\lambda}}(a) = 0 for all \lambda and \pi_{x}(a) \neq 0. Letting \tilde{D} = \{x_{\lambda} : \lambda \in A\} and \tilde{X}' = \tilde{D}, a \in \ker \pi_{D}, but a \not\in \ker \pi_{X'}, a contradiction. □

Let us now take A and X as in the previous lemma, and let J_x denote the ideal of \tilde{C}_{0,x}A of A; clearly J_x \subseteq \ker \pi_{X} for x \in X. A useful (and well-known) criterion for the upper semicontinuity of the functions x \to \|\pi_{x}(a)\|

**Lemma 2.3** The function x \to \|\pi_{x}(a)\| is upper semicontinuous on X for all a \in A if and only if J_x = \ker \pi_{X} for x \in X.

**Proof.** \(\Rightarrow\): Suppose a \in \ker \pi_{x}. Given \varepsilon > 0, there is a neighbourhood U of x such that \|\pi_{y}(a)\| \leq \varepsilon for y \in U. Let f : X \to [0, 1] be a continuous function such that f(x) = 1 and f(y) = 0 (y \not\in U). Then

\[
\|a - (1 - f)a\| = \sup_{y \in X} \|\pi_{y}(a) - (1 - f(y))\pi_{y}(a)\|
= \sup_{y \in X} |f(y)|\|\pi_{y}(a)\|
\leq \varepsilon.
\]

Since \varepsilon is arbitrary, a \in J_x. Since J_x \subseteq \ker \pi_{X}, J_x = \ker \pi_{X}.

\(\Leftarrow\): Let a \in A, x \in X and \varepsilon > 0. If b \in A and f \in C_{0}(X) are chosen so that f(x) = 0 and \|a - fb\| \leq \|\pi_{x}(a)\| + \varepsilon/2, there is a neighbourhood U of x such that |f(y)|\|b\| \leq \varepsilon/2 for y \in U. For such y,

\[
\|\pi_{y}(a)\| \leq \|\pi_{y}(a) - f(y)\pi_{y}(b)\| + \|f(y)\pi_{y}(b)\|
\leq \|a - fb\| + \varepsilon/2
\leq \|\pi_{x}(a)\| + \varepsilon.
\]

Since \varepsilon is arbitrary, the map x \to \|\pi_{x}(a)\| is upper semicontinuous at x. □

2.2 Continuity of tensor and crossed product bundles

Throughout this section X will be a locally compact Hausdorff space and \(\mathcal{A} = (X, \pi_{X} : A \to A_{x}, A)\) a continuous bundle of C*-algebras on X. If B is another C*-algebra, the product bundles \(\mathcal{A} \otimes B\) and \(\mathcal{A} \otimes_{\text{max}} B\) were defined in the introduction.

Now let G be a locally compact group, and let \(\alpha_{x}\) be a continuous action of G on \(A_{x}\) (by \(*\)-automorphisms) for x \in X. If for each a \in A and g \in G there is an element \(\alpha^{g}(a) \in A\) such that \(\pi_{x}(\alpha^{g}(a)) = \alpha^{g}_{x}(a_{x})\) for x \in X, the action \(\alpha\) of G on A is continuous, and \(\{\alpha_{x}\}_{x \in X}\) is said to be a continuous field of actions of G. In this case we can define natural bundle structures on X associated with the full and reduced crossed products A \(\rtimes_{\alpha} G\) and A \(\rtimes_{\alpha_{r}} G\) analogous to the tensor product bundles. For x \in X the morphism \(\pi_{x} : A \to A_{x}\) is a G-map (relative to the actions \(\alpha\) and \(\alpha_{x}\)), so that \(\pi_{x}\) extends naturally to
\(\ast\)-epimorphisms \(\pi^G_x : A \bowtie_{a_x} G \to A_x \bowtie_{a_x} G\) and \(\pi^{G,r}_x : A \bowtie_{a_x,r} G \to A_x \bowtie_{a_x,r} G\). We thus obtain bundles \(\mathcal{A} \bowtie_{a_x} G = (X, A_x \bowtie_{a_x} G, A \bowtie_{a_x} G)\) and \(\mathcal{A} \bowtie_{a_x,r} G = (X, A_x \bowtie_{a_x,r} G, A \bowtie_{a_x,r} G)\) which satisfy (i) and (iii) of Definition (1.1). When all the actions \(a_x\) are trivial, these bundles reduce to the tensor product bundles \(\mathcal{A} \otimes_{\text{max}} C^*(G)\) and \(\mathcal{A} \otimes C^r(G)\), respectively.

As is well known, the bundles \(\mathcal{A} \otimes_{\text{max}} B\) and \(\mathcal{A} \otimes B\) are upper semicontinuous (cf [14]). Short proofs of this fact follow easily from Lemma 2.3, and indeed for the maximal tensor product bundle, the result is a special case of the next lemma. The crossed product result is proved analogously. Let \(v\) be a C*-norm on the algebraic tensor product \(A \otimes B\), and let \(A \otimes B\) denote the completion. For \(x \in X\), the closure \(J_x \otimes B\) of \(J_x \otimes B\) in \(A \otimes B\) is an ideal. The quotient of \(A \otimes B\) by this ideal is naturally isomorphic to the completion of \(A_x \bowtie \mathcal{I}\) with respect to some C*-norm \(v_x\). Let \(\pi_x \otimes v_x\) id be the corresponding quotient morphism.

**Lemma 2.4** If \(c \in A \otimes B\), then the function \(x \to \|((\pi_x \otimes v_x) \text{id})(c)\|\) is upper semicontinuous.

**Proof.** Since \(\ker(\pi_x \otimes v_x \text{id}) = J_x \otimes B = C_{0,x}(X)A \otimes B = C_{0,x}(X)(A \otimes B)\), the result is immediate from Lemma 2.3.

Turning to the bundles \(\mathcal{A} \otimes B\) and \(\mathcal{A} \bowtie_{a_x,r} G\), these are known to be lower semicontinuous [14]; we include a short proof for the tensor product case which uses Lemma 2.2.

**Lemma 2.5** If \(c \in A \otimes B\), then the function \(x \to \|((\pi_x \otimes \text{id})(c))\|\) is lower semicontinuous.

**Proof.** Letting \(D \subseteq X\), \(X' = \bar{D}\) and \(A = \pi_{X'}(A)\), for \(x \in X'\), \(\pi_x\) factors through \(A_{x}\) that is, \(\pi_x = \pi_X \circ \pi_{X'}\), for \(x \in X'\), \(\pi_x\) is a \(\ast\)-epimorphism from \(A_{x}\) to \(A_{x}\). Let \(\bar{a} \in A\) with \(\bar{a}_x = 0\) for \(x \in D\). Then \(\bar{a} = \pi_{X'}(a)\) for some \(a \in A\), and \(\|\pi_x(a)\|\) = 0 for \(x \in D\), and hence for \(x \in X'\), since \(x \to \|\pi_x(a)\|\) is continuous. Thus \(\bar{a} = 0\), and \(\{\pi_x : x \in D\}\) is a faithful family of morphisms on \(A\), so that \(\{\pi_x \otimes \text{id} : x \in D\}\) is a faithful family of morphisms on \(A \otimes B\). Suppose \(c \in \ker(\pi_x \otimes \text{id})\) for \(x \in D\). Then \(\pi_x(\otimes \text{id})(\pi_X \otimes \text{id})(c) = 0\) for \(x \in D\), so that \((\pi_x \otimes \text{id})(c) = 0\), and \((\pi_x \otimes \text{id})(c) = 0\) for \(x \in X'\). The result now follows from Lemma 2.2.

**Remarks 2.6**
1. When \(B\) is a nuclear C*-algebra, the norms \(\| \cdot \|_{\text{max}}\) and \(\| \cdot \|_{\text{min}}\) coincide on \(A \otimes B\) and \(A_x \otimes B\) (\(x \in X\)), so that the bundles \(\mathcal{A} \otimes_{\text{max}} B\) and \(\mathcal{A} \otimes B\) coincide. For \(c \in A \otimes B\), the function \(x \to \|c_x\|\) is upper semicontinuous by Lemma 2.4 and lower semicontinuous by Lemma 2.5, hence continuous.

2. When \(G\) is an amenable group, \(B \bowtie_{a_x} G = B \bowtie_{a_x,r} G\) for any C*-algebra \(B\) [13], so that the bundles arising from the full and reduced crossed products of \(A\) by \(G\) coincide. Since one is upper, and the other lower, semicontinuous, the continuity of \(A \bowtie_{a_x} G\) follows.

