The framed braid group and representations

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Many approaches to the Jones polynomial have emerged in the last several years. Jones' original approach is through group representations. He constructs representations of B_n which are unitary when they arise from $\mathbf{C}^*$ algebras. E. Witten viewed the Jones polynomial through quantum mechanics and generalized it to an invariant of links in 3-manifolds. There is also the combinatorial viewpoint that considers skein relations.

The situation for Witten's 3-manifold invariant is similar. Witten's original approach is through quantum mechanics. This approach has been broadened and reinterpreted by several authors (Reshetikhin-Turaev [**R-T**], Kirby-Melvin [**K-M**], Cappell-Lee-Miller [**C-L-M**]). The Kirby-Melvin interpretation motivated W.B.R. Lickorish to develop a combinatorial approach [**L1**], [**L2**] and [**K-S1**]. The combinatorial approach uses the Kauffman bracket and the evaluation of the Jones polynomial at roots of unity (from Jones' viewpoint, unitary representations of B_n). However there is no development via group representations. In fact there is no group. This paper introduces a candidate for a group, the framed braid group. The second portion discusses its representation theory. In particular it is shown that framing information always separates from braiding information when examining irreducible unitary representations.

Framed Braids. We begin with the viewpoint that any orientable 3-manifold can be produced by surgery on a link [**L**]. The nature of the equivalence relation on links that yields 3-manifolds is described by the Kirby calculus in [**K**]. Unframed oriented links can be described in terms of braids: given a geometric braid β the link $\hat{\beta}$ is gotten by identifying the initial points and the terminal points of β. Every link can be obtained in this manner. If we assign framing numbers to the top of each strand of a braid we obtain a framed braid.

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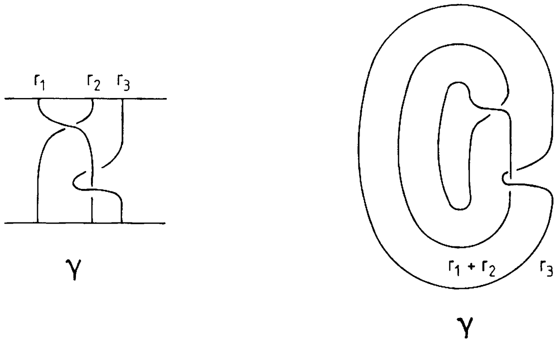

Figure 1

Let π be the natural map from the braid group to the symmetric group, $\pi : B_n \to \Sigma_n$. Then each braid β has a decomposition into cycles given by $\pi(\beta)$. If we begin with a framed braid we can observe which framings correspond to strands in the same cycle of $\pi(\beta)$. Hence we can close a framed braid to obtain a framed oriented link; close the braid and add the framings that correspond to a cycle, i.e., yield a link component (Figure 1). By forgetting the orientation we obtain a framed link description of a 3-manifold.

(*) Framed braids $\xrightarrow{\text{close}}$ Framed oriented links $\xrightarrow{\text{Forget orientation}}$ Framed links $\xrightarrow{\text{surgery}}$ 3-manifolds

Let B_n denote the braid group with the explicit generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$;

(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

B_n acts on $\{1, 2, \ldots, n\}$ through π, i.e., $\sigma(i) = \pi(\sigma)(i)$ for $\sigma \in B_n$. This paper follows the convention that the symmetric group acts from the right so that $(\sigma\tau)(i) = \tau(\sigma(i))$ for $\sigma, \tau \in B_n$. Note that $\pi(\sigma_i) = (i, i+1)$.

Definition. The framed braid group $\mathfrak{F}_n$ is the group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, $t_1, t_2, \ldots, t_n$ with the relations (1), (2) and additional relations

(3) $t_it_j = t_jt_i$ for all i, j;

(4) $\sigma_i t_j = t_{\sigma_i(j)}\sigma_i$.

