

# The stable homotopy theory of invertible gapped quantum spin systems

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## The main subject

The (smooth) homotopy set

$$[\mathcal{M}, IP_d] := \text{Map}_{\text{smooth}}(\mathcal{M}, IP_d) / \sim,$$

where  $H_1 \sim H_2$  if they are connected by a smooth deformation.

Temporary notation for the introduction:

$$IP_d := (\text{the space of all possible } d\text{-dimensional Hamiltonians}).$$

Requirement: **Short-range**, **gapped**, and **invertible**.

1. **Short-range**: Nearest neighborhood interaction, second-nearest, ...  
Interactions can extend infinitely far, but their rate decays rapidly.
2. **Gapped**:  $H$  has a (locally) unique ground state and a spectral gap above it.
3. **Invertible**: Explained later.

## Theorem (Kitaev's proposal '13, K.'25)

*The spaces  $\{IP_d\}_{d \in \mathbb{N}}$  form an  $\Omega$ -spectrum.*

Motivation: the set of SPT phases  $:= \pi_0(IP_d^G)$ .

# Definition: Observables, Hamiltonian, time evolution

## Our standpoint

We deal with infinite systems in functional-analytic way.

Benefit: Simple and rigorous formulation of the **bulk spectral gap**.

- ▶  $\Lambda$ : discrete metric space (typically  $\Lambda \subset \mathbb{R}^{d+l}$ ).
- ▶ We place a matrix algebra  $\mathcal{A}_x \cong M_n(\mathbb{C})$  on each  $x \in \Lambda$ .
- ▶ The observable algebra is  $\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{A}_x$ .
- ▶ A Hamiltonian is an infinitesimal generator of the time evolution on  $\mathcal{A}_\Lambda$ .
- ▶  $a \in \mathcal{A}_\Lambda$  is **almost local** if  $a$  is well approximated by  $\mathcal{A}_{B_r(x)}$ .

Precise definition: via  $\Pi_Z := (\text{id}_Z \otimes \text{tr}_Z c)$ ,

$$\|a\|_{x,f} := \max \left\{ \|a\|, \sup_{r \in \mathbb{R}_{>0}} (f(r))^{-1} \cdot \|a - \Pi_{B_r(x)}(a)\| \right\} < \infty$$

for any  $f(r) = \exp(-r^\mu)$  for  $0 \leq \mu < 1$ .

- ▶ If  $a$  is localized at  $x \in \Lambda$  and  $b$  is localized at  $y \in \Lambda$ , then  $\|[a, b]\|$  is as small as  $f(d(x, y))$ .

## Definition: Observables, Hamiltonian, time evolution

- ▶ Let  $H = \{H_x\}_{x \in \Lambda}$  such that each  $H_x$  is localized at  $x$  uniformly, i.e.,

$$\|H\|_f := \sup_{x \in \Lambda} \|H_x\|_{x,f} < \infty.$$

- ▶ Then, for  $a \in \mathcal{A}_\Lambda$  localized at  $y \in \Lambda$ ,  $\|[H_x, a]\|$  decays as  $x \rightarrow \infty$ , and hence the infinite sum

$$[H, a] := \sum_{x \in \Lambda} [H_x, a]$$

converges.

- ▶ It generates a derivation  $\text{ad}(H)$  on  $\mathcal{A}_\Lambda^{\text{al}}$ .
- ▶ By the analysis of **Lieb–Robinson bound**, such a derivation is integrated to  $\tau_H: \mathbb{R} \curvearrowright \mathcal{A}_\Lambda$ .
- ▶ This time evolution preserves the almost locality (with finite spread). In particular,  $\{\tau_{H,t}(H'_x)\}_x$  is also uniformly almost local.
- ▶ The notion of ground state and spectral gap is defined in terms of operator algebra (Gelfand–Naimark–Segal theory).

# Kitaev's pump

- ▶ The trivial (atomic) Hamiltonian  $h$  is initially fixed.
- ▶ The composite system  $H_1 \boxtimes H_2$  on  $\mathcal{A}_{\Lambda_1} \otimes \mathcal{A}_{\Lambda_2}$  (omitted).

