ELEMENTARY TOPOSES

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Introduction

The notion of topos embodies in it two aspects of mathematics, the one a geometric, the other a logical aspect. Toposes were introduced by the circle around Grothendieck as a generalization of the notion of topological space, for purposes in algebraic geometry. To a topological space is associated a topos, namely the category of set-valued sheaves over it.

Later, Lawvere maintained that the notion of topos could be viewed as a conceptual form of the notion of higher order language, or alternatively, as a world in which higher order notions could be interpreted (the germ of this viewpoint is found in [13] and [14]); it was substantiated by the work of Lawvere and Tierney [17] on elementary toposes (culminating in an independence proof of the continuum hypothesis). The notion of elementary topos, as defined by these authors, frees the notion of topos from any external form of infinities; for instance, an elementary topos is not required to have arbitrary (infinite) limits or colimits. This finitary-ness of the theory of elementary toposes conforms with the idea that an elementary topos also has the features of a syntactic object ("a language"). On the other hand, a language, though of finitary nature, should be able to "speak about" infinitary and higher order ideas. Elementary toposes have that feature.

In particular, the idea of "power set formation" exists inside an elementary topos, in the sense that there is an object Ω (or 2) so that to each object X there exists an object 2^X (in our notes denoted $X \pitchfork \Omega$) whose "elements" index the "subsets" of X.

The specific aims of these notes are first to develop part of the "classical" (Grothendieck-Verdier) theory of toposes in the setting of elementary toposes; in particular, the notion of "morphisme de topos", [9], what Lawvere - Tierney call <u>geometric morphisms</u>. Secondly to illustrate the logical aspects of elementary toposes, by developing the notion of "small category-object" and "topological space object" inside an arbitrary elementary topos, and carry out certain constructions related to such objects ("sheaf reflection for a presheaf object on a topological space object"), thereby producing new elementary toposes (generalizing the way classical sheaf theory out of a topological space produced a (classical) topos). To carry out this program, we lean heavily on techniques of the first four chapters (factorization of topos morphisms).

The main example of an elementary topos is $\mathcal{S}$, the category of sets. A "topological space object in $\mathcal{S}$" is just a topological space. Lawvere has pointed out that "an important technique is to lift constructions first understood for "the" category $\mathcal{S}$ of abstract sets to an arbitrary topos". This is true primarily for "logical" concepts, which in these notes occur mainly in Chapter 1 and Chapter 5. The reader should for every construction carried out in these chapters in his mind specialize the constructions to $\mathcal{S}$ in order to see <u>which</u> constructions in $\mathcal{S}$ actually are being

The source we have used for Chapter 1 and Chapter 3 is Tierney's lectures in Halifax, 1969-1970, in a joint seminar with Lawvere. The ideas of the remaining chapters are also largely due to Lawvere and Tierney (the results of Chapter 5 are thus stated in [16]), but we had to supply the relevant constructions ourselves.

We are also in debt to Julian Cole and Chr. Juul Mikkelsen for supplying ideas, examples, and curiosity, during the seminar which produced these notes (November 1970 - May 1971).

Conventions and Notation

We use almost entirely standard notation. Maps are composed the 'algebraic' way: 'f.g' means: f followed by g. Functors and other things that are applied on the left of their argument (like $F(A)$, $\exists_f(X)$, etc.), however, often are composed the other way.

As usual, $F \dashv G$ indicates that F is left adjoint to G. We use $\hat{}$ to denote passage (either way) along the adjointness isomorphism for exponential adjointness

$$\hom(A\times B,C) \cong \hom(A,C^B) = \hom(A,B \pitchfork C)$$

($B \pitchfork C$ being used for C^B for typographical reasons). Some concepts have double notation, like

$$t\colon\ 1 \longrightarrow \Omega\,, \quad \text{true:}\ 1 \longrightarrow \Omega;$$

likewise '$\rightsquigarrow$' and '$\longmapsto$' means the same (used when a set theoretic mapping is defined elementwise). 'Colimits' and 'Right limits' are used for the same; a reflection functor is a <u>left</u>

1. Exactness properties

We study exactness properties which categories $\underline{E}$ have, if they satisfy* axioms T1 - T3 below; such categories we shall here call elementary toposes, or just toposes, [16], [8].

T1 $\underline{E}$ has finite limits and finite colimits

T2 $\underline{E}$ has exponentiation

T3 $\underline{E}$ has a subobject classifier $1 \xrightarrow{\text{true}} \Omega$.

T2 means that for any $A \in |\underline{E}|$, $- \times A: \underline{E} \longrightarrow \underline{E}$ has a right adjoint, denoted $A \pitchfork -$ or $(-)^A$. The end-adjunction for the adjointness is denoted ev:

$$(A \pitchfork B) \times A \xrightarrow{ev} B.$$

T3 means the following: 1 denotes the terminal object. Ω is an object equipped with a map $1 \xrightarrow{\text{true}} \Omega$ so that:

for any monomorphism $f: A' \rightarrowtail A$ in $\underline{E}$, there is a unique $\varphi: A \longrightarrow \Omega$ ("characteristic function of f") making

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \Omega \\ {\scriptstyle f}\uparrow & * & \uparrow{\scriptstyle \text{true}} \\ A' & \longrightarrow & 1 \end{array}$$

a pull-back.

---

* Since limits, colimits, exponentials, and also subobject classifiers are unique (up to unique isomorphisms) when they exist, it makes sense to say that a category is a topos, rather than "it is equipped with topos structure". (For the uniqueness of Ω, see Remark 1.38 at the end of this chapter). For notational convenience, however, we assume that a definite choice of limits, colimits, exponentiation, and subobject classifier has been [illegible]

Note. Since 1 is terminal a) the map $A' \longrightarrow 1$ does not have to be specified further; b) any map with domain 1 is a monomorphism; in particular, 'true' is monic. Pulling a monic back along a map gives a monic; so a pull-back diagram of the form * above necessarily has f monic. We call (the equivalence class of) such f defined by the pull-back * "the subobject of A classified by φ".

The following concepts make sense in any category:

(1.1) $\dfrac{\text{monomorphism} \qquad \text{equalizer}}{\text{epimorphism} \qquad \text{coequalizer}}$

as well as

(1.2) $\dfrac{\text{equivalence relation} \qquad \text{kernel pair}}{\text{coequivalence relation} \qquad \text{cokernel pair}}$

The ones in block (1.1) are well known. For block (1.2):

Definition 1.1. An equivalence relation in a category $\mathcal{C}$ is a jointly monic pair

$$K \underset{k_1}{\overset{k_0}{\rightrightarrows}} A$$

so that for any $X \in |\mathcal{C}|$,

$$\hom(X,K) \xrightarrow[\hom(1,k_0),\hom(1,k_1)]{} \hom(X,A) \times \hom(X,A)$$

describes an equivalence relation on the set $\hom(X,A)$.

Note. If $\mathcal{C}$ has products, $K \xrightarrow{\langle k_0,k_1\rangle} A\times A$ is necessarily monic. If k_0,k_1 is an equivalence relation, we shall abuse language and call the single map $\langle k_0,k_1\rangle$ an equivalence relation.

Definition 1.2. A kernel pair for a map $f\colon A \longrightarrow B$ in $\mathcal{C}$ is a pair of maps $K \underset{k_1}{\overset{k_0}{\rightrightarrows}} A$, so that $k_0.f = k_1.f$, and which is universal with this property.

Note. k_0, k_1 may be obtained by the pull-back

$$\begin{array}{ccc} K & \xrightarrow{k_0} & A \\ {\scriptstyle k_1}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

Proposition 1.3. Any equalizer is monic; any kernel pair is an equivalence relation.

Proof. Trivial. - We shall prove that assuming T1 - T3, the converse of the proposition holds:

Proposition 1.4. Assuming T1 and T3, any monomorphism is an equalizer.

Proof. Let $f\colon A' \rightarrowtail A$ be monic, let φ be its characteristic function. The pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \Omega \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle true} \\ A' & \longrightarrow & 1 \end{array}$$

shows that f is the equalizer of φ and $A \longrightarrow 1 \xrightarrow{true} \Omega$.

Theorem 1.5. Assuming T1, T2, and T3, every equivalence relation is a kernel pair.

Before proving it, we develop some concepts for categories $\underline{E}$ satisfying T1-T3 (these axioms will be in force from now on).

Let $\varphi, \psi\colon A \longrightarrow \Omega$. Call $\varphi \leq \psi$ if the subobject of A classified by φ is smaller than the subobject of A classified by ψ (usual ordering of subobjects).

Proposition 1.6. In order that $\varphi \leq \psi\colon A \longrightarrow \Omega$, it is necessary and sufficient that for all $X \in |E|$ and all $x\colon X \longrightarrow A$

$$(x.\varphi = \mathrm{true}_X) \Longrightarrow (x.\psi = \mathrm{true}_X),$$

where 'true_X' denotes the map $X \to 1 \xrightarrow{\mathrm{true}} \Omega$.

Proof. Easy.

Note. If $\varphi, \psi\colon A \to \Omega$ and $\varphi \leq \psi$ and $\psi \leq \varphi$, then $\varphi = \psi$ (the corresponding thing for monomorphisms into A only holds up to isomorphism).

Proof of Theorem 1.5. Let $\varkappa$ be characteristic function for $K \xrightarrow{\langle k_0, k_1\rangle} A \times A$, where $k_0, k_1\colon K \rightrightarrows A$ is an equivalence relation. We shall see that k_0, k_1 in

$$(1.3) \qquad K \underset{k_1}{\overset{k_0}{\rightrightarrows}} A \xrightarrow{\hat{\varkappa}} A \pitchfork \Omega$$

is a kernel pair of $\hat{\varkappa}$.

First we prove that (1.3) commutes. By exponential adjointness it suffices to prove commutativity of

$$K \times A \underset{k_1 \times 1}{\overset{k_0 \times 1}{\rightrightarrows}} A \times A \xrightarrow{\varkappa} \Omega.$$

By the Note above, it suffices to see $k_0 \times 1.\varkappa \leq k_1 \times 1.\varkappa$, and the opposite inequality.

Let X be arbitrary in $|\underline{E}|$. Denote the equivalence relation on the set $\hom(X,A)$ given by k_0, k_1 by $\sim$. Let $X \xrightarrow{\langle r, a\rangle} K \times A$ have

$$\langle r, a\rangle . k_0 \times 1 . \varkappa = \mathrm{true}.$$

Left hand side is just $\langle r.k_0, a\rangle.\varkappa = \mathrm{true}$, so $r.k_0 \sim a$. Since $r.k_0 \sim r.k_1$, we get $r.k_1 \sim a$, whence

$$\langle r,a\rangle . k_1 \times 1 . \varkappa = \text{true}.$$

Proposition 1.6 now gives the desired inequality.

To see the universal property of (1.3), let $a_0, a_1\colon X \longrightarrow A$ have $a_0.\hat{\varkappa} = a_1.\hat{\varkappa}$. Passing to exponential adjoints, the two maps

$$X \times A \xrightarrow{a_i \times 1} A \times A \xrightarrow{\varkappa} \Omega \qquad i = 0,1$$

are equal. So for any $b\colon X \longrightarrow A$

$$\langle a_0, b\rangle . \varkappa = \langle a_1, b\rangle . \varkappa\colon$$

$$X \xrightarrow{\langle a_i, b\rangle} A \times A \xrightarrow{\varkappa} \Omega.$$

So for every $b \in \hom(X,A)$, $a_0 \sim b$ iff $a_1 \sim b$, whence $a_0 \sim a_1$, i.e. $\langle a_0, a_1\rangle . \varkappa = \text{true}$, i.e. $\langle a_0, a_1\rangle$ factors through $K \xrightarrow{\langle k_0, k_1\rangle} A \times A$.

<u>The notion of graph, and related notions</u>.

Let $f\colon A \longrightarrow B$ be any map in $\underline{E}$. Define Γ_f ("graph of f") to be the map

$$\Gamma_f\colon\ A \xrightarrow{\langle 1, f\rangle} A \times B.$$

It is clearly monic. Its characteristic function

$$\gamma_f\colon\ A \times B \longrightarrow \Omega$$

is denoted γ_f. We then have

$$\hat{\gamma}_f\colon\ A \longrightarrow B \pitchfork \Omega .$$

In case $f = 1_A$, there is special notation for Γ_f, γ_f, and $\hat{\gamma}_f$:

$$\triangle_A : \quad A \xrightarrow{\langle 1,1 \rangle} A \times A \qquad \text{("diagonal")}$$

$$\delta : \quad A \times A \longrightarrow \Omega \qquad \text{("Kronecker-}\delta\text{")}$$

$$\hat{\delta} = \{.\} : \quad A \longrightarrow A \pitchfork \Omega \quad \text{("singleton").}$$

Proposition 1.7. $\{.\}: \ A \longrightarrow A \pitchfork \Omega$ is monic.

Proof. By the proof of Theorem 1.5, the equivalence relation $\triangle$ is kernel pair for $\{.\}$. From the universal property of that kernel pair:

$$A \underset{1}{\overset{1}{\rightrightarrows}} A \xrightarrow{\{.\}} A \pitchfork \Omega$$

clearly follows: $\{.\}$ is monic.

The $\sim$-construction

Let $\xi : \ (A \pitchfork \Omega) \times A \longrightarrow \Omega$ be characteristic function for

$$A \xrightarrow{\langle \{.\}, 1 \rangle} (A \pitchfork \Omega) \times A.$$

Let $\tilde{A}$ be the equalizer

$$\tilde{A} \overset{e}{\rightarrowtail} A \pitchfork \Omega \underset{\hat{\xi}}{\overset{1}{\rightrightarrows}} A \pitchfork \Omega .$$

We shall see that $\{.\}: \ A \longrightarrow A \pitchfork \Omega$ factors through e, so that there is a map $A \xrightarrow{\eta_A} \tilde{A}$ with

$$\eta_A . e = \{.\}.$$

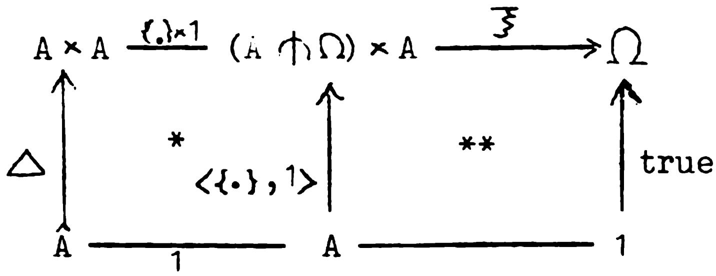
We must prove

$$\{.\}.\hat{\xi} = \{.\}.$$

Pass by exponential adjointness to

$$\{.\}\times 1 \, . \, \xi = \delta : \quad A \times A \longrightarrow \Omega .$$

To prove this equality, we must prove that these two maps in Ω classify the "same" (up to equivalence) subobject of $A \times$
By definition, δ classifies $A \overset{\Delta}{\rightarrowtail} A \times A$. To see what $\{$
classifies, form the pull-back



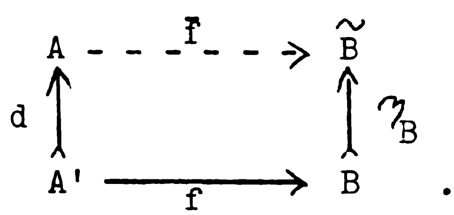
we have used here that we pull back along a composite by pull back along the parts; ** is a pull-back by definition of ξ
* is a pull-back because $\{.\}$ is monic (Proposition 1.7).

This produces

$$\eta_A : \ A \longrightarrow \tilde{A} .$$

Theorem 1.8. To any pair of maps (d,f) as in the pictu below (d monic), there is a unique $\tilde{f} : A \longrightarrow \tilde{B}$ making t diagram into a pull-back:

N.B.



("$\tilde{f}$ classifies the partial map (d,f): $A \dashrightarrow B$").

Proof. Consider $\Gamma_{(d,f)} = \langle d,f \rangle : \ A' \rightarrowtail A \times B$.
It is monic since d' is. Let $\gamma_{(d,f)}$ be its characteristic function $A \times B \longrightarrow \Omega$, and take its exponential adjoint

$\hat{\gamma}: \; A \longrightarrow B \pitchfork \Omega$. If we can prove

$$(1.4) \qquad \hat{\gamma} = \hat{\gamma}.\hat{\xi},$$

then by definition of $\tilde{B} \xrightarrow{e} B \pitchfork \Omega$ as an equalizer, $\hat{\gamma}$ can written $\bar{f}.e$ for a unique $\bar{f}: \; A \longrightarrow \tilde{B}$. To prove (1.4), pass adjoints; so we must prove

$$(1.5) \qquad \gamma_{(d,f)} = \hat{\gamma}\times 1 . \xi : \qquad A \times B \longrightarrow \Omega .$$

To prove this equality, we must prove that these two maps into Ω classify the "same" subobject of $A \times B$. By definition, the left hand side classifies $\langle d,f \rangle$; to see that the right hand si also does, pull back in two steps; we are through with the proof of (1.5) if we can prove that the left hand square in the diagra below is a pull-back:

$$\begin{array}{ccccc}
A \times B & \longrightarrow & B \pitchfork \Omega \times B & \xrightarrow{\;\xi\;} & \Omega \\
{\scriptstyle \langle d,f \rangle}\uparrow & & \uparrow{\scriptstyle \langle \{\cdot\},1 \rangle} & & \uparrow{\scriptstyle \text{true}} \\
A' & \longrightarrow & B & \longrightarrow & 1
\end{array}$$

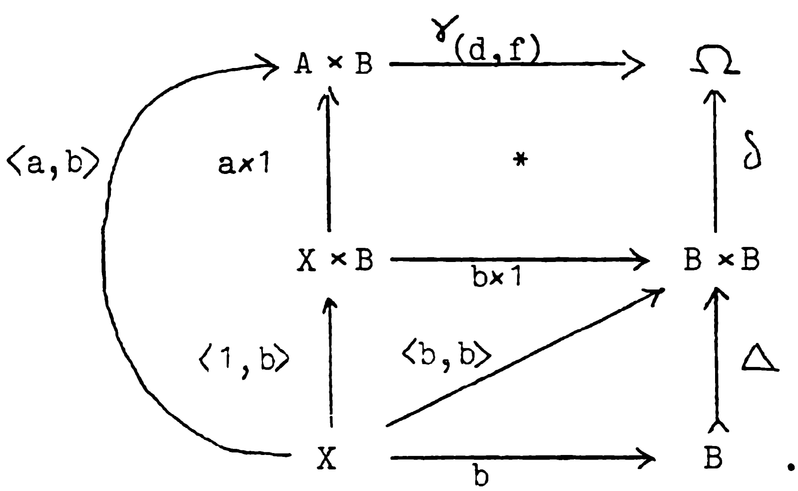
The fact that the left hand square is a pull-back is an immediat consequence of

<u>Lemma 1.9</u>. The diagram

$$\begin{array}{ccc}
A & \xrightarrow{\;\hat{\gamma}\;} & B \pitchfork \Omega \\
{\scriptstyle d}\uparrow & & \uparrow{\scriptstyle \{\cdot\}} \\
A' & \xrightarrow[\;f\;]{} & B
\end{array}$$

is a pull-back (where $\hat{\gamma}$ is the exponential adjoint of the chara teristic map γ of $\langle d,f \rangle: \; A' \rightarrowtail A \times B)$

<u>Proof</u>. Let $X \xrightarrow{a} A$, $X \xrightarrow{b} B$ have $a.\hat{\gamma} = b.\{.\}$. Pass to adjoints; we get that the square*(below) commutes

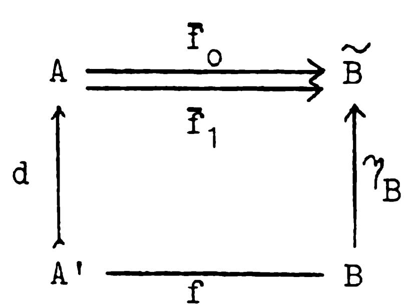


All the rest obviously commutes. Since δ classifies Δ, $\Delta.\delta$ factors through $1 \xrightarrow{\text{true}} \Omega$; therefore $\langle a,b\rangle.\gamma_{(d,f)}$ factors through true. But $\gamma_{(d,f)}$ classifies the subobject $A' \xrightarrow{\langle d,f\rangle} A \times B$, whence $\langle a,b\rangle$ factors through $\langle d,f\rangle$. This proves the lemma.

So we know (1.4) is valid, and $\bar{f}: A \longrightarrow \tilde{B}$ exists with $\bar{f}.e = \hat{\gamma}$. Since both $\hat{\gamma}$ and $\{.\}$ factor through e (as $\bar{f}$ and η_B, respectively), and e is monic, we get immediately from the Lemma 1.9 that the diagram N.B. is a pull-back, as desired.

Finally, we must prove uniqueness of such $\bar{f}$. Suppose both the diagrams

(1.6$_i$)



$i = 0,1$

are pull-backs. Consider $\bar{f}_i.e: A \longrightarrow B \pitchfork \Omega$ $(i=0,1)$, and the two maps corresponding to them under exponential adjointness

$$A \times B \underset{f_1}{\overset{f_0}{\rightrightarrows}} \Omega .$$

By symmetry, it suffices to prove $f_0 \leq f_1$. We apply Proposition 1.6. Let $\langle a,b\rangle : X \longrightarrow A \times B$ have $\langle a,b\rangle . f_0 = \mathrm{true}_X$. Then

$$\mathrm{true}_X = \langle a,b\rangle . f_0 = \langle 1,b\rangle . (a\times 1 . f_0) =$$
$$= \langle 1,b\rangle . (a.\hat{f}_0)^{\wedge} ;$$

since $\hat{f}_0$ factors as $\bar{f}_0 . e$ (by construction of f_0 from $\bar{f}_0$), $\hat{f}_0 . \hat{\xi} = \hat{f}_0$ (ξ defined on page 10); so the equation continues

$$= \langle 1,b\rangle . (a.\hat{f}_0 . \hat{\xi})^{\wedge} = \langle 1,b\rangle . a\times 1 . \hat{f}_0 \times 1 . \xi$$
$$= \langle a,b\rangle . \hat{f}_0 \times 1 . \xi = \langle a.\hat{f}_0 , b\rangle . \xi .$$

Since ξ classifies $\langle \{.\},1\rangle$, we get from this

(1.7) $$a.\hat{f}_0 = b.\{.\}.$$

Since (1.6_0) is a pull-back and $\tilde{B} \longrightarrow B \pitchfork \Omega$ is monic, also

$$\begin{array}{ccc} A & \xrightarrow{\hat{f}_0} & B \pitchfork \Omega \\ {\scriptstyle d}\uparrow & & \uparrow{\scriptstyle \{.\}} \\ A' & \xrightarrow[f]{} & B \end{array}$$

is a pull-back, so from (1.7) follows existence of an $a' : X \longrightarrow A'$ with $a'.d = a$, $a'.f = b$.

From commutativity of (1.6_1), we now conclude

$$a.\bar{f}_1 = b.\eta_B .$$

Multiplying on the right by e gives

$$a.\hat{f}_1 = b.\{.\}: \quad X \longrightarrow B \pitchfork \Omega.$$

Passing to exponential adjoints gives

$$a \times 1.f_1 = b \times 1.\delta$$

and thus

$$\langle a,b\rangle.f_1 = \langle 1,b\rangle.a \times 1.f_1 = \langle 1,b\rangle.b \times 1.\delta,$$

but the right-hand side here obviously is true_X. By Proposition 1.6 we conclude that $f_0 \leq f_1$. Similarly $f_1 \leq f_0$; thus $f_0 = f_1$, thus $\hat{f}_0 = \hat{f}_1$ and $\bar{f}_0 = \bar{f}_1$. This proves the uniqueness of $\bar{f}$; Theorem 1.8 is proved.

Remark. Since η_B is monic, given $A \xrightarrow{\bar{f}} B$, we can, by pulling η_B back along $\bar{f}$, produce a pair (d,f): ("a partial map" $A \dashrightarrow B$")

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & \tilde{B} \\ {\scriptstyle d}\uparrow & & \uparrow \\ A' & \xrightarrow[f]{} & B. \end{array}$$

So the Theorem actually asserts the existence of a 1-1 correspondence between maps $A \longrightarrow \tilde{B}$ and "equivalence classes of partial maps $A \dashrightarrow B$".

The uniqueness of $\tilde{f}$ has as a standard consequence that $\sim$ actually becomes a functor $\underline{E} \longrightarrow \underline{E}$, and η a natural transformation $id_{\underline{E}} \Longrightarrow \sim$.