The group $\mathfrak{F}_n$ is a semidirect product $\mathbf{Z}^n \rtimes B_n$ where the action of B_n on $\mathbf{Z}^n$ is given by $\sigma(r_1, r_2, \ldots, r_n) = (r_{\sigma(1)}, r_{\sigma(2)}, \ldots, r_{\sigma(n)})$. If $t_1^{r_1}t_2^{r_2}\cdots t_n^{r_n}\alpha \in \mathfrak{F}_n$ with $\alpha \in B_n$ then the r_i's are called *framings*. Note that $\sigma t_i = t_{\sigma^{-1}(i)}\sigma$ for $\sigma \in B_n$. The product and the inverse in this notation are given as follows. See Figure 2 for an example.

$$(t_1^{r_1}t_2^{r_2}\cdots t_n^{r_n}\alpha)(t_1^{s_1}t_2^{s_2}\cdots t_n^{s_n}\beta) = t_1^{r_1+s_{\alpha(1)}}t_2^{r_2+s_{\alpha(2)}}\cdots t_n^{r_n+s_{\alpha(n)}}\alpha\beta$$

and

$$(t_1^{r_1}t_2^{r_2}\cdots t_n^{r_n}\alpha)^{-1} = t_1^{-r_{\alpha^{-1}(1)}}t_2^{-r_{\alpha^{-1}(2)}}\cdots t_n^{-r_{\alpha^{-1}(n)}}\alpha^{-1}.$$

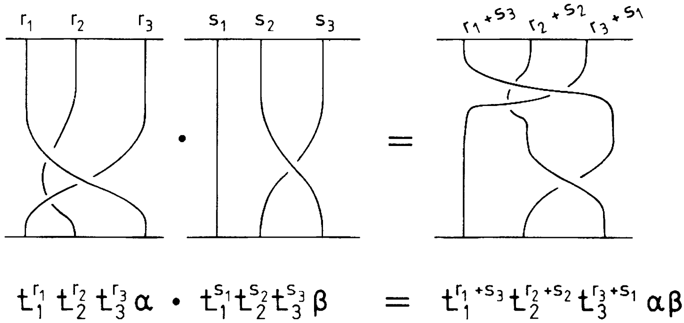

Figure 2

The subgroup $\mathbf{Z}^n$ is the subgroup of framings.

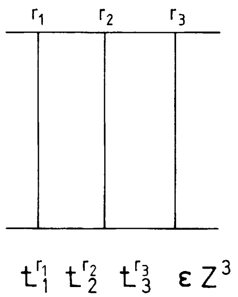

Figure 3

In [**K-S2**] the relationship between framed braids and 3-manifolds via the correspondence $(*)$ is determined. This relationship is given in the theorem below. Framing changes are given in terms of functions on the symmetric group. For $i < n$ let λ_i assign to each permutation in Σ_{n-1} a function from $\{1, 2, \ldots, n-1\}$ to $\{0, 1, \ldots, n-i\}$ as follows. Suppose $\tau \in \Sigma_{n-1}$ decomposes into a product of disjoint cycles $\tau_1\tau_2\cdots\tau_k$. If p occurs in some $\tau_j = (j_1, j_2, \ldots, j_m)$ then $\lambda_i(\tau)(p)$ is defined to be the number of elements in the intersection $\{j_1, j_2, \ldots, j_m\} \cap \{i, i+$

$1,\ldots,n-1\}$. Denote by $W_{n,j}$ the braid $\sigma_{n-1}\sigma_{n-2}\cdots\sigma_{j+1}\sigma_j^2\sigma_{j+1}\cdots\sigma_{n-2}\sigma_{n-1}$ in B_n.