If we apply the discussion preceding Lemma 2.4 to the norm \(\| \cdot \|_{\text{min}}\) on \(A \otimes B\), each quotient \((A \otimes B)/(J_x \otimes B)\) is naturally isomorphic to some completion.
$A_x \otimes_v B$ of $A_x \otimes B$. The norms $\| \cdot \|_{v_x}$ and $\| \cdot \|_{min}$ coincide exactly when the sequence

$$0 \rightarrow J_x \otimes B \rightarrow A \otimes B \rightarrow A_x \otimes B \rightarrow 0$$

is exact. This leads to the following useful criterion for $\mathcal{A} \otimes B$ to be continuous.

**Proposition 2.7** The bundle $\mathcal{A} \otimes B$ is continuous if and only if (*) is exact for $x \in X$.

**Proof.** $\Leftarrow$: Since exactness of (*) implies $\| \cdot \|_{v_x} = \| \cdot \|_{min}$ for $x \in X$, the bundles $(X, A_x \otimes_v B, A \otimes B)$ and $(X, A_x \otimes B, A \otimes B)$ coincide. Since these bundles are upper and lower semicontinuous, respectively, their continuity follows.

$\Rightarrow$: Let $c \in A \otimes B$, $x \in X$ and let $U$ be a neighbourhood of $x$. If $f : X \rightarrow [0, 1]$ is a continuous function such that $f(x) = 1$ and $f|_{X \setminus U} \equiv 0$,

$$\|fc\| = \sup_{y \in X} \|(\pi_y \otimes \text{id})(fc)\|$$

$$= \sup_{y \in X} f(y)\|(\pi_y \otimes \text{id})(c)\|$$

$$\leq \sup_{y \in U} \|(\pi_y \otimes \text{id})(c)\|$$

and

$$\|(\pi_x \otimes_v \text{id})(c)\| = \|(\pi_x \otimes_v \text{id})(fc)\|$$

$$\leq \|fc\|.$$ 

Thus

$$\|(\pi_x \otimes_v \text{id})(c)\| \leq \sup_{y \in U} \|(\pi_y \otimes \text{id})(c)\|$$

and, since $U$ is an arbitrary neighbourhood of $x$, the continuity assumption implies $\|(\pi_x \otimes_v \text{id})(c)\| \leq \|(\pi_x \otimes \text{id})(c)\|$, whence $\| \cdot \|_{v_x} = \| \cdot \|_{min}$. \hfill \Box

**Corollary 2.8** Let $\mathcal{A} = (X, A_x, A)$ be a continuous bundle of $C^*$-algebras. For a $C^*$-algebra $B$, the bundle $\mathcal{A} \otimes B$ is continuous when either $A$ or $B$ is exact.

**Proof.** When $B$ is exact, it is immediate that (*) is exact for all $x$, so that the result follows. If $A$ is exact, it has property C of Archbold and Batty by results of [1] and [11], and this implies that (*) is exact for arbitrary $B$ by [1, Theorem 3.4]. \hfill \Box

**Remarks 2.9** 1. We shall give a converse to this in Sect. 4.

2. If $A$ is exact, so is $A_x$ for $x \in X$, by [10].

### 3 The operation $\cdot \otimes_{max} B$

Recall that if $A$, $B$ and $C$ are $C^*$-algebras such that $A \subseteq B$, the inclusion map $A \otimes C \rightarrow B \otimes C$ extends to a $*$-homomorphism $i : A \otimes_{max} C \rightarrow B \otimes_{max} C$ which
is, in general, not injective. In what follows we write $A \otimes_{\text{max}} C \subseteq B \otimes_{\text{max}} C$ to indicate that $i$ is injective.

If $C$ is a nuclear $C^*$-algebra, $\|\cdot\|_{\text{max}} = \|\cdot\|_{\text{min}}$ on $A \otimes C$ and $B \otimes C$, so that $i$ is injective, since, by a basic property of the minimal norm, $A \otimes C \subseteq B \otimes C$. It is known, conversely, that if, for a given $C$, $i$ is injective for any $B$ and $A \subseteq B$, then $C$ is nuclear ([7], see also the proof of Theorem 3.1 below). It follows, that if $C$ is any nonnuclear $C^*$-algebra, there are $B$ and $A \subseteq B$ such that the corresponding map $i : A \otimes_{\text{max}} C \rightarrow B \otimes_{\text{max}} C$ is not a monomorphism. We now construct, given such $A$, $B$, and $C$, a continuous bundle $\mathcal{D} = ([0,1], \pi_x : D \rightarrow D_x, D)$ such that the bundle $\mathcal{D} \otimes_{\text{max}} C$ is not continuous.

Fix $x \in [0,1]$ and let $D_x = \{f \in C([0,1],B) : f(x) \in A\}$, $D_x = B$ for $x \in [0,1] \setminus \{x\}$; $D_x = A$ and let $\pi_x : D \rightarrow D_x$ be the map $f \mapsto f(x)$ ($x \in [0,1]$).

Then $\mathcal{D}_x = ([0,1],D_x,\pi_x)$ is a continuous bundle of $C^*$-algebras (it is, in fact, a subbundle of the trivial bundle $([0,1],B,C([0,1],B))$). By our assumption on $A$, $B$ and $C$, there is an $h$ in $A \otimes C$ such that $\|h\|_{A \otimes_{\text{max}} C} > \|h\|_{B \otimes_{\text{max}} C}$. Let $h = \sum_{i=1}^m a_i \otimes c_i$ and let $\tilde{a}_i$ be the function on $[0,1]$ taking the constant value $a_i$ ($i = 1,2,\ldots,m$). Then each $\tilde{a}_i \in D$, $\tilde{h} = \sum_{i=1}^m \tilde{a}_i \otimes c_i \in D \otimes C \subseteq D \otimes_{\text{max}} C$ and

$$\|(\pi_x \otimes_{\text{max}} \text{id})(\tilde{h})\| = \begin{cases} \|h\|_{B \otimes_{\text{max}} C} & (x \neq x) \\ \|h\|_{A \otimes_{\text{max}} C} & (x = x) \end{cases}.$$ 

Thus $\|\tilde{h}_x\| > \|\tilde{h}_x\|$ for $x \neq x$, and $\|\tilde{h}_x\|$ is constant on the set $[0,1] \setminus \{x\}$, so that the function $x \mapsto \|\tilde{h}_x\|$ is not lower semicontinuous at $x = x$ (it is clearly upper semicontinuous on $[0,1]$). It follows that the product bundle $\mathcal{D}_x \otimes_{\text{max}} C$ is not continuous.

**Examples 3.1** 1. To exhibit concrete examples of such bundles $\mathcal{D}_x$, we take $A = C = C^*_r(\mathbb{F}_2)$, the regular $C^*$-algebra of the free group $\mathbb{F}_2$ on two generators, and $B$ the Cuntz algebra $\mathcal{O}_2$. By a well-known result of Choi [4], $\mathcal{O}_2$ has a unital $C^*$-subalgebra $*$-isomorphic to $C^*_r(\text{PSL}_2(\mathbb{Z})) = C^*_r(\mathbb{Z}_2 \ast \mathbb{Z}_3)$. As many people have observed, since $\mathbb{Z}_2 \ast \mathbb{Z}_3$ has a subgroup isomorphic to $\mathbb{F}_2$, $C^*_r(\mathbb{F}_2)$ is $*$-isomorphic to a $C^*$-subalgebra of $C^*_r(\mathbb{Z}_2 \ast \mathbb{Z}_3)$ and hence of $\mathcal{O}_2$. Accordingly we identify $A$ with a closed $*$-subalgebra of $B$. Since $\mathcal{O}_2$ is nuclear, but $\|\cdot\|_{\text{max}} = \|\cdot\|_{\text{min}}$ on $C^*_r(\mathbb{F}_2) \otimes C^*_r(\mathbb{F}_2)$ [15], the corresponding map $i : A \otimes_{\text{max}} C \rightarrow B \otimes_{\text{max}} C = B \otimes C$ is not injective. In fact an explicit example of an $h \in A \otimes C$ such that $\|h\|_{\text{max}} = \|h\|_{\text{min}}$ is given in [17]. In the corresponding product bundle $\mathcal{D}_x \otimes_{\text{max}} C$, the function $x \mapsto \|\tilde{h}_x\|$ is not lower semicontinuous at $x = x$.

