Invertible Field Theory Models for SPTs in the Bott Spiral

Geometry, Topology & Physics (GTP) Seminar | CQTS, NYU Abu Dhabi

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Outline

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- Free versus Interacting SPTs
- Fermionic Groups and Classification of SPTs
- Modeling SPTs in the “Bott Spiral”
- The Free-to-Interacting Map
- Dimensional Reduction
- Extra: Computations

Symmetry-Protected Topological Phases

For SPT phases...

- $\mathcal{M}$ = space of Hamiltonian operators with specified symmetries
- Δ = space of gapless Hamiltonians
- phase $\in (\mathcal{M} \setminus \Delta)/\sim$

SPT phases can be *free* or, more generally, *interacting*...

- *free* Hamiltonians are a subset $\mathcal{M}_{\text{free}} \subset \mathcal{M}_{\text{int}}$

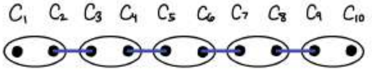
Our focus:

- How does $(\mathcal{M}_{\text{free}} \setminus \Delta)/\sim_{free}$ compare with $(\mathcal{M}_{\text{int}} \setminus \Delta)/\sim_{int}$?

Example: Free and Interacting Fermionic SPTs

The group of *free* $d = 1$ class BDI SPTs is $\mathbb{Z}$,
generated by the TRS Majorana chain:

$$\hat{H} = \frac{i}{2}\left(\sum_{\ell} \hat{c}_{2\ell}\hat{c}_{2\ell+1}\right).$$

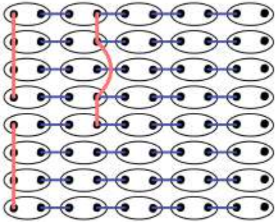


Interacting Trivialization [Fidkowski–Kitaev '10]

Eight copies of the TRS Majorana chain trivializes

$$\hat{H} = \sum_{\alpha=1}^{8} \hat{H}_{\alpha} \qquad \text{where} \qquad \hat{H}_{\alpha} = \frac{i}{2}\left(\sum_{\ell} \hat{c}^{\alpha}_{2\ell}\hat{c}^{\alpha}_{2\ell+1}\right)$$

with interaction term $\hat{c}^1_\ell \hat{c}^2_\ell \hat{c}^3_\ell \hat{c}^4_\ell + \hat{c}^5_\ell \hat{c}^6_\ell \hat{c}^7_\ell \hat{c}^8_\ell + \hat{c}^1_\ell \hat{c}^2_\ell \hat{c}^5_\ell \hat{c}^6_\ell + \ldots$



$\Longrightarrow$ The group of *interacting* BDI SPTs in $d = 1$ is $\mathbb{Z}/8$

Our [Debray–K.–Pacheco-Tallaj–Stehouwer] Approach

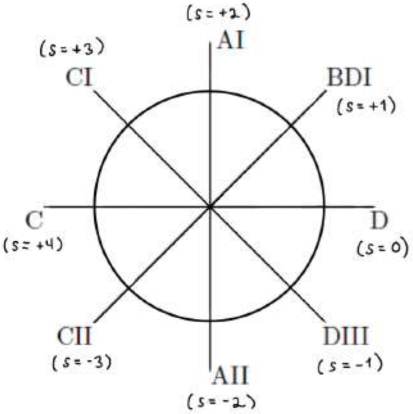
- Following [Freed–Hopkins, '16], we model the passage ($[H]_{free} \mapsto [H]_{int}$) from free to interacting SPTs as a homotopical map...

$$\begin{tikzcd}
\left\{\begin{matrix}\text{``unstable''}\\ \text{free ''}\end{matrix}\right\} \arrow[r] & \left\{\begin{matrix}\text{free fermion}\\ G\text{-SPTs}\end{matrix}\right\} \arrow[r] \arrow[d, "\simeq"] & \left\{\begin{matrix}\text{interacting fermion}\\ G\text{-SPTs}\end{matrix}\right\} \arrow[r] \arrow[d, "\simeq"', "\text{ansatz}"] & \left\{\begin{matrix}\text{intrinsically}\\ \text{interacting ''}\end{matrix}\right\} \\
\text{ker} \arrow[r] & KO_G^{d-2,\tau} \arrow[r, dashed, "F2I"] & I_{\mathbb{Z}}\Omega_{\xi}^{d+2} \arrow[r] & \text{coker}
\end{tikzcd}$$

- Today: construct $F2I$ for the "Bott spiral" of SPTs [Queiroz–Khalaf–Stern '16], for which $G = (\mathbb{Z}/2)^n$, compute its image, and discuss dimensional reduction

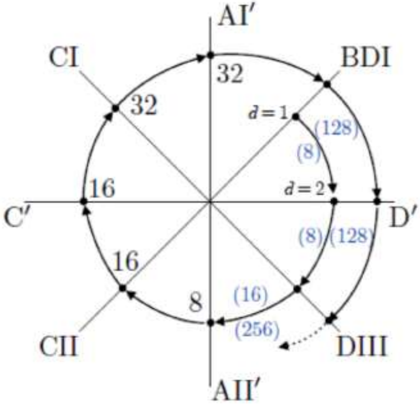
The “Bott Spiral” of Primed Phases [Queiroz–Khalaf–Stern, ’16]

“Bott Clock” of Free SPTs



- Periodic: $KO^{d+s-2} = KO^{(d+1)+(s-1)-2}$

“Bott Spiral” of Interacting SPTs



- Not periodic! $\mathbb{Z}/(2^n)$ for increasing n

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Fermionic Groups and Classification of SPTs

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K-Theory Classification of Free SPTs

Definition

Let A be a finite-dimensional superalgebra. Then

$$K_n(A) \cong (\mathrm{Mod}^{\mathrm{proj}}_{A \otimes C\ell_{-n}})/(\mathrm{Mod}^{\mathrm{proj}}_{A \otimes C\ell_{-n-1}}).$$

Ansatz

In spatial dimension d, free SPTs with A-symmetry are classified by $K_{2-d}(A)$.