Theorem 1. *Two framed braids represent homeomorphic 3-manifolds if and only if they are related by the equivalence relation generated by the following moves:*
(1) conjugation by framed braids;
(2) Markov move : for $\beta \in \mathfrak{F}_n$, $\beta\sigma_n \sim \beta \sim \beta\sigma_n^{-1}$;
(3) blow up : for $\beta \in \mathfrak{F}_n$, $t_{n+1}\beta \sim \beta \sim t_{n+1}^{-1}\beta$;
handle slide : for $\alpha, \beta \in \mathfrak{F}_{n-1}$,
$t_nW_{n,j}\alpha W_{n,i}^{-1}\beta \sim t_{n+1}t_n^{2\lambda+1}(W_{n,j}\sigma_nW_{n,j}\sigma_n^{-1})\alpha(W_{n,i}^{-1}\sigma_n^{-1}W_{n,i}\sigma_n)\beta\sigma_n^{-2}\sigma_{n-1}^{-1}$
where $\lambda = \lambda_i(\pi(\beta\alpha))(\alpha(n-1)) - \lambda_j(\pi(\alpha\beta))(n-1)$;
(3) orientation reversing : for $\alpha, \beta \in \mathfrak{F}_{n-1}$, $W_{n,j}\alpha W_{n,i}^{-1}\beta \sim W_{n,j}^{-1}\alpha W_{n,i}\beta$.

Figures 4, 5, 6 and 7 describe the moves 2, 3, 4 and 5.

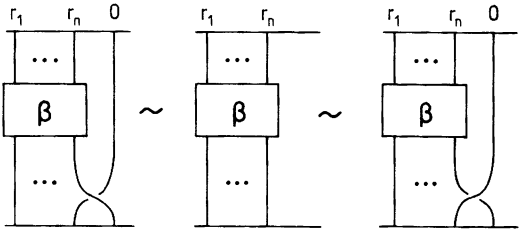

Figure 4

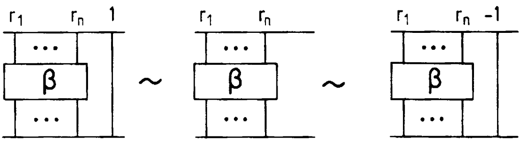

Figure 5

Unitary representations. In the semidirect product $\mathfrak{F}_n$, B_n acts on $\mathbf{Z}^n$ and so gives an action on the characters of $\mathfrak{F}_n$. Let $char(\mathbf{Z}^n)$ be the characters of $\mathbf{Z}$ (i.e., homomorphisms to S^1) and take $\chi \in char(\mathbf{Z}^n)$. If $\beta \in B_n$ then $\chi\cdot\beta \in char(\mathbf{Z}^n)$ by $\chi\cdot\beta(z) = \chi(\beta z\beta^{-1})$ As a space, $char(\mathbf{Z}^n)$ is $S^1\times S^1\times\cdots\times S^1$ the n-torus. The action of B_n on $\mathbf{Z}^n$ is by permuting the coordinates so B_n acts on $char(\mathbf{Z}^n)$ by permuting the S^1 factors, i.e., $(\omega_1,\omega_2,\cdots,\omega_n)\cdot\beta = (\omega_{\pi(\beta)(1)},\omega_{\pi(\beta)(2)},\cdots,\omega_{\pi(\beta)(n)})$. For $\chi \in char(\mathbf{Z}^n)$ let B_χ be the stabilizer of χ in B_n, $B_\chi \subset B_n$. Suppose $\chi = (\omega_1,\omega_2,\cdots,\omega_n)$. The n-tuple χ has k distinct entries $\{\eta_1,\eta_2,\cdots,\eta_k\}$ and η_j occurs with multiplicity n_j. From the action of Σ