If Part(A,B) denotes the set of equivalence classes of partial maps $A \dashrightarrow B$, the one-to-one correspondence λ mentioned in the above remark has naturality properties, for example, for $B \xrightarrow{g} C$

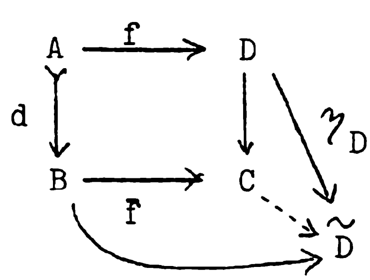
$$\begin{array}{ccc} \mathrm{Part}(A,B) & \longrightarrow & \mathrm{Part}(A,C) \\ \cong \big\downarrow \lambda & & \lambda \big\downarrow \cong \\ \hom(A,\tilde{B}) & \xrightarrow[\hom(1,\tilde{g})]{} & \hom(A,\tilde{C}), \end{array}$$

with top map "composing by g" in an obvious sense, commutes. In fact

<u>Exercise</u>. Define a category Part with objects the same as the objects of $\underline{E}$, and with hom-sets Part(A,B). Define a functor $\underline{E} \xrightarrow{i}$ Part (identity on objects). Define a functor $\sim$: Part $\longrightarrow \underline{E}$. Then $i \dashv \sim$; the front adjunction is η.

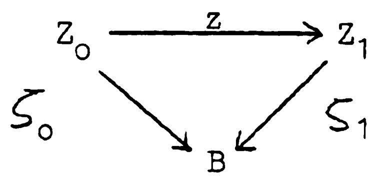
<u>Corollary 1.10</u>. (Push-out Theorem). The pushout of a mono by something is a mono, and the resulting diagram is a pull-back also.

<u>Proof</u>. Let the pushout diagram be the inner square in



Construct the pull-back diagram containing the partial map $(d,f)\colon B \dashrightarrow D$. Now the theorem easily follows.

For each pair of maps with common codomain *choose* a pull-back diagram for that pair. Then, for $f\colon A \longrightarrow B$, "pulling back along f" gives in fact a functor $f^*\colon \underline{E}/B \longrightarrow \underline{E}/A$ (where $\underline{E}/B$ is the (usual) comma-category: *objects* are maps $\zeta\colon Z \longrightarrow B$ to B; a *morphism* from $\zeta_0\colon Z_0 \longrightarrow B$ to $\zeta_1\colon Z_1 \longrightarrow B$ is a commutative triangle



).

Likewise, *composing with f* gives a functor the other way, denoted $\Sigma_f\colon \underline{E}/A \longrightarrow \underline{E}/B$.

Proposition 1.11. For any $f\colon A \longrightarrow B$, $\Sigma_f \dashv f^*$.

Proof. An easy diagram chase. (In fact, this proposition holds just assuming axiom T1).

The MAIN THEOREM of this chapter says that f^* has a *right* adjoint also.

Theorem 1.12. For any $f\colon A \longrightarrow B$, $f^*\colon \underline{E}/B \longrightarrow \underline{E}/A$ has a right adjoint $\Pi_f\colon \underline{E}/A \longrightarrow \underline{E}/B$.

Proof.(i) *Construction* of $\Pi_f(\xi)$, where $\xi\colon X \longrightarrow A$ is an object of $\underline{E}/A$. Let φ be so that the diagram

$$\begin{array}{ccc} B \times A & \xrightarrow{\varphi} & \tilde{A} \\ {\scriptstyle \langle f,1 \rangle} \uparrow & & \uparrow {\scriptstyle \eta_A} \\ A & \xrightarrow[1_A]{} & A \end{array}$$

is a pull-back (using Theorem 1.8). Let $\hat{\varphi}$ be its exponential adjoint. Form the pull-back of $1 \pitchfork \tilde{\xi}$ along $\hat{\varphi}$; this we define as $\Pi_f(\xi)$, i.e.

$$\begin{array}{ccc} \Pi_f(X) & \longrightarrow & A \pitchfork \tilde{X} \\ {\scriptstyle \Pi_f(\xi)} \downarrow & \text{p.b.} & \downarrow {\scriptstyle 1 \pitchfork \tilde{\xi}} \\ B & \xrightarrow[\hat{\varphi}]{} & A \pitchfork \tilde{A} \end{array} .$$

It depends in an obvious functorial way on $\xi \in \underline{E}/A$.

(ii) Verification (sketch). Let $Z \xrightarrow{\zeta} B$ be an object in $\underline{E}/B$. We produce a 1-1 correspondence

$$\hom_{\underline{E}/B}(\zeta, \Pi_f(\xi))$$

$$\cong \hom_{\underline{E}/A}(f^*\zeta, \xi)$$

by means of the following string of bijections; the bijection (a) is by the defining pull-back diagram for $\Pi_f(\xi)$, (b) by exponential adjointness, (c) by Theorem 1.8 and the naturality statement on page 16 ; (d) is obvious; (e) is by multiplying (d,k) on the left by j^{-1}:

$$\hom_{\underline{E}/B}(\zeta, \Pi_f(\xi)) \quad \cong$$

$(\tilde{\bar{\bar{a}}})$ $\{h\colon Z \longrightarrow A \pitchfork \tilde{X} \quad | \quad h.1 \pitchfork \tilde{\xi} = \zeta . \hat{\varphi}\}$

$(\tilde{\bar{\bar{b}}})$ $\{h\colon Z \times A \longrightarrow \tilde{X} \quad | \quad \hat{h}.\tilde{\xi} = \zeta \times 1 . \varphi\}$

$(\tilde{\bar{\bar{c}}})$ $\{$classes of partial maps

$$\begin{array}{ccc} Z \times A & & \\ \uparrow{\scriptstyle d} & & \\ D & \xrightarrow{k} & X \end{array}$$

with $(d, k.\xi)$ equivalent to the partial map

$$\begin{array}{ccccc} Z \times A & & & & \\ {\scriptstyle e}\uparrow & & & & \\ f^*Z & \xrightarrow[f^*\zeta]{} & A & \xrightarrow[1]{} & A, \end{array}$$

where e is the canonical inclusion of the pull-back-object f^*Z into $Z \times A$ $\}$

$(\tilde{\bar{\bar{d}}})$ $\{(d,k) \mid d\colon D \longrightarrow Z \times A,\ k\colon D \longrightarrow X$ so that there exists isomorphism $j\colon D \longrightarrow f^*Z$ with $j.e = d$ and $j.f^*\zeta = k.\xi\}$

$(\tilde{\bar{\bar{e}}})$ $\{k'\colon f^*Z \longrightarrow X$ with $k'.\xi = f^*\zeta\}$
$= \hom_{\underline{E}/A}(f^*\zeta, \xi)$.

The naturality of the correspondences is left to the reader.

Lemma 1.13. The obvious functor $\underline{E}/A \xrightarrow{\partial_0} \underline{E}$ preserves colimits, and preserves and reflects epimorphisms.

Proof. ∂_0 may be written $\underline{E}/A \xrightarrow[\Sigma_k]{} \underline{E}/1 \cong \underline{E}$, where $k\colon A \longrightarrow 1$ is the only such map. Since Σ_k is a left adjoint by Proposition 1.11, the preservation is immediate; ∂_0 reflects epics since it is faithful.

Alternative Proof. Obvious by inspection.

<u>Theorem 1.14</u>. (Pull-back theorem). If ζ is epic and the diagram below is a pull-back, then ξ is epic.

$$\begin{array}{ccc} X & \longrightarrow & Z \\ {\scriptstyle \xi}\downarrow & \text{p.b.} & \downarrow\!\downarrow{\scriptstyle \zeta} \\ A & \xrightarrow[f]{} & B \end{array}$$

"PULLING BACK AN EPIC GIVES AN EPIC".

<u>Proof</u>. By Lemma 1.13, ζ gives rise to an epimorphism in $\underline{E}/B$, namely

$$\begin{array}{ccc} Z & \xrightarrow{\zeta} & B \\ & {\scriptstyle \zeta}\searrow \quad \swarrow{\scriptstyle 1_B} & \\ & B & \end{array} .$$

Since f^* has a right adjoint, it takes epics in $\underline{E}/B$ to epics in $\underline{E}/A$. The theorem now easily follows.

Since, by T2, $A \times -$ and $- \times A$ have right adjoints (in both cases $A \pitchfork -$), it follows that

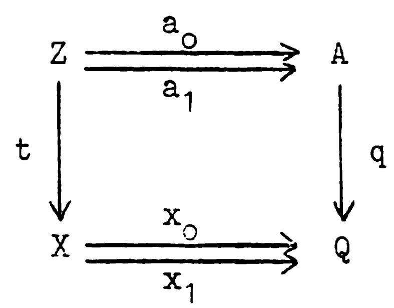
<u>Proposition 1.15</u>. If $q_i : A_i \longrightarrow Q_i$ is epic for $i = 0,1$, then

$$q_0 \times q_1 : \quad A_0 \times A_1 \longrightarrow Q_0 \times Q_1$$

is epic.

Definition 1.16. Let $x_0, x_1: X \longrightarrow Q$ and $q: A \longrightarrow Q$. Then a joint-pull-back of (x_0, x_1) along q is a pairwise commutative diagram

(1.8)



(i.e. $a_i.q = t.x_i$ for $i = 0,1$), which is universal with this property, i.e., if

$$Z' \overset{a_0'}{\underset{a_1'}{\rightrightarrows}} A, \quad Z' \xrightarrow{t'} X$$

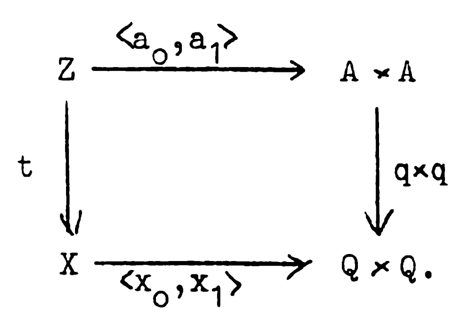
has $a_i'.q = t'.x_i$ for $i = 0,1$, then there is a unique $h: Z' \longrightarrow Z$ with

$$h.a_i = a_i' \qquad i = 0,1,$$

and $$h.t = t'.$$

Note. The two squares forming (1.8) will not in general be pull-back diagrams in the ordinary sense.

Proposition 1.17. With notation as in Definition 1.16, the maps a_0, a_1, t in (1.8) can be constructed by forming the pull-back diagram (in the ordinary sense)



<u>Definition 1.18</u>. A diagram

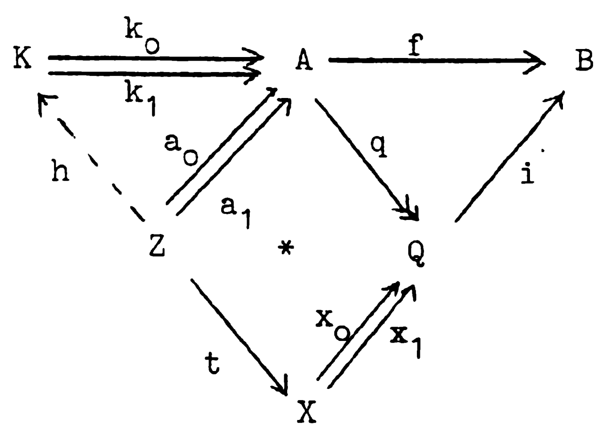
$$K \overset{k_0}{\underset{k_1}{\rightrightarrows}} A \xrightarrow{q} Q$$

is called <u>exact</u> if q is a coequalizer of k_0, k_1, and k_0, k_1 is a kernel pair of q.

Definition 1.16, Proposition 1.17 and Definition 1.18 refer to any category satisfying just T1. Returning to categories with T1, T2 and T3, we have

<u>Proposition 1.19</u>. Any map f can be factored as an epic followed by a monic. Such a factorization is unique up to a unique commuting isomorphism.

<u>Proof</u>. Let k_0, k_1 be a kernel pair of f, and q its coequalizer. Then it is clear that $k_0, k_1; q$ is exact, and that f factors as $q.i$ in the diagram below. We shall prove i monic.



Suppose $x_0.i = x_1.i$. Let $*$ be formed as joint-pull-back. We have

$$\begin{aligned} a_0.f &= a_0.q.i = t.x_0.i = t.x_1.i \\ &= a_1.q.i = a_1.f, \end{aligned}$$

whence there is an $h: Z \longrightarrow K$ with $h.k_i = a_i$ $(i = 0,1)$. Then

$$t.x_0 = a_0.q = h.k_0.q = h.k_1.q$$
$$= a_1.q = t.x_1.$$

Byt by Proposition 1.17 and Proposition 1.15, and the pull-back theorem, t is epic, whence $x_0 = x_1$. This proves i monic.

If $q'.i'$ is another epi-mono factorization of f, it is easy to produce z with $q.z = q'$ and $z.i' = i$. Then z is monic and epic. Using Proposition 1.4, it is easy to conclude that z is actually then an isomorphism. - Because q is epic, it is the only possible map with $q.z = q'$. This proves Proposition 1.19. The proof actually gave also (taking account again of Proposition 1.4)

Proposition 1.20. Every map can be factored as an coequalizer followed by an equalizer. Any epic map is a coequalizer, any monic map is an equalizer.

The Grothendieck-school (who first studied exactness properties in a non-additive setting) calls an equivalence relation effective if it has a coequalizer. By Theorem 1.5, all equivalence relations in a topos are effective. Also, they call things universal if they are preserved by pulling back. The joint-pull-back (Definition 1.16) of an equivalence relation is an equivalence relation for fairly trivial reasons, so that one might state that equivalence relations are universal and effective. The main exactness-property for toposes, however, is that equivalence relations are universally effective, that is the whole exact diagram, into which an equivalence relation (by effectivity) can be embedded, is preserved by pull-back.

The formal statement of this is the following theorem of which the first part is a restatement of Theorem 1.5.

Theorem 1.21. Let $K \overset{k_o}{\underset{k_1}{\rightrightarrows}} A$ be an equivalence relation. Then:

(i) k_o, k_1 can be embedded in an exact diagram

$$K \overset{k_o}{\underset{k_1}{\rightrightarrows}} A \xrightarrow{q} Q$$

("effectivity of equivalence relations"); and

(ii) if $f: \bar{Q} \longrightarrow Q$ is any map, there exists a diagram with exact columns

$$(1.10) \qquad \begin{array}{ccc} \bar{K} & \xrightarrow{f''} & K \\ \bar{k}_o \downdownarrows \bar{k}_1 & & k_o \downdownarrows k_1 \\ \bar{A} & \xrightarrow{f'} & A \\ \bar{q} \downarrow & & \downarrow q \\ \bar{Q} & \xrightarrow{f} & Q \end{array}$$

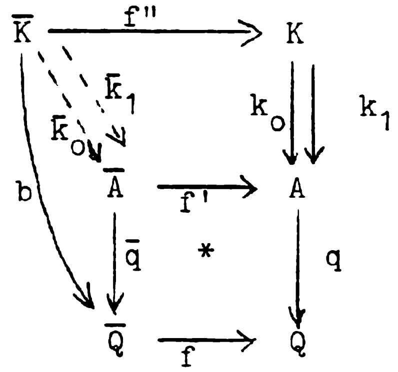
with the bottom square a pull-back and each of the squares

$$(1.11_i) \qquad \begin{array}{ccc} \bar{K} & \xrightarrow{f''} & K \\ \bar{k}_i \downarrow & & \downarrow k_i \\ \bar{A} & \xrightarrow{f'} & A \end{array} \qquad i = 0,1$$

a pull-back ("universality of the effectivity of an equivalence relation").

Proof of (ii). Consider the diagram

(1.12)



where the right-hand column is exact, the square * is a pull-back and where the outer square is a pull-back (of f along $k_0.q = k_1.q$). For $i = 0,1$, construct $\bar{k}_i$: $\bar{K} \longrightarrow \bar{A}$ so that

$$\bar{k}_i.\bar{q} = b$$

and

$$\bar{k}_i.f' = f''.k_i$$

which is possible since * is a pull-back. We then have the desired diagram in so far as the commutativities go. We must prove that (1.11_i) is a pull-back, and that the constructed left-hand column is exact. To prove, for example, that (1.11_0) is a pull-back, let there be given

$$\bar{a}: X \longrightarrow \bar{A} \quad \text{and} \quad k: X \longrightarrow K$$

with

(1.13) $$\bar{a}.f' = k.k_0.$$

Then

$$\bar{a}.\bar{q}.f = \bar{a}.f'.q = k.k_0.q,$$

so since the outer diagram in (1.12) is a pull-back, there exists $\bar{k}$: $X \longrightarrow \bar{K}$ with

(1.14) $$\bar{k}.b = \bar{a}.\bar{q}$$

and

(1.15) $$\bar{k}.f'' = k.$$

To prove that $\bar{k}.k_o = \bar{a}$, it suffices, by the universal property of the pull-back diagram *, to prove

$$\bar{k}.\bar{k}_o.\bar{q} = \bar{a}.\bar{q}$$

and

$$\bar{k}.\bar{k}_o.f' = \bar{a}.f'.$$

The first is immediate from (1.14). For the second,

$$\bar{k}.\bar{k}_o.f' = \bar{k}.f''.k_o = k.k_o = \bar{a}.f',$$

using (1.15) and the assumption (1.13). - The uniqueness of $\bar{k}$ is clear from the fact that the outer diagram in (1.12) is a pull-back. - To prove that the left-hand column is exact: First, $\bar{q}$ is epic, by the Pull-back Theorem. To prove that $\bar{k}_o, \bar{k}_1$ is the kernel pair for $\bar{k}$, let $x_i\colon X \longrightarrow \bar{A}$ $(i=0,1)$ have

$$x_o.\bar{q} = x_1.\bar{q}.$$

Then

$$x_o.f'.q = x_1.f'.q,$$

so since k_o, k_1 is kernel pair for q, we get $y\colon X \longrightarrow K$ with $y.k_i = x_i.f'$ $(i=0,1)$. Now $x_o.\bar{q}$ and y match up so as to give a map $z\colon X \longrightarrow \bar{K}$ (the outer diagram being a pull-back);

$$z.b = x_o.\bar{q}$$

and

$$z.f'' = y.$$

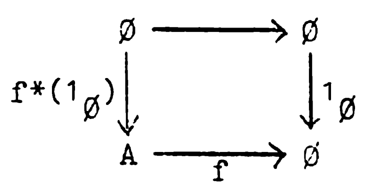
To prove that $z.\bar{k}_i = x_i$ $(i=0,1)$, it suffices (again because * is a pull-back) to prove the equations

$$x_o.\bar{q} = z.\bar{k}_o.\bar{q}$$
$$x_o.f' = z.\bar{k}_o.f'$$
$$x_1.\bar{q} = z.\bar{k}_1.\bar{q}$$
$$x_1.f' = z.\bar{k}_1.f'.$$

They are all proved in a straightforward way from previous equations; the proof of the third uses $x_o.\bar{q} = x_1.\bar{q}$ and $\bar{k}_o.\bar{q} = \bar{k}_1.\bar{q}$. - This proves the theorem.

Proposition 1.22. The initial object $\emptyset$ (existing by T1) is strict initial, i.e., any map $f\colon A \longrightarrow \emptyset$ is an isomorphism.

Proof. Clearly $\emptyset \longrightarrow B$ is initial in $\underline{E}/B$, for any B. By Main Theorem, 1.12, pulling back preserves initial objects. In particular



is a pull-back. But clearly, "pulling back" preserves isomorphisms. Since $1_\emptyset$ is iso, $f^*(1_\emptyset)$ is iso. Since $f^*(1_\emptyset).f = 1_\emptyset$ (what else could it be) f is the inverse for $f^*(1_\emptyset)$.

Corollary 1.23. Any map $\emptyset \longrightarrow B$ is monic.

Proof. For any A, there is at most one map $A \longrightarrow \emptyset$, namely the inverse for the only map $\emptyset \longrightarrow A$ (if it has an inverse).

For any A,B, the diagram (existing by T1)

$$\begin{array}{ccc} \emptyset & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle \mathrm{incl}_A} \\ B & \xrightarrow[\mathrm{incl}_B]{} & A+B \end{array}$$

is a pushout. By Push-out Theorem (Corollary 1.10) and by Corollary 1.23, incl_A, incl_B are monic, and the diagram is a pull-back. So

Proposition 1.24. The inclusions into a sum are monic, and their intersection is $\emptyset$ ("Sums (coproducts) are disjoint"). Since pulling-back has a right adjoint, pulling back a coproduct diagram gives a coproduct diagram, so: COPRODUCTS ARE DISJOINT AND UNIVERSAL.

Theorem 1.25. For any $A \in |\underline{E}|$, $\underline{E}/A$ is again a topos (satisfies T1,T2,T3).

Proof. $\partial_0 : \underline{E}/A \longrightarrow \underline{E}$ not only preserves colimits, but "constructs them", so $\underline{E}/A$ has finite colimits. Equalizers are constructed the same way. $A \xrightarrow{1_A} A$ is clearly a terminal object in $\underline{E}/A$, and pulling back over A gives binary categorical products in $\underline{E}/A$. So T1 is verified.

To verify T2, let $\xi : X \longrightarrow A$ be an object in $\underline{E}/A$. The functor $\xi \times - : \underline{E}/A \longrightarrow \underline{E}/A$ can be described as the composite

$$\underline{E}/A \xrightarrow{\xi^*} \underline{E}/X \xrightarrow{\Sigma_\xi} \underline{E}/A.$$

Both of the functors in this composite have right adjoints, by Proposition 1.11 and MAIN THEOREM 1.12. Hence the composite has a right adjoint.

To verify T3, check that $\Omega \times A \xrightarrow{\text{proj}} A$ is a subobject classifier in $\underline{E}/A$ provided we let 'true' be the map in $\underline{E}/A$:

$$\begin{array}{ccc} A \cong 1 \times A & \xrightarrow{\text{true} \times 1_A} & \Omega \times A \\ & {\scriptstyle 1_A} \searrow \quad \swarrow {\scriptstyle \text{proj}_A} & \\ & A & \end{array} .$$

Definition 1.26. An open object in $\underline{E}$ is an object U whose (unique!) map to 1 is monic.

Clearly U is open if and only if:

(1.16) For ANY $X \in |\underline{E}|$, THERE IS AT MOST ONE MAP $X \longrightarrow U$.

Any map $U \longrightarrow Y$ where U is open is monic. Let $\mathcal{O}_{\underline{E}}$ denote the full subcategory of open objects. It is a preordered class, by (1.16) the ordering being given by: $U \leq U'$ iff there exists a map $U \longrightarrow U'$ in $\underline{E}$. The subcategory $\mathcal{O}_{\underline{E}}$ of $\underline{E}$ is closed under forming products and exponentials in $\underline{E}$, as is easily seen, using (1.16)In particular, it satisfies T2. It also has coproducts: If U and V are open, form $U + V$ in $\underline{E}$, and take the image of the canonical map $U + V \longrightarrow 1$. So $\mathcal{O}_{\underline{E}}$ satisfies T1. (Only in trivial cases T3 is satisfied).

A partially ordered set satisfying T1 and T2 is called a Heyting-algebra or a Brouwerian lattice, or a pseudo-boolean algebra. Identifying isomorphic objects of $\mathcal{O}_{\underline{E}}$ gives a Heyting-algebra $\tilde{\mathcal{O}}_{\underline{E}}$, since $\mathcal{O}_{\underline{E}}$ is preordered and satisfies T1,T2. (Clearly, $\tilde{\mathcal{O}}_{\underline{E}}$ is a set, namely being isomorphic to $\hom_{\underline{E}}(1, \Omega)$, by T3). Conventional notation: in a Heyting algebra

$A \pitchfork B$	is denoted	$A \Rightarrow B$,
$A \times B$	is denoted	$A \wedge B$,
$A + B$	is denoted	$A \vee B$.

Proposition 1.27. Denote by $\mathcal{P}(A)$ the preordered class of subobjects of A. Then

$$\mathcal{P}(A) \simeq \mathcal{O}_{\underline{E}/A}.$$

Proof. Obvious.

In view of this and Theorem 1.25, we have

Proposition 1.28. For any $A \in |\underline{E}|$, $\mathcal{P}_{\sim}(A)$ is a Heyting algebra. ($\mathcal{P}_{\sim}(A)$ denoting $\mathcal{P}(A)$ modulo identification of isomorphic objects).

By slight abuse of language, we shall talk about $\mathcal{P}(A)$ itself as a Heyting algebra.

Let $\underline{E} \xrightarrow{R} \underline{E}'$ be a right adjoint functor. Then it preserves terminal object and monic maps, hence defines a functor by restriction $\mathcal{O}_{\underline{E}} \longrightarrow \mathcal{O}_{\underline{E}'}$. In particular, for $f: A \longrightarrow B$, the functors $\Pi_f: \underline{E}/A \longrightarrow \underline{E}/B$ and $f^*: \underline{E}/B \longrightarrow \underline{E}/A$ (which are right adjoints by Theorem 1.12 and Proposition 1.11) define, by restriction, functors $\forall_f: \mathcal{O}_{\underline{E}/A} \longrightarrow \mathcal{O}_{\underline{E}/B}$ and $f^{-1}: \mathcal{O}_{\underline{E}/B} \to \mathcal{O}_{\underline{E}/A}$, that is, functors with this notation in the big diagram below (p. 32).

Proposition 1.29. $f^{-1}: \mathcal{O}_{\underline{E}/B} = \mathcal{P}(B) \longrightarrow \mathcal{P}(A) = \mathcal{O}_{\underline{E}/A}$ has a left adjoint $\exists_f$.

Proof. Let $(A' \overset{a}{\rightarrowtail} A) \in \mathcal{P}(A)$. Let

$$A' \longrightarrow\!\!\!\!\!\rightarrow \exists_f(A') \rightarrowtail B$$

be a (chosen) epi-mono factorization of $a.f$.