Example

- For A one of the ten classes of superdivision algebras ($\mathbb{R}$, $C\ell_{-1}$, $C\ell_{-2}$, ...), this recovers the tenfold way
- Class BDI superconductors have $A \cong C\ell_{+1}$, generated by T.
  The TRS Majorana chain generates $K_{2-1}(C\ell_{+1}) \cong K_0(\mathbb{R}) \cong \mathbb{Z}$.

Tangential Structures I

Definition

A *stable tangential structure* is an object $(\xi\colon X \to BO) \in \mathsf{Spc}_{/BO}$.

A stable tangential structure on a manifold M is a lift...

$$\begin{tikzcd} & X \arrow[d, "\xi"] \\ M \arrow[ur, dashed] \arrow[r, "TM"] & BO. \end{tikzcd}$$

Examples

- $X = BSO \rightsquigarrow$ orientations
- $X = B\mathrm{Spin} \rightsquigarrow$ spin structures
- $X = B\mathrm{Pin}^{\pm} \rightsquigarrow$ $\mathrm{pin}^{\pm}$ structures

Tangential Structures II

Definition

The bordism group of d-manifolds with ξ-structure is

$$\Omega_d^\xi = (\{\text{closed } d\text{-dim'l } \xi\text{-manifolds}\}/\text{bordism}, \sqcup).$$

Pontryagin–Thom

Bordism for ξ-manifolds is encoded by the *Madsen–Tillmann spectrum*:

$$\pi_d(MT\xi) \cong \Omega_d^\xi.$$

Example

In dimension 2 with $\xi\colon B\mathrm{Pin}^- \to BO$, the group is $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$ generated by $[\mathbb{RP}^2]$.

Invertible Field Theory Classification of Interacting SPTs I

Basic Definition

An n-dimensional TQFT is a symmetric monoidal functor

$$\mathcal{Z} \colon \mathrm{Bord}_n \to \mathrm{Vect}_{\mathbb{C}}$$

from the *bordism category* Bord_n of closed $(n-1)$-manifolds and bordisms to the category $\mathrm{Vect}_{\mathbb{C}}$ of complex vector spaces.

- closed $(n-1)$-manifold $M \longmapsto \mathcal{Z}(M)$ the *state space* of the theory on M
- closed n-manifold $W \longmapsto \mathcal{Z}(W) \in \mathbb{C}$ the value of the *partition function* of $\mathcal{Z}$

Ansatz

Interacting SPT phases of a given symmetry type are classified by the deformation class of their low energy field theory, which is a *fully extended reflection-positive invertible field theory* on manifolds with certain tangential structure.

Invertible Field Theory Classification of Interacting SPTs II

Theorem [Freed–Hopkins '16, Grady '23]

Let ξ be a stable tangential structure.

$$\left\{ \begin{matrix} \text{fully extended reflection-positive} \\ (d+1)\text{-dim'l invertible } \xi\text{-theories} \end{matrix} \right\} / \text{deformation} \cong [MT\xi,\, \Sigma^{d+2} I_{\mathbb{Z}}]$$

- There is a short exact sequence

$$0 \longrightarrow (\mathrm{Hom}(\Omega^{\xi}_{d+1}, \mathbb{C}^{\times}))_{\mathrm{tor}} \longrightarrow [MT\xi, \Sigma^{d+2} I_{\mathbb{Z}}] \longrightarrow \mathrm{Hom}(\Omega^{\xi}_{d+2}, \mathbb{Z}) \longrightarrow 0$$

(where, recall, $\Omega^{\xi}_{d+1} = (\{\text{closed } (d+1)\text{-dim'l } \xi\text{-manifolds}\}/\text{bordism}, \sqcup)$)

Example

The interacting analog of the TRS Majorana chain is the Arf–Brown TQFT, which generates the group $(\mathrm{Hom}(\Omega_2^{\mathrm{Pin}^-}, \mathbb{C}^{\times}))_{\mathrm{tor}} \cong \Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$ [Debray–Gunningham, '18]

Connecting the Classifications

To construct the free-to-interacting map

$$\begin{tikzcd}
\left\{\begin{matrix}\text{free fermion}\\ G\text{-SPTs}\end{matrix}\right\} \arrow[r, "F2I"] \arrow[d, "\simeq"] & \left\{\begin{matrix}\text{interacting fermion}\\ G\text{-SPTs}\end{matrix}\right\} \arrow[d, "\simeq"', "\text{ansatz}"] \\
KO_G^{d-2,\tau} \arrow[r, dashed, "F2I"] & I_{\mathbb{Z}}\Omega_{\xi}^{d+2}
\end{tikzcd}$$

we need to know how to match up G and ξ...

