NOTE ON THE HOMOTOPY PROPERTIES OF THE COMPONENTS OF THE MAPPING SPACE X^{S^p}

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1. **Introduction.** Let X be a topological space and S^p be the polarized p-sphere with a fixed pole y_0. Following G. W. Whitehead [**10**], we shall denote by $G^p(X)$ the mapping space X^{S^p}, which is the totality of (continuous) maps of S^p into X endowed with compact-open topology. Let $\pi: G^p(X)\to X$ be defined by $\pi(f)=f(y_0)$, $(f\in G^p(X))$, and let $F^p(X, x)=\pi^{-1}(x)$ for each $x\in X$. Consider now the mapping space $B(X)$ consisting of all the maps of y_0 into X. There is a natural map $p: G^p(X)\to B(X)$ defined by $p(f)=f|y_0$ for every $f\in G^p(X)$. It is well known (cf. [**3**, pp. 83–84]) that p has the path lifting property. Clearly, the space X can be identified with $B(X)$ in a natural way. The map π is then identified with p. Consequently $\pi: G^p(X)\to X$ is a fibre map of $G^p(X)$ onto X having the absolute covering homotopy property [**3**, p. 82]. For each $x\in X$, the fibre in $G^p(X)$ over x is $F^p(X, x)$. The arc components of $F^p(X, x)$ are elements of the pth homotopy group $\pi_p(X, x)$ of X at x. Denote by $G^p_\alpha(X)$ the arc component of $G^p(X)$ which contains $\alpha=F^p_\alpha(X, x)\in\pi_p(X)$ (cf. [**10**]). If X is arcwise connected, then $G^p_\alpha(X)$ is also a fibre space over X. The restriction $\pi_\alpha=\pi|G^p_\alpha(X)$ is a fibre map of $G^p_\alpha(X)$ onto X. The homotopy properties of the various components $G^p_\alpha(X)$ of $G^p(X)$ have been studied by M. Abe (Jap. J. Math. vol. 16 (1940) pp. 169–176), G. W. Whitehead [**10**] and S. T. Hu [**2**]. The present note may be regarded as a continuation of these studies.

2. **H-space and H_*-space.** In what follows, we shall denote $G^p(X)$ by G^p and $F^p(X, x)$ by F^p whenever no confusion is likely to arise.

Let X be a topological space which admits a continuous multiplication $\mu(x, x')=x\cdot x'$. If $f: S\to X$ is a map of a space S into X, we denote by $x\cdot f$ the transformation defined by $(x\cdot f)(s)=x\cdot f(s)$ for each $s\in S$. Clearly $x\cdot f$ is a map (i.e. it is continuous).

By an H-space we mean a topological space X with a given continuous multiplication which has a homotopy unit $e\in X$ (see e.g. [**3**, pp. 80–81]).

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(2.1) THEOREM. *If X is an arcwise connected H-space, then $G^p_\alpha(X)$ and $G^p_\beta(X)$ have the same homotopy type for arbitrary α and β in $\pi_p(X)$, $p \geqq 1$.*

PROOF. It suffices to prove that $G^p_\alpha(X)$ and $G^p_0(X)$ have the same homotopy type, for any $\alpha \in \pi_p(X)$. According to [10], it remains to prove that $G^p_\alpha(X)$ admits a (global) cross-section. Choose an element $f \in G^p_\alpha \cap F^p(X, e)$. Then $\pi_\alpha(f) = e$. Define $\phi: X \to G^p$ by $\phi(x) = x \cdot f$. Then $\phi(e) = e \cdot f \simeq f \in G^p_\alpha$. Since X is arcwise connected, we have $\phi: X \to G^p_\alpha$. Now, $\pi_\alpha(\phi(x)) = \pi_\alpha(x \cdot f) = x \cdot e$, therefore $\pi_\alpha \phi \simeq \mathrm{id}_X$. Since π_α has the absolute covering homotopy property, there exists a covering homotopy, in particular, there is a map $\psi: X \to G^p_\alpha$ such that $\pi_\alpha \psi = \mathrm{id}_X$. This proves (2.1).

Following H. Wada [9], we call a topological space X an H_*-space if the following conditions are satisfied:

(i) A continuous multiplication $\mu(x, x') = x \cdot x'$ is defined for each pair of elements x, x' in X.