Let $(B' \overset{b}{\rightarrowtail} B) \in \mathcal{P}(B)$. If $\exists_f(A') \leq B'$, it is straightforward to produce a map $A' \longrightarrow f^{-1}(B')$ since $f^{-1}(B')$ is defined as a pull-back. Conversely, if

$$A' \leq f^{-1}(B'),$$

we mus prove $\exists_f(A') \leq B'$ by constructing $h\colon \exists_f(A') \longrightarrow B'$ so that $h.b$ equals $\exists_f(A') \rightarrowtail B$. Since q is epic, this equation is equivalent to $q.h.b = a.f$. But the composite map k from A' via $f^{-1}(B')$ to B' coequalizes the kernel pair of $a.f$, since b is monic, and the coequalizer of that kernel pair is precisely q, by the construction of epi-mono factorization; from this the existence of the desired h follows.

Proposition 1.30. The inclusion $i\colon \mathcal{O}_{\underline{E}} \subseteq \underline{E}$ has a left adjoint σ; σ preserves products.

Proof. Let $X \in \underline{E}$. Let $X \xrightarrow{\eta_X} \!\!\!\rightarrow \sigma X \rightarrowtail 1$ be an epi-mono factorization of the unique map $X \longrightarrow 1$. It is easy to see that η_X (being coequalizer of the kernel pair of $X \longrightarrow 1$) has the universal property required for a front adjunction. The last statement is a consequence of Proposition 1.15: A product of epis is epi.

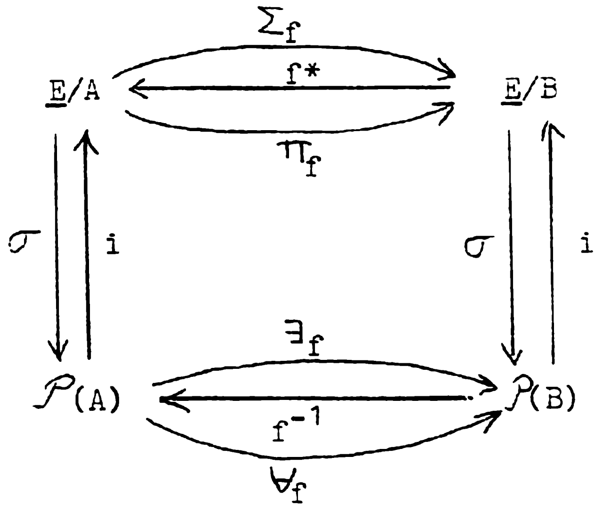
By Propositions 1.27 and 1.30, the canonical functor

$$\mathcal{P}(A) \simeq \mathcal{O}_{\underline{E}/A} \xrightarrow{-} \underline{E}/A$$

also has a left adjoint, denoted σ, which also preserves products.

Putting things together, we have the

DOCTRINAL DIAGRAM FOR $f: A \longrightarrow B$:



with

$$\Sigma_f \dashv f^* \dashv \Pi_f; \qquad \exists_f \dashv f^{-1} \dashv \forall_f$$

$$\sigma \dashv i$$

Proposition 1.31. In the doctrinal diagram the following equations hold up to isomorphism:

(1.17)	$\sigma.\exists_f$	=	$\Sigma_f.\sigma$
(1.18)	$\forall_f.i$	=	$i.\Pi_f$
(1.19)	$i.f^*$	=	$f^{-1}.i$
(1.20)	$\sigma.f^{-1}$	=	$f^*.\sigma$.

Proof. (1.18) and (1.19) hold by construction of $\forall_f$, f^{-1}. (1.17) follows by taking left adjoints on both sides of (1.19); (1.20) follows by taking left adjoints on both sides of (1.18).

Remark 1.32. If $f: A \rightarrowtail B$ is a monic map, let $ch(f): B \longrightarrow \Omega$ denote its characteristic map. Let $g: D \longrightarrow B$ be arbitrary. Then

$$ch(g^{-1}(f)) \quad = \quad g.ch(f).$$

Remark 1.33. We have already (prior to Proposition 1.7) given some notation which is natural when one has the main example $\underline{E}$ = (category of sets) in mind:

$$\Delta_A: \quad A \longrightarrow A \times A$$
$$\delta: \quad A \times A \longrightarrow \Omega$$
$$\{\cdot\}: \quad A \longrightarrow A \pitchfork \Omega.$$

The $\in$-relation itself comes the following way: Let ev_A denote the end-adjunction for the adjointness $- \times A \dashv A \pitchfork -$; in particular

$$ev_A: \quad (A \pitchfork \Omega) \times A \longrightarrow \Omega \ .$$

The subobject of $(A \pitchfork \Omega) \times A$ characterized by ev_A will be denoted $\in_A$.

Remark 1.34. Let $r: R \rightarrowtail X \times A$ be a monic map (we may in the set-case think of r as a <u>relation</u> from X to A; this viewpoint will be important later on). Let ch(r) be the characteristic map of r. The exponential adjoint of ch(r) is a map

$$\widehat{ch(r)}: \quad X \longrightarrow A \pitchfork \Omega,$$

and so $\widehat{ch(r)} \times 1$ is a map

$$\widehat{ch(r)} \times 1: \quad X \times A \longrightarrow (A \pitchfork \Omega) \times A.$$

Then

$$(\widehat{ch(r)} \times 1)^*(\in_A) = r;$$

this follows easily from the fact that we can get the pull-back diagram defining ch(r) in two steps using

$$ch(r) = \widehat{ch(r)} \times 1 . ev.$$

<u>Remark 1.35</u>. We may (by Remark 1.32) view $\underset{\sim}{\mathcal{P}}$ as a contravariant functor from $\underline{E}$ to sets represented by Ω. Hence, by the Eckmann-Hilton version of the Yoneda lemma, whatever algebraic structure can be put naturally on the $\underset{\sim}{\mathcal{P}}(A)$'s, can be put on Ω itself. This we carry out here; however, we must put the structure '$\Rightarrow$'$:\Omega\times\Omega\longrightarrow\Omega$ on Ω by other means, since we do not know on beforehand that $\Rightarrow$ is natural; this will rather be the conclusion of our making Ω into a Heyting algebra object which we now proceed to do. First we produce

$$(1.21) \qquad \wedge \; : \; \Omega\times\Omega \longrightarrow \Omega$$

as characteristic function of $\langle \mathrm{true},\mathrm{true}\rangle : \; 1 \longrightarrow \Omega\times\Omega$.
Then, for $X \rightarrowtail A$ and $Y \rightarrowtail A$ elements in $\underset{\sim}{\mathcal{P}}(A)$ we have that

$$\langle \mathrm{ch}(X),\mathrm{ch}(Y)\rangle . \wedge = \mathrm{ch}(X\cap Y).$$

So $\wedge$ makes Ω into a lower-semi-lattice object (with $\mathrm{true}\colon 1 \longrightarrow \Omega$ as maximal "element"). The "order relation" $\textcircled{\leq}$ on Ω is defined as the equalizer

$$\textcircled{\leq} \rightarrowtail \Omega\times\Omega \underset{\mathrm{proj}_0}{\overset{\wedge}{\rightrightarrows}} \Omega;$$

Let A be an object and $X \rightarrowtail A$, $Y \rightarrowtail A$ two subobjects thereof. Define a subobject $X \overset{\bullet}{\Rightarrow} Y$ of A by the pull-back diagram

$$\begin{array}{ccc}
\textcircled{\leq} & \rightarrowtail & \Omega\times\Omega \\
\uparrow & & \uparrow \langle \mathrm{ch}(X),\mathrm{ch}(Y)\rangle \\
X \Rightarrow Y & \longrightarrow & A
\end{array}$$

We claim that $X \dot{\Rightarrow} Y$ is the same subobject as $X \Rightarrow Y$ defined in connection with the Heyting algebra structure on $\mathcal{P}(A)$; to prove this we must prove that for any $z: Z \rightarrowtail A$

$$Z \leq X \dot{\Rightarrow} Y$$

iff

$$Z \cap X \leq Y.$$

If $Z \leq X \dot{\Rightarrow} Y$, it follows that $z.\langle ch(X), ch(Y) \rangle$ factors across $(\leq)$; this means (essentially by Remark 1.32) that

$$Z \cap X \leq Z \cap Y \leq Y.$$

If $Z \cap X \leq Y$, we conversely have that $z.\langle ch(X), ch(Y) \rangle$ factors across $(\leq)$, whence we have z that factors across the pullback of $\langle ch(X), ch(Y) \rangle$ with $(\leq)$, which is $X \dot{\Rightarrow} Y$. This proves the claim.

If we now denote the characteristic map of $(\leq)$ by ' $\Rightarrow$ ':

(1.22) $$\Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$$

it follows that

$$ch(X \Rightarrow Y) = ch(X \dot{\Rightarrow} Y) = \langle ch(X), ch(Y) \rangle . \Rightarrow .$$

The maps (1.21) and (1.22) (together with easily defined minimal element and join-operation) make Ω into a Heyting algebra-object.

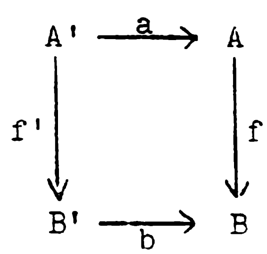
The reason why it is easy to get $\wedge : \Omega \times \Omega \longrightarrow \Omega$ representing 'intersection' in $\mathcal{P}(A)$ is that for $f: B \longrightarrow A$,

$$f^{-1}: \underset{\sim}{\mathcal{P}}(A) \longrightarrow \underset{\sim}{\mathcal{P}}(B)$$

clearly preserves intersection. The same is not obvious for $\dot{\Rightarrow}$ in the subobject lattices. However, now that we have produced a representing operation (1.22), we can conclude from this that

(1.23) $$f^{-1}(X \dot{\Rightarrow} Y) = f^{-1}(X) \dot{\Rightarrow} f^{-1}(Y).$$

Remark 1.36. ("Beck condition" in the terminology of [15]).
Let



be a pull-back. Let $r: R \rightarrowtail A \in \underset{\sim}{\mathcal{P}}(A)$. Then

$$b^{-1}(\exists_f(r)) = \exists_{f'}(a^{-1}(r))$$

in $\underset{\sim}{\mathcal{P}}(B')$. To prove this again uses the "pulling back in two steps-technique", and Proposition 1.31.

Remark 1.37. ("Frobenius reciprocity", [15]). Let $f: B \longrightarrow A$, let $X \rightarrowtail A \in \underset{\sim}{\mathcal{P}}(A)$ and let $Z \rightarrowtail B \in \underset{\sim}{\mathcal{P}}(B)$. Then in (A)

$$\exists_f(Z \cap f^{-1}(X)) = X \cap \exists_f(Z).$$

This is a consequence of (1.23), together with the adjointnesses:

$$\exists_f \dashv f^{-1}; \qquad - \cap X \dashv X \Rightarrow -$$

and

$$- \cap f^{-1}(X) \dashv f^{-1}(X) \Rightarrow - \quad .$$

Remark 1.38. A subobject classifier is unique (up to isomorphism). For, suppose true: $1 \longrightarrow \Omega$ and true': $1 \longrightarrow \Omega'$ are both subobject classifiers. Being maps out of a terminal object, both "true" and "true'" are monic maps; in particular true': $1 \longrightarrow \Omega'$ has a characteristic map $\Omega' \longrightarrow \Omega$, similarly, true: $1 \longrightarrow \Omega$ has a characteristic map $\Omega \longrightarrow \Omega'$. These two characteristic maps are easily seen to be mutually inverse (using the fact that id: $\Omega \longrightarrow \Omega$ is characteristic map for true: $1 \longrightarrow \Omega$).

The following Corollary of a recent embedding theorem of Barr will be of great use in Chapter 5.

Theorem 1.39. Let $\underline{E}$ be an elementary topos. Then there exists a small category $\mathbb{C}$ and a functor

$$\beta : \underline{E} \longrightarrow \mathfrak{S}^{\mathbb{C}}$$

(where $\mathfrak{S}^{\mathbb{C}}$ denotes the category of functors from $\mathbb{C}$ to the category $\mathfrak{S}$ of sets) which satisfies

(i) β is full and faithful

(ii) β preserves and reflects finite inverse limit diagrams

(iii) β preserves and reflects exact diagrams (Definition 1.18).

Proof. Every map has a kernel pair by T1; every equivalence relation (and so every kernel pair) has a coequalizer by Theorem 1.5 (and Proposition 1.3). Pull-backs of epic maps are epic maps by the Pull-back Theorem 1.14, and epic maps are coequalizers, by Proposition 1.20. Thus $\underline{E}$ is an exact category in the sense of Barr [1], and by his embedding theorem for exact categories

([1], Section 3, Corollary), there exists a functor β with the properties (i) - (iii).

The force of Theorem 1.39 lies in Statement (iii) since (i) and (ii) (except for smallness of $\mathfrak{T}$) can be accomplished by the Yoneda embedding. Note that (iii) does not say that β preserves arbitrary coequalizer diagrams

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{q} Q$$

but it says that β will preserve this coequalizer diagram <u>provided</u> f,g is an equivalence relation.

In particular, we can easily conclude that β will preserve epi-mono factorizations. The Frobenius reciprocity (Remark 1.37), for instance, can also be proved by this technique.

2. Left exact cotriples

In this chapter we shall give a method of constructing elementary toposes. Let $\underline{E}$ be an elementary topos, and let

$$\mathbb{C} = (C, \varepsilon, \psi)$$

be a left exact cotriple on $\underline{E}$. (By saying that $\mathbb{C}$ is left exact we just mean that the functor part $C\colon \underline{E} \longrightarrow \underline{E}$ is a left exact functor, i.e. commutes with finite inverse limits. The notion of cotriple is as in Eilenberg-Moore [5], $\varepsilon\colon\ C \longrightarrow 1_{\underline{E}}$ denoting the co-unit and $\psi\colon\ C \longrightarrow C^2$ the comultiplication.) We denote by $\underline{E}_{\mathbb{C}}$ the category of $\mathbb{C}$-co-algebras and their homomorphism (the "universal cogenerator" for C in the terminology of [5]).

We have a functor

$$\lambda_*\colon\ \underline{E} \longrightarrow \underline{E}_{\mathbb{C}}\colon \quad Y \longmapsto (CY, \psi_Y) \qquad f \longmapsto Cf$$

which has a left adjoint

$$\lambda^*\colon\ \underline{E}_{\mathbb{C}} \longrightarrow \underline{E}\colon \quad (X, \xi) \longmapsto X.$$

Lemma 2.1. The category $\underline{E}_{\mathbb{C}}$ has right limits.

Proof. Let $\alpha \longmapsto (X_\alpha, \xi_\alpha)$ be a diagram in $\underline{E}_{\mathbb{C}}$, and let μ be the canonical map

$$\varinjlim_\alpha C(X_\alpha) \longrightarrow C(\varinjlim_\alpha X_\alpha).$$

Then $(\varinjlim_\alpha X_\alpha,\ \ \varinjlim_\alpha \xi_\alpha \cdot \mu)$ is a $\mathbb{C}$-coalgebra and is a right limit of the diagram in $\underline{E}_{\mathbb{C}}$.

Lemma 2.2. The category $\underline{E}_{\mathbb{C}}$ has finite left limits.

Proof. Let $\alpha \longmapsto (X_\alpha, \xi_\alpha)$ be a finite diagram in $\underline{E}_{\mathbb{C}}$. As $\mathbb{C}$ is left exact, we have a canonical isomorphism

$$C(\varprojlim_{\alpha} X_\alpha) \xrightarrow{\mu'} \varprojlim_{\alpha} C(X_\alpha).$$

Then $(\varprojlim X_\alpha, \varprojlim_\alpha \xi_\alpha \, \mu'^{-1})$ is a $\mathbb{C}$-coalgebra and is a left limit of the diagram in $\underline{E}_{\mathbb{C}}$.

Corollary 2.3. The functor $\lambda^*\colon \underline{E}_{\mathbb{C}} \longrightarrow \underline{E}$ is left exact.

For any X and Y in $\underline{E}$ we have a map

$$\sigma_{X,Y}\colon \; C(X \pitchfork Y) \longrightarrow CX \pitchfork CY$$

adjoint to

$$C(X \pitchfork Y) \times CX \xrightarrow{\cong} C((X \pitchfork Y) \times X) \xrightarrow{C(ev)} CY,$$

where the first map is the canonical isomorphism given by the left exactness of C. It is clear that $\sigma_{X,Y}$ gives a natural map.

For any two $\mathbb{C}$-coalgebras (A,α) and (B,β), define $(B,\beta) \pitchfork (A,\alpha)$ by the requirement that

$$(B,\beta) \pitchfork (A,\alpha) \longrightarrow (C(B \pitchfork A), \psi_{B \pitchfork A}) \underset{\theta'}{\overset{\theta}{\rightrightarrows}} (C(B \pitchfork CA), \psi_{B \pitchfork CA})$$

be an equalizer diagram in $\underline{E}_{\mathbb{C}}$, where θ and θ' are the homomorphisms given by

$$C(B \pitchfork A) \xrightarrow{C(1_B \pitchfork \alpha)} C(B \pitchfork CA)$$

and

$$C(B \pitchfork A) \xrightarrow{\psi_{B \pitchfork A}} C^2(B \pitchfork A) \xrightarrow{C(\sigma_{B,A})} C(CB \pitchfork CA) \xrightarrow{C(\beta \pitchfork 1_{CA})} C(B \pitchfork CA)$$

respectively.

In this way we get a functor $(-) \pitchfork (-) : \mathbb{E}_{\mathbb{C}}^{op} \times \mathbb{E}_{\mathbb{C}} \longrightarrow \mathbb{E}_{\mathbb{C}}$.

Lemma 2.4. $\mathbb{E}_{\mathbb{C}}$ is Cartesian closed.

Proof. Let $(A,\alpha), (B,\beta)$ and (D,δ) be $\mathbb{C}$-coalgebras and let $D \times B \xrightarrow{f} A$ be a map with exponential adjoint $D \xrightarrow{\hat{f}} B \pitchfork A$. Let g denote the composite

$$D \xrightarrow{\delta} CD \xrightarrow{C\hat{f}} C(B \pitchfork A).$$

Then g is a homomorphism $(D,\delta) \longrightarrow (C(B \pitchfork A), \psi_{B \pitchfork A})$. We propose to show that f is a homomorphism if and only if g equalizes θ and θ', i.e. g factors through the canonical map

$$(B,\beta) \pitchfork (A,\alpha) \longrightarrow (C(B \pitchfork A), \psi_{B \pitchfork A}),$$

by $(D,\delta) \xrightarrow{h} (B,\beta) \pitchfork (A,\alpha)$ let us say. Then h will be the exponential adjoint of f in $\mathbb{E}_{\mathbb{C}}$.

Now f is a homomorphism if and only if the diagram

$$\begin{array}{ccc} D \times B & \xrightarrow{f} & A \\ {\scriptstyle \delta \times \beta}\downarrow & & \downarrow{\scriptstyle \alpha} \\ C(D \times B) & \xrightarrow[Cf]{} & CA \end{array}$$

commutes. We have omitted reference to the canonical isomorphisms expressing left exactness of C. We denote the composites $f.\alpha$ and $(\delta \times \beta).Cf$ by $\emptyset$ and $\emptyset'$ respectively. Their exponential adjoints, $\hat{\emptyset}$ and $\hat{\emptyset}'$ are given by

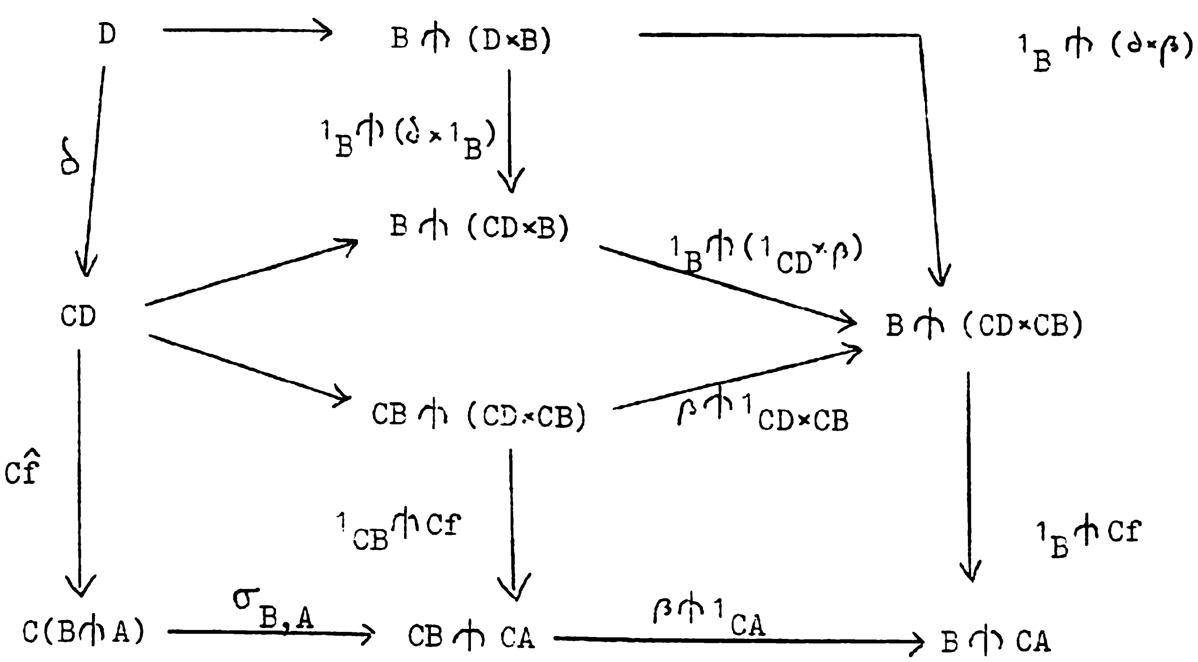
$$\hat{\phi}: \quad D \xrightarrow{\hat{f}} B \pitchfork A \xrightarrow{1_B \pitchfork \alpha} B \pitchfork CA$$

and

$$\hat{\phi}': \quad D \xrightarrow{u} B \pitchfork (D \times B) \xrightarrow{1_B \pitchfork (\delta \times \beta)} B \pitchfork (CD \times CB) \xrightarrow{1_B \pitchfork Cf} B \pitchfork CA$$

where the first map u is the front adjunction for exponential adjointness.

Consider the commutative diagram



(The lower left-hand square commutes, using naturality of σ and an easily deduced equation between u, , σ and the canonical isomorphism $C(D \times B) \cong CD \times CB$.)

We deduce that $\hat{\emptyset}' = \delta.C\hat{f}.\sigma_{B,A}.(\beta \pitchfork 1_{CA})$.

Now f is a homomorphism if and only if $\emptyset = \emptyset'$ if and only if $\hat{\emptyset} = \hat{\emptyset}'$ if and only if the homomorphisms

$$D \xrightarrow{\delta} CD \xrightarrow{C\hat{\emptyset}} C(B \pitchfork CA)$$

and

$$D \xrightarrow{\delta} CD \xrightarrow{C\hat{\emptyset}'} C(B \pitchfork CA)$$

agree. But this is precisely the condition that g equalizes θ and θ'. This proves the Lemma.

Let $C\Omega \xrightarrow{\lambda} \Omega$ classify $1 \xrightarrow{C(t)} C\Omega$. Since the diagram

$$\begin{array}{ccccc} C\Omega & \xrightarrow{\psi_\Omega} & C^2\Omega & \xrightarrow{C\lambda} & C\Omega \\ \downarrow{\scriptstyle \psi_\Omega} & & \downarrow{\scriptstyle \psi_{C\Omega}} & & \downarrow{\scriptstyle \psi_\Omega} \\ C^2\Omega & \xrightarrow{C\psi_\Omega} & C^3\Omega & \xrightarrow{C^2\lambda} & C^2\Omega \end{array}$$

commutes, we have a homomorphism

$$(C\Omega, \psi_\Omega) \xrightarrow{\psi_\Omega . C\lambda} (C\Omega, \psi_\Omega).$$

Define $(\bar{\Omega}, \omega) \longrightarrow (C\Omega, \psi_\Omega)$ to be the equalizer of $\psi_\Omega . C\lambda$ and $1_{C\Omega}$.

<u>Lemma 2.5</u>. The $\mathbb{C}$-coalgebra $(\bar{\Omega}, \omega)$ is a subobject classifier for $\mathbf{E}_{\mathbb{C}}$.