Fermionic Group Symmetries

The fermionic group formalism encodes symmetries compatibly for both KO_{G_b}-theory (the free side) and (Anderson-dual) spin bordism (the interacting side).

Definition [Benson, '88]

A *fermionic group* is the data of

- a topological group G_b
- an extension ω by $\mathbb{Z}/2^F$ ← generated by fermion parity $(-1)^F$

$$1 \to \mathbb{Z}/2^F \to G_f \to G_b \to 1$$

- a grading $\theta\colon G_f \to \mathbb{Z}/2^T$ ← labels which symmetries are antiunitary

Fermionic Groups $\rightsquigarrow$ Superalgebras

Definition [Benson, '88, §7]

Let (G_b, θ, ω) be a finite fermionic group. Its *fermionic group algebra* is the real superalgebra

$$\mathbb{R}^f[G_f] := \frac{\mathbb{R}[G_f]}{((-1)^F + 1)}$$

graded by θ.

Fact [Karoubi]

If V is a rank zero G_b-vector bundle with $w_1 V = \theta$ and $w_2 V = \omega$, then

$$K_0(\mathbb{R}^f[G_f]) \cong KO^0_{G_b}(S^V).$$

- Recall that G_f is the $\mathbb{Z}/2$-extension of G_b classified by ω.

Fermionic Groups ⇝ Tangential Structures

Let (G_b, θ, ω) be as before. Define $H(G_f)$ as the pullback

$$\begin{tikzcd}
BH(G_f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & BO \arrow[d, "(w_1,\, w_1^2+w_2)"] \\
BG_b \arrow[r, "(\theta, \omega)"] & B\mathbb{Z}/2 \times B^2\mathbb{Z}/2.
\end{tikzcd}$$

Then setting ξ to the top map $BH(G_f) \to BO$ allows us to define $H(G_f)$ tangential structures. These are related to spin structures...

Theorem [Debray–K.–Stehouwer]

There is a symmetric monoidal functor

$$(\mathrm{FermGrp},\ \hat{\times}) \to (MT\mathrm{Spin}\text{-}\mathrm{Mod},\ \wedge)$$
$$(G_b, \theta, \omega) \mapsto MTH(G_f).$$

Examples: $\mathrm{Pin}^{\pm}$

Example

Consider twisting by $\sigma \to B\mathbb{Z}/2$. Write $x = w_1\sigma \in H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$.

- $\mathrm{Pin}^+(1) := (\mathbb{Z}/2,\, x,\, 0) \in \mathsf{FermGrp}$

$\rightsquigarrow$ $\Omega_n^{\mathrm{Pin}^-} \simeq \Omega_n^{\mathrm{Spin}}((B\mathbb{Z}/2)^{\sigma-1})$ on the bordism side

$\rightsquigarrow$ $\mathbb{R}^f[\mathrm{Pin}^+(1)] \cong C\ell_{+1}$ on the superalgebra side, and $K_0(C\ell_{+1}) \cong KO^{-1}$

Example

Similarly, for $3\sigma \to B\mathbb{Z}/2$, or equivalently, $-\sigma \to B\mathbb{Z}/2$,

- $\mathrm{Pin}^-(1) := (\mathbb{Z}/2,\, x,\, x^2) \in \mathsf{FermGrp}$

$\rightsquigarrow$ $\Omega_n^{\mathrm{Pin}^+} \simeq \Omega_n^{\mathrm{Spin}}((B\mathbb{Z}/2)^{3\sigma-3})$ on the bordism side

$\rightsquigarrow$ $\mathbb{R}^f[\mathrm{Pin}^-(1)] \cong C\ell_{-1}$ on the superalgebra side, and $K_0(C\ell_{-1}) \cong KO^{1}$

Discrete versus Continuous Symmetry Groups I

Definition

Generalizing the previous examples, define $E_{\ell,k} := (\mathrm{Pin}^+(1))^{\hat{\times}\ell} \hat{\times} (\mathrm{Pin}^-(1))^{\hat{\times}k}$.

- Then $\mathbb{R}^f[E_{\ell,k}] \cong \mathbb{C}\ell_{\ell,k}$ [Albuquerque–Majid, '02]
- And $K_{2-d}(C\ell_{\ell,k}) \cong KO^{d+\ell-k-2}$ recovers the tenfold way

Example

Class AI has two anticommuting time-reversal symmetries $\mathcal{T}_1, \mathcal{T}_2$, which could be encoded using $E_{2,0}$. Equivalently, can use $\mathcal{T}_1, \mathcal{T}_1\mathcal{T}_2$. But if the unitary symmetry $\mathcal{T}_1\mathcal{T}_2$ is meant to encode charge, then it should be enlarged to a continuous $U(1)$, leading to $\mathrm{Pin}^+(2)$. We have $E_{2,0} \subset \mathrm{Pin}^+(2)$ and $K_0(E_{2,0}) \cong KO^{-2} \subset K_0(\mathrm{Pin}^+(2))$.

Discrete versus Continuous Symmetry Types II

The choice of discrete versus continuous has a profound effect on the interacting classification...

Definition

Spin-(ℓ, k) is the stable tangential structure $H(E_{\ell,k})$.