(ii) There is a fixed element e in X, satisfying

$$x \cdot e = x, \qquad \text{for all } x \in X.$$

(iii) To each $x \in X$, there is an inverse $x^{-1} \in X$, defined continuously by x, such that

$$x \cdot x^{-1} = e, \qquad \text{for all } x \in X.$$

(iv) For each pair of elements x, x' in X, we have

$$x^{-1} \cdot (x \cdot x') = x'.$$

With these conditions Wada was able to prove that

(ii') e is unique,

(iii') x^{-1} is uniquely defined by x and $x^{-1} \cdot x = e$,

(v) $(x^{-1})^{-1} = x$ and, consequently, $x \cdot (x^{-1} \cdot x') = x'$, for arbitrary x and x' in X.

We remark that an H_*-space need not to be an H-space.

The following theorem resembles a construction of Wada [9], where he deals with mapping space of an H_*-space into itself.

(2.2) THEOREM. *Let X be an H_*-space. The mapping space $G^p(X)$ is homeomorphic to $X \times F^p(X, e)$ for each $p \geqq 1$.*

PROOF. Let $g \in G^p(X)$ be an arbitrary map of S^p into X. Then $x \cdot g$ is defined and continuous. Hence $x \cdot g \in G^p(X)$. Clearly $g = e \cdot g = x^{-1} \cdot (x \cdot g) = x \cdot (x^{-1} \cdot g)$ for any $x \in X$. Let

$$\phi: G^p(X) \to X \times F^p(X, e),$$

and

$$\psi: X \times F^p(X, e) \to G^p(X),$$

be defined as follows: Let y_0 be the pole of S^p. For each $g \in G^p(X)$, let $\hat{g} = g(y_0) \in X$. Then define

$$\phi(g) = (\hat{g}, \hat{g}^{-1} \cdot g), \qquad (g \in G^p(X))$$

and

$$\psi(x, f) = x \cdot f, \qquad (x \in X, f \in F^p(X, e)).$$

(A) ϕ and ψ are bijective:
For any $g \in G^p(X)$, we have

$$\psi\phi(g) = \psi(\hat{g}, \hat{g}^{-1} \cdot g) = \hat{g} \cdot (\hat{g}^{-1} \cdot g) = g.$$

On the other hand,

$$\begin{aligned} \phi\psi(x, f) = \phi(x \cdot f) &= ((x \cdot f)\hat{}, ((x \cdot f)\hat{})^{-1} \cdot (x \cdot f)) \\ &= (x \cdot \hat{f}, (x \cdot \hat{f})^{-1} \cdot (x \cdot f)) \\ &= (x \cdot e, (x \cdot e)^{-1} \cdot (x \cdot f)) \\ &= (x, x^{-1} \cdot (x, f)) \\ &= (x, f). \end{aligned}$$

Hence both ϕ and ψ are one-to-one, onto.

(B) ϕ and ψ are continuous:

Suppose K be a compact set in S^p and U an open set in X. We shall denote by (K, U) be the subset of $G^p(X)$ consisting of all mappings which send K into U. Let H be an arbitrary neighborhood of $(\hat{g}, \hat{g}^{-1} \cdot g)$. Then $H \supset U_0 \times [(K_1, U_1) \cap \cdots \cap (K_n, U_n)]$ for some open sets $U_0, U_1, \cdots, U_n$ in X and compact sets $K_1, \cdots, K_n$ in S^p. Denote $g(K_i)$ by K_i', then K_i' is compact, $i = 1, 2, \cdots, n$. Corresponding to each $k_i^\alpha \in K_i'$, there exist open sets W_i^α containing $\hat{g}^{-1}$ and V_i^α containing k_i^α such that $W_i^\alpha \cdot V_i^\alpha \subset U_i$, since the multiplication in X is continuous. The collection $\{V_i^\alpha\}$ forms an open covering of K_i'. There is a finite subcovering $\{V_i^{\alpha_1}, \cdots, V_i^{\alpha_{m_i}}\}$ of K_i'. Let $W_i = \bigcap_{j=1}^{m_i} W_i^{\alpha_j}$ and $V_i = \bigcup_{i=1}^{m_i} V^{\alpha_i}$. Then W_i is an open neighborhood of $\hat{g}^{-1}$; V_i is an open neighborhood of K_i' and $W_i \cdot V_i \subset U_i$.