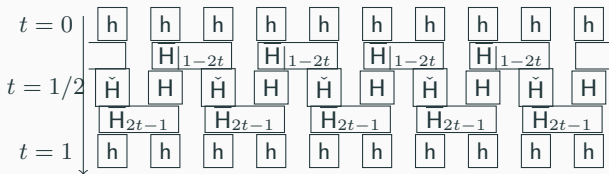
## Definition

$H$  is invertible if it has a homotopy inverse with respect to  $\boxtimes$ .

I.e.,  $\exists \check{H}$ : gapped Hamiltonian, and

$\exists \bar{H}(t)$ : smooth homotopy  $H \boxtimes \check{H} \simeq h$  of gapped Hamiltonians.

The **Kitaev pump** is  $\kappa_d(H, \check{H}, \bar{H}) \in \Omega IP_{d+1}$ .



Kitaev (also Xiong and Gaiotto–Johnson–Freyd) proposed a construction of a homotopy inverse of  $\kappa_d$ . We follow their approach, paying careful attention to the analytic details.

# Modulating Hamiltonian

What we want: the “modulating Hamiltonian” associated to a smooth path  $H_t$ :



In particular, when we have  $H(0) = H(1) = h$ , the associated modulating Hamiltonian should be localized near  $\mathbb{R}^d \times \{0\}$ .



An easy idea: place  $H_{\chi(x_d)}$  on  $\mathbf{x}$ , i.e.,  $\tilde{H}_{\mathbf{x}} = H_{\chi(x_d), \mathbf{x}}$  ( $\chi$ : chopping function).

## Stumbling block 1

Such an easy construction may violate the spectral gap.

Solution: Use Hastings' adiabatic theorem.

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For  $G(t) = \{G_x(t)\}$  uniformly almost local,  $\exists$  parallel transport

$$\alpha(G; t): \mathcal{A}_\Lambda \rightarrow \mathcal{A}_\Lambda, \quad \frac{d}{dt}\alpha(G; t)(a) = [G, \alpha(G; t)(a)].$$

## Theorem (Hastings' adiabatic process '04)

$H(t)$ : 1-parameter family of Hamiltonians with the ground state  $\omega_t$ .

Then  $\exists G_H(t)$  s.t.  $\omega_t = \omega_0 \circ \alpha(G_H; t)^{-1}$ .

Idea: Truncate  $G_H$  at the right half space and transport  $\omega_0$  by it.

Then,  $\tilde{\omega} := \omega_0 \circ \alpha(\Pi_R(G_H); 1)^{-1}$  interpolates  $\omega_0$  at the left and  $\omega_1$  at the right.



$\omega_0$

$\omega_1$

## Stumbling block 1'

Adiabatic interpolation can handle the ground state but **cannot control the Hamiltonian itself.**

While the topology of the Hamiltonian and that of the ground state are considered to be the same, in some context there can be subtle differences.

E.g.  $H \in \Omega IP_{d+1}$ . A Hamiltonian  $\tilde{H} = \alpha(\Pi_R(G_H); 1)(H(0))$  obtained by the truncated adiabatic transport has the ground state  $\tilde{\omega}$ .



But  $\tilde{H}$  may not be approximated by a  $d$ -dimensional Hamiltonian.

## Theorem (K.'25)

*The adiabatic interpolation lifts to an interpolation of Hamiltonians.*

## Stumbling block 2

Hastings' adiabatic theorem is valid only for **smooth** paths of Hamiltonians. However, topology is a framework to deal with continuous maps.

Solution: We formulate  $IP_d$  as a **diffeological space** (sheaf on Man) instead of a topological space.

Diffeological space: a set  $X$  equipped with the additional structure given by the distinguished subsets of smooth maps

$$C^\infty(\mathcal{M}, X) \subset \text{Map}(\mathcal{M}, X)$$

respecting some axioms.

## Theorem (Madsen–Weiss '03)

*There is a topological space  $|\text{Sing } X|$  such that  $[\mathcal{M}, |\text{Sing } X|]_{\text{cont}} \cong [\mathcal{M}, X]_{\text{sm}}$ .*

Define  $\mathcal{JP}_d$  as diffeological space  $\rightsquigarrow IP_d := |\text{Sing } \mathcal{JP}_d|$ .