- A spin-(ℓ, k) structure on a manifold M is a spin structure on $TM + L_1 + \cdots + L_\ell - L_{\ell+1} - \cdots - L_{\ell+k}$ for some line bundles L_i

Example

For class AI, the discrete fermionic group $E_{2,0}$ gives rise to spin-$(2, 0)$-structures, while the continuous fermionic group $\mathrm{Pin}^+(2)$ gives rise to $\mathrm{Pin}^{\tilde{c}-} = \mathrm{Pin}^- \ltimes_{\pm 1} \mathrm{U}(1)$.

$$\Omega_2^{\mathrm{Pin}^{\tilde{c}-}} \cong \mathbb{Z} \ \ncong \ (\mathbb{Z}/2)^{\oplus w_{\mathrm{AI}}} \cong \Omega_2^{E_{2,0}}$$

Discrete versus Continuous Tenfold Way Fermionic Groups

Class	Cont. K_f	Disc. G_f	$\mathbb{R}^f[G_f]$	$H(K_f)$	$H(G_f)$
CII	$\mathrm{Pin}^-(3)$	$E_{0,3}$	$C\ell_{-3}$	$\mathrm{Pin}^{h-} := \mathrm{Pin}^- \times_{\pm 1} SU(2)$	$\mathrm{Spin}\text{-}(0,3)$
AII	$\mathrm{Pin}^-(2)$	$E_{0,2}$	$C\ell_{-2}$	$\mathrm{Pin}^{\tilde{c}+} := \mathrm{Pin}^+ \ltimes_{\pm 1} \mathrm{U}(1)$	$\mathrm{Spin}\text{-}(0,2)$
DIII	$\mathrm{Pin}^-(1)$	$E_{0,1}$	$C\ell_{-1}$	Pin^+	$\mathrm{Spin}\text{-}(0,1)$
D	$\mathrm{Spin}(1)$	$E_{0,0}$	$\mathbb{R}$	Spin	Spin
BDI	$\mathrm{Pin}^+(1)$	$E_{1,0}$	$C\ell_1$	Pin^-	$\mathrm{Spin}\text{-}(1,0)$
AI	$\mathrm{Pin}^+(2)$	$E_{2,0}$	$C\ell_2$	$\mathrm{Pin}^{\tilde{c}-} := \mathrm{Pin}^- \ltimes_{\pm 1} \mathrm{U}(1)$	$\mathrm{Spin}\text{-}(2,0)$
CI	$\mathrm{Pin}^+(3)$	$E_{3,0}$	$C\ell_3$	$\mathrm{Pin}^{h+} := \mathrm{Pin}^+ \times_{\pm 1} SU(2)$	$\mathrm{Spin}\text{-}(3,0)$
C	$SU(2)$	Q_8	$C\ell_4$	$\mathrm{Spin}^h := \mathrm{Spin} \times_{\pm 1} SU(2)$	$\mathrm{Spin} \times_{\pm 1} Q_8$

- column $H(K_f)$ recovers the groups $H(s)$ of [Freed–Hopkins, '16]

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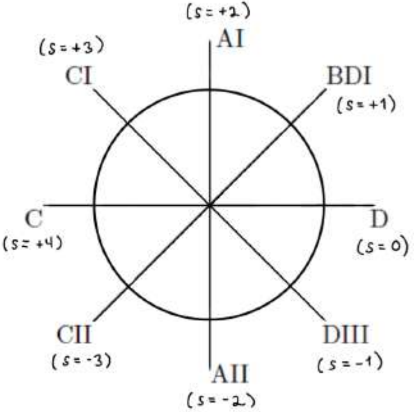
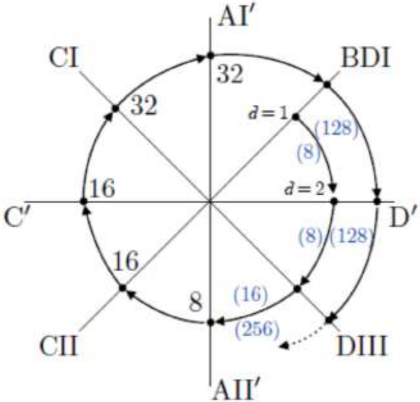
The “Bott Spiral” of Primed Phases [Queiroz–Khalaf–Stern, ’16]

“Bott Clock” of Free SPTs



- Periodic: $K_i(C\ell_{1,1}\hat{\otimes}A) \cong K_i(A)$

“Bott Spiral” of Interacting SPTs

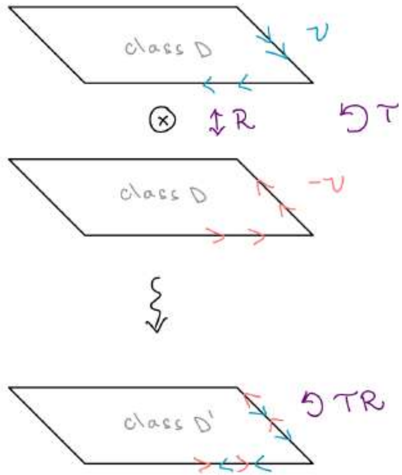


- Not periodic! $\mathbb{Z}/(2^n)$ for increasing n

Primed Altland–Zirnbauer Classes and [Queiroz–Khalaf–Stern, '16]

- *Primed* phases are made by by stacking two opposite chiral phases and imposing a unitary $\mathbb{Z}/2$ symmetry [Yao–Ryu, '13 and Qi, '13]

- Queiroz–Khalaf–Stern calculated the groups of interacting phases connected to certain free fermion primed phases



Our Model for Primed Symmetry Classes

Definition

Let G_f be a fermionic group. The primed version is

$$G_f' := G_f \mathbin{\hat{\times}} \mathrm{Pin}^+(1) \mathbin{\hat{\times}} \mathrm{Pin}^-(1).$$

Consequently...