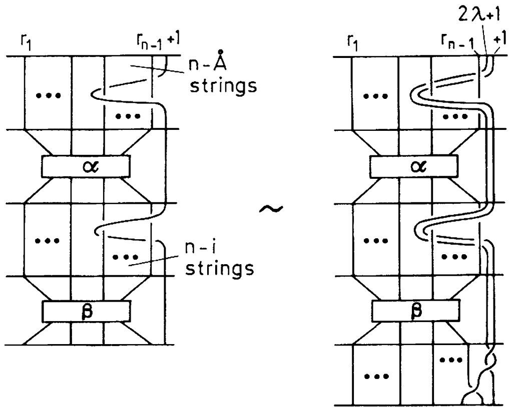

Figure 6

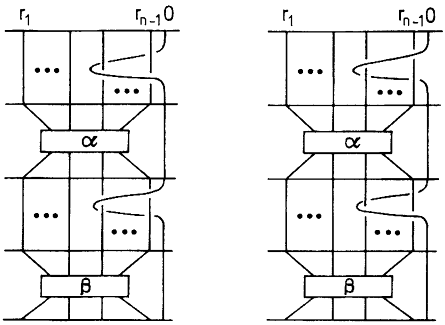

Figure 7

on $char(\mathbf{Z}^n)$ we see that the stablizer of χ in Σ_n is isomorphic to $\Sigma_\chi = \Sigma_{n_1} \oplus \Sigma_{n_2} \oplus \cdots \oplus \Sigma_{n_k}$. Since B_n acts via $\pi : B_n \to \Sigma_n$ we have that $B_\chi = \pi^{-1}(\Sigma_\chi)$. This subgroup consists of elements that preserve some collection of subsets of the initial points. For example if $\chi \in char(\mathbf{Z}^3)$ and $\omega_1 = \omega_2 \neq \omega_3$ then $\beta \in B_\chi$ for the β in Figure 8 preserves the sets $\{1.2\}$ and $\{3\}$.

The induced map $\pi : B_n/B_\chi \to \Sigma_n/\Sigma_\chi$ is a bijection so that $|B_n/B_\chi| = \binom{n}{n_1, n_2, \cdots, n_k}$ and B_χ is of finite index in B_n.

We now consider the general theory of representations of semidirect products $G = N \rtimes S$ with N abelian. A complete discussion is given in [**B-R**] chapter 17. In the semidirect product G, S acts on N and so on $char(N)$. In order to describe

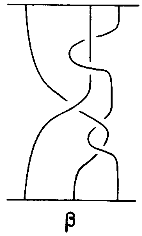

Figure 8

the unitary representations of G we need G to satisfy a technical condition: G is a regular semidirect product if $char(N)$ contains a countable family $Z_1, Z_2, Z_3, \cdots$ of Borel subsets, each a union of G orbits such that every orbit in $char(N)$ is the intersection of members of the subfamily containing that orbit.

Suppose G is a regular semidirect product of N and S. Let $\chi \in char(N)$ and denote by S_χ the stabilizer of χ under the S action. Let λ be a representation of S_χ. If $a \in N$ and $b \in S$ then let $\chi \cdot \lambda(ab) = \chi(a)\lambda(b)$. If $a' \in N$, $b' \in S$, $g = ab$ and $g' = a'b'$ then

$$\begin{aligned}
\chi \cdot \lambda(gg') &= \chi \cdot \lambda(aba'b') \\
&= \chi \cdot \lambda(aba'b^{-1}bb') \\
&= \chi(a)\chi(ba'b^{-1})\lambda(b)\lambda(b') \\
&= \chi(a)\chi(a')\lambda(b)\lambda(b') \\
&= \chi \cdot \lambda(ab)\chi \cdot \lambda(a'b') \\
&= \chi \cdot \lambda(g)\chi \cdot \lambda(g')
\end{aligned}$$

so $\chi \cdot \lambda$ is a representation of the subgroup $N \rtimes S_\chi = G_\chi$. We can now form the representation

$$\rho = \mathrm{ind}_{G_\chi}^{G}(\chi \cdot \lambda)$$

a representation of the whole group G. The following theorems are results from Mackey's theory of induced representations (see **[B-R]** pages 508-509).

Theorem 2. *Let G be a regular semidirect product $N \rtimes S$ of separable, locally compact groups N and S with N abelian.*

(1) Let ρ be an irreducible unitary representation of G. Then there is a character χ of N and an irreducible unitary representation λ of S_χ such that ρ is unitary equivalent to $\mathrm{ind}_{G_\chi}^{G}(\chi \cdot \lambda)$.