<u>Proof</u>. First note that $1 \xrightarrow{C(t)} C\Omega$ equalizes $\psi_\Omega . C\lambda$ and $1_{C\Omega}$, and so lifts to a homomorphism

$$1 \xrightarrow{\bar{t}} (\bar{\Omega}, \omega).$$

Suppose $(X', \xi') \xrightarrow{i} (X, \xi)$ is a monic in $\mathbf{E}_{\mathbb{C}}$, and let

$X' \overset{i}{\rightarrowtail} X$ be classified by $X \xrightarrow{\varphi} \Omega$ in $\underline{E}$. Define $\check{\varphi}$ to be the composite

$$X \xrightarrow{\xi} CX \xrightarrow{C\varphi} C\Omega$$

so that we have a homomorphism $(X,\xi) \xrightarrow{\check{\varphi}} (C\Omega, \psi_\Omega)$. Now we show that $\check{\varphi}$ equalizes $\psi_\Omega . C\lambda$ and $1_{C\Omega}$:-

Consider the commutative diagram

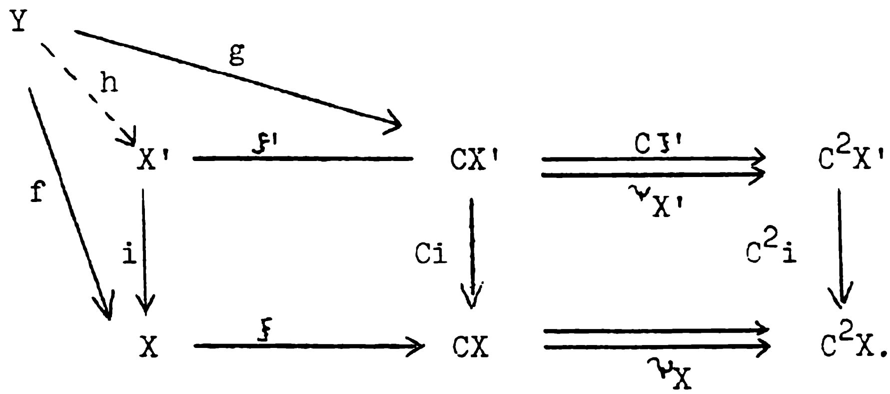
$$(2.1) \qquad \begin{array}{ccccccc} X' & \xrightarrow{\xi'} & CX' & \longrightarrow & 1 & \longrightarrow & 1 \\ {\scriptstyle i}\downarrow & & {\scriptstyle Ci}\downarrow & & {\scriptstyle Ct}\downarrow & & {\scriptstyle t}\downarrow \\ X & \xrightarrow{\xi} & CX & \xrightarrow{C\varphi} & C\Omega & \xrightarrow{\lambda} & \Omega . \end{array}$$

The right-hand two squares are pullbacks. Now we show that the left-hand square is also :-

Consider the diagram

$$\begin{array}{ccccc} X' & \xrightarrow{\xi'} & CX' & \underset{\psi_{X'}}{\overset{C\xi'}{\rightrightarrows}} & C^2X' \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\xi} & CX & \underset{\psi_X}{\overset{C\xi}{\rightrightarrows}} & C^2X . \end{array}$$

The rows are equalizer diagrams, split by the co-units. Let $Y \xrightarrow{f} X$, $Y \xrightarrow{g} CX'$ be maps in $\underline{E}$ such that $f.\xi = g.Ci$.



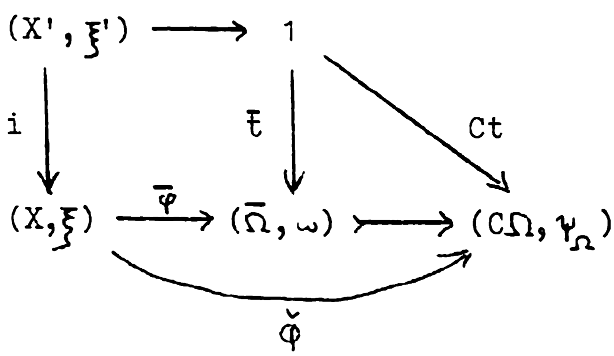
Since C^2i is monic, g equalizes $C\xi'$ and $\psi_{X'}$. Hence there exists $Y \xrightarrow{h} X'$ such that $h.\xi' = g$. As ξ is monic $hi = f$. Because i is monic, h is unique with this property. Hence the left-hand square is a pullback.

To return to the diagram (2.1), we see that $\xi.C\varphi.\lambda$ and φ both classify the same subobject of X and hence are equal. This gives us that $\check{\varphi}$ equalizes $\psi_{\Omega}.C\lambda$ and $1_{C\Omega}$, and hence lifts to a homomorphism

$$(X,\xi) \xrightarrow{\bar{\varphi}} (\bar{\Omega},\omega)$$

which we shall show is the classifying map of $(X',\xi') \longrightarrow (X,\xi)$ in $\underline{E}_{\mathbb{C}}$.

From the commutative diagram



we get that

$$\text{(2.2)}\qquad \begin{array}{ccc} (X',\xi') & \longrightarrow & 1 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{t}} \\ (X,\xi) & \longrightarrow & (\bar{\Omega},\omega) \end{array}$$

commutes. Now we show that it is a pullback diagram. Let $(Y,y) \xrightarrow{f} (X,\xi)$ be a homomorphism such that $f.\check{\phi}$ factors through $\bar{t}$. Then $f.\varphi$ factors through t, and so there exists a map $Y \xrightarrow{h} X'$ such that $h.i = f$. But f and i are homomorphisms. Since $C(i)$ is monic, h is a homomorphism.

Suppose now that ρ is any homomorphism $(X,\xi) \longrightarrow (\bar{\Omega},\omega)$ which substituted for $\bar{\varphi}$ in (2.2) makes (2.2) into a pullback. Since the forgetful λ^* preserves pullbacks, the left and square in (2.3) below is a pullback in $\underline{E}$

$$\text{(2.3)}\qquad \begin{array}{ccccccc} X' & \longrightarrow & 1 & \longrightarrow & & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle \bar{t}} & & & & \downarrow{\scriptstyle t} \\ X & \longrightarrow & \bar{\Omega} & \xrightarrow[i]{} & C\Omega & \xrightarrow[\varepsilon_\Omega]{} & \Omega \end{array} .$$

If we can prove that the right hand square is also a pullback, the total diagram will be, which means that $\rho.i.\varepsilon_\Omega$ will classify $X' \rightarrowtail X$, whence $\rho.i.\varepsilon_\Omega = \varphi$. From this it will follow by adjointness that the homomorphism $\rho.i$ equals the homomorphism $\check{\phi} = \bar{\varphi}.i$, and since i is monic, $\rho = \bar{\varphi}$. - To prove that the right hand square is a pullback, is immediate, using that i equalizes id and $\psi.C\lambda$ so that

$$\begin{aligned} i.\varepsilon_\Omega &= i.\psi.C\lambda.\varepsilon_\Omega \\ &= i.\psi.\varepsilon_{C\Omega}.\lambda \\ &= i.\lambda \end{aligned} .$$

Putting together the information in the above lemmas, we have the theorem:

Theorem 2.6. Let $\mathbb{C}$ be a left exact cotriple on an elementary topos $\underline{E}$ and let $\underline{E}_{\mathbb{C}}$ denote the category of $\mathbb{C}$-coalgebras. Then $\underline{E}_{\mathbb{C}}$ is an elementary topos, and there is a functor

$$\underline{E} \longrightarrow \underline{E}_{\mathbb{C}}$$

with a left exact left adjoint (namely the forgetful functor.) (We shall in Chapter 4 introduce the terminology "<u>map</u> between toposes" for situations like this.)

We remark that the forgetful functor $\underline{E}_{\mathbb{C}} \longrightarrow \underline{E}$ reflects isomorphisms.

We give, without proof, an example of a left exact cotriple on a topos $\underline{E}$. Let $(X,\mathcal{O})$ be a topological space, ($\mathcal{O}$ being the set of open subsets of X). Let $\underline{E}$ be the topos $\mathcal{S}/X$ ("sheaves over X when equipped with the discrete topology"). Each open subset U of X defines an object $U \longrightarrow X$ in $\mathcal{S}/X$; let also $\mathcal{O}$ denote the full subcategory determined by these. Then the "density cotriple" or "model induced cotriple" (see e.g. [18] or [ ]) for $\mathcal{O} \subseteq \mathcal{S}/X$ is left exact; the topos arising out of this situation by Theorem 2.6 is precisely the category of sheaves on the topological space X. - We shall return to this situation in the context of an arbitrary topos instead of $\mathcal{S}$, in Chapter 5.

3. Topology on a topos

Definition 3.1. By a topology on a topos $\underline{E}$ is meant a left-exact, pull-back-natural closure operator on each lattice of subobjects $\mathcal{P}(A)$, $A \in |\underline{E}|$.

Explanation. To say that $\overline{\cdot}$ is a closure operator on $\mathcal{P}(A)$ means that for any $A' \rightarrowtail A$ in $\mathcal{P}(A)$

$$(A' \rightarrowtail A) \leq (\overline{A' \rightarrowtail A})\,; \quad (\overline{\overline{A' \rightarrowtail A}}) = (\overline{A' \rightarrowtail A})$$

(and that $A' \leq A'' \leq A$ implies that $\bar{A}' \leq \bar{A}''$; this latter condition is here implied by left-exactness. To say that $\overline{\cdot}$ is left exact means $\bar{A}' \cap \bar{A}'' = \overline{A' \cap A''}$; as usual we write A' for the element of $\mathcal{P}(A)$ determined by $A' \rightarrowtail A$.)

Finally, pull-back-naturality states that for $f\colon A \longrightarrow B$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{P}(B) & \xrightarrow{f^{-1}(.)} & \mathcal{P}(A) \\ \overline{\cdot}\downarrow & & \downarrow\overline{\cdot} \\ \mathcal{P}(B) & \xrightarrow[f^{-1}(.)]{} & \mathcal{P}(A), \end{array}$$

in other words, $\overline{f^{-1}(B')} = f^{-1}\overline{(B')}$.

Since here the (contravariant) functor $\mathcal{P}(-)\colon \underline{E} \longrightarrow \mathcal{S}$ is represented by the object Ω :

$$\mathcal{P}(A) \cong \hom_{\underline{E}}(A, \Omega)$$

we get, by (the Eckmann-Hilton version of) the Yoneda principle that the topology is determined by, and determines, a morphism

$j: \Omega \longrightarrow \Omega$ which satisfies

$$\begin{aligned} true.j &= true: \; 1 \longrightarrow \Omega \\ j.j &= j: \; \Omega \longrightarrow \Omega \\ (j \times j).\wedge &= \wedge.j: \; \Omega \times \Omega \longrightarrow \Omega . \end{aligned}$$

We indicate how the correspondences go, and leave the further details to the reader.

(i) Given the closure operators $\bar{\ }$, apply $\bar{\ }$ to the subobject $1 \xrightarrow{true} \Omega$ of Ω; this gives a new subobject of Ω; let j be its characteristic map.

(ii) Given j; let $A' \rightarrowtail A$ be a subobject of A with characteristic map $\alpha: A \longrightarrow \Omega$. Take $\overline{A'}$ to be the subobject of A classified by $A \xrightarrow{\alpha} \Omega \xrightarrow{j} \Omega$.

Definition 3.2. $A' \rightarrowtail A$ is called a dense subobject if $\overline{A'} = A$; it is called a closed subobject if $\overline{A'} = A'$.

Let $J \rightarrowtail \Omega$ be the subobject of Ω classified by j (in other words, J is the closure of $1 \xrightarrow{true} \Omega$). Let $\Omega_j \rightarrowtail \Omega$ be the equalizer of $\Omega \underset{j}{\overset{id}{\rightrightarrows}} \Omega$. (Since $j.j = j$, one could equivalently define Ω_j as the image of j).

Proposition 3.3. $A' \rightarrowtail A$ is dense if and only if its characteristic map $A \xrightarrow{\alpha} \Omega$ factors through J. It is closed if and only if α factors through Ω_j.

Proof. Omitted because it is easy.

Lemma 3.4. Let $C \rightarrowtail B$ and $B \rightarrowtail A$ be dense subobjects of B and A, respectively. Then the composite $C \overset{h}{\rightarrowtail} B \overset{k}{\rightarrowtail} A$ is a dense subobject of A.

Proof. Clearly $k^{-1}(C \rightarrowtail A) = C \rightarrowtail B$ in $\mathcal{P}(B)$, so

$$\overline{C \rightarrowtail B} = \overline{k^{-1}(C \rightarrowtail A)} = k^{-1}(\overline{C \rightarrowtail A}),$$

but $\overline{C \rightarrowtail B} = B$ since C is dense in B. Therefore $B = k^{-1}(\overline{C \rightarrowtail A})$, in particular $B \leq k^{-1}(\overline{C \rightarrowtail A})$. Apply the adjointness $\exists_k \dashv k^{-1}$ to get out of this

$$\exists_k B \leq \overline{C \rightarrowtail A} \qquad \text{in} \quad \mathcal{P}(A).$$

Clearly $\exists_k B = B \rightarrowtail A$, so

$$B \rightarrowtail A \leq \overline{C \rightarrowtail A}.$$

Since closure is monotone, we get

$$\overline{B \rightarrowtail A} \leq \overline{\overline{C \rightarrowtail A}} = \overline{C \rightarrowtail A}$$

but since B is dense in A, the left-hand side is A, the largest subobject of A. So also the right-hand side is A; this says that C is dense in A.

Lemma 3.5. If $B \overset{k}{\rightarrowtail} A$ is a closed subobject, then $\exists_k : \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$ commutes with the closure operators.

Proof. Let $C \rightarrowtail B$. Then

$$\overline{C \rightarrowtail B} = (\overline{C \rightarrowtail B}) \cap B = (\overline{C \rightarrowtail B \rightarrowtail A}) \cap (B \rightarrowtail A)$$

(since $- \cap B$ is just "pulling back along k" which commutes with closure). Since B is closed in A, this in turn equals

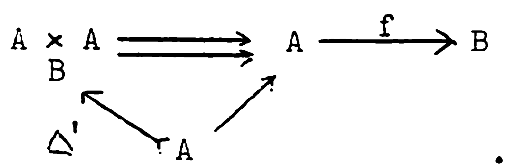
$$= \overline{C \rightarrowtail A} \cap \overline{B \rightarrowtail A}$$

$$= \overline{(C \rightarrowtail A) \cap (B \rightarrowtail A)} \quad \text{by left exactness of } \bar{\ } .$$

This clearly is just $\overline{C \rightarrowtail A}$ since $C \leq B$. But $C \rightarrowtail A$ is $\exists_k(C \rightarrowtail B)$.

<u>Corollary 3.6</u>. Let $A' \rightarrowtail A$ be a subobject. Then $A' \rightarrowtail \overline{A'}$ is dense.

<u>Definition 3.7</u>. A morphism $f\colon A \longrightarrow B$ is called <u>almost monic</u> with respect to the topology, provided the canonical map Δ' (which is monic) is dense:



Note that if the topology is trivial $(\overline{A' \rightarrowtail A} = A' \rightarrowtail A)$, then "almost monic" $\Leftrightarrow$ "monic".

<u>Lemma 3.8</u>. A map $f\colon A \longrightarrow B$ is almost monic if and only if for any pair of maps $G \overset{g_0}{\underset{g_1}{\rightrightarrows}} A$ with $g_0.f = g_1.f$, there is a dense subobject $G' \overset{k}{\rightarrowtail} G$ with $k.g_0 = k.g_1$.

<u>Proof</u>. Exercise.

<u>Definition 3.9</u>. A map $f\colon A \longrightarrow B$ is called <u>almost epic</u> with respect to the topology provided its image $\mathrm{Im}(f) \rightarrowtail B$ is dense.

<u>Definition 3.10</u>. A map which is almost monic as well as almost epic will be called <u>bidense</u>.

Note that since a monic map is almost monic, and since it is almost epic if and only if it is dense, we have for monic maps: bidense $\Leftrightarrow$ dense. For the trivial topology mentioned above, a map is bidense if and only if it is epic and monic (i.e. invertible, by Proposition 1.20, essentially).

Let Σ denote the class of bidense maps in $\underline{E}$ with respect to the topology. We shall prove that Σ "admits a calculus of right fractions", meaning that we have a nice way of inverting them (to be described).

Lemma 3.11. In the following diagram, let $s \in \Sigma$

$$\begin{array}{ccc} P & \xrightarrow{f'} & A \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ C & \xrightarrow[f]{} & B \end{array} .$$

Assume the diagram is a pull-back. Then also $t \in \Sigma$ ("Σ IS STABLE UNDER PULL-BACK").

Proof. Since pulling-back along f preserves epi-mono factorization (essentially by Theorem 1.14, "pull-back theorem", and also preserves the closure operator, it is clear that 's almost epic' implies 't almost epic'. To prove t almost monic apply Lemma 3.8: Suppose $g_0.t = g_1.t$ $(g_i: D \longrightarrow P)$. Therefore $g_0.f'.s = g_1.f'.s$; since s is almost monic, there is a dense monic $D' \overset{d}{\rightarrowtail} D$ with

$$d.g_0.f' = d.g_1.f'.$$

Since also $\quad d.g_0.t = d.g_1.t,$ it follows from uniqueness of a map into a pull-back that

$$d.g_o = d.g_1.$$

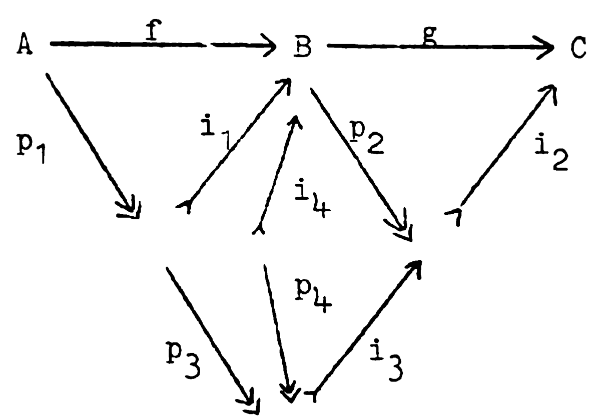
Lemma 3.12. $\sum$ is stable under composition.

Proof. By Lemma 3.4, dense monics compose. Using Lemma 3.8 twice, one easily sees that almost-monic maps compose. It is also true that almost-epic maps compose; note first that since the contravariant functor $\underset{\sim}{\mathcal{P}}(\ )$ from $\underline{E}$ to sets is representable: $hom_{\underline{E}}(-,\Omega)$, it sends epic maps in $\underline{E}$ into monomorphisms in sets. From this we may easily conclude:

(3.1) PULLING BACK ALONG AN EPIC MAP PRESERVES AND REFLECTS THE NOTION OF DENSITY,

"reflects" means: if q is epic and $q^{-1}(A')$ is dense in the domain of q then A' itself already was dense in the codomain of q.

Suppose that f and g are composable and almost epic; consider the diagram



with p_1, i_1 epi-mono-factorization of f, p_2, i_2 epi-mono-factorization of g, and p_3, i_3 epi-mono factorization of $i_1.p_2$, and with p_4, i_4 a pull-back diagram for p_2, i_3.

Since $i_1.p_2 = p_3.i_3$ we get from the pull-back property that i_1 factors through i_4. Since i_1 is dense and $i_1 \leq i_4$. i_4 is dense. Since $i_4 = p_2^{-1}(i_3)$, we get from (3.1) above that i_3 is dense. Since i_2 is dense, $i_3.i_2$ is dense by Lemma 3.4. Since $p_1.p_3$, $i_3.i_2$ is an epi-mono factorization of $f.g$, we conclude that $f.g$ is almost epic. Lemma 3.12 is proved.

Since an invertible map clearly is bidense, we may sum Lemmas 3.8, 3.11 and 3.12 up in:

The class Σ of bidense maps satisfies

(3.2) all isomorphisms are in Σ

(3.3) Σ is stable under composition

(3.4) each diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & B \\ & & \uparrow{\scriptstyle s} \\ & & A \end{array}$$

with $s \in \Sigma$ may be completed to a commutative diagram

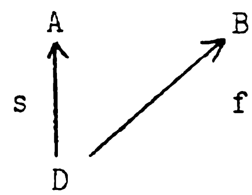
$$\begin{array}{ccc} C & \xrightarrow{f} & B \\ {\scriptstyle s'}\uparrow & & \uparrow{\scriptstyle s} \\ C' & \xrightarrow{f'} & A \end{array} \qquad \text{with } s' \in \Sigma$$

(3.5) If f,g are morphisms so that $f.s = g.s$ for some $s \in \Sigma$, then there exists a (monic) $s' \in \Sigma$ with $s'.f = s'.g$.

These are the four conditions (dualized) of Gabriel-Zisman, [6], p.12.

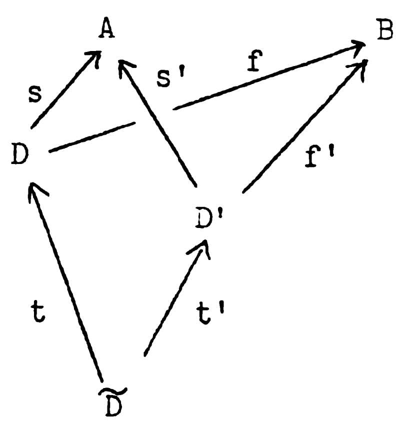
We briefly sketch how the class Σ gives rise to a category $\underline{E}[\Sigma^{-1}]$ and a functor $\underline{E} \xrightarrow{P} \underline{E}[\Sigma^{-1}]$ (a full account may be found in Gabriel-Zisman).

$\underline{E}[\Sigma^{-1}]$ has the same objects as $\underline{E}$; the hom "sets" (they will actually be sets in our case) are defined by letting $\hom_{E[\Sigma^{-1}]}(A,B)$ be the set of equivalence classes of pairs (s,f)



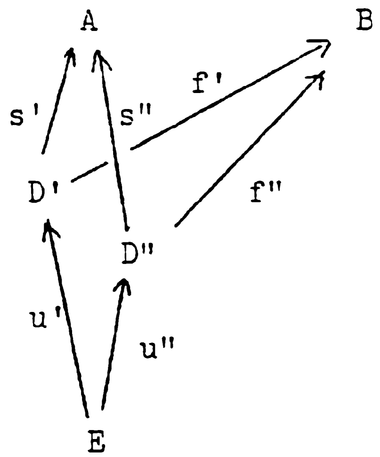
with $s \in \Sigma$

under the equivalence relation: $(s,f) \equiv (s',f')$ if there exists a commutative diagram of form



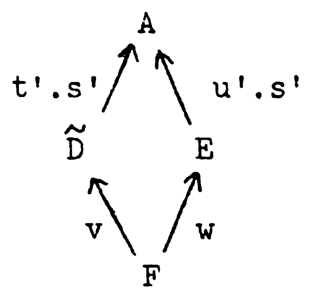
with $t.s = t'.s' \in \Sigma$.

To prove that this actually is an equivalence relation, one needs (3.3)-(3.5). For instance, to prove transitivity, suppose that besides the relation $(s,f) \equiv (s',f')$ displayed in the diagram above, we also have $(s',f') \equiv (s'',f'')$ by means of the diagram



with $u'.s' = u".s" \in \Sigma$.

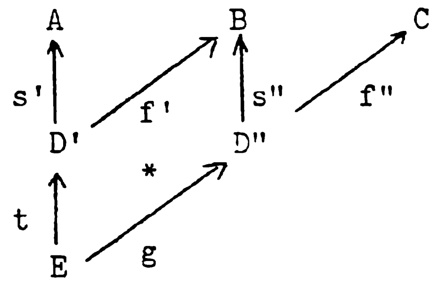
Then use (3.4) to produce a commutative square having $t'.s'$ and $u'.s'$ as the two "upper" arrows:



* with $v.t'.s' = w.u'.s' \in \Sigma$.

Then use (3.5) to produce a monic $g\colon G \longrightarrow F$ in Σ with $g.v.t' = g.w.u'$, which is possible by (3.1). Finally, by (3.3), $g.v.t'.s' \in \Sigma$; the pair $g.v.t'$, $g.w.u'$ proves that $(s,f) \equiv (s",f")$.

Composition in $\underline{E}[\Sigma^{-1}]$ is defined by letting $(s',f').(s",f")$ be the outer pair of legs in the diagram



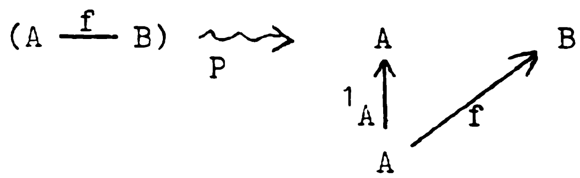
s' ∈ , s" ∈ .

where the square * is formed according to (3.4), with $t \in \Sigma$ and therefore, by (3.3), with $t.s' \in \Sigma$.

We leave it to the reader to prove well-definedness, associativity, etc. Finally, the functor

$$P\colon\ \underline{E} \longrightarrow \underline{E}[\Sigma^{-1}]$$

is the identity map on objects; on maps:



(by (3.2), $1_A \in \Sigma$). Note that

$$(s',f').P(f) = (s',f'.f)$$

whenever it makes sense.

Everything we have done so far with Σ only depends on the properties (3.2)-(3.5) of Σ, not on the special properties we assume on $\underline{E}$. We quote the following theorem which holds in <u>any</u> category $\underline{E}$ (for a Σ with the properties (3.2)-(3.5)):

<u>Theorem</u> (Gabriel Zisman, Prop. I.3.1). The functor

$$P\colon\ \underline{E} \longrightarrow \underline{E}[\Sigma^{-1}]$$

commutes with <u>finite</u> inverse limits ("is left exact").