$\rightsquigarrow$ $\mathbb{R}^f[G_f'] \cong \mathbb{R}^f[G_f] \hat{\otimes} C\ell_{1,1} \sim_M \mathbb{R}^f[G_f]$

$\rightsquigarrow$ $\Omega_n^{H(G_f')} \cong \Omega_n^{H(G_f)}((B\mathbb{Z}/2)^{\sigma-1} \wedge (B\mathbb{Z}/2)^{1-\sigma}) \ncong \Omega_n^{H(G_f)}$

Example: Morita Variance

Consider $G_f = \mathbb{R}$, modeling class D.

- Then $G_f' = \mathrm{Pin}^+(1) \hat{\times} \mathrm{Pin}^-(1)$ models class D′.
- $\mathbb{R}^f[\mathbb{R}] \cong \mathbb{R}$ and $\mathbb{R}^f[\mathrm{Pin}^+(1) \hat{\times} \mathrm{Pin}^-(1)] \cong C\ell_{1,1} \sim_M \mathbb{R}$
- $\Omega_n^{H(\mathbb{R})} = \Omega_n^{\mathrm{Spin}}$, while $\Omega_n^{H(\mathrm{Pin}^+(1) \hat{\times} \mathrm{Pin}^-(1))} \cong \Omega_n^{\mathrm{DPin}}$
- Spin structures and dpin structures are different. For example, $\mathbb{RP}^2$ is *not* spin, but since $T\mathbb{RP}^2 + \sigma - 1$ admits a spin structure, $\mathbb{RP}^2$ *is* dpin.
- The bordism groups are quite different $\rightsquigarrow$

d	Ω_d^{Spin}	Ω_d^{DPin}
0	$\mathbb{Z}$	$\mathbb{Z}/2$
1	$\mathbb{Z}/2$	$\mathbb{Z}/2$
2	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$
3	0	$\mathbb{Z}/8$
4	$\mathbb{Z}$	$(\mathbb{Z}/2)^2$
5	0	0
6	0	$(\mathbb{Z}/2)^2$

[K-PM-T-D, '20]

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KO_{2-d}(\mathbb{R}^f[G_f]) \arrow[r, dashed, "F2I"] & I_{\mathbb{Z}}\Omega^{d+2}_{H(G_f)}
\end{tikzcd}$$

we need to know how to go between (twisted) spin manifolds and K-theory classes...

The Free-to-Interacting Map for Primed Phases

The free-to-interacting map is dual to a (twisted) Atiyah–Bott–Shapiro index map

$$\begin{aligned} \Omega_n^{\mathrm{Spin}} &\xrightarrow{\ ABS\ } KO_n \\ [M] &\longmapsto \not{D}_M \curvearrowright \overline{\mathcal{C}^\infty(\mathcal{S})} \end{aligned}$$

For primed phases, like dpin, this index map takes the following form:

$$\begin{tikzcd}
\Omega_n^{\mathrm{DPin}} \arrow[r, "\cong"] \arrow[ddrr, bend right, "ABS_{(1,1)}"'] & \Omega_n^{\mathrm{Spin}}\big((B\mathbb{Z}/2)^{\sigma-1} \wedge (B\mathbb{Z}/2)^{1-\sigma} \big) \arrow[d, "\mathrm{sm}_\sigma \circ c"] & \\
& \Omega_{n-1}^{\mathrm{Spin}}((B\mathbb{Z}/2)^{\sigma-1}) \arrow[r, "ABS"] & KO_{n-1}((B\mathbb{Z}/2)^{\sigma-1}) \arrow[d, "\lambda_\sigma"] \\
& & KO_n
\end{tikzcd}$$

Computational Results

Theorem [Debray–K.–Stehouwer]

Let $\widetilde{m} := \lfloor (d - \ell - k + i + 1)/8 \rfloor$, where $i \in \{0, 1, 2, 3\}$ with $i \equiv \ell \mod 4$. For d large enough, the image of the type-$E_{\ell,k}$ free-to-interacting map

$$F2I_{\ell,k} \colon KO^{d+\ell-k-2} \to \mho^{d+2}_{\mathrm{Spin}}(ME_{\ell,k})$$

is

$$\mathrm{im}\,(F2I_{\ell,k}) \cong \begin{cases} \mathbb{Z}/2^{4+4\widetilde{m}-i} & d \equiv \ell + k - 2i + 2 \text{ mod } 8 \\ \mathbb{Z}/2^{5+4\widetilde{m}-i} & d \equiv \ell + k - 2i + 6 \text{ mod } 8 \\ \mathbb{Z}/2 & d \equiv \ell + k - 2i \text{ mod } 8 \text{ or } \ell + k - 2i + 1 \text{ mod } 8. \end{cases}$$

- Here, $\widetilde{m}$ tracks the number of times we have gone around the spiral.