## Theorem (K.'25)

*Madsen–Weiss theorem can be made  $G$ -equivariant for  $G$ : compact Lie group.*

## Stumbling block 3

We want both  $\vartheta_d \circ \kappa_d \simeq \text{id}$  and  $\kappa_d \circ \vartheta_d \simeq \text{id}$ . The former is easy to be verified, but the latter is not. Indeed, **the latter is probably wrong**.

Evidence: The equivalence fails for free fermions (C\*-algebra K-theory).

Q. Why does  $\kappa_d \circ \vartheta_d \simeq \text{id}$  fail in  $1D$ ?

A. A boundary-gapped (or right-half supported) Hamiltonian is not necessarily contained in the trivial phase.



Q. How does it resolved?

A. We allow virtual dimensional spatial expansion. E.g.



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# Internal Degrees of freedom

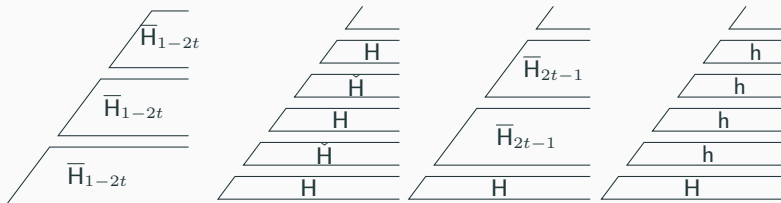
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is a 1D lattice (by regarding the  $y$ -direction is 'internal').

This enables us to apply the Carlsson–Petersen–Higson–Roe Eilenberg swindle:



$H$  supported on the right half is in the trivial phase!

(1) Kapustin–Sopenko higher order Berry phase is a natural transform

$$\text{Ch}_{\text{KS}}: [\mathcal{M}, IP_d] \rightarrow H^{d+2}(\mathcal{M}; \mathbb{Q}).$$

- ▶ This coincides with a part of the Chern–Dold character.
- ▶ The existence of integral lifts

$$\text{Ch}_{\text{KS}}^{\mathbb{Z}}: [\mathcal{M}, IP_d] \rightarrow H^{d+2}(\mathcal{M}; \mathbb{Z})$$

in  $d \leq 2$  by Artymowicz–Kapustin–Sopenko '24 is reproved.

(2) There is a fermionic version  $fIP_d$ .

- ▶ (free fermions)  $\subset$  (fermions) gives rise to a morphism of spectra

$$Q: \Sigma^{-2}KO \rightarrow fIP.$$

Definition: **Araki's quasi-free second quantization** (in preparation).

- ▶ Truncated spectrum  $fIP\langle -2, \infty \rangle$  satisfies the weak equivalences

$$cAlg_{\mathbb{C}}^{\times} \simeq fIP\langle -2, \infty \rangle \simeq (\Sigma^{-2}KO)\langle -2, 2 \rangle,$$

which answers to a question by Freed.

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(3) The homotopy groups of  $n - d \geq -1$  are completely determined.

$n - d$	$\cdots$	$-3$	$-2$	$-1$	$0$	$1$	$2$	$3$	$\cdots$
$\pi_n(IP_d)$	$\cdots$	?	?	0	0	0	$\mathbb{Z}$	0	$\cdots$
$\pi_n(fIP_d)$	$\cdots$	?	?	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\cdots$
$\pi_n(KO_{d+2})$	$\cdots$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\cdots$

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(4) We get an appropriate refinement of cohomology-valued SPT index:

$$\pi_0(IP_d^G) = [\text{pt}, IP_d^G] \rightarrow [EG, IP_d^G] \cong [BG, IP_d] \xrightarrow{\text{if } d \leq 2} [BG, K(\mathbb{Z}, d+2)].$$