We return to a topos $\underline{\underline{E}}$ with a topology .

<u>Definition 3.13</u>. An object $F \in |\underline{\underline{E}}|$ will be called a <u>sheaf</u> for the topology if for any dense map $X' \overset{d}{\rightarrowtail} X$, the mapping of sets

$$\hom_{\underline{\underline{E}}}(X,F) \xrightarrow{\hom(d,1)} \hom_{\underline{\underline{E}}}(X',F)$$

is bijective. (If it is injective, F will be called a <u>separated</u> object).

So F is a sheaf if diagrams

$$\begin{array}{ccc} X' & \overset{\text{dense}}{\rightarrowtail} & X \\ & \searrow & \\ & & F \end{array}$$

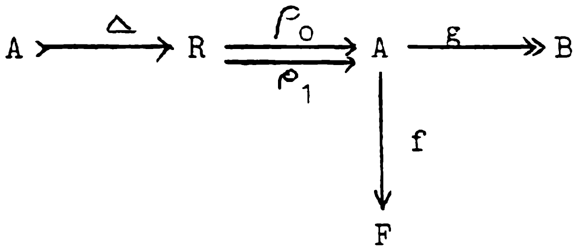
can be filled out $X \longrightarrow F$ in a unique way.

<u>Proposition 3.14</u>. F is a sheaf if and only if for every bidense $g\colon A \longrightarrow B$, the set mapping

$$(3.6) \qquad \hom_{\underline{\underline{E}}}(B,F) \xrightarrow[\hom(g,1)]{} \hom_{\underline{\underline{E}}}(A,F)$$

is bijective.

<u>Proof</u>. $\Leftarrow$ is trivial, since a dense monic is bidense. To prove $\Rightarrow$, take an epi-mono factorization of g. This gives a factorization of (3.6). Since the monic part of g is dense, we only have to prove the bijectivity of (3.6) for an epic map which is almost monic. Let g be a such, and let ρ_0, ρ_1 be its kernel pair

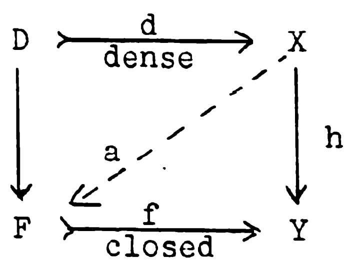


Now $\Delta.\rho_0.f = f = \Delta.\rho_1.f$. By assumption, $\Delta : A \rightarrowtail R$ is dense. Since f is a sheaf, we therefore conclude $\rho_0.f = \rho_1.f$. Since g is a cokernel of $\rho_0.\rho_1$, f factors uniquely across g.

We shall now prove that <u>the sheaves form a reflective sub-category</u> of $\underline{E}$; for that, we can forget all the fraction-calculus stuff which will be used later to prove that the reflection functor is left exact.

We first prove a general "closure-theoretic" lemma.

<u>Lemma 3.15</u>. ("Dense-closed-square"). Suppose we have a commutative square in $\underline{E}$ (straight arrows)



with d dense and f closed. Then there is a map $X \longrightarrow F$ (broken arrow in the diagram) making both triangles commute.

<u>Proof</u>. By commutativity of the diagram

$$\exists_h(d) \leq f$$

hence by $\exists_h \dashv h^{-1}$

$$d \leq h^{-1}(f)$$

hence

$$\bar{d} \leq \overline{h^{-1}(f)} = h^{-1}(\bar{f}) = h^{-1}(f)$$

using monotonicity of closure, naturality of closure, and closedness of f. But $\bar{d} = X$, since d is dense, so

$$X \leq h^{-1}(f)$$

whence

$$\exists_h(X) \leq f.$$

So a exists with a.f = h. The other commutativity follows since f is monic. This proves the lemma.

Note that

(3.7) If $F \rightarrowtail X'$ is closed and $X' \rightarrowtail X$ arbitrary monic, then $\bar{F}^{(X)} \cap X' = F$.

This is an immediate consequence of naturality of $\bar{\ }$.

Lemma 3.16. An object A is separated (Definition 3.13) if and only if $A \overset{\Delta}{\rightarrowtail} A \times A$ is closed, if and only if $A \xrightarrow{\{\cdot\}} A ⫛ \Omega$ factors through $A ⫛ \Omega_j$.

Proof. Equivalence of the last two conditions is immediate from Proposition 3.3 and the definition of $\{\cdot\}$ as the exponential adjoint of the characteristic map of Δ. - To prove the two first conditions equivalent: Suppose A is separated. Let $\bar{A} \xrightarrow{\langle\rho_0,\rho_1\rangle} A \times A$ be the closure of $A \overset{\Delta}{\rightarrowtail} A \times A$. Then ρ_0, ρ_1 agree on $A \leq \bar{A}$, which is dense by Corollary 3.6, whence by separatedness $\rho_0 = \rho_1$, whence $A = \bar{A}$.

Suppose on the other hand that $\Delta : A \longrightarrow A\times A$ is closed. Let $X \xrightarrow{\varkappa_i} A$ $(i=0,1)$ agree on the dense subobject $D \overset{d}{\rightarrowtail} X$ of X. We then have a commutative square

$$\begin{array}{ccc} D & \overset{d}{\rightarrowtail} & X \\ \downarrow & & \downarrow{\scriptstyle \langle \varkappa_0, \varkappa_1 \rangle} \\ A & \underset{\Delta}{\longrightarrow} & A\times A \end{array} \quad ;$$

the result follows from the "dense-closed-square" (Lemma 3.15). The four main lemmas now are

I Ω_j is a sheaf

II $A \pitchfork Y$ is a sheaf if Y is

III The closure of $A \overset{\Delta}{\rightarrowtail} A\times A$ is an equivalence relation

IV If Y is a sheaf and $X \rightarrowtail Y$ is closed, then X is a sheaf.

Proof of I. Let $X' \overset{d}{\rightarrowtail} X$ be dense and let $\varphi : X' \longrightarrow \Omega_j$ be given. φ classifies a closed subobject F of X'. Form $\overline{F}^{(X)}$, which as a closed subobject of X has characteristic function $\psi : X \longrightarrow \Omega_j$. By (3.7) $\overline{F}^{(X)} \cap X' = F$, whence $d.\psi = \varphi$. This proves existence of extension of φ over de. To prove uniqueness, let $d.\gamma = \varphi$, where $\gamma : X \longrightarrow \Omega_j$ classifies the closed subobject $G \rightarrowtail X$. Since $d.\gamma = \varphi$, $G \cap X' = F$. We then have

$$G = \overline{G}^{(X)} = \overline{G}^{(X)} \cap \overline{X'}^{(X)} = \overline{G \cap X'}^{(X)} = \overline{F}^{(X)}$$

$G = \overline{G}^{(X)}$	$= \overline{G}^{(X)} \cap \overline{X'}^{(X)}$	$= \overline{G \cap X'}^{(X)}$	$= \overline{F}^{(X)}$
↑	↑	↑	↑
assumption on G	X' dense	left exactness of closure	since $G \cap X' = F$

which proves uniqueness of ψ.

<u>Proof of II</u>. Let $X' \overset{d}{\rightarrowtail} X$ be dense. Since pulling back a dense monic gives a dense monic (by naturality of closure) $X' \times A \overset{d\times 1}{\rightarrowtail} X \times A$ is dense. By naturality of the fundamental adjointness we have a commutative diagram of sets

$$\begin{array}{ccc}
\hom_{\underline{E}}(X, A \pitchfork Y) & \xrightarrow{(d,1)} & \hom_{\underline{E}}(X', A \pitchfork Y) \\
\cong \downarrow & & \downarrow \cong \\
\hom_{\underline{E}}(X \times A, Y) & \xrightarrow[(d\times 1,1)]{} & \hom_{\underline{E}}(X' \times A, Y)
\end{array}$$

Since $d \times 1$ is dense and Y is a sheaf, the bottom map is a bijection, hence so is the top map, so $A \pitchfork Y$ is a sheaf.

<u>Proof of III</u>. By definition of "equivalence relation in a category" we should prove that for any "test" object X the relation $\sim$ on $\hom_{\underline{E}}(X,A)$ given by

$$(X \xrightarrow{x_0} A) \sim (X \xrightarrow{x_1} A)$$

iff

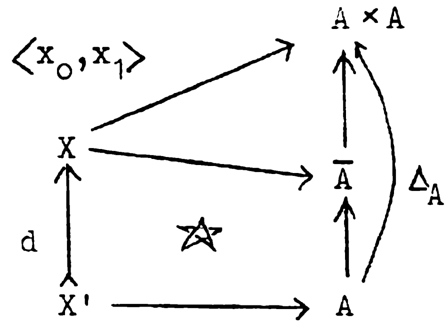
(3.8) $\quad X \xrightarrow{\langle x_0, x_1 \rangle} A \times A$ factors through

$$\overline{A \overset{\Delta}{\rightarrowtail} A \times A}\;(A \times A)$$

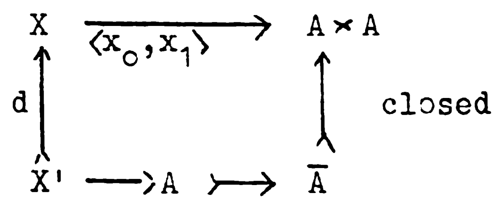
is an "ordinary" equivalence relation. We shall prove that (3.8) is equivalent to

(3.9) $\quad x_0$ and x_1 agree on a dense subobject of X

from which it will be immediate that we have an equivalence relation (transitivity follows from the fact that (by left exactness of closure) an intersection of two dense subobjects is again a dense subobject). - To prove the equivalence of (3.8) and (3.9), suppose (3.8) holds. Let ✭ below be a pullback

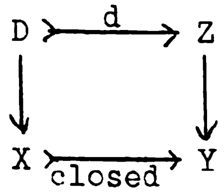


Since A is dense in $\bar{A}$ (Corollary 3.6), d is dense, and by commutativity of the outer diagram x_0 and x_1 agree on X'. - Conversely, suppose (3.9) holds for x_0, x_1, i.e. x_0, x_1 agree on a dense subobject X', so we have a commutative



apply "dense-closed-square-" Lemma 3.15 to get $X \longrightarrow \bar{A}$ as desired.

Proof of IV. Let $D \overset{d}{\rightarrowtail} Z$ be dense and let $D \rightarrowtail X$ be given. Since Y is a sheaf, we can extend $D \longrightarrow X \longrightarrow Y$ to Z so as to get a commutative square



;

now apply "dense-closed-square" Lemma 3.15 to get $Z \longrightarrow X$. Uniqueness is clear.

Proposition 3.17. For any A there is an epic and almost monic map $A \twoheadrightarrow SA$, where SA is separated.

Proof. Let $\bar{A} \xrightarrow{\langle \rho_0, \rho_1 \rangle} A \times A$ be the closure of $\Delta_A \colon A \rightarrowtail A \times A$. By III, (ρ_0, ρ_1) is an equivalence relation; let $q\colon A \twoheadrightarrow SA$ be its coequalizer; then (ρ_0, ρ_1) is the kernel pair of q, which means that

$$\begin{array}{ccc} \bar{A} & \longrightarrow & SA \\ {\scriptstyle \langle \rho_0, \rho_1 \rangle}\downarrow & & \downarrow{\scriptstyle \Delta_{SA}} \\ A \times A & \xrightarrow[q \times q]{} & SA \times SA \end{array}$$

is a pull-back. Now $q \times q$ is epic by Proposition 1.15, and $\langle \rho_0, \rho_1 \rangle$ is a closed monic by construction. The proof establishing that "pulling back along an epic map reflects the notion of density" (3.1) also will give that "pulling back along an epic reflects the notion of closedness." Hence Δ_{SA} is closed, so SA is separated by Lemma 3.16.

Proposition 3.18. For any separated object S there is a dense monic map $S \rightarrowtail F$ into a sheaf.

Proof. Since S is separated, $\{\cdot\}\colon S \longrightarrow S \pitchfork \Omega$ factors through $S \pitchfork \Omega_j$ by Lemma 3.16, and is monic by Proposition 1.7. By I and II, $S \pitchfork \Omega_j$ is a sheaf. Let $\bar{S}$ be the closure of $\{\cdot\}\colon S \longrightarrow S \pitchfork \Omega_j$. By IV, $\bar{S}$ is a sheaf, and S is dense in $\bar{S}$ (by Corollary 3.6). This proves the Proposition.

Proposition 3.19. Let $Sh = Sh_j(\underline{E})$ denote the full subcategory of $\underline{E}$ determined by the sheaves. Then Sh is a <u>reflective</u> subcategory.

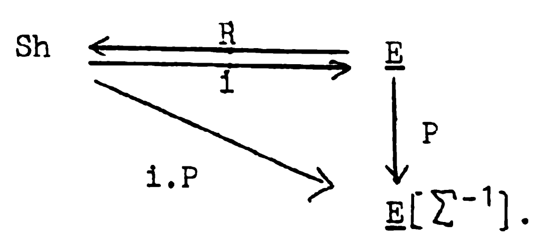
Proof. Let $A \in |\underline{E}|$ be arbitrary. By Proposition 3.17 we have a bidense map from A to a separated object SA and by Proposition 3.18 we have a bidense map $SA \longrightarrow \overline{SA}$ into a sheaf. But bidense maps compose, by Lemma 3.12. So we have a bidense map $A \xrightarrow{\rho_A} \overline{SA}$ where $\overline{SA}$ is a sheaf. If $A \xrightarrow{f} F$ is arbitrary, and F is a sheaf, f factors in a unique way over ρ_A, by Proposition 3.14; thus ρ_A has the required universal property.

By the universal property of ρ_A we can make the assignment $A \rightsquigarrow \overline{SA}$ into a functor $\underline{E} \xrightarrow{R} Sh$, with R left adjoint to the inclusion functor i: $Sh \longrightarrow \underline{E}$, with ρ_A as front adjunction.

Theorem 3.20. Sh is a reflective subcategory of $\underline{E}$; the reflection functor R is left exact; and Sh is a topos with Ω_j as its subobject classifier.

Proof. The first statement we have proved. The idea in the proof of the left exactness of R is to compare it with P: $\underline{E} \longrightarrow \underline{E}[\Sigma^{-1}]$ which by Gabriel-Zisman's Theorem (page 57 here) <u>is</u> left exact;

Consider the diagram of categories



Since ρ_A: $A \longrightarrow iR(A)$ is bidense and natural in A, one easily gets a natural isomorphism of functors $R.i.P \simeq P$. If we can prove i.P to be an equivalence of categories, left exactness of R will follow from left exactness of P. To prove i.P an equivalence, produce a right inverse H for i.P by

$$A \overset{H}{\rightsquigarrow} R(A)$$

$$\begin{array}{ccc} A & & B \\ {\scriptstyle s}\uparrow & \nearrow_{f} & \\ C & & \end{array} \overset{H}{\rightsquigarrow} R(s)^{-1}.R(f)$$

$(s \in \Sigma)$

which makes sense, since R applied to a bidense map gives an isomorphism in Sh - this one can see since the bottom map in

$$X \xrightarrow[\text{bidense}]{f} Y \qquad \begin{array}{ccc} \hom_{\underline{E}}(Y,F) & \xrightarrow[\cong]{(f,1)} & \hom_{\underline{E}}(X,F) \\ \downarrow \cong & & \cong \downarrow \\ \hom_{Sh}(RY,F) & \xrightarrow[(Rf,1)]{} & \hom_{Sh}(RX,F) \end{array}$$

F a sheaf

has to be a bijection, since the vertical ones are bijections by adjointness and the top map is a bijection by Proposition 3.14. Clearly, $i.P.H \cong 1_{Sh}$. Also, one can use ρ_A to get $H.i.P \cong 1_{\underline{E}[\Sigma^{-1}]}$, proving that i.P is an equivalence.

Finally, we prove that Sh is a topos. Since R preserves finite inverse limits, Sh will have such. Since R preserves colimits, Sh will have at least as many colimits as $\underline{E}$ has.

That Sh has exponentiation is easy from II and the fact that $i\colon \text{Sh} \longrightarrow \underline{E}$ preserves products. (An argument may be found in Day [3]). The fact that Ω_j classifies subobjects is seen as follows.

Suppose $F \rightarrowtail G$ is monic in Sh, then it is monic in $\underline{E}$, since the inclusion functor $\text{Sh} \longrightarrow \underline{E}$ is a right adjoint. Let $\varphi\colon G \longrightarrow \Omega$ be the characteristic function. Now F is a <u>closed</u> subobject of G in $\underline{E}$; for, $F \overset{d}{\rightarrowtail} \overline{F}$ is dense, and F being a sheaf, it has a right inverse, whence $F = \overline{F}$ inside G.*) So φ factors across $\Omega_j \longrightarrow \Omega$ (Proposition 3.3). Conversely, if $\varphi\colon G \longrightarrow \Omega_j$ is given in Sh, then $G \longrightarrow \Omega_j \longrightarrow \Omega$ classifies a <u>closed</u> subobject of G, by Proposition 3.3. That subobject is a sheaf, by IV.

This proves the Theorem.

<u>Left exact triples</u>.

Let $\mathbf{T} = ((T,\eta,\mu)$ be a left exact triple on $\underline{E}$. We define a map

$$j\colon \Omega \longrightarrow \Omega$$

as the composite

$$\Omega \xrightarrow{\eta_\Omega} T\Omega \xrightarrow{ch(T(true))} \Omega$$

(note that $T(true)\colon T(1) \longrightarrow T\Omega$ is monic, since T is left exact).

*) Here we use that G is separated.

Proposition 3.21. The map j defines a topology on $\underline{E}$, which in terms of closure operations to $X' \rightarrowtail X$ associates the upper map in the pull-back

$$(3.10) \qquad \begin{array}{ccc} \overline{X'} & \rightarrowtail & X \\ \downarrow & & \downarrow \\ TX' & \rightarrowtail & TX \end{array}$$

(the lower map being monic, since T is left exact). We call this topology the topology induced by $\underline{T}$.

Proof. We first prove that the map $j: \Omega \longrightarrow \Omega$ classifies the operation on $\mathcal{P}(X)$ in (3.10). Let $x: X' \rightarrowtail X$ represent a subobject; let its characteristic map be $\xi: X \rightarrow \Omega$. The map

$$X \xrightarrow{\xi} \Omega \xrightarrow{\eta_\Omega} T\Omega \xrightarrow{ch(T(true))} \Omega$$

equals

$$X \xrightarrow{\eta} TX \xrightarrow{T(\xi)} T\Omega \xrightarrow{ch(T(true))} \Omega;$$

to see what subobject of X it classifies, pull $true: 1 \longrightarrow \Omega$ back along it, which can be done in three steps. In the first step we get (by definition of $ch(T(true))$) just $T(true): T(1) \longrightarrow T(\Omega)$. Since

$$\begin{array}{ccc} X & \longrightarrow & \Omega \\ \uparrow & & \uparrow{\scriptstyle true} \\ X' & \longrightarrow & 1 \end{array}$$

is a pull-back, and T is left exact, $T(true)$ pulls back along $T(\xi)$ to $T(X') \rightarrowtail T(X)$, which finally pulls back along η_X to what we in (,) denoted $\overline{X'}$.

It is clear that $X' \leq \overline{X'}$ in $\mathcal{P}_J(X)$. To prove that $\overline{\overline{X'}} \leq \overline{X'}$ is equivalent to proving $j.j = j$.

Note first that even though

$$** \qquad \begin{array}{ccc} \Omega & \xrightarrow{\eta_\Omega} & T\Omega \\ \text{true}\uparrow & \cong & \uparrow T(\text{true}) \\ 1 & \xrightarrow[\eta_1]{} & T1 \end{array}$$

is not in general a pull-back, T applied to it is a pull-back, because $T(\eta_\Omega)$ is monic (split monic, in fact, by μ_Ω). Now consider $j.j$:

$$j.j = \eta_\Omega.ch(T(\text{true})).\eta_\Omega.ch(T(\text{true}))$$

$$= \eta_\Omega.T(\eta_\Omega).T(ch(T(\text{true}))).ch(T(\text{true}))$$

by straightforward naturality arguments. Pulling true back along this composite in four steps yields first $T(\text{true})$, by definition, then $T^2(\text{true})$ since T preserves pull-backs, then $T(\text{true})$ since T of ** is a pull-back, and finally $\eta_\Omega^*(T(\text{true}))$. This is precisely what we get by pulling true back along $\eta_\Omega.ch(T(\text{true})) = j$. So by uniqueness of characteristic maps, $j.j = j$.

Finally, we must prove the left exactness of the closure operator. This is straightforward using the fact that, for $X' \rightarrowtail X$ and $X'' \rightarrowtail X$ two subobjects of X, we have (by left exactness of T) a pull-back

$$\begin{array}{ccc} T(X'\cap X'') & \longrightarrow & TX' \\ \downarrow & & \downarrow \\ TX'' & \longrightarrow & TX \end{array}$$

which makes it easy to construct a map $\overline{X'} \cap \overline{X''} \longrightarrow T(X'\cap X'')$, and therefore also a map $\overline{X'} \cap \overline{X''} \longrightarrow \overline{X'\cap X''}$.

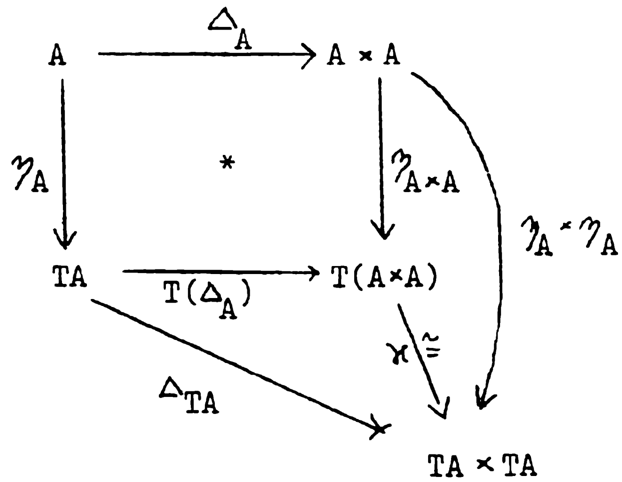
This concludes the proof of Proposition 3.21.

It is well-known that each idempotent triple T, η, μ on a category $\underline{E}$ (that is, μ_A is an isomorphism for each A) is isomorphic to one arising out of a reflective subcategory $\underline{E}'$ of $\underline{E}$ (as the reflection followed by the inclusion). In fact, $\underline{E}'$ may be taken to be the full subcategory of $\underline{E}$ determined by those A for which η_A is an isomorphism.

Suppose that $\mathbf{T} = (T, \eta, \mu)$ is an idempotent and left exact triple on $\underline{E}$; let j be the topology associated to it by Proposition A , and let $\underline{E}'$ be the reflective subcategory associated to T as above. Then

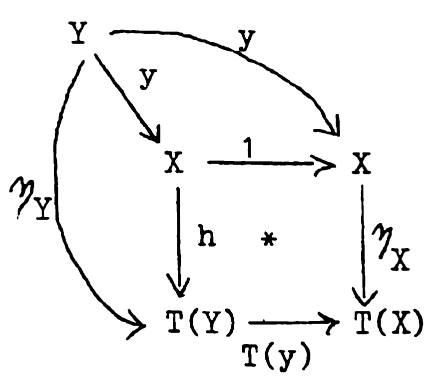
__Proposition 3.22__. The subcategories sh_j and $\underline{E}'$ of $\underline{E}$ are equal. In particular, each reflective subcategory of $\underline{E}$ with exact reflection functor is the category of sheaves for a suitable topology j on $\underline{E}$.

__Proof__. We shall prove, first, that A is separated with respect to j iff η_A is monic. So let A be separated, so $\Delta_A\colon A \rightarrowtail A\times A$ is closed in $\mathcal{R}(A\times A)$ which means that $*$ in the following commutative diagram is a pullback



Since the canonical map κ: $T(A\times A) \longrightarrow TA \times TA$ is an isomorphism, the outer diagram is a pull-back as well, whence for trivial diagrammatic reasons η_A is monic. Conversely, if η_A is monic, the outer diagram in the above is a pull-back, whence * is, whence Δ_A is closed, whence A is separated.