Example Free-to-Interacting Map

Example

Consider $d = 2$. The F2I map for the class D superconductor is an isomorphism:

$$\mathbb{Z} \cong KO^0(\mathrm{pt}) \longrightarrow \mathbb{Z} \cong I_{\mathbb{Z}}\Omega^3_{\mathrm{Spin}}.$$

The F2I map for the class D′ superconductor, meanwhile, is reduction mod 8:

$$\mathbb{Z} \cong KO^0(\mathrm{pt}) \longrightarrow \mathbb{Z}/(2^3) \cong I_{\mathbb{Z}}\Omega^3_{\mathrm{DPin}}.$$

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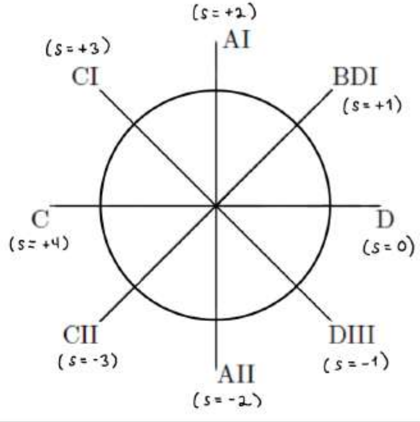
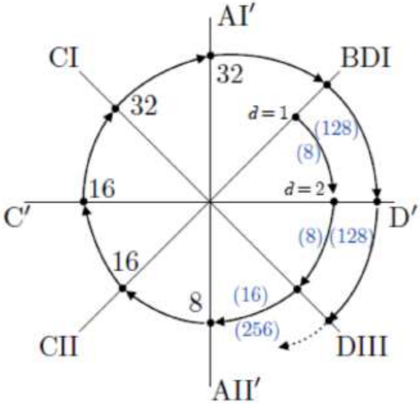


“Bott Spiral” of Interacting SPTs



- Periodic: $K_i(C\ell_{1,1}\hat{\otimes}A) \cong K_i(A)$
- Not periodic! $\mathbb{Z}/(2^n)$ for increasing n

Dimensional Reduction

- We use "dimensional reduction" in the sense of [Ryu–Schnyder–Furusaki–Ludwig '10, Teo–Kane, '10, Queiroz–Khalaf–Stern '16, ...] where it takes...

$$\mathrm{SPT}(\dim d, \text{type } s) \rightsquigarrow \mathrm{SPT}(\dim d+1, \text{type } s-1)$$

- We define this for IFTs by generalizing *Smith homomorphism* models for symmetry breaking [Hason–Komargodski–Thorngren, '20, Debray–Devalapurkar–K.–Liu–Pacheco-Tallaj–Thorngren, '23 & '24]
- We need two different types of maps...

Spiral Maps

Write $ME_{\ell,k} := ((B\mathbb{Z}/2)^{\sigma-1})^{\wedge \ell} \wedge ((B\mathbb{Z}/2)^{1-\sigma})^{\wedge k}$.

Definition

For $\ell \geq 0$ and $k > 0$, set

$$\phi_{\ell,k}\colon ME_{\ell,k} = ME_{\ell,k-1} \wedge (B\mathbb{Z}/2)^{1-\sigma} \xrightarrow{\mathrm{id}\wedge e_\sigma} ME_{\ell,k-1} \wedge \Sigma(B\mathbb{Z}/2)_+ \xrightarrow{\mathrm{id}\wedge c} \Sigma ME_{\ell,k-1},$$

$$\begin{tikzcd}
\psi_{\ell,k}\colon ME_{\ell,k} = ME_{\ell,k-1} \wedge (B\mathbb{Z}/2)^{1-\sigma} \arrow[r, "\mathrm{id}\wedge e_\sigma"] & ME_{\ell,k-1} \wedge \Sigma(B\mathbb{Z}/2)_+ \arrow[d, "\mathrm{id}\wedge\Delta"] & \\
& ME_{\ell,k-1} \wedge \Sigma ME_{1,1} \arrow[r, equal] & \Sigma ME_{\ell+1,k}.
\end{tikzcd}$$

Definition

The *spiral maps* are the induced maps on twisted spin IFTs.

Example Spiral Map

Consider dimension 3 and spin-$(1,1)$, a.k.a. dpin:

- On bordism...

  $$\Omega_3^{\mathrm{DPin}} \cong \mathbb{Z}/8 \xrightarrow[\cong]{\phi} \mathbb{Z}/8 \cong \Omega_2^{\mathsf{Pin}^-}$$

  $$(\mathbb{RP}^3, L_1, L_2) \longmapsto (\mathbb{RP}^2, L_1)$$

  where $L_1 = L_2 = \sigma$.

- On IFTs...

  $$\mho^2_{\mathsf{Pin}^-} \cong \mathbb{Z}/8 \xrightarrow[\cong]{\mathrm{sp}^{\phi}_{1,1}} \mho^3_{\mathrm{DPin}} \cong \mathbb{Z}/8.$$

Spiral Maps in General

Theorem

Restricted to $\operatorname{im} F2I$ in the relevant degrees,

$$\mathrm{sp}^{\phi}_{\ell,k} \colon \mho^{d+1}_{\mathsf{Spin}}(ME_{\ell,k-1}) \to \mho^{d+2}_{\mathsf{Spin}}(ME_{\ell,k})$$

is an *isomorphism* and

$$\mathrm{sp}^{\psi}_{\ell,k} \colon \mho^{d+1}_{\mathsf{Spin}}(ME_{\ell+1,k}) \to \mho^{d+2}_{\mathsf{Spin}}(ME_{\ell,k}).$$

is *multiplication by 2*.