2. Example 1 can be modified so that the resulting bundle on $[0,1]$ has the additional property that the fibre algebras are constant. Let $E_{2i} = C^*_r(\mathbb{F}_2)$, $E_{2i-1} = C^*_r(\mathbb{F}_2)$ ($i = 1,2,\ldots$) and let $E$ be the infinite spatial tensor product $\bigotimes_{i=1}^{\infty} E_i$. Letting $A_1 = E \otimes A$ and $B_1 = E \otimes B$, $A_1$ is a $C^*$-subalgebra of $B_1$ and $E \cong E \otimes A_1 \cong E \otimes B_1$. Let $h \in A \otimes C$ be such that

---

1 If $a$ and $b$ are the generators of $\mathbb{Z}_2 \ast \mathbb{Z}_3$, with $a^2 = b^3 = 1$, the elements $bab$ and $ababa$ generate such a subgroup.
\[ \|h\|_{A \otimes_{\text{max}} C} > \|h\|_{B \otimes_{\text{max}} C} \] and let \( k = 1_E \otimes h \in A_1 \). Then \( \|k\|_{A_1 \otimes_{\text{max}} C} = \|h\|_{A \otimes_{\text{max}} C} \), since \( A \otimes_{\text{max}} C \cong (E \otimes A) \otimes_{\text{max}} C \subseteq (E \otimes A) \otimes_{\text{max}} C = A_1 \otimes_{\text{max}} C \). Similarly, \( \|k\|_{B_1 \otimes_{\text{max}} C} = \|h\|_{B \otimes_{\text{max}} C} \), so that \( \|k\|_{A_1 \otimes_{\text{max}} C} > \|k\|_{B_1 \otimes_{\text{max}} C} \). The above construction now yields a bundle \( \mathcal{B}_x = ([0, 1], D_x , D_z) \) with \( D_x \cong E \) for \( 0 \leq x \leq 1 \). The element \( k \in D_x \otimes_{\text{max}} C \) constructed from \( k \) as above has the property that the function \( x \mapsto \|k_x\| \) is not lower semicontinuous at \( x = \alpha \).

3. By taking the restricted direct product of the bundles \( \mathcal{B}_x \) for all \( x \in [0, 1] \), we obtain a continuous bundle \( ([0, 1], D_x, D) \) with constant fibre such that \( D \otimes_{\text{max}} C \) contains, for each \( x \in [0, 1] \), an \( h \) such that \( x \mapsto \|h_x\| \) is not lower semicontinuous at \( x = \alpha \). If instead of taking the direct sum we take the infinite spatial tensor product of the \( \mathcal{B}_x \), the resulting bundle has the same properties, and in addition each fibre is simple (since \( \mathcal{O}_2, C^*_r(\mathbb{F}_2) \) and hence \( E \) are simple).

The method used in constructing these examples also yields the following characterisation of nuclearity.

**Theorem 3.2** For a C*-algebra \( C \) the following conditions are equivalent:

(i) \( C \) is nuclear;

(ii) if \( A \subseteq B \) are C*-algebras, then \( A \otimes_{\text{max}} C \subseteq B \otimes_{\text{max}} C \);

(iii) if \( A \subseteq B \) are separable C*-algebras, then \( A \otimes_{\text{max}} C \subseteq B \otimes_{\text{max}} C \);

(iv) for every continuous C*-bundle \( \mathcal{A} = (\mathbb{N}, A_x, A) \), with \( A \) separable, the bundle \( \mathcal{A} \otimes_{\text{max}} C \) is continuous;

(v) for every continuous C*-bundle \( \mathcal{A} = (X, A_x, A) \) over a locally compact space \( X \), the bundle \( \mathcal{A} \otimes_{\text{max}} C \) is continuous.

**Proof.** (i) \( \Rightarrow \) (v): This follows immediately from Remark 2.6(1).

(v) \( \Rightarrow \) (iv): Trivial.

(iv) \( \Rightarrow \) (iii): Let \( A \subseteq B \) be separable C*-algebras, and let \( D \) be the algebra of sequences \( \{a_i\} : a_i \in B \) \((i = 1, 2, \ldots, \lim_{i \to \infty} a_i \text{ exists and is in } A)\). Then \( D \) is a C*-algebra, and putting \( D_i = B \) \((i \in \mathbb{N})\), \( D_\infty = A \), and taking \( \pi_i : D \to D_i \) and \( \pi_\infty : D \to D_\infty \) to be the morphisms \( (a_j) \to a_i \) and \( (a_j) \to \lim_{i \to \infty} a_i \), respectively, \( \mathcal{D} = (\mathbb{N}, \pi_n : D \to D_n, D) \) is a continuous C*-bundle over \( \mathbb{N} \) with separable bundle algebra \( D \).

By hypothesis the product bundle \( \mathcal{D} \otimes_{\text{max}} C \) is continuous. Let \( d \in A \otimes C \), so that \( d = \sum_{j=1}^m a^j \otimes c^j \) for suitable \( a^j \in A \) and \( c^j \in C \), let \( \tilde{d} \) be the element of \( D \) with constant entry \( a_j \), for \( 1 \leq j \leq m \), and let \( \tilde{d} \) be the element of the product bundle with constant entry \( d \). Then

\[
\|(\pi_n \otimes \text{id})(\tilde{d})\| = \|(\pi_n \otimes \text{id})(\sum \tilde{a}^j \otimes c^j)\| = \begin{cases} \| \sum a^j \otimes c^j \|_{B \otimes_{\text{max}} C} & (n = 1, 2, \ldots) \\ \| \sum a^j \otimes c^j \|_{A \otimes_{\text{max}} C} & (n = \infty) \end{cases}.
\]

By the continuity of the function \( n \to \|(\pi_n \otimes \text{id})(\tilde{d})\| \) at \( \infty \), it follows that

\[
\|d\|_{A \otimes_{\text{max}} C} = \|d\|_{B \otimes_{\text{max}} C}
\]

as required.
(iii) \implies (ii): The separable C*-subalgebras of B form a net \( \{B_\lambda\}_{\lambda \in \Lambda} \) partially ordered by inclusion. If \( B_\lambda \subseteq B_\mu \), then \( B_\lambda \otimes_{\text{max}} C \subseteq B_\mu \otimes_{\text{max}} C \) by hypothesis. We can thus form the inductive limit C*-algebra \( \text{lim}_{\rightarrow}(B_\lambda \otimes_{\text{max}} C) \), which is the completion \( B \otimes_{\text{C}} C \) of \( B \odot C \) relative to some C*-norm \( v \). For \( x \in B \odot C \) there is a \( B_\lambda \) such that \( x \in B_\lambda \odot C \). If \( A \subseteq B \) and \( x \in A \odot C \), we can choose \( B_\lambda \) to be contained in \( A \). Thus
\[
\|x\|_{A \otimes_{\text{max}} C} \leq \|x\|_{B_\lambda \otimes_{\text{max}} C} = \|x\|_{B \otimes_{\text{max}} C} \leq \|x\|_{B \otimes_{\text{max}} C},
\]
since \( B_\lambda \odot C \subseteq A \odot C \). Since \( A \odot C \subseteq B \odot C \), \( \|x\|_{B \otimes_{\text{max}} C} \leq \|x\|_{A \otimes_{\text{max}} C} \). Thus \( x \) has the same norm considered either as an element of \( A \otimes_{\text{max}} C \) or \( B \otimes_{\text{max}} C \). It follows that \( A \otimes_{\text{max}} C \subseteq B \otimes_{\text{max}} C \).

(ii) \implies (i): This follows from [7, Theorem 6.2], but for completeness we outline a short proof. If \( \pi \) is a faithful nondegenerate representation of \( C \) on a Hilbert space \( H \), let \( A = \pi(C)' \) and \( B = B(H) \). The map \( a \otimes c \to a \pi(c) \) extends by linearity and continuity to a representation \( \tilde{\pi} \) of \( A \otimes_{\text{max}} C \) on \( H \). Since by hypothesis \( A \otimes_{\text{max}} C \subseteq B \otimes_{\text{max}} C \), there is a representation \( \sigma \) of \( B \otimes_{\text{max}} C \) on a Hilbert space \( K \supseteq H \) such that for \( x \in A \otimes_{\text{max}} C \), \( E_H \sigma(x)E_H |_{H} = \tilde{\pi}(x) \), where \( E_H \) is the orthogonal projection of \( K \) onto \( H \). The map \( \rho \) of \( B(H) \) into itself given by \( \rho(x) = E_H \sigma(x \otimes 1)E_H \) is then a norm one projection of \( B(H) \) onto \( A \). Thus \( A \), and hence \( A' = \pi(C)'' \) are injective. In particular, \( C^{**} \) is injective, and so \( C \) is nuclear.