Let $N = (y_0, U_0 \cap W_1^{-1} \cap \cdots \cap W_n^{-1}) \cap (K_1, V_1) \cap \cdots \cap (K_n, V_n)$, where W_i^{-1} denotes, of course, the set $\{w^{-1} | w \in W_i\}$. By the continuity of the inverse, N is a neighborhood of g in G^p. It is now readily seen that $\phi(N) \subset H$. This proves the continuity of ϕ.

Next, let $U = (K_1, U_1) \cap (K_2, U_2) \cap \cdots \cap (K_n, U_n)$ be a basic open neighborhood of $\psi(x, f) = x\cdot f$. Then $x\cdot f(K_i) \subset U_i$. By a similar argument as above, one proves that there exist open neighborhoods W_i of x and V_i of $f(K_i)$ such that $W_i \cdot V_i \subset U_i$. Then

$$\psi[(W_1 \cap \cdots \cap W_n) \times ((K_1, V_1) \cap \cdots \cap (K_n, V_n) \cap F^p)] \subset U.$$

Hence ψ is continuous and the proof of (2.2) is completed.

(2.3) COROLLARY. *If X is an arcwise connected H_*-space, then G^p_α and $X \times F^p_\alpha$ are homeomorphic.*

PROOF. Since X is arcwise connected, G^p_α is a fibre space over X. By replacing G^p and G^p_α and π by π_α in the proof of (2.2), we obtain that G^p_α is homeomorphic to $X \times \pi_\alpha^{-1}(e)$. Being a component, G^p_α is connected hence $\pi_\alpha^{-1}(e)$ contains only one component F^p_α. This proves (2.3).

As a by-product of the proof of (2.3) we have:

(2.4) COROLLARY. *Every arcwise connected H_*-space is n-simple, for $n \geqq 1$.*

(2.5) COROLLARY. *If X is an arcwise connected H_*-space, then G^p_α and G^p_β have the same homotopy type for arbitrary α and β in $\pi_p(X)$. Furthermore*

$$\pi_q(G^p_\alpha) \approx \pi_{p+q}(X) + \pi_q(X), \qquad (q \geqq 1).$$

PROOF. Since G. W. Whitehead [10] proved that F^p_α and F^p_β have the same homotopy type for any α and β in $\pi_p(X)$, the first part of (2.5) follows from (2.3). The Hurewicz isomorphism $\pi_q(F^p_\alpha) \approx \pi_{p+q}(X)$ (cf. [10]) completes the proof.

(2.6) COROLLARY. *Let $X = S^r$. Then G^p_α is homeomorphic to $S^r \times F^p_\alpha$ when $r = 1$, 3 or 7. Conversely, if $G^p_{i_r}$ and $S^r \times F^p_0$ have the same homotopy type then $r = 1$, 3 or 7, where $i_r \in \pi_r(S^r)$ is represented by the identity map $S^r \to S^r$.*

PROOF. This follows from Wada [8] and a recent result of Adams [1].

(2.6) PROPOSITION. *If X is a H-space, then for each $\alpha \in \pi_p(X)$,*

$$\pi_q(G^p_\alpha)/\pi_{p+q}(X) \approx \pi_q(X),$$

where $\pi_{p+q}(X)$ is, of course, imbedded in $\pi_q(G^p_\alpha)$ isomorphically.

PROOF. According to G. W. Whitehead [10] (see also [11]), we have the following diagram:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \pi_q(F^p_\alpha) & \xrightarrow{i_*} & \pi_q(G^p_\alpha) & \xrightarrow{j_*} & \pi_q(G^p_\alpha, F^p_\alpha) & \xrightarrow{\partial} & \pi_{q-1}(F^p_\alpha) & \longrightarrow \cdots \\
& \updownarrow & & & & \updownarrow \pi_* & \searrow & \updownarrow H & \\
& \pi_{p+q}(X) & & & & \pi_q(X) & \xrightarrow[\rho_\alpha]{} & \pi_{p+q-1}(X), &
\end{array}$$

where π_* denotes the isomorphism induced by the projection π, H denotes the Hurewicz isomorphism and ρ_α is defined by $\rho_\alpha(\beta) = -[\alpha, \beta]$. Since ρ_α is always trivial when X is an H-space, (2.7) follows from the exactness of the sequence.