(2) If $\chi \in char(N)$ and λ is an irreducible unitary representation of S_χ then $ind_{G_\chi}^{G}(\chi \cdot \lambda)$ is an irreducible unitary representation of G.

The framed braid group is a regular semidirect product by,

Proposition 3. *If $G = N \rtimes S$, N a finitely generated abelian group and if S acts on N through a finite group then G is a regular semidirect product.*

Proof. If N is finite then so is $char(N)$ and the Z_i's are the orbits. If not then $char(N)$ is a finite union of tori, $S^1 \times \cdots \times S^1$. Since this space is second countable, let $\{\mathfrak{O}_i\}_{i=1}^{\infty}$ be a countable base and take $Z_i = G(\mathfrak{O}_i)$. □

As a corollary to Theorem 2 we have,

Corollary 4. *If ρ is an irreducible unitary representation of $\mathfrak{F}_n$ then there is a character χ of $\mathbf{Z}^n$ and an irreducible representation λ of B_χ such that*

$$\rho = \mathrm{ind}_{\mathbf{Z}^n \rtimes B_\chi}^{\mathbf{Z}^n \rtimes B_n}(\chi \cdot \lambda).$$

The $\mathbf{Z}^n$ subgroup is purely framing phenomena and the B_χ subgroup is purely braiding phenomena so the representation theory of $\mathfrak{F}_n$ separates these aspects.

Some examples. These examples are motivated by the Kirby-Melvin interpretation of Witten's invariants **[K-M]**. Let $c : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, r\}$ so that $c(i)$ is a natural number and let T_j be a braid of m_j strings. For example see Figure 9.

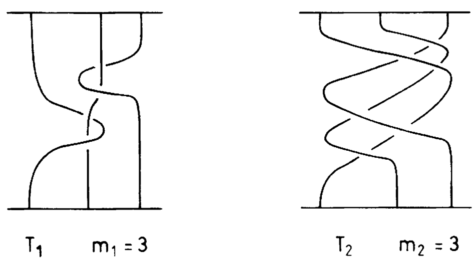

Figure 9

Let A_n be the Temperley-Lieb algebra considered by Jones in **[J]**. This is a finite dimensional $*$-algebra with generators denoted by $e_1, e_2, \cdots, e_n$. Let $\gamma_q : \mathbf{C}[B_q] \to A_q$ be the map $\gamma_q(\sigma_i) = A + A^{-1}e_i$ for A a root of unity. This defines a unitary representation of B_q. In particular, $\gamma_q(T_{c(i)})$ is unitary. Let $\mathfrak{z}_i : \mathbf{Z} \to A_{m_{c(i)}}$ be defined by $\mathfrak{z}_i(k) = \gamma_{m_{c(i)}}(T_{c(i)}^k)$ and define a unitary representation of $\mathbf{Z}^n$ by

$\mathfrak{Z} : \mathbf{Z}^n \to A_{\Sigma_{i=1}^n m_{c(i)}}$ by

$$\mathbf{Z}^n \xrightarrow{\oplus \mathfrak{Z}_i} \oplus A_{m_{c(i)}} \subset A_{\Sigma_{i=1}^n m_{c(i)}}.$$

So if $n = 3$ and $c(1) = c(2) = 1$ and $c(3) = 2$ then $\mathfrak{Z}(1, 3, 2)$ is shown in Figure 10.