Remarks 3.3 A further condition (iv)', obtained by replacing \( \hat{N} \) by [0,1] in condition (iv) of Theorem 3.2, is easily seen to be equivalent to conditions (i) – (iv) of the theorem using the construction preceding Examples 3.1.

4 The operation \( \otimes_{\text{min}} B \)

4.1 Constructing separable continuous bundles of C*-algebras

Let \( B_1, B_2, \ldots \) be a sequence of separable unital C*-algebras and \( A \) a separable unital C*-subalgebra of \( \bigoplus_{i=1}^{\infty} B_i = \{(b_i) : b_i \in B_i (i = 1, 2, \ldots), \sup \|b_i\| < \infty\} \) containing the ideal \( I_0 = \{(a_i) \in \bigoplus B_i : \lim_{i \to \infty} \|a_i\| = 0\} \) of zero sequences. Let \( \pi_i \) be the coordinate projection of \( \bigoplus B_i \) onto \( B_i \) (1 = 1, 2, \ldots).

Lemma 4.1 There is a sequence \( k_1 < k_2 < \ldots \) of natural numbers such that if \( \sigma_i = \bigoplus_{l=k_i}^{k_{i+1}-1} \pi_l \), then \( \lim_{i \to \infty} \|\sigma_i(a)\| \) exists for \( a \in A \).

Proof. Let \( A_\infty = A/I_0 \), with quotient map \( \pi : A \to A_\infty \), and let \( a^1, a^2, \ldots \) be a sequence dense in \( A \). We define the sequence \( k_1 < k_2 < \ldots \) of natural numbers so that for \( i = 1, 2, \ldots \),
\[
\left\| \bigoplus_{l=k_i}^{k_{i+1}-1} \pi_l(a^j) \right\| - \|\pi(a^j)\| \leq (1/i)\|a^j\|.
\]
for $1 \leq j \leq i$. To do this we actually define inductively two sequences $k_i$ and $k'_i$, with $1 = k_0 = k'_0 < k_1 < k'_1 < k_2 < k'_2 < \ldots$, such that for $i \geq 1$ and $1 \leq j \leq i$

$$\left\| \bigoplus_{l=k_i}^{\infty} \pi_l(a') \right\| - \left\| \pi(a') \right\| \leq \left( \frac{1}{i} \right) \|a'\|
$$

and

$$\left\| \bigoplus_{l=k_i}^{r} \pi_l(a') \right\| - \left\| \bigoplus_{l=k_i}^{\infty} \pi_l(a') \right\| \leq \left( \frac{1}{i} \right) \|a'\|
$$

for $r \geq k'_i$. If $k_i$ and $k'_i$ are defined, since $\|\pi(a')\| = \limsup \|\pi_i(a')\|$ for $a \in \bigoplus B_i$, choose $k_{i+1} > k'_i$ such that

$$\sup_{l \geq k_{i+1}} \|\pi_l(a')\| \leq \left( 1 + \frac{1}{i+1} \right) \|\pi(a')\|
$$

for $1 \leq j \leq i + 1$, and choose $k'_{i+1} > k_{i+1}$ so that if $r \geq k'_{i+1}$,

$$\left\| \bigoplus_{l=k_{i+1}}^{\infty} \pi_l(a') \right\| - \left\| \bigoplus_{l=k_{i+1}}^{r} \pi_l(a') \right\| \leq \frac{1}{i+1} \|a'\|
$$

for $1 \leq j \leq i$.

For $i = 1, 2, \ldots$ let $\sigma_i = \bigoplus_{l=k_i}^{k_{i+1}-1} \pi_l$. From the construction of the sequence $\{k_i\}$ it follows that for $i = 1, 2, \ldots$, $\lim_{j \to \infty} \|\sigma_j(a')\|$ exists and equals $\|\pi(a')\|$. Thus $\lim_{j \to \infty} \|\sigma_j(a)\|$ exists for $a \in A$. \hfill $\Box$

Let $A_i = \sigma_i(A) = \bigoplus_{l=k_i}^{k_{i+1}-1} B_l$. We can regard $A$ as a $C^*$-subalgebra of $\bigoplus_{i=1}^{\infty} A_i$. Since $A$ contains the ideal $I_0$ of zero-sequences in $\bigoplus B_i$, $I_0$ coincides with the ideal of sequences in $\bigoplus_{i=1}^{\infty} A_i$ tending to zero at infinity. Since $A$ is unital, it follows that $C(\hat{N}) \subseteq A$, where $C(\hat{N})$ is identified with the algebra of sequences in $\bigoplus A_i$ whose entries are scalar multiples of the identity. Letting $A_\infty = A/I_0$ and $\sigma_\infty = \pi$, we have

Corollary 4.2 The algebra $A$ is a $C(\hat{N})$-module, and $(\hat{N}, \sigma_n : A \to A_n, A)$ is a continuous bundle of $C^*$-algebras.

Let $M_i$ be the $C^*$-algebra of $i \times i$ complex matrices, let $M = \bigoplus_{i=1}^{\infty} M_i$ and let $I_0$ be the ideal of zero-sequences in $M$. It was shown in [14] that a $C^*$-algebra $C$ is exact if and only if the sequence

$$0 \to I_0 \otimes C \to M \otimes C \to (M/I_0) \otimes C \to 0 \quad (*)$$

is exact.

Proposition 4.3 Let $C$ be an inexact $C^*$-algebra. Then there is a continuous $C^*$-bundle $\mathcal{A} = (\hat{N}, A_n, A)$ with $A$ separable such that $\mathcal{A} \otimes C$ is not continuous.
Proof. If \( C \) is not exact, \((*)\) is not exact, which implies that there is an \( x \in \ker(\pi \otimes \text{id}) \) such that \( x \notin I_0 \otimes C \), where \( \pi : M \to M/I_0 \) is the quotient map. Let \( A \) be a separable unital \( C^* \)-subalgebra of \( M \) containing \( I_0 \) and such that \( x \in A \otimes C \). Applying Lemma 4.1 and Corollary 4.2 with \( B_i = M_i \) \((i = 1, 2, \ldots \)) there is a continuous bundle \((\hat{\mathbb{N}}, \sigma_n : A \to A_n, A)\) such that \( I_0 = \ker \sigma_\infty \). Since

\[
0 < \text{dist}(x, I_0 \otimes C) = \limsup_n \| (\sigma_n \otimes \text{id})(x) \|
\]

and \((\sigma_\infty \otimes \text{id})(x) = 0\), the function \( n \to \| (\sigma_n \otimes \text{id})(x) \| \) is not upper semicontinuous at \( \infty \). \( \square \)

Examples 4.4 1. Let \( G \) be a countable residually finite group with property T, for example \( SL_3(\mathbb{Z}) \). Then there is a sequence \( \pi_1, \pi_2, \ldots \) of mutually inequivalent irreducible unitary representations on finite dimensional Hilbert spaces \( H_1, H_2, \ldots \) such that any finite dimensional irreducible unitary representation of \( G \) is equivalent to one of the \( \pi_i \). If \( C = \bigoplus_{i=1}^\infty B(H_i) \), then the \( C^* \)-subalgebra \( A = C^*((\bigoplus \pi_k)(G)) \) of \( C \) contains the ideal \( J_0 \) of zero sequences in \( C \), and the sequence

\[
0 \to J_0 \otimes A \to A \otimes A \to (A/J_0) \otimes A \to 0
\]

is not exact (see [18] for details). Taking \( B_i = B(H_i) \) \((i = 1, 2, \ldots )\), the above construction yields a continuous bundle \( \mathcal{A} = (\hat{\mathbb{N}}, A_n, A) \) with each \( A_i \) a direct sum of finitely many of the \( B_i \), for \( i < \infty \), and \( A_\infty = A/J_0 \), such that \( \mathcal{A} \otimes A \) is not continuous. It is unknown whether, for a specific choice of \( G \) such as \( SL_3(\mathbb{Z}) \), the representations \( \pi_1, \pi_2, \ldots \) can be chosen in such a way that for all \( a \in A \), \( \lim_{i \to \infty} \| \pi_i(a) \| \) exists. If this is the case, the sequences \( \{ \pi_i \} \) and \( \{ \sigma_i \} \) coincide. In this example all the \( A_i \) for \( i < \infty \) are finite dimensional, hence exact. It is not at present known whether \( A/J_0 \) is exact for any choice of \( G \). It is known that \( C_\alpha^* (SL_n(\mathbb{Z})) \) is a subalgebra of a nuclear \( C^* \)-algebra, for \( n \geq 2 \) (cf [11]), and it can be shown that for any \( G \) of the type under consideration, the canonical morphism \( C^*(G) \to C_\alpha^*(G) \) has a factorisation

\[
C^*(G) \to A \to A/J_0 \to C^*_\alpha(G).
\]