3. **The sphere S^r.** Let $X = S^r$, an r-sphere, then we have the following exact sequence

$$(3.1) \quad \cdots \xrightarrow{\rho_\alpha} \pi_{p+q}(S^r) \xrightarrow{\mu} \pi_q(G^p_\alpha) \xrightarrow{\nu} \pi_q(S^r) \xrightarrow{\rho_\alpha} \pi_{p+q-1}(S^r) \xrightarrow{\mu} \cdots.$$

The following propositions are fairly obvious.

(3.2) Proposition. *Let $X = S^r$ and $\alpha \in \pi_p(S^r)$. Since $\pi_q(S^r) = 0$ for $q < r$ we have*

$$\pi_q(G^p_\alpha) \approx \pi_{p+q}(S^r), \qquad (q < r - 1).$$

(3.3) Corollary. $\pi_1(G^r_\alpha) \approx Z_2$ *for* $r \geqq 3$.

Since $\pi_{r+2}(S^r) \approx Z_2$, for $r \geqq 3$, we have

(3.4) Corollary. $\pi_1(G^{r+1}) \approx Z_2$, $r \geqq 3$.

Denote the image of $\rho_\alpha \colon \pi_q(S^r) \to \pi_{p+q-1}(S^r)$ by J^{p+q-1}_α and the kernel of ρ_α by K^p_α. Denote the image of $\mu \colon \pi_{p+q}(S^r) \to \pi_q(G^p_\alpha)$ by P_α. Then

(3.5) Proposition (Hu) [2]. *For $X = S^r$ and $\alpha \in \pi_p(S^r)$*

(a) $\pi_q(G^p)/P^q_\alpha \approx K^q_\alpha$, $(q > 1)$,

(b) $\pi_{p+q}(G^p_\alpha)/J^{p+q}_\alpha \approx P^q$, $(q > 1)$,

(c) $\pi_{r-1}(G^q_\alpha) \approx \pi_{p+r-1}(S^r)/J^{p+r-1}_\alpha$,

(d) $\pi_{r+3}(G^p_\alpha)$ *has a subgroup* $P^{r+3}_\alpha \approx \pi_{p+r+3}(S^r)$, $(r \geqq 6)$,

(e) $\pi_{r+4}(G^p) \approx \pi_{p+r+4}(S^r)/J^{p+r+4}$, $(r \geqq 6)$.

Since for $r \geqq 7$, $\pi_{r+4}(S^r) = \pi_{r+5}(S^r) = 0$. It follows that

(3.6) Proposition. If $r > 7$, *for each* $\alpha \in \pi_p(S^r)$,

$$\pi_{r-1}(G^7) \approx \pi_{r-2}(G^8) \approx \cdots \approx 0.$$

And,

(3.7) Proposition. *For $r \geqq 7$, $\alpha \in \pi_p(S^r)$,*

$$\pi_{r+6-p}(G^p) \approx \pi_{r+6-p}(S^r).$$

We now proceed to prove the main theorem of this section. Consider the following sequence

$$\pi_r(S^r) \xrightarrow{\rho_\alpha} \pi_{2r-1}(S^r) \xrightarrow{E} \pi_{2r}(S^{r+1}), \tag{3.8}$$

where E denotes the Freudenthal suspension. By the delicate suspension theorem, the kernel of E is a cyclic subgroup generated by $[\iota_r, \iota_r]$. If r is even, it is infinite cyclic; if r is odd $\neq 1, 3, 7$, it is cyclic of order 2.

(3.9) Lemma (Hu). *For $X = S^2$ and $\alpha \in \pi_2(S^2)$, we have*

$$\pi_1(G_\alpha^2) \approx Z_{2m},$$

where m is the absolute value of the degree of α.

Proof. Since $\pi_{2r}(S^{r+1}) = \pi_4(S^3) \approx Z_2$. From (3.8) $\pi_3(S^2)/\text{Ker } E \approx Z_2$. Let γ be a generator of the free cyclic group $\pi_3(S^2)$. Then $[\iota_2, \iota_2] = \pm 2$. We can choose γ so that $[\iota_2, \iota_2] = -2\gamma$. Let $\alpha \in \pi_2(S^2)$. By linearity of the Whitehead product $\rho_\alpha(\iota_2) = -[\alpha, \iota_2] = -m[\iota_2, \iota_2] = 2m\gamma$. In other words J_α^3 is generated by $2m\gamma$. From (3.5(c)), we have $\pi_1(G_\alpha^2) \approx Z_{2m}$. This proves (3.9).