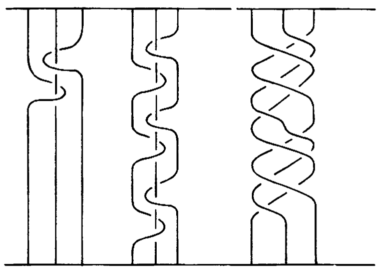

Figure 10

As a representation of $\mathbf{Z}^n$, $\mathfrak{Z} = \Sigma a_j \chi_j$ for $\chi_j \in char(\mathbf{Z}^n)$. Let P be the subset of the powerset of $\{1, \cdots, n\}$ given by $\{c^{-1}(1), c^{-1}(2), \cdots, c^{-1}(r)\}$ and $B_P \subset B_n$ the subgroup that preserves these sets of points, i.e., $\{\alpha \in B_n |$ if $x \in c^{-1}(i)$ then $\pi(\alpha)(x) \in c^{-1}(i)\}$. Now $B_P \cong \pi^{-1}(\Sigma_{|c^{-1}(1)|} \oplus \Sigma_{|c^{-1}(2)|} \oplus \cdots \in \Sigma_{|c^{-1}(r)|})$. The functions c and c' yield conjugate subgroups if and only if there is a $\delta \in \Sigma_n$ with $|c^{-1}(i)| = |c'^{-1}(\delta(i))|$, i.e., c and c' determine the same partition of the number n. If $G = \beta B_P \beta^{-1}$ for $\beta \in B_n$ then let $c' = c \circ \pi(\beta)$ and P' be the resulting set of subsets of $\{1, \cdots, n\}$. We have that $G = B_{P'}$. So the collection of subgroups conjugate to B_P is $\{B_{P'} | P'$ and P determine the same partition of $n\}$. This observation is important if one wishes to consider characters of representations. If $a_j \neq 0$ then $B_{\chi_j} \subset B_P$ and we can construct representations of $\mathbf{Z}^n \rtimes B_P$ by constructing representations λ_P of B_P. Let λ_P be the composition

$$B_n \to B_{\Sigma_{i=1}^n m_{c(i)}} \xrightarrow{\gamma_{\Sigma_{i=1}^n m_{c(i)}}} A_{\Sigma_{i=1}^n m_{c(i)}}$$

where the map between braid groups makes $c(i)$ parallel copies of the i^{th} string as shown below. Then the $a_j \chi_j \cdot \lambda_P$ are representations of $\mathbf{Z}^n \rtimes B_P$ and determine representations of $\mathfrak{F}_n$.

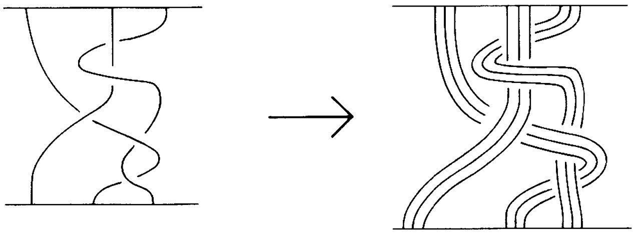

Figure 11

References

[B-R] A.O. Barut and R. Raczka, *Theory of group representations and applications*, World Scientific Publishing Co. Pte. Ltd., Singapore, 1986.

[B] J. Birman, *Braids, Links and Mapping Class Groups*, Ann. Math. Stud., vol. 82, Princeton University Press, Princeton, New Jersey, 1975.

[C-L-M] S. Cappell, R. Lee and E. Miller, *Invariants of 3-manifolds from conformal field theory, preprint.*

[J] V.F.R. Jones, *Index for subfactors*, Inventiones Math. **72** (198), 1–25.

[K] R. C. Kirby, *A calculus for framed links in* S^3, Inventiones Math. **45** (1978), 35–56.

[K-M] R. C. Kirby and P. Melvin, *On the 3-manifold invariants of Witten and Reshetikhin-Turaev for* $sl(2, \mathbf{C})$*, preprint.*

[K-S1] K.H. Ko and L. Smolinsky, *A Combinatorial matrix in 3-manifold theory*, Pacific J. Math., to appear.

[K-S2] ——, *The framed braid group and 3-manifolds, preprint.*

[L] W. B. R. Lickorish, *A representation of orientable, combinatorial 3-manifolds*, Ann. Math. **76** (1962), 531–540.

[L1] ——, *3-Manifold invariants from the combinatorics of the Jones polynomial*, Pacific J. Math., to appear.

[L2] ——, *3-Manifolds and the Temperley-Lieb algebra, preprint.*

[R-T] N. Yu. Reshetikhin and V.G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Inventiones Math., to appear.

[W] E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), 351–399.