This reproves

- ▶  $\pi_0(IP_1^G) \cong H^2(G; \mathbb{T})$  (Gu–Wen '09, Pollmann–Berg–Tener–Oshikawa '09, Ogata '19, Kapustin–Sopenko–Yang '20, Carvalho–de Roeck–Jappens '24)
- ▶  $\pi_0(fIP_1^G) \cong \mathbb{Z}/2 \oplus H^1(G; \mathbb{Z}/2) \oplus H^2(G; \mathbb{T})$ . (Bourne–Ogata '20)
- ▶  $\pi_0(IP_2^G) \rightarrow H^3(G; \mathbb{T})$  (Ogata '22, Sopenko '22)
- ▶ Invariant of  $2D$  fermionic SPT phases taking value in 'supercohomology'.  
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- ▶ Invariant of 2D fermionic SPT phases taking value in 'supercohomology'. (Gu–Wen '14, Ogata '23)

(5) There are 3 ways to construct a model so far (in preparation).

- ▶ Lattice Dijkgraaf–Witten model (Gu–Wen '13, Ogata '21, Tasaki).  
DW:  $\Sigma^{-2}H\mathbb{Z} \rightarrow IP$ .
- ▶ Araki's quasi-free second quantization of free fermions (BdG Hamiltonians).  
Q:  $\Sigma^{-2}KO \rightarrow fIP$ .
- ▶ Matrix product state in 1D (Beaudry–Hermele–Pflaum–Qi–Spiegel–Stephen '25).  
MPS:  $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3) \rightarrow IP_1^{\mathbb{Z}}$ .

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(6) We can place quantum spin systems on a more general metric space  $X$  than  $\mathbb{R}^d$ .

- ▶ The space

$$IP(X) := (\text{invertible gapped Hamiltonians on a lattice of } X)$$

gives a **coarse homotopy theory**, which violate the local topology of  $X$ .

- ▶ We follow the idea of G. Yu in coarse geometry replacing 'finite' range to 'infinitesimal'.
- ▶ **Definition:** A **localization flow** of Hamiltonians on  $X$  is a family  $\{H(s)\}_{s \in [1, \infty)}$  whose range decays as  $s \rightarrow \infty$ .
- ▶ Also inspired from matrix product renormalization group flow.
- ▶  $X \mapsto \pi_0(IP_{\text{loc}}(X))$  is a **generalized homology functor**, which agrees with the one coming from the  $\Omega$ -spectrum  $IP$ .

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(7) The crystallographic symmetry can be taken into account.

- ▶ The spaces  $IP^\Gamma(X)$  and  $IP_{\text{loc}}^\Gamma(X)$  is defined.
- ▶ They form  $\Omega$ -spectra via  $IP_d^\Gamma(X) := IP^\Gamma(X \times \mathbb{R}^d)$ .
- ▶ We are interested in  $IP^\Gamma(X)$ , besides  $IP_{\text{loc}}^\Gamma(X)$  is computable. E.g.  
 $\pi_0(IP_{\text{loc}}^{\mathbb{Z}^d}(\mathbb{R}^d)) \cong \pi_0(IP_{\text{loc}}(\mathbb{T}^d))$ .
- ▶  $[\mathcal{M}, IP_{\text{loc}}^\Gamma(X)] \rightarrow [\mathcal{M}, IP^\Gamma(X)]$  is split injective (IP Baum–Connes conj.).

- ▶ Kitaev's conjecture: the space  $IP_d$  of short-range, gapped, and invertible Hamiltonians in quantum spin system from an  $\Omega$ -spectrum.
- ▶ Invertible gapped Hamiltonian has a rigorous definition based on functional analysis and  $C^*$ -algebra.
- ▶ We can follow the original argument of Kitaev, as well as the idea by Higson–Roe–Yu in 1990's coarse geometry.
- ▶ Stumbling block 1: Construction of modulating Hamiltonian, a gapped domain walls associated to an adiabatic path.
  - ↪ Use Hastings' adiabatic theorem.
- ▶ Stumbling block 2: Hastings' theory works only for smooth paths.
  - ↪ Formulate  $IP$  as diffeological spaces.
- ▶ Stumbling block 3: We need an Eilenberg swindle argument.
  - ↪ Relax the notion of  $d$ -dimensional lattice.
- ▶ Many comments around the theorem.

Thank you for your attention.