The Bott Spiral Unraveled

$$\begin{tikzcd}
\mho^{i}_{3,0} \arrow[r, "\phi"] & \mho^{i+1}_{3,1} \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mho^{i+2}_{3,2} \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mho^{i+3}_{3,3} \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mho^{i+4}_{3,4} \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mho^{i+5}_{3,5} \arrow[r, "\phi"] \arrow[d, "\psi"] & \cdots \\
\mho^{i+1}_{2,0} \arrow[r, "\phi"] & \mho^{i+2}_{2,1} \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mho^{i+3}_{2,2} \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mho^{i+4}_{2,3} \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mho^{i+5}_{2,4} \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mho^{i+6}_{2,5} \arrow[r, "\phi"] \arrow[d, "\psi"] & \cdots \\
\mho^{i+2}_{1,0} \arrow[r, "\phi"] & \mho^{i+3}_{1,1} \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mho^{i+4}_{1,2} \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mho^{i+5}_{1,3} \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mho^{i+6}_{1,4} \arrow[r, "\phi"] \arrow[d, "\psi"'] & \mho^{i+7}_{1,5} \arrow[r, "\phi"] \arrow[d, "\psi"] & \cdots \\
 & \mho^{i+4}_{0,1} \arrow[r, "\phi"] & \mho^{i+5}_{0,2} \arrow[r, "\phi"] & \mho^{i+6}_{0,3} \arrow[r, "\phi"] & \mho^{i+7}_{0,4} \arrow[r, "\phi"] \arrow[d, "\psi\circ(\mathbf{2.37})"'] & \mho^{i+8}_{0,5} \arrow[r, "\phi"] \arrow[d, "\psi\circ(2.38)"] & \cdots \\
 & & & & \mho^{i+8}_{3,0} \arrow[r, "\phi"'] \arrow[d, "\psi"] & \mho^{i+9}_{3,1} \arrow[r, "\phi"] \arrow[d, "\psi"] & \cdots \\
 & & & & \vdots & \vdots &
\end{tikzcd}$$

The Spiral from $d = 1$ Class BDI$'$

$$\begin{tikzcd}
\mathrm{BDI}' & \mathrm{D}'' & \mathrm{DIII}' & \mathrm{AII}' & \mathrm{CII} & \mathrm{C}' & \mathrm{CI} & \mathrm{AI}' & \mathrm{BDI}' & \\
\mho^2_{2,1} \arrow[r, "\phi"] & \mho^3_{2,2} \arrow[r, "\psi"] & \mho^4_{1,2} \arrow[r, "\phi"] & \mho^5_{1,3} \arrow[r, "\psi"] & \mho^6_{0,3} \arrow[r, "\phi"] & \mho^7_{0,4} \cong \mho^7_{4,0} \arrow[r, "\psi"] & \mho^8_{3,0} \arrow[r, "\phi"] & \mho^9_{3,1} \arrow[r, "\psi"] & \mho^{10}_{2,1} \arrow[r] & \cdots \\
\cup & \cup & \cup & \cup & \cup & \cup & \cup & \cup & \cup & \\
\mathbb{Z}/4 \arrow[r, "\cong"] & \mathbb{Z}/4 \arrow[r, "\times 2"] & \mathbb{Z}/8 \arrow[r, "\cong"] & \mathbb{Z}/8 \arrow[r, "\times 2"] & \mathbb{Z}/16 \arrow[r, "\cong"] & \mathbb{Z}/16 \arrow[r, "\times 2"] & \mathbb{Z}/32 \arrow[r, "\cong"] & \mathbb{Z}/32 \arrow[r, "\times 2"] & \mathbb{Z}/64 \arrow[r] & \cdots
\end{tikzcd}$$

- Need discrete symmetry types
- Need additional primes on first three classes
- Need exceptional choice for class C and to use the isomorphisms $E_{4,0} \cong E_{0,4}$
- Observe “interacting Bott periodicity”

Related Articles

- Arun Debray, CK, and Luuk Stehouwer. *Unraveling the Bott Spiral*. Coming soon.
- Omar Antolín Camarena, Arun Debray, CK, Natalia Pacheco-Tallaj, Daniel Sheinbaum, and Luuk Stehouwer. *Weak Topological Phases in the Presence of Interactions*. arXiv: `2410.10031 [math-ph,cond-mat.str-el,hep-th]`.

Thank you!