2. To obtain an example of a continuous bundle \( \mathcal{A} = (\hat{\mathbb{N}}, A_n, A) \) with all the \( A_n \) exact but \( \mathcal{A} \otimes B \) not continuous for some \( B \), we use results from [11]. Recall [11] that for any \( C^* \)-algebra \( A \), \( C(A) = \text{cone}(A) \) is the unification of the \( C^* \)-algebra \( C_0((0, 1]) \otimes A \). Thus \( C(A) \subseteq C([0, 1]) \otimes A \). If \( A \) is exact, then so are \( A \), \( C([0, 1]) \otimes A \) and hence \( C(A) \). When \( G \) is a countable group, the regular \( C^* \)-algebra \( C^*_\alpha(G) \) is known to be exact in many cases, in particular when \( G \) is amenable, \( G \) is a free group \( \mathbb{F}_n \) \((n = 2, 3, \ldots, \infty) \) or \( G \cong SL_n(\mathbb{Z}) \) \((n = 2, 3, \ldots) \) [11]. (It is conjectured, but, so far as we know, unproven, that \( C^*_\alpha(G) \) is exact for all countable \( G \)).

Let \( G \) be a countable, nonamenable residually finite group. By [11, Sect. 7] (see also [16]) there is an ideal \( J \) of the full group \( C^* \)-algebra \( D = C^*(\mathbb{F}_\infty) \) such that \( D/J \cong C^*_\alpha(G) \) \((= B, \text{ say})\), and such that the sequence

\[
0 \to J \otimes D \to D \otimes D \to B \otimes D \to 0
\]
is not exact. By [11, Proposition 5.1 and Lemma 6.1], there are a sequence
\( n_1, n_2, \ldots \) in \( \mathbb{N} \) and a C*-subalgebra \( A \subseteq M = \bigoplus_{i=1}^{\infty} M_{n_i} \) such that, if \( I_0 \) is the
ideal of zero sequences in \( M, I_0 \subseteq A, A/I_0 \cong C(B) \) and the sequence
\[
0 \to I_0 \otimes D \to A \otimes D \to C(B) \otimes D \to 0
\]
is not exact. Applying the above construction, we obtain a continuous C*
bundle \( \mathcal{A} = (\hat{\mathbb{N}}, \hat{A}_n, \hat{A}) \) on \( \hat{\mathbb{N}} \), with the \( A_n \) finite dimensional for \( n < \infty \)
and \( A_\infty \cong C(B) \) such that \( \mathcal{A} \otimes D \) is not continuous. Choosing \( G \) so that
\( C_\pi^*(G) \) is exact, for example taking \( G = SL_2(\mathbb{Z}) \), all the fibre algebras \( A_n (n = 1, 2, \ldots, \infty) \) are exact. Since the product bundle is not exact, it follows by
Corollary 2.8, that \( A \) is not exact. Thus for a continuous C*-bundle \( (X, A_x, A) \),
exactness of \( A_x \) for all \( x \) does not imply that \( A \) is exact, in general. We shall
give necessary and sufficient conditions under which exactness of \( A_x \) for all \( x \)
does imply exactness of \( A \) in Theorem 4.6.

3. Example 2 can be modified to give a continuous C*-bundle \( \hat{\mathcal{A}} = (\hat{\mathbb{N}}, \hat{A}_n, \hat{A}) \) with constant exact fibre algebra such that \( \hat{\mathcal{A}} \otimes D \) is not continuous.
Let \( D_i \) \( (i = 1, 2, \ldots) \) be a sequence of C*-algebras consisting of \( A_1, A_2, \ldots, A_\infty \),
each occurring an infinite number of times. Since spatial tensor products and
inductive limits of exact C*-algebras are exact, it follows that the algebras
\( C = \bigotimes_{i=1}^{\infty} D_i, \hat{A}_n = C \otimes A_n (n \in \mathbb{N}) \) and \( \hat{A} = C \otimes A \) are exact. Then \( A_n \cong C \)
\( (n = 1, 2, \ldots, \infty) \) and \( \hat{\mathcal{A}} = C \otimes \hat{\mathcal{A}} = (\hat{\mathbb{N}}, \text{id}_C \otimes \pi_n : \hat{A} \to \hat{A}_n, \hat{A}) \) is continuous,
by Lemma 2.8, but \( \hat{\mathcal{A}} \otimes D \) is not continuous, since it contains the discontinuous
bundle of Example 2 as a subbundle.

4.2 A characterisation of exactness

The techniques of the previous section give the following general result.

Theorem 4.5 For a C*-algebra \( C \) the following conditions are equivalent:
(i) \( C \) is exact;
(ii) The sequence
\[
0 \to I_0 \otimes C \to M \otimes C \to (M/I_0) \otimes C \to 0
\]
is exact;
(iii) for every continuous C*-bundle \( \mathcal{A} = (X, A_x, A) \) with \( X \) locally compact,
the bundle \( \mathcal{A} \otimes C \) is continuous;
(iv) for every continuous C*-bundle \( \mathcal{A} = (\hat{\mathbb{N}}, A_x, A) \) with \( A \) separable, the
bundle \( \mathcal{A} \otimes C \) is continuous.

Proof. (ii) \( \iff \) (i): this is just [9, Theorem 1.1].
(i) \( \Rightarrow \) (iii): This follows by Corollary 2.8.
(iii) \( \Rightarrow \) (iv): Trivial.
(iv) \( \Rightarrow \) (i): This follows immediately from Proposition 4.3. \( \square \)
4.3 A characterisation of continuous C*-bundles with exact bundle
C*-algebras

Let \((X, \pi_x : A \to A_x, A)\) be a continuous bundle of C*-algebras. If \(A\) is exact, then by Remark 2.9(2), each \(A_x\) is exact, and, as remarked in the proof of Corollary 2.8, for any C*-algebra \(B\) and \(x \in X\), the sequence
\[
0 \to J_x \otimes B \to A \otimes B \to A_x \otimes B \to 0
\]  
(*)
is exact. Effros and Haagerup [6, Theorem 3.2] showed that (*) is exact for arbitrary \(B\) precisely when the quotient map \(\pi_x\) is locally liftable, that is, for any finite dimensional operator subsystem \(Z \subseteq A_x\), there is a completely positive isometry \(i_{Z,x} : Z \to A\) such that \(\pi_x \circ i_{Z,x} = \text{id}_Z\).

If, conversely, each \(A_x\) is exact, we have already seen that \(A\) need not be exact, and the question that arises is under what circumstances we can conclude that \(A\) is exact. This is answered by the following theorem. Here the amalgamated minimal tensor product \(A \otimes_{C_0(X)} B\) of the C*-algebras of continuous bundles \((X, \pi_x : A \to A_x, A)\) and \((X, \sigma_x : B \to B_x, B)\) over the same space \(X\) is the algebra \((\bigoplus_{x \in X} (\pi_x \otimes \sigma_x))(A \otimes B)\).

**Theorem 4.6** Let \(\mathcal{A} = (X, A_x, A)\) be a continuous bundle of C*-algebras on a locally compact Hausdorff space \(X\), such that each \(A_x\) is exact. The following conditions are equivalent:

(i) the algebra \(A\) is exact;
(ii) the map \(\pi_x : A \to A_x\) is locally liftable for \(x \in X\);
(iii) for every C*-algebra \(B\), the bundle \(\mathcal{A} \otimes B\) is continuous;
(iv) for every continuous C*-bundle \(\mathcal{B} = (X, \sigma_x : B \to B_x, B)\) on \(X\), the bundle \(\mathcal{A} \otimes_{C_0(X)} \mathcal{B} = (X, \pi_x \otimes \sigma_x : A \otimes_{C_0(X)} B \to A_x \otimes B_x, A \otimes_{C_0(X)} B)\) is continuous;
(v) for every continuous C*-bundle \(\mathcal{B} = (Y, \sigma_y : B \to B_y, B)\) on a locally compact space \(Y\), the bundle \(\mathcal{A} \otimes \mathcal{B} = (X \times Y, \pi_x \otimes \sigma_y : A \otimes B \to A_x \otimes B_y, A \otimes B)\) is continuous.