(3.10) Lemma. *For $X = S^4$ and $\alpha \in \pi_4(S^4)$, we have*

$$\pi_3(G_\alpha^4) \approx Z_{24m} + Z_{12},$$

where m is the absolute value of the degree of α.

Proof. $\pi_{2r}(S^{r+1}) \approx \pi_8(S^5) \approx Z_{24}$ and $\pi_{2r-1}(s^r) = \pi_7(S^4) \approx Z + Z_{12}$. One generator of Ker E is determined as follows:

From a theorem of characteristic map [5, p. 121], that

$$[\iota_4, \iota_4] = 2[q] - \epsilon E[\xi],$$

where $\epsilon = \pm 1$ depends on the convention of orientation, $[q]$ denotes the homotopy class of the Hopf map $q: S^7 \to S^4$ and $[\xi]$ a generator of $\pi_6(S^3)$ represented by the characteristic map $\xi: S^6 \to S^3$ of the fibre bundle Sp(2) over S^7 with Sp(1) as fibre. Hence in $\pi_8(S^5)$ we have

$$E^2[\xi] = \epsilon 2E[q],$$

(E^2 denotes the iterated suspension). This implies that $\pi_8(S^5)$ has $E[q]$ as a generator. Hence

$$\pi_7(S^4)/\text{Ker } E \approx Z_{24} + Z_{12}.$$

A similar argument as used in (3.9) yields

$$\pi_3(\overset{4}{G_\alpha}) \approx Z_{24m} + Z_{12}.$$

(3.11) LEMMA. *For* $X=S^6$ *and* $\alpha\in\pi_6(S^6)$, *we have*

$$\pi_5(\overset{6}{G_\alpha}) \approx Z_m,$$

where m is the absolute value of the degree of α.

PROOF. Since $\pi_{2r}(S^{r+1})=\pi_{12}(S^7)=0$ and $\pi_{2r-1}(S^5)=\pi_{11}(S^6)\approx Z$. Ker $E=J_\alpha^{11}$. Hence we can choose the generator γ of $\pi_{11}(S^6)$ such that $\gamma=-[\iota_6, \iota_6]$, consequently $\rho_\alpha(\iota_6)=m\gamma$, or $\pi_5(G_\alpha^6)\approx Z_m$ by (3.4(c)).

(3.12) LEMMA. *For* $X=S^8$ *and* $\alpha\in\pi_8(S^8)$, *we have*

$$\pi_7(G^8) \approx Z_{240m} + Z_{120},$$

where m is the absolute value of the degree of α.

PROOF. $\pi_{2r}(S^{r+1})=\pi_{16}(S^9)\approx Z_{240}$ and $\pi_{2r-1}(S^r)=\pi_{15}(S^8)\approx Z+Z_{120}$. Since $[\iota_6, \iota_6]=2[q']-\epsilon E[\xi']$, where $[q']$ denote the homotopy class represented by the Hopf map $q': S^{15}\rightarrow S^8$ and $\xi'\in\pi_{14}(S^7)$ has nonzero Hopf invariant, we have

$$E^2[\xi'] = 2\epsilon E[q'].$$

Using the same argument as in (3.10), one proves (3.12).

(3.13) LEMMA. *For* $X=S^{10}$ *and* $\alpha\in\pi_{10}(S^{10})$, *we have*

$$\pi_9(G^{10}) \approx Z_m + Z_2 + Z_2 + Z_2,$$

where m denotes the absolute value of the degree of α.

(3.14) LEMMA. *For* $X=S^{12}$ *and* $\alpha\in\pi_{12}(S^{12})$, *we have*

$$\pi_{11}(G^{12}) \approx Z_m + Z_8 + Z_{27} + Z_7,$$

where m denotes the absolute value of the degree of α.

The proof of (3.13) follows from the table in Toda [6] the first row and a similar argument as before; for a proof of (3.14), one uses the third row of the above mentioned table.

(3.15) LEMMA. *For* $X=S^{14}$ *and* $\alpha\in\pi_{14}(S^{14})$, *we have*

$$Z_{13}(G^{14}) \approx Z_m + Z_3,$$

where m denotes the absolute value of the degree of α.

PROOF. Since $\pi_{17}(S^{14})\approx Z+Z_3$ and $\pi_{18}(S^{15})\approx Z_3$ and the suspension E sends Z into 0 in $\pi_{18}(S^{15})$ (Toda [6]). The proof is immediate.