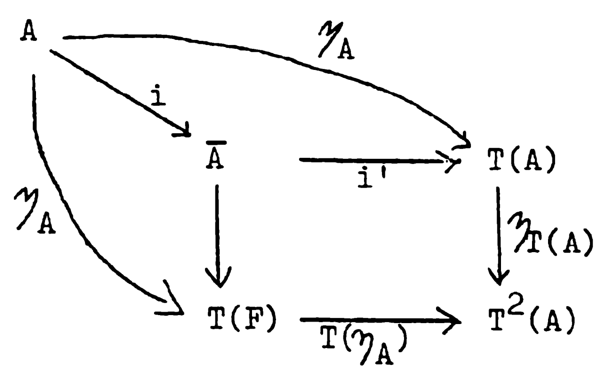
We now prove that η_A is an isomorphism iff A is a sheaf (with respect to j). Let η_A be iso. Let f: $Y \longrightarrow A$ be an arbitrary map and y: $Y \rightarrowtail X$ a dense monic map. We must prove that f extends over y. Now, y being dense means that the square * in the following diagram is a pull-back (for a suitable h: $X \longrightarrow T(Y)$):



It is now easy to prove, using naturality of η, that the composite

$$X \xrightarrow{h} T(Y) \xrightarrow{T(f)} T(A) \xrightarrow{\eta_F^{-1}} A$$

is the desired extension of f over y. (Uniqueness of the extension follows, since we know already that A is separated). Note that in particular every object of the form T(X) is a sheaf (using the idempotency). Conversely, let A be a sheaf. Since A is then separated, η_A is monic, whence we can form the closure $\bar{A}$ of the subobject $\eta_A\colon A \rightarrowtail T(A)$ of T(A). Since T(A) is a sheaf and in particular separated, and since the inclusion $i\colon A \rightarrowtail \bar{A}$ is dense, it is easy to conclude that i is an isomorphism. We then remark that we have a commutative diagram



with the square being a pull-back. By idempotency of T, η, μ $T(\eta_A)$ is an isomorphism, whence i' is an isomorphism, whence $\eta_A = i.i'$ is also an isomorphism. This proves the Proposition.

A triple on a topos, whose functor part commutes with finite products has already a cartesian closed category as its category of algebras, according to [10]. Even for left exact triples, though, the algebra category does not seem to be a topos in general. It <u>is</u> a topos if the triple further is idempotent, by the Proposition just proved. Chr.J. Mikkelsen proved that the coalgebras on the algebras for a left exact triple $\mathbb{T}$ form a topos, equivalent to sh_j (j induced by $\mathbb{T}$).

Left exact triples arise whenever we have morphisms of toposes (this concept will be defined in the next chapter).

Another type of example is the double-negation topology: For any object A define a map

$$\neg : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$$

by putting

$$\neg A' = (A' \Rightarrow \emptyset)$$

where $A' \rightarrowtail A$ is a subobject of A, $\emptyset$ denotes the minimal subobject of A (which is represented by the unique map from the initial object to A), and $\Rightarrow$ denotes the exponentiation in the cartesian closed category $\mathcal{P}(A)$ (compare p. 12).

For general closed-category reasons

$$A' \rightsquigarrow \neg\neg A' \quad (= (A' \Rightarrow \emptyset) \Rightarrow \emptyset)$$

is a strong triple on the closed category $\mathcal{P}(A)$; note, namely, that the contravariant functor $- \Rightarrow X$ for any X is right adjoint to itself, whence the composite $(- \Rightarrow X) \Rightarrow X$ is a strong triple; in particular, putting $X = \emptyset$, we get the triple $\neg\neg$ on $\mathcal{P}(A)$. Since $\mathcal{P}(A)$ is cartesian closed and partially ordered, this implies that $\neg\neg$ commutes with $\wedge$, so is left exact. Finally it is pull-back natural, by two applications of (1.23) and the fact that, by Theorem 1.12, $\emptyset$ is preserved by pullback; so it is a topology.

(Th proof that $\neg\neg$ commutes with $\wedge$ can also be seen by the usual methods of intuitionistic propositional calculus).

The category of sheaves for $\neg\neg$ will have its Ω to be not only a Heyting algebra object, but a Boolean algebra object. (It is not false to say that this is related to Kolmogoroff's idea [12] of embedding classical mathematics in intuitonistic mathematics).

4. Maps of toposes

If $\underline{E}$ and $\underline{E}'$ are elementary toposes, we define a map (or morphism) of toposes

$$\underline{E} \xrightarrow{f} \underline{E}'$$

to be a functor f_*: $\underline{E} \longrightarrow \underline{E}'$ having a left exact left adjoint f^*: $\underline{E}' \longrightarrow \underline{E}$. Composition of maps is to be given by composition of functors. We call f_* the direct image part of f, and f^* the inverse image part of f. In this way we get a 2-category: the 0-cells are elementary toposes, the 1-cells are maps of toposes, and the 2-cells are natural maps between the direct image parts. We shall generally only be interested in concepts "up to 2-isomorphism" in this 2-category; two elementary toposes are 2-isomorphic if and only if they are equivalent as categories.

As an example of a map of toposes between elementary toposes consider a map $X \xrightarrow{\alpha} Y$ in an elementary topos $\underline{E}$. We have already seen that the categories $\underline{E}/X$ and $\underline{E}/Y$ are elementary toposes and that the functor "pullback along α":-

$$\alpha^*: \underline{E}/Y \longrightarrow \underline{E}/X$$

has a left adjoint Σ_α and a right adjoint Π_α. Since α^* is left exact and left adjoint to Π_α, we have a map of toposes

$$E/\alpha: \underline{E}/X \longrightarrow \underline{E}/Y$$

with Π_α for direct image and α^* for inverse image. Strictly speaking, $\underline{E}/\alpha$ is only defined up to 2-isomorphism; in any case, we have a pseudo-functor $\alpha \longmapsto \underline{E}/\alpha$ from $\underline{E}$ to the 2-category of elementary toposes.

If j is a topology on an elementary topos $\underline{E}$, we have seen that the inclusion functor $sh_j(\underline{E}) \longrightarrow \underline{E}$ has a left exact left adjoint, namely sheafification. This gives us a map of elementary toposes $sh_j(\underline{E}) \longrightarrow \underline{E}$; which we shall refer to as the canonical map for the topology j.

<u>Theorem 4.1</u>. Let j be a topology on an elementary topos $\underline{E}'$ and let $i\colon sh_j(\underline{E}') \longrightarrow \underline{E}'$ be the canonical map for j. Then a map of toposes $f\colon \underline{E} \longrightarrow \underline{E}'$ factors through i if and only if f^* takes j-bidense maps to isomorphisms.

<u>Proof</u>. Let $K \xrightarrow{g} L$ be a j-bidense map in $\underline{E}'$ and let X be an object of $\underline{E}$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\underline{E}'}(L;f_*(X)) & \xrightarrow{\mathrm{Hom}_{\underline{E}}(g,f_*(X))} & \mathrm{Hom}_{\underline{E}'}(K,f_*(X)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\underline{E}}(f^*(L),X) & \xrightarrow{\mathrm{Hom}_{\underline{E}}(f^*(g),X)} & \mathrm{Hom}_{\underline{E}}(f^*(K),X) \end{array}$$

in which the vertical maps are adjunction isomorphisms. The top map, $\mathrm{Hom}_{\underline{E}'}(g,f_*(X))$ is an isomorphism for all j-bidense maps g if and only if $f_*(X)$ is a j-sheaf. The bottom map, $\mathrm{Hom}_{\underline{E}}(f^*(g),X)$, is an isomorphism for all j-bidense maps g if f^* takes j-bidense maps to isomorphisms.

Suppose f does factor through i. Then f^* takes j-bidense maps to isomorphisms because i^* does. Conversely, if f^* takes j-bidense maps to isomorphisms, $f_*(X)$ is a j-sheaf for all X in $\underline{E}$, by the argument above. Hence there is a functor

$$u_*\colon\ \underline{E} \longrightarrow sh_j(\underline{E}')$$

such that $f_* = i_* u_*$. Define $u^*\colon\ sh_j(\underline{E}') \longrightarrow \underline{E}$ to be the composite

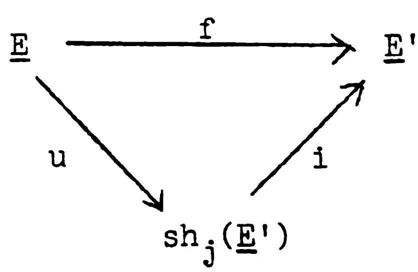
$$sh_j(\underline{E}') \xrightarrow{i_*} \underline{E}' \xrightarrow{f^*} \underline{E}.$$

Then u^* is left exact because i_* and f^* are. The natural bijections

$$\mathrm{Hom}_{sh_j(\underline{E}')}(Y, u_*(X)) \cong \mathrm{Hom}_{\underline{E}'}(i_*(Y), f_*(X)) \cong$$

$$\cong \mathrm{Hom}_{\underline{E}}(f^* i_*(Y), X) \cong \mathrm{Hom}_{\underline{E}}(u^*(Y), X)$$

show that u^* is left adjoint to u_*. Hence we have a factorization



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We recall from Chapter 3 that if T is a left exact triple on an elementary topos $\underline{E}'$, with unit $1_{\underline{E}'} \xrightarrow{\eta} T$, then the map

$$\Omega' \xrightarrow{\eta_{\Omega'}} T(\Omega') \xrightarrow{ch(T(true'))} \Omega'$$

is a topology on $\underline{E}'$. We call this the topology induced by T. If $\underline{E} \xrightarrow{f} \underline{E}'$ is a map of toposes, the composite functor

$$\underline{E}' \xrightarrow{f^*} \underline{E} \xrightarrow{f_*} \underline{E}'$$

has a natural triple structure. This triple is left exact because f^* and f_* both are. Hence we obtain a topology on $\underline{E}'$, which we call the topology on $\underline{E}'$ induced by f.

Consider the commutative diagram

$$\begin{array}{ccccc}
N & \xrightarrow{\varphi} & \Omega' & & \\
\downarrow{\scriptstyle \eta_N} & & \downarrow{\scriptstyle \eta_{\Omega'}} & \searrow{\scriptstyle j} & \\
f_*f^*N & \xrightarrow{f_*f^*\varphi} & f_*f^*\Omega' & \xrightarrow{\lambda} & \Omega' \\
\uparrow{\scriptstyle f_*f^*(\beta)} & & \uparrow{\scriptstyle f_*f^*(\mathrm{true}')} & & \uparrow{\scriptstyle \mathrm{true}'} \\
f_*f^*M & \longrightarrow & 1 & \longrightarrow & 1
\end{array}$$

where λ is the characteristic map of

$$1 \xrightarrow{\simeq} f_*f^*(1) \xrightarrow{f_*f^*(\mathrm{true}')} f_*f^*(\Omega').$$

The bottom square are both pullbacks. The closure of $M \rightarrowtail^{\beta} N$ for the topology j has φj for characteristic map, and so is given by the pullback of $f_*f^*(\beta)$ along η_N. By hypothesis, $f^*(\beta)$ is an isomorphism, so β is j-dense.

Conversely, suppose $M \overset{\beta}{\rightarrowtail} N$ is j-dense. Consider the commutative diagram

$$\begin{array}{ccccccccc}
f^*N & \xrightarrow{f^*\varphi} & f^*\Omega' & \xrightarrow{f^*\eta_{\Omega'}} & f^*f_*f^*\Omega' & \xrightarrow{f^*\lambda} & f^*\Omega' & \xrightarrow{\mu} & \Omega \\
\uparrow{\scriptstyle f^*(\beta)} & & \uparrow{\scriptstyle f^*(\mathrm{true}')} & & \uparrow{\scriptstyle f^*f_*f^*(\mathrm{true}')} & & \uparrow{\scriptstyle f^*(\mathrm{true}')} & & \uparrow{\scriptstyle \mathrm{true}} \\
f^*M & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & 1
\end{array}$$

where μ is the characteristic map of

$$1 \xrightarrow{\sim} f^*(1) \xrightarrow{f^*(\mathrm{true}')} f^*\Omega'.$$

The right hand square is a pullback by definition of μ, the second square on the left is a pullback because $f^*\eta_{\Omega'}$ is monic, and the remaining two squares are pullbacks because f^* preserves pullbacks. The top row is $f^*(\varphi j).\mu$ and by hypothesis φj is the composite

$$N \longrightarrow 1 \xrightarrow{\text{true}'} \Omega' .$$

Thus we have pullback diagrams

$$\begin{array}{ccccccc}
f^*N & \longrightarrow & 1 & \xrightarrow{f^*(\text{true}')} & f^*\Omega' & \xrightarrow{\mu} & \Omega \\
{\scriptstyle f^*(\beta)}\uparrow & & \uparrow & & \uparrow {\scriptstyle f^*(\text{true}')} & & \uparrow {\scriptstyle \text{true}} \\
f^*M & = & 1 & = & 1 & = & 1 \; .
\end{array}$$

Hence $f^*(\beta)$ is an isomorphism, and the proof is complete.

Theorem 4.3. (Factorization Theorem). Every map of toposes

$$\underline{E} \xrightarrow{f} \underline{E}'$$

where $\underline{E}$ and $\underline{E}'$ are elementary toposes, has a factorization

$$\begin{array}{ccc}
\underline{E} & \xrightarrow{f} & \underline{E}' \\
{\scriptstyle a}\searrow & & \nearrow{\scriptstyle b} \\
& \underline{F} &
\end{array}$$

where b_* is full and faithful and a^* reflects isomorphisms.

Proof. Take $\underline{F} = sh_j(\underline{E}')$ where j is the topology on $\underline{E}'$ induced by f, and where b is the canonical map for j. Then b_* is full and faithful. By the preceding two theorems, there exists a map of toposes $\underline{E} \xrightarrow{a} \underline{F}$ such that $ba = f$. Let g be a map in $\underline{F}$ such that $a^*(g)$ is an isomorphism. Since $a^*(g) = f^*b_*(g)$, $b_*(g)$ is j-bidense. Hense $g = b^*b_*(g)$ is an isomorphism, and so a^* reflects isomorphisms.

Note that the topologies on $\underline{E}'$ induced by f and b are the same, and that the topology on $\underline{F}$ induced by a is the trivial one, i.e. every object of $\underline{F}$ is a sheaf for it.

Dually, note that because $b^*b_* \cong 1_{\underline{F}}$, the cotriples on $\underline{E}$ induced by f and by a coincide, and the cotriple on $\underline{F}$ induced by b is the trivial one, namely $1_{\underline{F}}$.

Lemma 4.4. Let $\underline{E} \xrightarrow{f} \underline{E}'$ be a map of toposes, where $\underline{E}$ and $\underline{E}'$ are elementary toposes, such that f_* is full and faithful, and f^* reflects isomorphisms. Then f is a 2-isomorphism (that is, an equivalence).

Proof. For any object X of $\underline{E}'$, consider the front adjunction

$$\eta_X: \ X \longrightarrow f_*f^*X.$$

Because f_* is full and faithful, the end adjunction $\varepsilon: \ f^*f_* \longrightarrow 1_{\underline{E}}$ is an isomorphism, and so $f^*(\eta_X)$ is an isomorphism. Since f^* reflects isomorphisms, η_X is an isomorphism.

Corollary 4.5. If $\underline{E}$ and $\underline{E}'$ are elementary toposes and $\underline{E} \xrightarrow{f} E'$ is a map of toposes such that f_* is full and faithful, then there is an equivalence of categories

$$a_*: \ \underline{E} \xrightarrow{\sim} sh_j(\underline{E}')$$

such that $f_* = b_*a_*$, where j is the topology on $\underline{E}'$ induced by f and b_* is the inclusion of the j-sheaves.

Of course, this Corollary may also be inferred from Proposition 3.22.

Corollary 4.6. Let $\underline{E}_1, \underline{E}_2, \underline{E}'$ be elementary toposes, and let $f_i\colon \underline{E}_i \longrightarrow \underline{E}'$ be maps of toposes for $i = 1,2$. If f_{2*} is full and faithful, a necessary and sufficient condition that f_1 should factor through f_2 is that for every map α in $\underline{E}'$ for which $f_2^*(\alpha)$ is an isomorphism, we should have that $f_1^*(\alpha)$ is an isomorphism.

Theorem 4.7. Let

$$\begin{array}{ccccc}
\underline{E}_1 & \xrightarrow{a_1} & \underline{F}_1 & \xrightarrow{b_1} & \underline{E}_1' \\
{\scriptstyle u}\downarrow & & \vdots\,{\scriptstyle v} & & \downarrow{\scriptstyle w} \\
\underline{E}_2 & \xrightarrow{a_2} & \underline{F}_2 & \xrightarrow{b_2} & \underline{E}_2'
\end{array}$$

be a 2-commutative diagram of elementary toposes and maps of toposes, such that a_1^* reflects isomorphisms and b_{2*} is full and faithful. Then there is a map of toposes $\underline{F}_1 \xrightarrow{v} \underline{F}_2$ making the whole diagram 2-commutative.

Proof. Let α be a map in $\underline{E}_2'$ such that $b_2^*(\alpha)$ is an isomorphism. Then $u^*a_2^*b_2^*(\alpha)$ is an isomorphism; hence $a_1^*b_1^*w^*(\alpha)$ is an isomorphism; hence $b_1^*w^*(\alpha)$ is an isomorphism.

Corollary 4.8. Let $\underline{E}$ and $\underline{E}'$ be elementary toposes and let $\underline{E} \xrightarrow{f} \underline{E}'$ be a map of toposes. Then the factorization of f into $\underline{E} \xrightarrow{a} \underline{F} \xrightarrow{b} \underline{E}'$ where a^* reflects isomorphisms and b_* is full and faithful, is unique up to 2-isomorphism.

Proof. In the theorem above, take $u = 1_{\underline{E}}$, $w = 1_{\underline{E}'}$ and suppose that a_i^* reflects isomorphisms and b_{i*} is full and faithful for $i = 1,2$. Then v^* reflects isomorphisms, because $a_1^*v^* = a_2^*$, and v_* is full and faithful because $b_{2*}v_* = b_{1*}$.

In view of this result, we will refer to the factorization of a topos map, with the usual abuses of language.

Suppose that $X \xrightarrow{f} Y$ is a map in an elementary topos $\underline{E}$. We have seen that we get a map of toposes

$$\underline{E}/f\colon\ \underline{E}/X \longrightarrow \underline{E}/Y.$$

To study the factorization of this map, we have the following lemmas.

Lemma 4.9. If f is monic, $(\underline{E}/f)_*$ is full and faithful.

Proof. $(\underline{E}/f)^* = f^*$ has a left adjoint, Σ_f, given by composition with f. Since f is monic, the diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback, so $f^* \Sigma_f \simeq 1_{\underline{E}/X}$. By adjointness, $f^* \Pi_f \simeq 1_{\underline{E}/X}$, so $\Pi_f = (\underline{E}/f)_*$ is full and faithful.

Lemma 4.10. If f is epic, f^* reflects isomorphisms.

Proof. Let

$$\begin{array}{ccc} K & \xrightarrow{\alpha} & L \\ & \searrow \ \swarrow & \\ & Y & \end{array}$$

be a map in $\underline{E}/Y$ such that $f^*(\alpha)$ is an isomorphism. We have a commutative diagram

$$\begin{array}{ccc} f^*(K) & \xrightarrow{f^*(\alpha)} & f^*(L) \\ \downarrow & & \downarrow \\ K & \xrightarrow{\alpha} & L \end{array}$$

where the vertical maps, induced by $X \xrightarrow{f} Y$, are epic, since pullbacks of epics are epic. Since $f^*(\alpha)$ is an isomorphism, it follows that α is epic. Let

$$R \overset{k_o}{\underset{k_1}{\rightrightarrows}} K$$

be the kernel pair of α. We have a commutative diagram

$$\begin{array}{ccccc} f^*(R) & \overset{f^*(k_o)}{\underset{f^*(k_1)}{\rightrightarrows}} & f^*(K) & \xrightarrow{f^*(\alpha)} & f^*(L) \\ \theta \downarrow & & \downarrow & & \downarrow \\ R & \overset{k_o}{\underset{k_1}{\rightrightarrows}} & K & \xrightarrow[\alpha]{} & L \end{array}$$

in which the squares are pullbacks and the vertical maps are epics. The top line is an equalizer diagram so $f^*(k_o) = f^*(k_1)$; so $\theta k_o = \theta k_1$. Since θ is epic, $k_o = k_1$ and so α is an isomorphism.

Theorem 4.11. If $X \overset{a}{\twoheadrightarrow} I \overset{b}{\rightarrowtail} Y$ is an epi-mono factorization of $X \xrightarrow{f} Y$ in $\underline{E}$, then

$$\underline{E}/X \xrightarrow{\underline{E}/a} \underline{E}/I \xrightarrow{\underline{E}/b} \underline{E}/Y$$

is the factorization of $\underline{E}/f$.

Recall that if $\mathbb{C}$ is a left exact cotriple on a topos $\underline{E}$, then by Theorem 2.6 we have a topos map $\underline{E} \longrightarrow \underline{E}_{\mathbb{C}}$, where $\underline{E}_{\mathbb{C}}$ is the category of coalgebras for $\mathbb{C}$.

We remark that because the forgetful functor $E_{\mathbb{C}} \longrightarrow E$ reflects isomorphisms, the topology on $E_{\mathbb{C}}$ induced by the topos map $E \longrightarrow E_{\mathbb{C}}$ is the trivial one.

The following Proposition characterizes topos maps that arise from left exact cotriples (just as Corollary 4.5 characterizes topos maps arising from a topology j).

<u>Proposition 4.12</u>. Let $\underline{E} \xrightarrow{f} \underline{E}'$ be a map of toposes, such that f^* reflects isomorphisms. Let $\mathbb{C}$ be the cotriple on $\underline{E}$ induced by the adjoint pair f_*, f^*, and let $\underline{E} \xrightarrow{\lambda} \underline{E}_{\mathbb{C}}$ be the canonical map of toposes, such that λ^* is the forgetful functor. Then there is a 2-isomorphism $\underline{E}_{\mathbb{C}} \xrightarrow{b} \underline{E}'$ such that the diagram

$$\begin{array}{ccc} \underline{E} & \xrightarrow{f} & \underline{E}' \\ & \searrow^{\lambda} \quad \nearrow_{b} & \\ & \underline{E}_{\mathbb{C}} & \end{array}$$

commutes.

<u>Proof</u>. Since f^* reflects isomorphisms and is left exact the adjoint pair f_*, f^* satisfies the conditions for the dual of Beck's tripleability criterion. The conclusion states precisely that there is an equivalence of categories $b^*: \underline{E}' \longrightarrow \underline{E}_{\mathbb{C}}$ such that the diagram

$$\begin{array}{ccc} \underline{E}' & \xrightarrow{b^*} & \underline{E}_{\mathbb{C}} \\ & {}_{f^*}\searrow \quad \swarrow_{\lambda^*} & \\ & \underline{E} & \end{array}$$

commutes. The functor b^* takes an object X of $\underline{E}'$ to the $\mathbb{C}$-coalgebra $(f^*X, f^*\eta_X)$, where $id_{E'} \xrightarrow{\eta} f_*f^*$ is the front adjunction.

<u>Proposition 4.13</u>. If a map of toposes $\underline{E} \xrightarrow{f} \underline{E}'$ factorizes $\underline{E} \xrightarrow{a} \underline{F} \xrightarrow{b} \underline{E}'$, where a^* reflects isomorphisms and b_* is full and faithful, then, up to natural equivalence, we may interpret the category $\underline{F}$ either as the category of sheaves in $\underline{E}'$ for the topology on $\underline{E}'$ induced by f, or as the category of coalgebras in $\underline{E}$ for the left exact cotriple on $\underline{E}$ induced by f.

5. Category theory in toposes

Let $\underline{E}$ be a category with 1 and with pull-back; choose for each pair of maps with common codomain one of all the (isomorphic) pull-back diagrams for that pair, and call the chosen ones canonical (actually, we ought to have done the same when defining f^* etc.).

For each pair (A,B) of objects of $\underline{E}$, we denote by $\mathrm{Span}(A,B)$ the category whose objects are diagrams of form ("spans"):

$$(5.1) \qquad \begin{array}{ccc} & X & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\ A & & B \end{array} \qquad \text{(by abuse denoted } X\text{)}$$

and where a morphism from (5.1) to

$$\begin{array}{ccc} & X' & \\ {\scriptstyle\alpha'}\swarrow & & \searrow{\scriptstyle\beta'} \\ A & & B \end{array}$$

is a map $X \xrightarrow{x} X'$ with $x.\beta' = \beta$, $x.\alpha' = \alpha$. Clearly, since $\underline{E}$ has products

$$\mathrm{Span}(A,B) \cong \underline{E}/A \times B.$$

We construct a functor in two variables denoted $\divideontimes$:

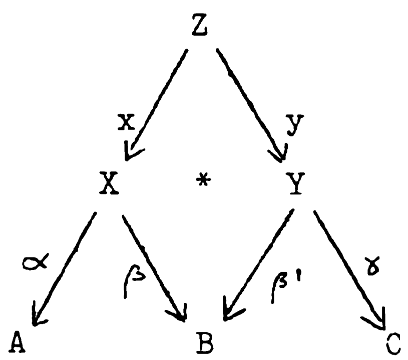
$$\mathrm{Span}(A,B) \times \mathrm{Span}(B,C) \xrightarrow{\ast} \mathrm{Span}(A,C)$$

by assigning to the pair of objects

(5.2)

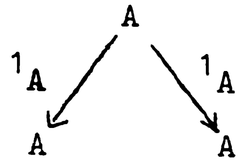
the outer diagram in

(5.3)



where * is the canonical pullback for β, β'. (We leave it to the reader to define $\ast$ on morphisms).

By I_A we denote the span



more generally, for a map $A \xrightarrow{f} B$, we denote by $[1,f]$, and by $[f,1]$ the two spans

$$\begin{array}{ccc} & A & \\ {}^{1_A}\swarrow & & \searrow{}^{f} \\ A & & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & A & \\ {}^{f}\swarrow & & \searrow{}^{1_A} \\ B & & A \end{array}$$

respectively (so that $I_A = [1_A, 1_A]$).