To prove this theorem we need a number of preparatory results.

**Lemma 4.7** Let \(A\) and \(B\) be C*-algebras. For \(c \in A \otimes B\), \(\|c\|_{\text{min}}\) is the supremum of
\[
\left[ \frac{(f \otimes g)(d^* c d)}{(f \otimes g)(d^* d)} \right]^{\frac{1}{2}}
\]
over all pure states \(g \in P(A), \ g \in P(B)\) and \(d \in A \otimes B\) such that \((f \otimes g)(d^* d) \neq 0\).

**Proof.** Let \((\pi_f, H_f, \xi_f)\) be the triple consisting of an irreducible representation of \(A\) on a Hilbert space \(H_f\) with cyclic unit vector \(\xi_f\) corresponding to \(f \in P(A)\) by the GNS construction. Since the families \(\{\pi_f : f \in P(A)\}\) and \(\{\pi_g : g \in P(B)\}\) of irreducible representations of \(A\) and \(B\), respectively, are faithful, the family
\[
\{\pi_f \otimes \pi_g : f \in P(A), g \in P(B)\}
\]
is faithful on $A \otimes B$ (cf [15]). If $0 \neq c \in A \otimes B$, and $\varepsilon > 0$, there are $f \in P(A)$ and $g \in P(B)$ such that $\|(\pi_f \otimes \pi_g)(c)\| \geq \|c\|_{min} - \varepsilon/2$. There is a $d \in A \otimes B$ such that $\xi = ((\pi_f \otimes \pi_g)(d))(\xi_f \otimes \xi_g)$ is a unit vector and

$$\|(\pi_f \otimes \pi_g)(c)\| \xi_2 \geq \|(\pi_f \otimes \pi_g)(c)\| - \varepsilon/2 \geq \|c\|_{min} - \varepsilon .$$

Thus $(f \otimes g)(d^*d) = 1$ and

$$\|c\|_{min}^2 \geq (f \otimes g)(d^*c^*cd) \geq (\|c\|_{min} - \varepsilon)^2 .$$

Since $\varepsilon$ is arbitrary, the result follows.

**Lemma 4.8** Let $(X, \pi_x : A \rightarrow A_x, A)$ be a continuous bundle of C*-algebras on a locally compact space $X$, and let $D \subseteq X$. If $x_0 \in \bar{D}$, $f \in P(A_{x_0})$, $a_1, \ldots, a_n \in A$ and $\varepsilon > 0$, then there are $x \in D$ and $\tilde{f} \in S(A_x)$ such that

$$|\tilde{f}(\pi_x(a_i)) - f(\pi_{x_0}(a_i))| \leq \varepsilon \quad (1 \leq i \leq n) .$$

**Proof.** Letting $A_D = \pi_D(A)$ and defining $\tilde{\pi}_x : A_D \rightarrow A_x$ by $\tilde{\pi}_x \circ \pi_D = \pi_x$ for $x \in D$, $\lim_{x \rightarrow x_0} \|\pi_x(a)\| = \|\pi_{x_0}(a)\|$ for $a \in A$. Thus $A_{x_0}$ is a quotient of $A_D$. Let $\pi_0 : A_D \rightarrow A_{x_0}$ be the quotient map. Taking $S \subseteq S(A_D)$ to be the set of states on $A_D$ of form $\varphi \circ \tilde{\pi}_x$ for $x \in D$, $\varphi \in S(A_x)$, then for $a \in A_D$, $a \geq 0 \iff f(a) \geq 0$ ($f \in S$). If $f$ is a pure state of $A_{x_0}$, then $f \circ \pi_0$ is a pure state of $A_D$ and, by [5, 3.4.1], $f \circ \pi_0$ is in the weak closure of $S$. Thus there are an $x \in D$ and an $\tilde{f} \in S(A_x)$ with the required properties.

**Proposition 4.9** Let $(X, \pi_x : A \rightarrow A_x, A)$ and $(Y, \sigma_y : B \rightarrow B_y, B)$ be continuous bundles of C*-algebras over the locally compact Hausdorff spaces $X$ and $Y$. Then the function $(x, y) \rightarrow \|(\pi_x \otimes \sigma_y)(c)\|$ is lower semicontinuous on $X \times Y$ for each $c \in A \otimes B$.

**Proof.** Suppose that $\|(\pi_x \otimes \sigma_y)(c)\|$ is not lower semicontinuous for some $c \in A \otimes B$ at $z_0 = (x_0, y_0)$. Then there are an $\varepsilon > 0$ and a net $\{z_\lambda\}_{\lambda \in \Lambda}$ converging to $z_0$ such that $\|c_{z_\lambda}\| \leq \|c_{z_0}\| - 3\varepsilon$ for all $\lambda$. Let $d \in A \otimes D$ with $\|c - d\| \leq \varepsilon$. Then

$$\|d_{z_\lambda}\|_{min} \leq \|d_{z_0}\|_{min} - \varepsilon \quad (\lambda \in \Lambda) . \quad (*)$$

By Lemma 4.7 there are $h_0 \in A_{x_0} \otimes B_{y_0}$, $f \in P(A_{x_0})$ and $g \in P(B_{y_0})$ such that $(f \otimes g)(h_0^*h_0) = 1$ and

$$\left[(f \otimes g)(h_0^*d_{z_0}^*d_{z_0}h_0)\right]^{1/2} \geq \|d_{z_0}\| - \varepsilon/3 .$$

Choose $h \in A \otimes B$ such that $h_{z_0} = h_0$. There are $a_1, a_2, \ldots, a_m, a_{m+1}, \ldots, a_n \in A$ and $b_1, b_2, \ldots, b_m, b_{m+1}, \ldots, b_n \in B$ such that $h^*d^*dh = \sum_{i=1}^m a_i \otimes b_i$ and $h^*h = \sum_{i=m+1}^n a_i \otimes b_i$.

Let $\varepsilon' > 0$, where a suitably small value of $\varepsilon'$ will be chosen later in the proof. If $z_\lambda = (x_\lambda, y_\lambda)$ for $\lambda \in \Lambda$, then $\lim_{\lambda \rightarrow \Lambda} x_\lambda = x_0$. If $U$ is a neighbourhood
of \(x_0\) in \(X\) and \(D\) is the set \(\{x_i : x_i \in U\}\), applying Lemma 4.8, there are an \(x \in D\) and \(\bar{f} \in S(A_x)\) such that
\[
|\bar{f}(\pi_x(a_i)) - f(\pi_{x_0}(a_i))| \leq \varepsilon' \quad (1 \leq i \leq n).
\]
It follows that there is a refinement \(\{z_{\lambda}\}_{\lambda \in A'}\) of the net \(\{z_i\}_{i \in A}\), with \(A' \subseteq A\), converging to \(z_0\) and such that for each \(\lambda \in A'\) there is an \(f_\lambda \in S(A_{x_\lambda})\) satisfying
\[
|f_\lambda(\pi_{x_\lambda}(a_i)) - f(\pi_{x_0}(a_i))| \leq \varepsilon' \quad (1 \leq i \leq n).
\]
Repeating this argument for subsets \(D \subseteq Y\) of form \(\{y_i : y_i \in V\}\), where \(V\) is a neighbourhood of \(y_0\) in \(Y\), we get a refinement \(\{z_i\}_{i \in A''}\) of the net \(\{z_i\}_{i \in A'}\), with \(A'' \subseteq A'\), converging to \(z_0\) and such that for each \(\lambda \in A''\) there is a \(g_\lambda \in S(B_{x_\lambda})\) satisfying
\[
|g_\lambda(\sigma_{y_\lambda}(b_i)) - g(\sigma_{y_0}(b_i))| \leq \varepsilon' \quad (1 \leq i \leq n).
\]
It is now easy to see that, for a suitable choice of \(\varepsilon'\), determined by \(|a_1|, \ldots, |a_n|, |b_1|, \ldots, |b_n|, (f \otimes g)(h_0^*d_0^*dh)\) and \(\varepsilon\), for any \(\lambda \in A''\)
\[
\left[\frac{(f_\lambda \otimes g_\lambda)(h_{z_\lambda}^*d_{z_\lambda}^*d_{z_\lambda}h_{z_\lambda})}{(f_\lambda \otimes g_\lambda)(h_{z_\lambda}^*h_{z_\lambda})}\right]^{\frac{1}{2}} \leq \left[\frac{(f \otimes g)(h_0^*d_0^*d_0h_0)}{(f \otimes g)(h_0^*h_0)}\right]^{\frac{1}{2}} - \varepsilon / 3.
\]
It follows that \(\|d_{z_\lambda}\|_{\min} \geq \|d_{z_0}\|_{\min} - 2\varepsilon / 3\), which contradicts (\(*\)). Thus the given function is lower semicontinuous. \(\square\)

**Lemma 4.10** Let \(\mathcal{A} = (X, \pi_x : A \to A_x, A)\) and \(\mathcal{B} = (Y, \sigma_y : B \to B_y, B)\) be continuous bundles of C*-algebras.