(3.16) LEMMA. *For $X=S^r$, $\alpha\in\pi_r(S^r)$ and r odd, $\neq 1, 3, 7$. Then*
(a) $\pi_{r-1}(G^r)\approx\pi_{2r-1}(S^r)$ *when α is of even degree,*
(b) $\pi_{r-1}(G^r)\approx\pi_{2r-1}(S^r)/Z_2$ *when α is of odd degree.*

PROOF. It suffices to prove that there is a nonzero element in J_α^{2r-1} when α is of odd degree and $r\neq 1, 3, 7$. In fact, in this case $\rho_\alpha(\iota_r)\neq 0$. (3.16) follows.

(3.17) LEMMA (HU). *Let X be any space. If $\alpha, \beta\in\pi_p(X)$, $\alpha+\beta=0$. Then G_α^p and G_β^p are homeomorphic.*

PROOF. Let $\theta: S^p\rightarrow S^p$ be a homeomorphism which reverses the orientation and leaves the pole y_0 fixed. Then a homeomorphism h of G_α^p onto G_β^p is given by $h(f)=f\cdot\theta$ for each $f\in G_\alpha^p$.

(3.18) THEOREM. *Let $X=S^r$. Let $\alpha, \beta\in\pi_r(S^r)$. Then for $r=2, 4, 6, 8, 10, 12, 14$, the components G_α^r and G_β^r have the same homotopy type if and only if $\alpha=\pm\beta$. When r is odd $\neq 1, 3, 7$, the components G_α^r and G_β^r are of different homotopy type if* deg $\alpha-$deg β *is odd.*

PROOF. The first part of the theorem follows from Lemmas (3.9) through (3.17). The remaining part follows from the fact that if r is odd then $\pi_p(S^r)$ is finite for $p>n$ [**4**].

(3.19) COROLLARY. *Let $X=S^r$ and $\alpha, \beta\in\pi_r(S^r)$ are of odd and even degree respectively. Then*:

$$\pi_4(G_\alpha^5) = 0, \qquad \pi_4(G_\beta^5) \approx Z_2,$$
$$\pi_8(G_\alpha^9) \approx Z_2 + Z_2, \qquad \pi_8(G_\beta^9) \approx Z_2 + Z_2 + Z_2,$$
$$\pi_{10}(G_\alpha^{11}) \approx Z_2 + Z_9, \qquad \pi_{10}(G_\beta^{11}) \approx Z_2 + Z_2 + Z_9,$$
$$\pi_{12}(G_\alpha^{13}) = 0, \qquad \pi_{12}(G_\beta^{13}) \approx Z_2,$$
$$\pi_{14}(G_\alpha^{15}) \approx Z_2 + Z_2 \ \textit{or}\ Z_4, \qquad \pi_{14}(G_\beta^{15}) \approx Z_4 + Z_2.$$

(3.20) PROPOSITION (HU). *When r is even and $\alpha\in\pi_r(S^r)$, $r\neq 0$. Then*

$$\pi_r(G_\alpha^r) \approx \pi_{2r}(S^r)/J_\alpha^{2r}.$$

PROOF. Since $K_\alpha^r=0$, the result follows from (3.4(a)) and (3.4(b)).

(3.21) PROPOSITION. *If $E:\pi_p(S^r)\rightarrow\pi_{p+1}(S^{r+1})$ is an injection, then for $q+s=p$ and $q>1$*

$$\pi_q(G_\alpha^s)/\pi_p(S^r) \approx \pi_q(S^r),$$

where $\pi_p(S^r)$ is imbedded in $\pi_q(G_\alpha^s)$.

PROOF. Since $E[\alpha, \beta]=0$, $J_\alpha^p \subset \operatorname{Ker} E=0$. From (3.4)(a) and (b), $\pi_q(G_\alpha^s)/\pi_{q+s}(S^r) \approx K_\alpha^q$. But $K_\alpha^q \approx \pi_q(S^r)$. This proves (3.21).

For $q<r$, $\pi_q(S^r)=0$, we have $\pi_q(G_\alpha^s) \approx \pi_p(S^r)$. This reduces to (3.2).

(3.22) COROLLARY. *If* $q+s=p<2r-1$, *then*

$$\pi_q(G_\alpha^s)/\pi_p(S^r) \approx \pi_q(S^r).$$

PROOF. This follows from (3.21).

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