Proposition 5.1. $\divideontimes$ is associative up to canonical isomorphisms, and the I_A's are two-sided units up to canonical isomorphisms. (Further, these isomorphisms are coherent).

This can easily be seen from the universal properties. Coherence means just that all diagrams formed out of canonical isomorphisms commute.

Proposition 5.2. Every span in Span(A,B) is isomorphic to one of form $[f,1] \divideontimes [1,g]$.

Proof. For the span (5.1), say, it suffices to take $[\alpha, 1_X] \divideontimes [1_X, \beta]$.

For fixed span $X \in \text{Span}(A,B)$, and any C, $X \divideontimes -$ is a functor:

$$X \divideontimes - : \quad \text{Span}(B,C) \longrightarrow \text{Span}(A,C).$$

Similarly $- \divideontimes X: \quad \text{Span}(C,A) \longrightarrow \text{Span}(C,B).$

Proposition 5.3. For $f: B \longrightarrow C$ any map, the following diagram (in Cat) commutes up to isomorphism

$$\begin{array}{ccc} \text{Span}(A,B) & \xrightarrow{\;-\divideontimes[1,f]\;} & \text{Span}(A,C) \\ \cong\downarrow & & \downarrow\cong \\ \underline{E}/A \times B & \xrightarrow[\Sigma_{1\times f}]{} & \underline{E}/A \times C \end{array}$$

Proof. This is trivial. Starting with the object (5.1), both ways round give, up to isomorphism, just

$$\begin{array}{ccc} & X & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta.f} \\ A & & C \end{array}$$

Proposition 5.4. For $f\colon B \longrightarrow C$ any map, the following diagram (in Cat) commutes up to isomorphism

$$\begin{array}{ccc} \mathrm{Span}(A,C) & \xrightarrow{-\circledast[f,1]} & \mathrm{Span}(A,B) \\ \cong\downarrow & & \downarrow\cong \\ \underline{E}/A\times C & \xrightarrow[(1\times f)^*]{} & \underline{E}/A\times B \end{array}$$

Proof. This is somewhat harder, but still a straightforward diagram chase with finite left limits. One may, therefore, for instance prove it for $\underline{E}$ = the category of sets, and then apply the Yoneda embedding. We omit the details.

We now assume that $\underline{E}$ is an (elementary) topos. Then there is a right adjoint for $(1\times f)^*$. Since any isomorphism in Cat has a right adjoint (= the inverse), we conclude from Proposition 5.4 that for $B \xrightarrow{f} C$

$$-\circledast[f,1]\colon\ \mathrm{Span}(A,C) \longrightarrow \mathrm{Span}(A,B)$$

has a right adjoint. Since $\sum_{1\times f} \dashv (1\times f)^*$, also $-\circledast[1,f]\colon\ \mathrm{Span}(A,B) \longrightarrow \mathrm{Span}(A,C)$ has a right adjoint. Consider now an arbitrary span X. By Proposition 5.2, it is isomorphic to one of the form $[f,1]\circledast[1,g]$; thus $-\circledast X$ (by associativity of $\otimes$) is isomorphic to a functor of form

$$(-\otimes[f,1]).(-\otimes[1,g]),$$

and each of the functors in this composite has a right adjoint; therefore

Theorem 5.5. In a topos $\underline{E}$, for any $X \in \mathrm{Span}(A,B)$, $-\otimes X$: $\mathrm{Span}(C,A) \longrightarrow \mathrm{Span}(C,B)$ has a right adjoint. Similarly $X\otimes -$ has a right adjoint.

Definition 5.6. (Benabou, [2]). A <u>bicategory</u> $\mathcal{B}$ is a structure of the following kind

(a) a class $\mathcal{B}_0$ whose elements are called 0-cells, or objects

(b) for each pair $A,B \in \mathcal{B}_0$ a category $\mathcal{B}(A,B)$ whose <u>objects</u> are called 1-cells, and whose maps are called 2-cells

(c) for each $A \in \mathcal{B}_0$, a specified 1-cell $I_A \in [\mathcal{B}_0(A,A)]$ and for each triple $A,B,C \in \mathcal{B}_0$, a functor in two variables, denoted $\otimes$

$$\mathcal{B}(A,B) \times \mathcal{B}(B,C) \longrightarrow \mathcal{B}(A,C);$$

$\otimes$ should be associative up to coherent isomorphisms, and the I_A's should be units up to isomorphisms (coherent).

A bicategory is called <u>biclosed</u> if for any $X \in \mathcal{B}(A,B)$ and any C,

$$-\otimes X\colon \mathcal{B}(C,A) \longrightarrow \mathcal{B}(C,B)$$

and

$$X\otimes -\colon \mathcal{B}(B,C) \longrightarrow \mathcal{B}(A,C)$$

have right adjoints.

The spans in $\underline{E}$ form a bicategory $\mathcal{B}$ with $\mathcal{B}_0 = |\underline{E}|$ and with $\mathcal{B}(A,B) = \mathrm{Span}(A,B)$. We denote it by SPAN($\underline{E}$). Theorem 5.5 says that

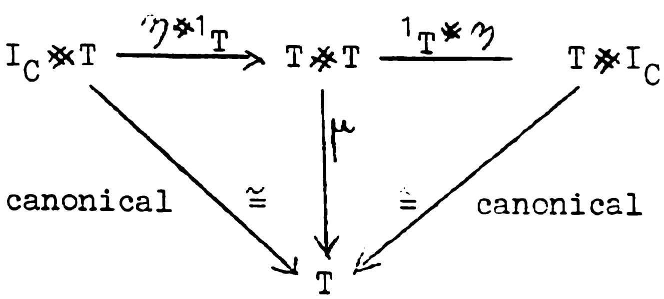
if $\underline{E}$ is a topos,
SPAN($\underline{E}$) is a biclosed bicategory.

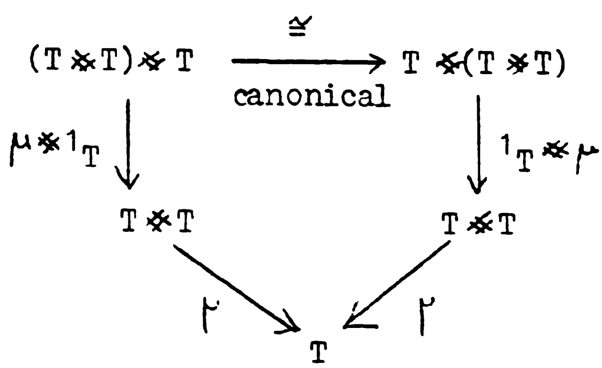
If $C \in \mathcal{B}_0$ in a bicategory $\mathcal{B}$, we shall state

<u>Definition 5.7</u>. A <u>monad in</u> $\mathcal{B}$ <u>on</u> C is a triple (T,η,μ) where $T \in \mathcal{B}(C,C)$ and where

$$\eta: I_C \longrightarrow T \qquad \mu: T * T \longrightarrow T;$$

η and μ are required to satisfy the usual unit- and associative laws: the following diagrams in $\mathcal{B}(C,C)$ commute





If $D \in \mathcal{B}_0$ is any object, the functor

$$T \ast - : \mathcal{B}(C,D) \longrightarrow \mathcal{B}(C,D)$$

inherits a triple structure from the monad structure of T; denote this triple $T^{\cdot}$.

We shall be interested in the category of algebras for the triple $T^{\cdot} = T \ast -$ in the bicategory $\mathcal{B} = \mathrm{SPAN}(\underline{E})$ ($\underline{E}$ a topos). Since $\mathrm{SPAN}(\underline{E})$ is a <u>biclosed</u> bicategory, the functor part of the triple $T^{\cdot}$ has a right adjoint $T_{\cdot}$. It follows from a theorem of Eilenberg and Moore ([5], Proposition 3.3) that $T_{\cdot}$ carries the structure of a cotriple, and that there is an isomorphism φ between the category of algebras for $T^{\cdot}$ and the category of coalgebras for $T_{\cdot}$, and in fact so that φ commutes with the two underlying functors. Now $T_{\cdot}$ being a right adjoint functor is left exact (even left continuous), so the category of coalgebras for it form a topos, by Theorem 2.5, and in fact so that the underlying functor is the inverse - image part of a topos morphism. Therefore we have, by the isomorphism φ, that the category of algebras for $T^{\cdot}$ form a topos in such a way that the underlying-functor

$$\mathcal{B}(C,D)^{T^{\cdot}} = \mathrm{Span}(C,D)^{T^{\cdot}} \longrightarrow \mathrm{Span}(C,D) \simeq \underline{E}/C \times D$$

is the inverse image part of a topos map.

For $D = 1$ we shall denote $\mathrm{Span}(C,D)^{T^{\cdot}}$ by the symbol $\mathcal{E}$-Mod(T) "the category of left T-modules"; an object in this category is an object M in Span $(C,1) \simeq \underline{E}/C$, equipped with a structure map

$$\xi : T \ast M \longrightarrow M.$$

The above arguments give in particular

Theorem 5.9. Let $\underline{E}$ be a topos, and let $C \in |\underline{E}|$; let T, η, μ be a monad in SPAN($\underline{E}$) on the object C. Then 1-Mod(T) is a topos, and the canonical functor

$$\text{1-Mod}(T) \longrightarrow \underline{E}/C$$

is the inverse image part of a topos map $\underline{E}/C \longrightarrow$ 1-Mod(T).

Of course, a similar result is true for "right-modules". Let us remark that, for $\underline{E}$ = sets, a monad T on C in SPAN($\underline{E}$)

$$C \xleftarrow{\;d_0\;} T \xrightarrow{\;d_1\;} C$$

may be interpreted as a small category $\mathbb{C}$ with $|\mathbb{C}| = C$, and with total set of morphisms equal to T (d_0 and d_1 being interpreted as domain and codomain respectively). Left and right modules over T then correspond to functors

$$\mathbb{C}^{op} \longrightarrow \mathcal{S}, \qquad \mathbb{C} \longrightarrow \mathcal{S}$$

respectively. (This way of viewing small categories is due, we believe, to Benabou). The reader should keep this description of functors $\mathbb{C}^{op} \longrightarrow \mathcal{S}$ as left modules over a monad in mind for the rest of the chapter, for heuristic purposes.

Having proved that the category of modules over a monad in SPAN($\underline{E}$) form a topos, we turn to a special class of monads. These can be described as monads in another bicategory associated to $\underline{E}$, the bicategory of relations which is simpler than the bicategory of spans:

Definition 5.10. A relation from A to B is a span $A \xleftarrow{\alpha} X \xrightarrow{\beta} B$ so that $\langle \alpha, \beta \rangle : X \longrightarrow A \times B$ is monic. The full subcategory of $\mathrm{Span}(A,B)$ determined by the relations is denoted $\mathrm{Rel}(A,B)$. It is a preordered class (in fact, isomorphic to $\mathcal{P}(A \times B)$).

We organize the relations into a bicategory, by associating to the pair of relations (5.2) (assuming they are relations) the pair of maps

$$\bar{Z} \xrightarrow{z} A \times B \xrightarrow{proj_0} A, \quad \bar{Z} \xrightarrow{z} A \times B \xrightarrow{proj_1} B$$

where $z: \bar{Z} \longrightarrow A \times B$ is the monic part of a (chosen) epi-mono factorization of

$$\langle x.\alpha, y.\gamma \rangle : \; Z \longrightarrow A \times C$$

(notation as in (5.3)). So, briefly, the composite of relations is the image of their composition as spans. It agrees with the well known composition of relations in the set case. We denote the composite of relations X, Y by the symbol $X \circ Y$. - The unit spans are relations, and are units for the composition $\circ$. We get in fact a bicategory $\mathrm{REL}(\underline{E})$. We write $X \leq X'$ for "there is a map from X to X' in the category (preordered class) $\mathrm{Rel}(A,B)$".

Proposition 5.11. Let $T_0 \in |\underline{E}|$, and let $T \in \mathrm{Rel}(T_0, T_0)$. Then

(i) T carries at most one structure as monad in the bicategory $\mathrm{SPAN}(\underline{E})$

(ii) T carries at most one structure as monad in the bicategory $\mathrm{REL}(\underline{E})$

(iii) T carries a structure as monad in $\mathrm{SPAN}(\underline{E})$ if and only if it does in $\mathrm{REL}(\underline{E})$, which is again the case if and only if the two given maps from T to T_0 make T_0 into a preordered object.

__Proof__. Let the span T be

$$T_0 \xleftarrow{t_0} T \xrightarrow{t_1} T_0.$$

Then (i) and (ii) follow immediately because $\langle t_0,t_1\rangle$ is monic. If T is made into a REL-monad by $\mu'\colon\ T\circ T \longrightarrow T,$ then it is made into a SPAN-monad by

$$T \boxtimes T \xrightarrow{q} T\circ T \xrightarrow{\mu'} T$$

where the epic map q displayed is the one defining $T\circ T$. Conversely, suppose T is a SPAN-monad by means of $\mu\colon\ T\boxtimes T \longrightarrow T$. Denote the two maps making $T\boxtimes T$ into a span by t_0', t_1'. Then q is coequalizer of the kernel pair for $\langle t_0', t_1'\rangle$. But

$$\mu\,.\langle t_0,t_1\rangle = \langle t_0', t_1'\rangle$$

and $\langle t_0,t_1\rangle$ is monic, so μ coequalizes the kernel pair for $\langle t_0', t_1'\rangle$, thus factors across q. Next, we should argue for the unit structure $\eta\colon\ I_A \longrightarrow T$. These arguments are trivial. Finally, for the last assertion of the Proposition, note that a map of spans $\mu\colon\ T\boxtimes T \longrightarrow T$ constructively shows that, for each $D\in|\underline{E}|$

$$\hom(D,T) \underset{\hom(D,t_1)}{\overset{\hom(D,t_0)}{\rightrightarrows}} \hom(D,T_0)$$

defines a transitive relation on the set $\hom(D,T_0)$. Conversely, if the relation $\langle t_0,t_1\rangle$ is a transitive relation, one easily constructs a span map $T\boxtimes T \longrightarrow T$.

Similarly, a map $\eta\colon\ I_{T_0} \longrightarrow T$ expresses the reflexiveness of the relation defined by t_0, t_1.

The proposition illustrates that composition of relations give a convenient way of describing properties of relations. We shall give a few more examples. To a relation X from A to B

$$A \xleftarrow{x_0} X \xrightarrow{x_1} B,$$

we let X^{-1} denote the relation from B to A:

$$B \xleftarrow{x_1} X \xrightarrow{x_0} A.$$

In particular, for $T \in \mathrm{Rel}(T_0, T_0)$ as above

T is symmetric iff $T = T^{-1}$

T is reflexive iff $I \leq T$

T is transitive iff $T \circ T \leq T$.

Definition 5.12. A $T \in \mathrm{Rel}(T_0, T_0)$ is called a directed preorder provided T is reflexive and transitive, and

$$T \circ T^{-1} \leq T^{-1} \circ T. \tag{5.4}$$

In the set case, interpreting $T \subseteq T_0 \times T_0$ as an order relation '$\leq$', (5.4) says: two elements which have a common upper bound have a common lower bound. For sets with a maximal element this is the usual (downward) directedness. (One way of defining the notion "T has a maximal element" is by postulating the existence of a map ν: $1 \longrightarrow T_0$ so that

$$(\mathrm{id}\colon\ T_0 \longrightarrow T_0) \leq (T_0 \longrightarrow 1 \xrightarrow{\nu} T_0)$$

in the ordering induced by t_0, t_1 on $\mathrm{hom}(T_0, T_0)$.)

Remark. If T is a monad, then T^{-1} is a monad as well. One may then view (5.4) as a distributivity of T over T^{-1}. Then it is not surprising that the composite $T^{-1} \circ T$ turns out to be a monad as well, that is, a preordered object. In fact

Proposition 5.13. Let T be a directed preorder that is $T \circ T^{-1} \leq T^{-1} \circ T$. Then $T^{-1} \circ T$ is an equivalence relation. It is minimal among equivalence relations containing T.

Proof. We must prove $T^{-1} \circ T$ reflexive, symmetric and transitive. Reflexive:

$$I_{T_o} = I_{T_o} \circ I_{T_o} \leq T^{-1} \circ T$$

since $I_{T_o} \leq T^{-1}$, $I_{T_o} \leq T$ and $\circ$ is monotone in each of its arguments. Symmetric:

$$(T^{-1} \circ T)^{-1} = T^{-1} \circ (T^{-1})^{-1} = T^{-1} \circ T$$

by the obvious inversion rule for relations: $(X \circ Y)^{-1} = Y^{-1} \circ X^{-1}$. Finally transitivity (ignoring the non-associativity of $\circ$)

$$T^{-1} \circ T \circ T^{-1} \circ T \leq T^{-1} \circ T^{-1} \circ T \circ T \leq T^{-1} \circ T,$$

the first inequality by directedness (applied to the two middle factors), the last inequality by transitivity of T^{-1} and T. This proves that $T^{-1} \circ T$ is an equivalence. An equivalence relation R containing T contains T^{-1} as well by symmetry. Therefore it contains $T^{-1} \circ T$, by transitivity. This proves the Proposition.

We define π_0 of a preordered object $T = (T_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} T_0)$ to be the coequalizer of d_0, d_1.

A functor $\beta: \underline{E} \longrightarrow \underline{E}'$ between toposes which satisfy the conditions (i)-(iii) of Theorem 1.39 (a B-functor, for short) takes preordered objects in $\underline{E}$ to preordered objects in $\underline{E}'$, but in general it does not preserve π_0. However,

Lemma 5.14. Let $\beta: \underline{E} \longrightarrow \underline{E}'$ be a B-functor. Let T be a directed preorder in $\underline{E}$. Then (up to isomorphism)

$$\pi_0(\beta(T)) = \beta(\pi_0(T)).$$

Proof. Since β preserves pull-backs and epi-mono factorizations, it preserves composition of relations:

$$\beta(R \circ S) = \beta(R) \circ \beta(S).$$

Since directedness is expressed in terms of composition of relations, if T is directed then so is $\beta(T)$. Therefore, the equivalence relations generated by T and $\beta(T)$, respectively, are by Proposition 5.13 $T^{-1} \circ T$ and $\beta(T)^{-1} \circ \beta(T)$. Since β preserves composition of relations, and inversion of relations, we see that β takes the equivalence relation generated by T to the equivalence relation generated by $\beta(T)$. The main property (iii) of a B-functor, however, is that it preserves coequalizers of equivalence relations. Since the coequalizer of a relation is the same as the coequalizer of the equivalence relation generated by it, the lemma follows.

For technical reasons, we need the following description of the category of left modules over a preordered object $T = (T_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} T_0)$. (A similar construction works for right modules; the assumption that the monad T is a preorder is only to make things simpler). For such a T we define the category S Fib (T) to have for objects diagrams in $\underline{E}$ of the form

(5.7)
$$\begin{array}{ccc} M_1 & \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} & M_0 \\ \phi_1 \downarrow & & \downarrow \phi_0 \\ T_1 & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & T_0 \end{array}$$

("split, discrete fibrations over T"), such that

(i) δ_0, δ_1 makes M_0 into a preordered object

(ii) $\delta_0.\phi_0 = \phi_1.d_0$ and $\delta_1.\phi_0 = \phi_1.d_1$

(iii) the diagram $\delta_1, \phi_0; \ \phi_1, d_1$ is a pull-back.

The morphisms in the category S Fib (T) are pairs of maps

$$M_0 \longrightarrow M'_0, \quad M_1 \longrightarrow M'_1$$

compatible with ϕ_0 and ϕ_1, respectively, as well as with the δ's.

Proposition 5.15. There is an equivalence of categories

$$\text{S Fib (T)} \simeq \text{1-Mod(T)}.$$

Proof. Let an object in S Fib (T) be given, say the one displayed in (5.7). We make $M = (M_0 \xrightarrow{\phi} T_0)$ into a left T-module, by noting that by (iii) $M_1 = T * M$; for structural map $T * M \longrightarrow M$ we take δ_0. The verification that the associative law for the structural map can be deduced from the transitive law for the preorder relation (δ_0, δ_1) is fairly easy in the set-case, and it suffices to prove the statement for this case, since only left limits are involved (in essence by the Yoneda Lemma). We omit details. (The reader will see the technique illustrated in the proof of Lemma 5.16). Conversely, let $M: M_o \xrightarrow{\emptyset_o} T_o$ be a left T-module by means of $\xi: T \ast M \longrightarrow M$. Denote the underlying object of $T \ast M$ by M_1; M_1 then sits in a pull-back diagram

$$\begin{array}{ccc} M_1 & \longrightarrow & M_o \\ \downarrow & & \downarrow \emptyset_o \\ T_1 & \xrightarrow[d_1]{} & T_o \end{array} .$$

Denote the left-hand arrow $\emptyset_1$ and the top arrow δ_1; let δ_o denote ξ; we then have a diagram like (5.7). The transitive law for δ_o, δ_1 now comes from the associative law for ξ (again the verification can take place in the category of sets, by Yoneda's lemma). The two processes described are clearly mutually inverse.

Lemma 5.16. Suppose T is a directed preorder (Definition 5.12), and suppose M is a left T-module. Then the preorder structure on M constructed in Proposition 5.15 makes M into a directed preorder.

Proof. Let us prove it first for the case where $\underline{E} = \mathcal{S}$, the category of sets. Suppose we have a diagram (5.7) satisfying (i)-(iii). We may identify T_1 with the set of pairs (t,t') with $t \leq t'$ (the ordering being the one given by d_o, d_1). Now M_1 being a pull-back may be identified by the set of triples (t,t',m) with $\emptyset_o(m) = t'$ and $(t,t') \in T_1$, so we may as well identify M_1 with the set of pairs (t,m) with $t \leq \emptyset_o(m)$. Suppose $m,m' \in M_o$ have $m \lesssim \bar{m}$ and $m' \lesssim \bar{m}$, the ordering here being the one given on M_o by δ_o, δ_1. For $t = \emptyset_o(m)$, $t' = \emptyset_o(m')$, $\bar{t} = \emptyset_o(\bar{m})$, this means that

$$\delta_o(t,\bar{m}) = m$$

and

$$\delta_o(t',\bar{m}) = m'.$$

Also, $\emptyset_o$ being order-preserving, we have $t \lesssim \bar{t}$ and $t' \lesssim \bar{t}$; by directedness of T we therefore have a $\underline{t} \in T_o$ with

$$\underline{t} \leq t \quad \text{and} \quad \underline{t} \lesssim t'.$$

we have $(\underline{t},m) \in M_1$, since

$$\underline{t} \leq t = \emptyset_o(m);$$

similarly $(\underline{t},m') \in M_1$.

Both $\delta_o(\underline{t},m)$ and $\delta_o(\underline{t},m')$ are smaller than $\bar{m}$ in M_o since $\delta_\bullet(\underline{t},m) \leq \delta_1(\underline{t},m) = m \lesssim \bar{m}$ (similarly for $\delta_o(\underline{t},m')$. Both $\delta_o(\underline{t},m)$ and $\delta_o(\underline{t},m')$ go by $\emptyset_o$ to $\underline{t}$. But M_1 being a pull-back asserts that there is precisely one ordered pair $n \lesssim \bar{m}$ in M_o with $\emptyset_o(n) = \underline{t}$. Thus

$$\delta_o(\underline{t},m) = \delta_o(\underline{t},m');$$

and $\delta_o(\underline{t},m) \leq m$ and $\delta_o(t,m') \lesssim m'$. This proves directedness of M in the set-case. Then it is also easy to prove the assertion for the case $\underline{E} = \mathcal{S}^{\mathbb{C}}$, since all constructions used here "take place pointwise". Finally for the general case, apply a B-functor $\underline{E} \longrightarrow \mathcal{S}^{\mathbb{C}}$ just as in Lemma 5.14; such a functor exists by Theorem 1.39.

For any preordered object T we have a functor

$$\pi_0^{!}:\ S\,Fib\,(T) \longrightarrow \underline{E}$$

which to a "fibration" as in (5.7) associates π_0 of the preordered object $M_1 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} M_0$.

<u>Theorem 5.17</u>. Suppose T is a directed preorder in $\underline{E}$ (Definition 5.12), and suppose that it has a maximal element. Then $\pi_0^{!}:\ S\,Fib\,(T) \longrightarrow \underline{E}$ is left exact.

<u>Proof</u>. The theorem is true for $\underline{E} = \mathcal{S}$, as is well known, [9] and can easily be checked, therefore also for a category of form $\mathcal{S}^{\mathbb{C}}$. Take a B-functor $\beta:\ \underline{E} \longrightarrow \mathcal{S}^{\mathbb{C}}$ (possible by Theorem 1.39). Then T goes to a preordered object βT, and we get in fact (just because β is left exact) a functor

$$S\,Fib\,(T) \xrightarrow{\ \beta'\ } S\,Fib\,(\beta T),$$

which is left exact. By Lemma 5.16 and Lemma 5.14 the diagram

$$\begin{array}{ccc} S\,Fib\,(T) & \xrightarrow{\ \pi_0^{!}\ } & \underline{E} \\ {\scriptstyle \beta'}\downarrow & & \downarrow{\scriptstyle \beta} \\ S\,Fib\,(\beta T) & \xrightarrow[\ \pi_0^{!}\ ]{} & \mathcal{S}^{\mathbb{C}} \end{array}$$

commutes up to isomorphism, and the lower $\pi_0^{!}$ is left exact. But β reflects left exactness being a B-functor, whence the upper $\pi_0^{!}$ is left exact. This proves the theorem.