1. If \(\mathcal{A} \otimes \mathcal{B} = (X \times Y, \pi_x \otimes \sigma_y : A \otimes B \to A_x \otimes B_y, A \otimes B)\) is a continuous bundle, so is \(\mathcal{A} \otimes \mathcal{B}\).

2. If \(A_x\) is exact for \(x \in X\) and \(\mathcal{A} \otimes \mathcal{B}\) is a continuous bundle, then the bundle \(\mathcal{A} \otimes \mathcal{B}\) is continuous.

**Proof.** 1. Let \(c \in A \otimes B\). The function \((x, y) \to \|((\pi_x \otimes \sigma_y)(c))\|\) is in \(C_0(X \times Y)\) and for fixed \(x\),
\[
\sup_{y \in Y} \|((\pi_x \otimes \sigma_y)(c))\| = \|((\pi_x \otimes \text{id})(c))\|.
\]
If \(\varepsilon > 0\), a simple compactness argument shows that there is a neighbourhood \(U\) of \(x\) such that
\[
\|((\pi_{x'}) \otimes \sigma_y)(c))\| \leq \|((\pi_x \otimes \sigma_y)(c))\| + \varepsilon
\]
for \(x' \in U\) and \(y \in Y\). Thus
\[
\|((\pi_{x'}) \otimes \text{id})(c))\| \leq \|((\pi_x \otimes \text{id})(c))\| + \varepsilon
\]
for \(x' \in U\). This shows that the map \(x \to \|((\pi_x \otimes \text{id})(c))\|\) is upper semicontinuous. Since this map is also lower semicontinuous, by Lemma 2.5, it is continuous.
2. For fixed $x$ exactness of $A_x$ implies that $A_x \otimes \mathcal{B}$ is continuous, by Corollary 2.8. For $c \in A \otimes B$, the function $(x, y) \mapsto \|(\pi_x \otimes \sigma_y)(c)\|$ is lower semicontinuous, by Proposition 4.9. So to prove (2), it is enough to show that this function is also upper semicontinuous for all $c$.

Suppose that for some $c$ the function is not upper semicontinuous at $(x_0, y_0)$. Then there are an $\varepsilon > 0$ and a net $\{(x_\lambda, y_\lambda)\}_{\lambda \in \Lambda}$ such that $(x_\lambda, y_\lambda) = \lim_\lambda (x_\lambda, u_\lambda)$ and

$$\|(\pi_{x_\lambda} \otimes \sigma_{y_\lambda})(c)\| \geq \|(\pi_{x_0} \otimes \sigma_{y_0})(c)\| + \varepsilon$$

for $\lambda \in \Lambda$. Since the bundle $A_{x_0} \otimes \mathcal{B}$ is continuous, there is a neighbourhood $U$ of $y_0$ in $Y$ such that

$$\|(\pi_{x_0} \otimes \sigma_y)(c)\| \leq \|(\pi_{x_0} \otimes \sigma_{y_0})(c)\| + \varepsilon/3$$

for $y \in U$. Let $f : Y \to [0, 1]$ be a continuous function such that $f|_{Y \setminus U} \equiv 0$, $f(y_0) = 1$. Replacing $c$ by $(1 \otimes f)c$ and refining the net $\{(x_\lambda, y_\lambda)\}$, we can assume that

$$\sup_{y \in Y} \|(\pi_{x_0} \otimes \sigma_y)(c)\| \leq \|(\pi_{x_0} \otimes \sigma_{y_0})(c)\| + \varepsilon/3$$

and

$$\|(\pi_{x_\lambda} \otimes \sigma_{y_\lambda})(c)\| \geq \|(\pi_{x_0} \otimes \sigma_{y_0})(c)\| + 3\varepsilon/4$$

for $\lambda \in \Lambda$. For some neighbourhood $V$ of $x_0$ in $X$

$$\|(\pi_x \otimes \text{id})(c)\| \leq \|(\pi_{x_0} \otimes \text{id})(c)\| + \varepsilon/3$$

$$\leq \|(\pi_{x_0} \otimes \sigma_{y_0})(c)\| + 2\varepsilon/3$$

for $x \in V$. Choose $\lambda$ so that $(x_\lambda, y_\lambda) \in V \times U$. Then

$$\|(\pi_{x_0} \otimes \sigma_{y_0})(c)\| + 3\varepsilon/4 \leq \|(\pi_{x_\lambda} \otimes \sigma_{y_\lambda})(c)\| \leq \|(\pi_{x_\lambda} \otimes \text{id})(c)\| \leq \|(\pi_{x_0} \otimes \sigma_{y_0})(c)\| + 2\varepsilon/3.$$

This is a contradiction. So $(x, y) \mapsto \|(\pi_x \otimes \sigma_y)(c)\|$ is upper semicontinuous for all $c$, and (2) follows.

\[\square\]

**Proposition 4.11** Let $(X, \pi_x : A \to A_x, A)$ be a continuous bundle of $C^*$-algebras such that each $A_x$ is exact and, for any $C^*$-algebra $B$, the sequence

$$0 \to J_x \otimes B \to A \otimes B \to A_x \otimes B \to 0$$

is exact. Then $A$ is exact.
Proof. 1. We prove first that if $\pi : A \to B(H)$ is any factor representation of $A$ and $J = \ker \pi$, then $A/J$ is exact and the sequence

$$0 \to J \otimes B \to A \otimes B \to (A/J) \otimes B \to 0 \quad (***)$$

is exact for any $B$. Extending $\pi$ to a representation $\bar{\pi}$ of $M(A)$ on $H$, $\bar{\pi}$ restricted to $C_0(X)$ is a character of $C_0(X)$ since $\pi$ is a factor representation of $A$. This implies that for some $x \in X$, $J_x \subseteq J$. Thus $A/J$ is a quotient of $A_x$. Since the latter is exact, so is $A/J$.

Let $c$ be in the kernel of the quotient morphism $J \otimes B \to (J/J_x) \otimes B$. Since $(J/J_x) \otimes B \subseteq (A/J_x) \otimes B = A_x \otimes B$, $c \in \ker(\pi_x \otimes \text{id}) = J_x \otimes B$, since (*) is exact. It follows that in the commutative diagram

$$
\begin{array}{cccc}
0 & \to & J_x \otimes B & \to & J \otimes B & \to & (J/J_x) \otimes B & \to & 0 \\
\downarrow & & \downarrow{id} & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & J_x \otimes B & \to & A \otimes B & \to & (A_x) \otimes B & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 & \to & (A/J) \otimes B & \to & (A/J) \otimes B & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

the rows and and outer columns are exact. (The exactness of the right-hand column follows from [1, Theorem 3.4], since $A_x$, being exact, has property C [10]). It is now easy to see, by a standard diagram-chasing argument, \(^2\) that the centre column, i.e. (**), is exact.