Outline

- Free versus Interacting SPTs
- Fermionic Groups and Classification of SPTs
- Modeling SPTs in the “Bott Spiral”
- The Free-to-Interacting Map
- Dimensional Reduction
- Extra: Computations

Idea of the Computation I

- Computing

  $$F2I_{\ell,k} \colon KO^{d+\ell-k-2} \to \mho^{d+2}_{\mathrm{Spin}}(ME_{\ell,k})$$

  relies on computing

  $$ABS_{\ell,k} \colon MT\mathrm{Spin} \wedge ME_{\ell,k} \to \Sigma^{k-\ell}KO,$$

  which is determined by the restriction to

  $$ME_{\ell,k} \to \Sigma^{k-\ell}KO$$

- Study $ko_*(ME_{\ell,k})$ via the Adams spectral sequence

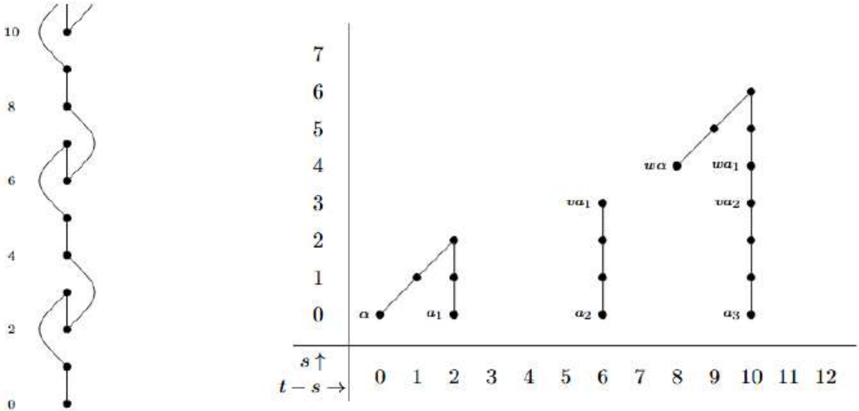
Idea of the Computation II

Recall $ME_{\ell,k} := ((B\mathbb{Z}/2)^{\sigma-1})^{\wedge\ell} \wedge ((B\mathbb{Z}/2)^{1-\sigma})^{\wedge k}$.

$H^*(ME_{\ell,k}; \mathbb{Z}/2)$ has a nice $\mathcal{A}(1)$-module structure in terms of the following...

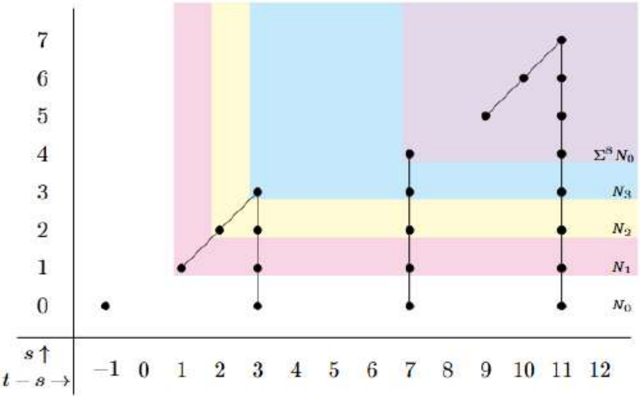
- $N_0 := \Sigma^{-1} H^*((B\mathbb{Z}/2)^{1-\sigma}; \mathbb{Z}/2)$
- $N_1 := H^*((B\mathbb{Z}/2)^{\sigma-1}; \mathbb{Z}/2)$
- $N_2 \subset \Sigma^{-2}\widetilde{H}^*(BA_4; \mathbb{Z}/2)$ unique non-free summand
- $N_3 \subset \Sigma^{-3}\widetilde{H}^*(BA_4 \wedge B\mathbb{Z}/2; \mathbb{Z}/2)$ unique non-free summand

Idea of the Computation III



$\mathcal{A}(1)$-module N_1 and the E_2 page of the Adams spectral sequence

Idea of the Computation IV



E_2 page of the Adams spectral sequence for shifts of the $\mathcal{A}(1)$-modules N_i

Idea of the Computation V

Lemma [Yu, '95]

Up to suspensions, there are stable isomorphisms of $\mathcal{A}(1)$-modules

$$N_i \otimes N_j \simeq N_{i+j \bmod 4}$$

Proposition [Debray–K.–Stehouwer]

Let $i = \ell \bmod 4$ and $j = \lfloor \ell/4 \rfloor$. If at least one of ℓ, k is positive, then

$$H^*(ME_{\ell,k}; \mathbb{Z}/2) \simeq \Sigma^{4j+k} N_i.$$

- Differentials and extension problems are all trivial for the N_i's
- We can ignore *Whitney summands*, i.e. summands that are trivialized upon inverting β
- A version of Margolis homology is effective for computing the remainder

Computational Results

Theorem [Debray–K.–Stehouwer]

Let $\widetilde{m} := \lfloor (d - \ell - k + i + 1)/8 \rfloor$, where $i \in \{0, 1, 2, 3\}$ with $i \equiv \ell \mod 4$. For d large enough, the image of the type-$E_{\ell,k}$ free-to-interacting map

$$F2I_{\ell,k} \colon KO^{d+\ell-k-2} \to \mho^{d+2}_{\mathrm{Spin}}(ME_{\ell,k})$$

is

$$\operatorname{im}(F2I_{\ell,k}) \cong \begin{cases} \mathbb{Z}/2^{4+4\widetilde{m}-i} & d \equiv \ell + k - 2i + 2 \bmod 8 \\ \mathbb{Z}/2^{5+4\widetilde{m}-i} & d \equiv \ell + k - 2i + 6 \bmod 8 \\ \mathbb{Z}/2 & d \equiv \ell + k - 2i \bmod 8 \text{ or } \ell + k - 2i + 1 \bmod 8. \end{cases}$$