<u>Theorem 5.18</u>. Suppose T is a directed preorder in $\underline{E}$, and suppose it has a maximal element. Then there is a topos morphism

$$p\colon\ \underline{E} \longrightarrow \text{1-Mod}(T) \simeq \text{S Fib}(T)$$

with $p^* = \pi_0^!$.

<u>Proof</u>. By Theorem 5.17 we just have to produce a right adjoint p_* for $\pi_0^!$. For p_* take

$$p_*(A) = T_0 \times A$$

with the order-relation given on $\hom(D, T_0 \times A)$

$$D \xrightarrow{\langle t, a\rangle} T_0 \times A \ \leq\ D \xrightarrow{\langle t', a'\rangle} T_0 \times A$$

iff $t_0 \leq t_1$ in $\hom(D, T_0)$ and $a = a'$. The order preserving map $T_0 \times A \longrightarrow T_0$ is just the projection. This actually produces a functor $\underline{E} \longrightarrow \text{S Fib}(T)$. The reader may check that it has the required properties.

Suppose that $\tau\colon\ T' \longrightarrow T$ is a map of preordered objects. Then we can produce a functor $\tau^*\colon\ \text{S Fib}(T) \longrightarrow \text{S Fib}(T')$ by a straightforward pull-back procedure; given an object M in $\text{S Fib}(T)$:

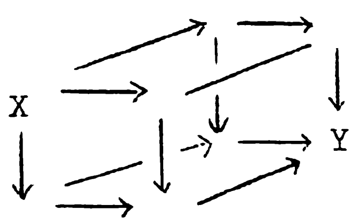
$$\begin{array}{ccc} M_1 & \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} & M_0 \\ \phi_1 \downarrow & & \downarrow \phi_0 \\ T_1 & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & T_0 \end{array}$$

we produce an object M' in $\text{S Fib}(T')$ by letting M_1' and M_0' be given by the pull-back diagrams, respectively:

$$\begin{array}{ccc} M_i' & \longrightarrow & M_i \\ {\scriptstyle \phi_i'}\downarrow & & \downarrow{\scriptstyle \phi_i} \\ T_i' & \underset{\tau_i}{\longrightarrow} & T_i \end{array} \qquad (i=1,0).$$

Functorality of pulling back gives rise to maps $\partial_0', \partial_1' : M_1' \to M_0'$, which the reader may check makes M' into an object in $S\,Fib\,(T')$; pull-back condition (iii) for M' is verified because of the general

Box-Lemma 5.19. Suppose there is given a commutative diagram



in which the two vertical square with Y as vertex are pull-backs and where one of the vertical squares with X as vertex is a pull-back. Then also the remaining vertical square is a pull-back.

Proof. This is true in any category, and the proof is straightforward diagram chasing.

We thus get a functor

$$(5.9) \qquad 1\text{-Mod}(T) \cong S\,Fib\,(T) \xrightarrow{\tau^*} S\,Fib\,(T') \cong 1\text{-Mod}(T').$$

In the set case, where 1-Mod(T) may be identified with $\mathcal{S}^{\mathbb{T}^{op}}$ (where $\mathbb{T}$ is T viewed as a preordered set), τ^* becomes identified with

$$\mathcal{S}^{\tau} : \mathcal{S}^{\mathbb{T}^{op}} \longrightarrow \mathcal{S}^{\mathbb{T}'^{op}}.$$

This functor is known to have a left as well as a right adjoint (the so called Kan-extensions along τ). In particular (5.9) will in this case be the "inverse image" part of a topos-morphism

$$\tau : \text{1-Mod}(T') \longrightarrow \text{1-Mod}(T).$$

We shall in the Appendix sketch that τ^* in the case of an arbitrary topos $\underline{E}$ also has adjoints on both sides. The reader may, however, by diagram chase, verify directly that τ^* is at least left exact (this is not surprising, since it is defined by a pull-back procedure).

Consider for a moment the category $\text{Ord}(\underline{E})$ of preordered objects in $\underline{E}$; it comes with a forgetful functor U to $\underline{E}$ (which sends $T_1 \rightrightarrows T_0$ to T_0) and it is very easy to equip $\text{Ord}(\underline{E})$ with finite inverse limits in such a way that U preserves them (e.g. letting the product of the underlying objects carry the "product ordering"). Also, if T is a preordered object $T_1 \rightrightarrows T_0$, and $\alpha : S \rightarrowtail T_0$, there is a maximal order-relation on S making the monic map α order-preserving. We shall talk about the ordering induced by T on S via the inclusion α. The functor U has a left adjoint D,

$$D(X) = X \underset{id}{\overset{id}{\rightrightarrows}} X$$

"putting the discrete order-relation on X". The functor D is full and faithful, and we omit it from notation.

For any object $X \in |\underline{E}|$, the object $X \pitchfork \Omega$ carries a canonical ordering induced by the ordering $(\leq)$ on Ω (in fact, the functor $X \pitchfork - : \underline{E} \longrightarrow \underline{E}$ carries preordered objects to preordered objects, being left exact).

We shall consider the following kind of structures: let $X \in |\underline{E}|$, and let $\alpha : \mathcal{O} \rightarrowtail X \pitchfork \Omega$ be a subobject of $X \pitchfork \Omega$, that is, in the set case, "a family of subsets of X". We shall give conditions which in the set case specialize to: "$\mathcal{O}$ is a basis for a (point-set) topology on the set X".

To this end, consider the diagram, where the square is a pull-back:

$$\begin{array}{ccc} \mathcal{G} & \rightarrowtail & \epsilon_X \\ \downarrow & & \downarrow \\ \mathcal{O} \times X & \xrightarrow{\alpha \times 1} & (X \pitchfork \Omega) \times X \\ \downarrow \text{proj} & & \\ X & & \end{array} \tag{5.10}$$

Here ϵ_X is as in Chapter 1 the subobject classified by the evaluation map ev: $(X \pitchfork \Omega) \times X \longrightarrow \Omega$. Let $X \pitchfork \Omega$ carry the canonical ordering, let X carry the discrete ordering, and let $(X \pitchfork \Omega) \times X$ carry the product ordering. Now $\mathcal{G}$ is a subobject of $(X \pitchfork \Omega) \times X$, so we can endow it with the ordering induced from that on $(X \pitchfork \Omega) \times X$.

Definition 5.20. A pair $X, \mathcal{O}$ (where $\mathcal{O}$ is a subobject of $X \pitchfork \Omega$) will be called a space-basis provided

(i) the maximal element of $\hom(1, X \pitchfork \Omega)$ (which exists uniquely) factors through $\mathcal{O} \rightarrowtail X \pitchfork \Omega$

(ii) the ordered object $\mathcal{G}$ constructed out of $X, \mathcal{O}$ in (5.10) is a directed preorder.

Note that in the set-case $\mathcal{C}$ consists of pairs (U,x) where $x \in U \in \mathcal{O}$; the ordering on $\mathcal{C}$ is given by

$$(U,x) \leq (U',x')$$

iff

$$U \subseteq U' \quad \text{and} \quad x = x'.$$

Two pairs (U,x) and (U',x') have a common upper bound iff $x = x'$ (take (X,x)). So the directedness condition in this case says: "$(x \in U$ and $x \in U')$ implies $(\exists U''(U'' \subseteq U$ and $U'' \subseteq U'$ and $x \in U''))$".

In the set case, the conditions in Definition 5.20 thus say: " $\mathcal{O}$ is a basis for a topology on X, and $X \in \mathcal{O}$".

Consider the left-hand vertical column in (5.10). By construction, it is a map between ordered objects. The ordering on X being discrete, implies that the order relation on $\mathcal{C}$ may be viewed as an order relation on $\mathcal{C} \longrightarrow X$ in the topos $\underline{E}/X$. Denote the ordered object thus obtained by $\mathcal{C}'$.

Lemma 5.21. Suppose $X, \mathcal{O}$ is a space-basis. Then the $\mathcal{C}'$ obtained by the above procedure is an ordered object in $\underline{E}/X$ and satisfies

(i) it has a maximal element

(ii) it is a directed preorder.

Also, the category $S\,Fib(\mathcal{C})$ is isomorphic to the category $S\,Fib(\mathcal{C}')$.

Proof. The first statement is obvious. The maximal element claimed in (i) is a map $X \longrightarrow \mathcal{G}$ which we get (using that the square in (5.10) is a pull-back), from maps

$$X \longrightarrow 1 \xrightarrow{\text{maximal}} \mathcal{O} \subseteq X \pitchfork \Omega$$

and

$$X \xrightarrow{1} X.$$

The remaining statements follow from the fact that the obvious functor $\underline{E}/X \longrightarrow \underline{E}$ preserves and reflects pull-backs and epi-mono factorizations, and in particular composition of relations.

We are now in a position to describe a functor 1-Mod($\mathcal{O}$) $\longrightarrow \underline{E}/X$ which for $\underline{E}$ = sets, and $\mathcal{O}$= (set of all open subsets of a topological space X), specializes to the construction (Godement [7], p. 110); it is the construction which to a pre-sheaf M over X associates the (underlying set of) the espace etalé of the sheaf associated to M.

Consider the composite order-preserving map

$$\mathcal{G} \rightarrowtail \mathcal{O} \times X \xrightarrow{\text{proj}} \mathcal{O}.$$

As in (5.9), we get a left exact functor

(5.11) $\quad$ 1-Mod($\mathcal{O}$) $\longrightarrow$ 1-Mod($\mathcal{G}$) $\simeq$ S Fib ($\mathcal{G}$) $\simeq$ S Fib ($\mathcal{G}'$),

using Lemma 5.21. By the same lemma, $\mathcal{G}'$ is a directed preorder with maximal element in $\underline{E}/X$, whence, by Theorem 5.17, we get a left exact

(5.12) $\quad$ $\pi_0^!$: S Fib ($\mathcal{G}'$) $\longrightarrow \underline{E}/X$.

Composing (5.11) with (5.12), we get the desired left exact functor $1\text{-Mod}(\mathcal{O}) \longrightarrow \underline{E}/X$.

By Theorem 5.18, (5.12) has a right adjoint, and by Appendix the map (5.11) (which is of the form "τ^*" (5.9)) has a right adjoint, whence

__Theorem 5.22.__ Let $X, \mathcal{O}$ be a space basis (so $\mathcal{O} \rightarrowtail X \pitchfork \Omega$. Then there is a morphism of toposes

$$g: \ \underline{E}/X \longrightarrow 1\text{-Mod}(\mathcal{O})$$

(whose inverse image part g^* is the composite of (5.11) and (5.12)).

__Definition 5.23.__ Let $X, \mathcal{O}$ be a space basis, and let g be the topos morphism described in Theorem 5.22. Consider the canonical factorization of g; we call the middle category for this factorization the <u>category of sheaves on X, </u>, denote $sh(X,\mathcal{O})$; thus we have a diagram of topos morphisms

$$\begin{array}{ccc} \underline{E}/X & \xrightarrow{\quad g \quad} & 1\text{-Mod}(\mathcal{O}). \\ & \searrow \qquad \nearrow_{i} & \\ & sh(X,\mathcal{O}) & \end{array}$$

We shall now see that the construction of Definition 5.23 specializes to well-known ones in the case, where $\underline{E}$ = (sets), and $\mathcal{O}$ is the set of all open subsets of a topological space X. First, $1\text{-Mod}(\mathcal{O})$ can, by the remarks after (5.9), be identified with the category $\mathcal{S}^{\mathcal{O}^{op}}$ of contravariant set-valued functors from $\mathcal{O}$ to $\mathcal{S}$, that is, the usual category of presheaves on X. The functor in (5.11) associates to such a presheaf $M: \ \mathcal{O}^{op} \longrightarrow \mathcal{S}$ the functor

$$N: \mathcal{C} \longrightarrow \mathcal{S}$$

given on objects by

$$N((U,x)) = M(U).$$

The discrete split fibration P over $\mathcal{C}$ associated to N has as its P_0

$$\coprod_{(U,x) \in |\mathcal{C}|} M(U)$$

which we may view as an ordered object P' in $\mathcal{S}/X$.

Forming $\pi_0^!$ of this object in $\underline{E}/X$ means just that for each $x \in X$ we should identify

$$m \in M(U) = N(U,x) \quad \text{to} \quad m' \in M(U') = N(U',x)$$

if there is an order relation

$$(U,x) \geq (U',x)$$

so that m "restricts to" m' under this - which just means that m "restricts to" m' in the original presheaf M. But this is precisely to say that m and m' define the same germ at the point x. Thus g* associates to the presheaf M the set of germs of elements in the M(U)'s (U∈).

We shall invoke results from the classical foundation of sheaf theory to prove that sh(X,$\mathcal{O}$) is the same category as the classical category of sheaves on X, here denoted SH(X,$\mathcal{O}$). Constructions of classical sheaf theory together with the set theoretic description of g* given above gives us the diagram

(5.12)
$$\begin{array}{ccc} \mathcal{S}/X & \xleftarrow{\quad g^* \quad} & \text{1-Mod}(\mathcal{O}) = (\text{Presheaves on } X) \\ \nwarrow{\scriptstyle h^*} & & {\scriptstyle r}\swarrow\!\!\nearrow{\scriptstyle i} \\ \text{Etale}(X) & \underset{\simeq}{\xleftarrow{\quad k \quad}} & SH(X,\mathcal{O}) \end{array}$$

in which 'Etale(X)' denotes the full subcategory of the category of topological spaces over X consisting of local homeomorphisms, and where the functor h^* just forgets the topology, and where finally $r \dashv i$ is the classical description of the "associated sheaf" functor. The diagram involving r commutes up to isomorphism, in fact, one classically constructs r by the "germ functor" $\bar{g}$: 1-Mod$(\mathcal{O}) \longrightarrow$ Etale(X) followed by k^{-1}; with this notation g^* is just $\bar{g}$ followed by h^*.

Now, classically, r is left exact, whence, by Proposition 3.22, $SH(X,\mathcal{O})$ is an elementary topos, and i is a full and faithful topos map. Since g^* has a right adjoint g_*, and i is full and faithful, one easily gets a right adjoint for $h^* k$, and since k is an equivalence, we also get a right adjoint for h^*; denote it h_*. Finally, one can see that "the forgetful functor" h^* reflects isomorphisms and is left exact, so the diagram (5.12) of functors is part of a diagram of topos maps

$$\begin{array}{ccc} \mathcal{S}/X & \xrightarrow{\quad g \quad} & \text{1-Mod}(\mathcal{O}) \\ \searrow{\scriptstyle h} & & \nearrow{\scriptstyle i} \\ \text{Etale}(X) & \underset{\simeq}{\longrightarrow} & SH(X,\mathcal{O}) \end{array}$$

with i full and faithful and h^* reflecting isomorphisms. From the uniqueness of such factorization (Corollary 4.8) and Definition 5.23 we conclude: $sh(X,\mathcal{O}) \simeq SH(X,\mathcal{O})$.

Remark 5.24. To complete the description of the "topological space object in a topos $\underline{E}$", we have to state (in finite terms) a property on a space basis $(X,\mathcal{O})$, namely a property which in the case $\underline{E} = \mathcal{S}$ specializes to the property: $\mathcal{O}$ is closed under arbitrary unions (which is the property distinguishing the notion: topological space from the notion: space-basis. This property can be expressed in terms of the "internal union formation for an object X" which is a map

$$\mu : (X \pitchfork \Omega) \pitchfork \Omega \longrightarrow X \pitchfork \Omega$$

constructed by specifying that $\hat{\mu} : [(X \pitchfork \Omega) \pitchfork \Omega] \times X \longrightarrow \Omega$ should be the characteristic map of that subobject of $[(X \pitchfork \Omega) \pitchfork \Omega] \times X$ which is the image along the map

$$\langle proj_0, proj_2 \rangle : [(X \pitchfork \Omega) \pitchfork \Omega] \times [X \pitchfork \Omega] \times X \longrightarrow [(X \pitchfork \Omega) \pitchfork \Omega] \times X$$

of the subobject

$$(\epsilon_{X \pitchfork \Omega} \times X) \cap ((X \pitchfork \Omega) \pitchfork \Omega \times \epsilon_X).$$

In fact, one can prove [11] that μ is the multiplication part of a strong and commutative triple structure on a certain covariant functor which on objects is just $- \pitchfork \Omega$.

APPENDIX

The biclosed bicategory of profunctors over a topos $\underline{E}$.

Let $\mathbb{A}$ and $\mathbb{B}$ be monads in the bicategory SPAN($\underline{E}$), with underlying spans

$$A = (A_0 \xleftarrow{\partial_0} A_1 \xrightarrow{\partial_1} A_0)$$

$$B = (B_0 \xleftarrow{\partial_0} B_1 \xrightarrow{\partial_1} B_0),$$

respectively. A profunctor $\mathbb{M}$ from $\mathbb{A}$ to $\mathbb{B}$ is an $\mathbb{A}$-$\mathbb{B}$-"bimodule", that is, a span

$$M = (A_0 \longleftarrow M \longrightarrow B_0)$$

which, as object over A_0 has a left $\mathbb{A}$-module structure, and as object over B_0 has a right $\mathbb{B}$-module structure; these structures should commute with each other in an obvious sense. We write $\mathrm{Prof}(\mathbb{A},\mathbb{B})$ for the class of such profuctors $\mathbb{M}$; it is actually a category with $\mathbb{A}$-$\mathbb{B}$ bimodule homomorphisms as maps. We define a composition of profunctors, denoted $\otimes$:

$$\mathrm{Prof}(\mathbb{A},\mathbb{B}) \times \mathrm{Prof}(\mathbb{B},\mathbb{C}) \longrightarrow \mathrm{Prof}(\mathbb{A},\mathbb{C}),$$

namely we let $\mathbb{M} \otimes \mathbb{N}$ have as underlying span the coequalizer in $\mathrm{Span}(A_0, C_0)$:

$$M \circledast B \circledast N \underset{g}{\overset{f}{\rightrightarrows}} M \circledast N \longrightarrow \mathbb{M} \otimes \mathbb{N}$$

where the two maps f and g indicate right action of B on M, and left action of B on N, respectively. Now both f and g are left A-module homomorphisms, and (essentially because $A \circledast -$ preserves coequalizers, having a right adjoint), $M \otimes N$ inherits a left $\mathbb{A}$-module structure from $M \circledast N$. Similarly, $M \otimes N$ gets a right $\mathbb{C}$-module structure .

In this way, the class of all profunctors is organized as a bicategory PROF($\underline{E}$) (whose objects are the monads, or the category objects, of $\underline{E}$). We claim that PROF($\underline{E}$) is a biclosed bicategory. This amounts to giving functors

$$(\mathrm{Prof}(\mathbb{A},\mathbb{B}))^{op} \times \mathrm{Prof}(\mathbb{A},\mathbb{C}) \xrightarrow{-\backslash\backslash-} \mathrm{Prof}(\mathbb{B},\mathbb{C})$$

and

$$\mathrm{Prof}(\mathbb{C},\mathbb{B}) \times (\mathrm{Prof}(\mathbb{A},\mathbb{B}))^{op} \xrightarrow{-//-} \mathrm{Prof}(\mathbb{C},\mathbb{A})$$

where for instance

$$\mathbb{M}\backslash\backslash- : \ \mathrm{Prof}(\mathbb{A},\mathbb{C}) \longrightarrow \mathrm{Prof}(\mathbb{B},\mathbb{C})$$

should be a right adjoint to $\mathbb{M} \otimes -$). We shall use a similar notation for the biclosed bicategory structure on SPAN($\underline{E}$)

$$(\mathrm{Span}(A,B))^{op} \times \mathrm{Span}(A,C) \xrightarrow{-\backslash-} \mathrm{Span}(B,C)$$
$$\mathrm{Span}(C,B) \times (\mathrm{Span}(A,B))^{op} \xrightarrow{-/-} \mathrm{Span}(C,A).$$

Now suppose, for instance, that

$$\mathbb{K} \in \mathrm{Prof}\,(\mathbb{A},\mathbb{B}) \,, \quad \mathbb{L} \in \mathrm{Prof}\,(\mathbb{C},\mathbb{B}).$$

Then $\mathbb{L} // \mathbb{K} \in \mathrm{Prof}\,(\mathbb{C},\mathbb{A})$ is defined as the equalizer

$$\begin{array}{ccccc} \mathbb{L}//\mathbb{K} & \rightarrowtail & L/K & \xrightarrow{1/\varkappa} & L/(K \circledast B) \\ & & {\scriptstyle y}\searrow & * & \nearrow{\scriptstyle \lambda/1} \\ & & & (L \circledast B)/(K \circledast B) & \end{array}$$

where λ and $\varkappa$ are the actions of B on L and K , respectively, and where y is gotten by the adjointness

$$- \circledast (K \circledast B) \dashv -/(K \circledast B)$$

from the map

$$(L/K) \circledast (K \circledast B) \xrightarrow{\cong} ((L/K) \circledast K) \circledast B \xrightarrow{ev \circledast 1} L \circledast B$$

(ev being the end-adjunction for the adjointness $- \circledast K \dashv -/K$). We endow the objects in the triangle * with left $\mathbb{C}$-, right $\mathbb{A}$-module structures in such a way that the maps forming the triangle become module homomorphisms. (For instance, L/K is given the right A-module structure

$$(L/K) \circledast A \longrightarrow L/K$$

which we get by $- \circledast K \dashv -/K$ adjointness from the map

$$(L/K) \circledast A \circledast K \xrightarrow[1 \circledast \mu']{} (L/K) \circledast K \xrightarrow[ev]{} L,$$

μ' denoting the A-action on K.) In this way, $\mathbb{L}/\mathbb{K}$ inherits a bimodule structure.

All this provides a sketch of how the biclosed structure on PROF($\underline{E}$) is constructed.

Now let $f\colon \mathbb{A} \longrightarrow \mathbb{B}$ be a functor between two category objects (= monads in SPAN($\underline{E}$)) in $\underline{E}$. We have clearly

$$\mathrm{Prof}(\mathbb{A},1) \simeq 1\text{-}\mathrm{Mod}(\mathbb{A}) \simeq \mathrm{S\text{-}Fib}(\mathbb{A})$$

(and similarly for $\mathbb{B}$), where 1 is the trivial category object $1 \rightrightarrows 1$. We shall construct a profunctor $\underline{f} \in \mathrm{Prof}(\mathbb{A},\mathbb{B})$ so that

$$\begin{array}{ccc} \text{S-Fib}(\mathbb{B}) & \xrightarrow{f^*} & \text{S-Fib}(\mathbb{A}) \\ \cong\downarrow & & \downarrow\cong \\ \text{Prof}(\mathbb{B},1) & \xrightarrow[\underline{f}\otimes -]{} & \text{Prof}(\mathbb{A},1) \end{array}$$

commutes (f^* being the functor defined as in (5.9). Since by biclosedness of PROF($\underline{E}$), $\underline{f}\otimes-$ has a right adjoint, then so has f^*. It remains to produce $\underline{f}$. If we display $f, \mathbb{A}$, and $\mathbb{B}$ as follows

$$\begin{array}{ccc} A_1 & \overset{\partial_1}{\underset{\partial_0}{\rightrightarrows}} & A_0 \\ f_1\downarrow & & \downarrow f_0 \\ B_1 & \overset{\partial_1}{\underset{\partial_0}{\rightrightarrows}} & B_0 \end{array}$$

then the desired bimodule $\underline{f}$ sits in the pull-back diagram

$$\text{(A.1)} \qquad \begin{array}{ccc} \underline{f} & \xrightarrow{h} & A_0 \\ k\downarrow & & \downarrow f_0 \\ B_1 & \xrightarrow[\partial_0]{} & B_0, \end{array}$$

and $\underline{f}$ becomes an object in $\text{Span}(A_0, B_0)$ by means of the two maps $h\colon \underline{f} \longrightarrow A_0$ and $k.\partial_1\colon \underline{f} \longrightarrow B_0$. The right $\mathbb{B}$-module structure on $\underline{f}$ is given as follows: We should produce

$$\underline{f} \underset{B_0}{\times} B_1 \longrightarrow \underline{f};$$

$\underline{f}$ being a pull-back, it suffices to produce maps from $\underline{f}$ into

B_1 and A_o. The map into A_o is obvious; the map into B_1 is

$$\underline{f} \times_{B_o} B_1 \xrightarrow{k \times 1} B_1 \times_{B_o} B_1 \xrightarrow{\mu} B_1,$$

μ being the multiplication of the monad $\mathbb{B}$. The left A-module structure on $\underline{f}$ is constructed in a similar way. - In the set case, $\underline{f}$ has as fibre over objects $a \in A_o$, $b \in B_o$ the set $\mathrm{Hom}_{\mathbb{B}}(f(a),b)$. The reader may check commutativity of (A.1) in this case.

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