2. Let

$$0 \to K \to B \to B/K \to 0$$

be a short exact sequence of $C^*$-algebras. If $\pi$ is an irreducible representation of $A \otimes B$ such that $\pi(A \otimes K) = \{0\}$, the restrictions $\pi_1$ and $\pi_2$ of $\pi$ to $A$ and $B$, respectively, are factor representations, and $\pi_2(K) = \{0\}$. Let $J = \ker \pi_1$. Then $J \otimes B \subseteq \ker \pi$ and, since the sequence (***) is exact,

$$(A \otimes B)/(J \otimes B) \cong (A/J) \otimes B$$

canonically, and $\pi = \bar{\pi} \circ (\varphi \otimes \text{id}_B)$, where $\varphi : A \to A/J$ is the quotient morphism and $\bar{\pi}$ is an irreducible representation of $(A/J) \otimes B$. By part (1) of the proof $A/J \cong \pi_1(A)$ is exact, and moreover $A \otimes K \subseteq \ker \pi$, which implies that $(A/J) \otimes K \subseteq \ker \bar{\pi}$. Thus the sequence

$$0 \to (A/J) \otimes K \to (A/J) \otimes B \to (A/J) \otimes (B/K) \to 0$$

\(^2\) Known variously as the $3 \times 3$- and 9-lemma
is exact, and so $\tilde{\pi} = \tilde{\pi} \circ (id_{A/J} \otimes \psi)$, where $\psi : B \to B/K$ is the quotient morphism and $\tilde{\pi}$ is an irreducible representation of $(A/J) \otimes (B/K)$.

Now $\varphi \otimes \psi = (id_{A/J} \otimes \psi) \circ (\varphi \otimes id_B) = (\varphi \otimes id_B) \circ (id_A \otimes \psi)$, and so $\pi = \tilde{\pi} \circ (id_{A/J} \otimes \psi) \circ (\varphi \otimes id_B) = (\tilde{\pi} \circ (\varphi \otimes id_B)) \circ (id_A \otimes \psi)$. If $x \in (A \otimes B) \setminus (A \otimes K)$, we can choose $\pi$ so that $\pi(x) \neq 0$ and $A \otimes K \subseteq \ker \pi$. Then $(\tilde{\pi} \circ (\varphi \otimes id_B))(x) = \pi(x) \neq 0$, which implies that $(id_A \otimes \psi)(x) \neq 0$, i.e. $x \notin \ker(id_A \otimes \psi)$. Since $x$ is arbitrary, it follows that $\ker(id_A \otimes \psi) = A \otimes K$, which is equivalent to the exactness of the sequence

$$0 \to A \otimes K \to A \otimes B \to A \otimes (B/K) \to 0.$$ 

Since $B$ and $K \triangleleft B$ are arbitrary, $A$ is exact. $\square$

**Proof of Theorem 4.6 (i) $\Rightarrow$ (iii):** By [10], if $A$ is exact it has property C. This implies [1, Theorem 3.4] that the sequence

$$0 \to J_x \otimes B \to A \otimes B \to A_x \otimes B \to 0 \quad (*)$$

is exact for any $B$ and $x \in X$. By Proposition 2.7 the bundle $\mathcal{A} \otimes B$ is continuous.

(iii) $\Rightarrow$ (v): This follows immediately from Lemma 4.10(2).

(v) $\Rightarrow$ (iv): Since the bundle $\mathcal{A} \otimes C_0(X) \mathcal{B}$ is obtained from the bundle $\mathcal{A} \otimes \mathcal{B}$ by restriction to the diagonal $\{(x,x) : x \in X\} \subseteq X \times X$, the result is immediate.

(iv) $\Rightarrow$ (iii): Let $\mathcal{B}$ be the trivial $C^*$-bundle on $X$ with fibre $B$. Then the product bundle $\mathcal{A} \otimes C_0(X) \mathcal{B}$, which is continuous by assumption, is just $\mathcal{A} \otimes B$.

(iii) $\Rightarrow$ (i): By Proposition 2.7, the continuity of the bundle $\mathcal{A} \otimes B$ implies the exactness of the sequence

$$0 \to J_x \otimes B \to A \otimes B \to A_x \otimes B \to 0 \quad (*)$$

for each $x \in X$. Since each $A_x$ is exact, Proposition 4.11 implies that $A$ is exact.

(iii) $\Leftrightarrow$ (ii): As remarked earlier, the exactness of $(*)$ for all $B$ is equivalent to the map $\pi_x$ being locally liftable, by [6, Theorem 3.2]. $\square$

**Remark 4.12** Another route from (v) to (i) is the following. If $\mathcal{B} = (Y, \sigma_y : B \to B_y, B)$ is a continuous bundle, then the continuity of the product bundle $\mathcal{A} \otimes \mathcal{B}$ implies that of the bundle $A \otimes \mathcal{B}$, by Lemma 4.10(1). By Theorem 4.5 $A$ is exact.

### 5 Concluding remarks

1. Recall that a continuous $C^*$-bundle $\mathcal{B} = (X, B_x, B)$ is trivial when $B = C_0(X, A)$, $B_x = A$ ($x \in X$) for some $C^*$-algebra and $\pi_x(f) = f(x)$ for $f \in C_0(X)$. Thus $\mathcal{B} = \sim_x \otimes A$, where $\sim_x$ is the trivial bundle on $X$ with fibre $\sim$.

For any $C^*$-algebra $C$, $\mathcal{B} \otimes C \cong (A \otimes \sim_x) \otimes C \cong \sim_x \otimes (A \otimes C)$, so that $\mathcal{B} \otimes C$ is itself continuous. If $\mathcal{B}_1$ is a subbundle of $\mathcal{B}$, it follows that $\mathcal{B}_1 \otimes C$ is a subbundle of $\mathcal{B} \otimes C$.

2. The exactness of the sequence

$$0 \to J_x \otimes B \to A \otimes B \to A_x \otimes B \to 0 \quad (*)$$

for each $x \in X$ implies that $A$ is exact.

3. The exactness of $(*)$ for all $B$ is equivalent to the map $\pi_x$ being locally liftable, by [6, Theorem 3.2].
\( \mathcal{B}_1 \otimes C \) is also continuous. If \( \mathcal{B}_1 \) is a continuous bundle for which \( \mathcal{B}_1 \otimes C \) is not continuous for some \( C \), as is the case for the bundles of Examples 4.4, \( \mathcal{B}_1 \) cannot be trivial, i.e. a subbundle of a trivial bundle. In particular, if \( \mathcal{B}_1 = (X, \pi_x : A \to A_x, A) \), there cannot exist a family \( \sigma_x \) of representations of the algebras \( A_x \) on a fixed Hilbert space \( H \) such that the function \( x \to \sigma_x(\pi_x(a)) \) is norm-continuous. This contrasts with a recent result of Blanchard [3] that for any continuous \( C^* \)-bundle \( \mathcal{B}_1 = (X, A_x, A) \) with \( A \) separable, there is a family of representations \( \sigma_x : A_x \to B(H) \) such that the function \( x \to \sigma_x(\pi_x(a)) \) is strongly continuous for \( a \in A \). It would be interesting to know under what circumstances a continuous bundle will be subtrivial. There is some evidence to suggest that this will be the case if all the fibre algebras are nuclear.

2. We have not considered continuity conditions for crossed product bundles of form \( \mathcal{A} \bowtie_a G \) and \( \mathcal{A} \bowtie_{a,r} G \). As noted in Sect. 2, when the actions are trivial, these bundles reduce to the tensor product bundles \( \mathcal{A} \otimes_{max} C^*(G) \) and \( \mathcal{A} \otimes C^*_r(G) \), respectively. By Theorem 3.2, \( \mathcal{A} \otimes_{max} C^*(G) \) is continuous for arbitrary continuous \( \mathcal{A} \) if and only if \( C^*(G) \) is nuclear, which is the case for discrete \( G \) if and only if \( G \) is amenable, by [12, Theorem 4.2]. By Remark 2.6 (2), amenability of \( G \) implies the continuity of any crossed product bundle \( \mathcal{A} \bowtie_a G = \mathcal{A} \bowtie_{a,r} G \).

The position for reduced crossed products is less clear. By Theorem 4.5, \( \mathcal{A} \otimes C^*_r(G) \) is continuous for arbitrary continuous \( \mathcal{A} \) if and only if \( C^*_r(G) \) is exact. As we mentioned earlier, it appears to be still unknown whether \( C^*_r(G) \) is exact for all discrete \( G \). If \( \mathcal{A} = (X, A_x, A) \) is a continuous \( C^* \)-bundle and \( x \to \alpha_x \) is a continuous field of actions of a discrete group \( G \) on \( \mathcal{A} \), then by reasoning similar to the proof of Proposition 2.7 it can be shown that the bundle \( \mathcal{A} \bowtie_a G \) is continuous if and only if each of the sequences

\[
0 \to J_x \bowtie_{a,r} G \to A \bowtie_{a,r} G \to A_x \bowtie_{a,r} G \to 0
\]

is exact. Exactness questions of this type will be considered in another paper.

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