Quivers and BPS states in 3d and 4d

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Based on 2508.09729 with
P. Longhi, D. Noshchenko, S. Park, & P. Sułkowski

NYU Abu Dhabi, October 22, 2025

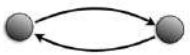
Outline

1. Motivation
2. Dimension door from two perspectives
   - Physics
   - Topology
3. Entering the dimension door: wall-crossing and unlinking
   - Chambers, dualities, and equivalent quivers
   - Structure of wall-crossing and unlinkings

So close, yet so far

BPS states in 3d $\mathcal{N} = 2$

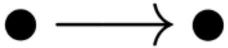
Symmetric quivers



Ekholm, P.K., Longhi, Reineke, Stošić, Sułkowski

BPS states in 4d $\mathcal{N} = 2$

Directed quivers



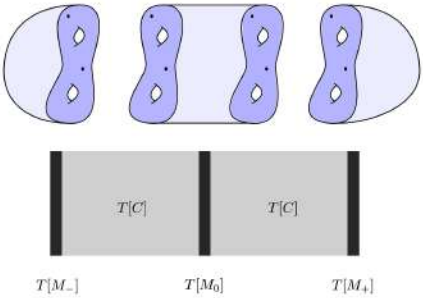
Gaiotto, Moore, Neitzke, Alim, Cecotti, Cordova, Espahbodi, Rastogi, Vafa...

How to travel between dimensions?

Outline

M-theory setup (6 = 2+4 = 3+3)

- Spacetime: $T^*M \times S^1 \times \mathbb{R}^4$
- Two M5-branes wrap $M \times S^1 \times \mathbb{R}^2$



- Coupled 3d-4d system
  - $C \sim \partial M_{\pm/0}$ gives 4d $\mathcal{N} = 2$ theory $T[C]$ on $S^1 \times \mathbb{R}^2 \times$ interval
  - $M_{\pm/0}$ gives 3d $\mathcal{N} = 2$ theory $T[M_{\pm/0}]$ on $S^1 \times \mathbb{R}^2$

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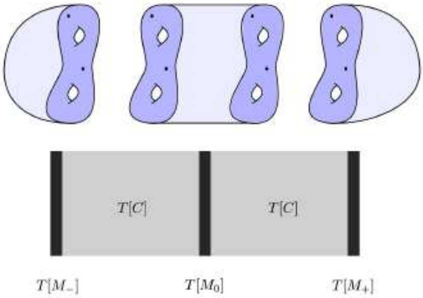

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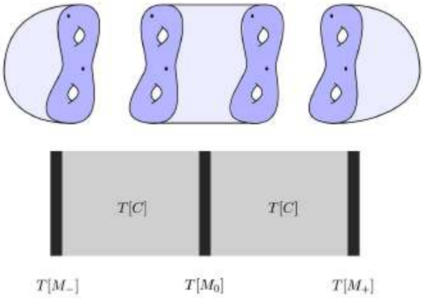

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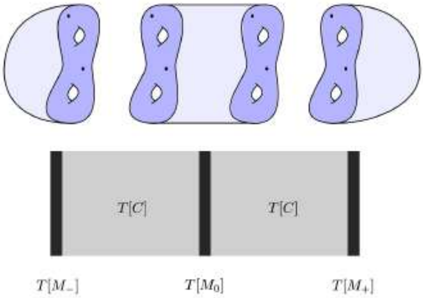

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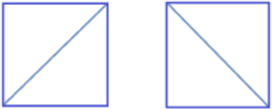
BPS states in 4d

- Argyres-Douglas theories: $C = \mathbb{C}$ with $n+3$ roots of unity $\sim$ $(n+3)$-gon
- They are associated to the A_n-type quivers:

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n}{\bullet}$$

- BPS states are given by flips of the triangulation



[Gaiotto-Moore-Neitzke]

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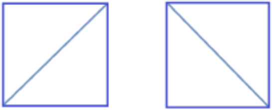

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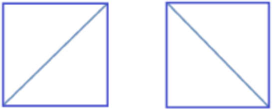

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Example: BPS states for the pentagon

- Example: Argyres-Douglas theory associated to A_2 quiver

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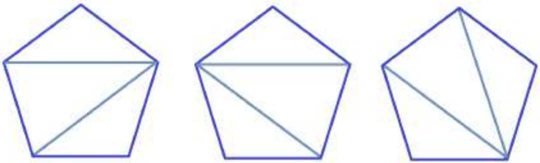
- $n = 2 \Rightarrow C = \mathbb{C}$ with 5 roots of unity $\sim$ pentagon



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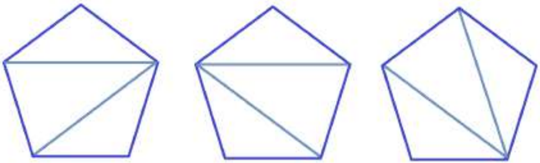

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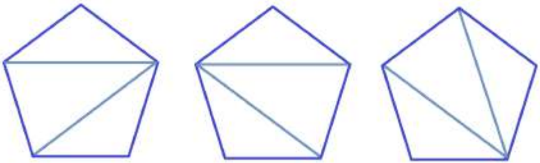

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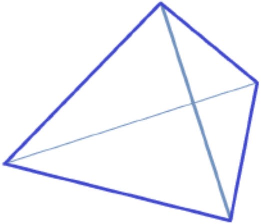
BPS states in 3d

- We want M_0 such that $\partial M_0 \sim$ polygons
- BPS states are given by tetrahedra



[Cecotti-Cordova-Vafa, Dimofte-Gaiotto-Gukov, Terashima-Yamazaki]

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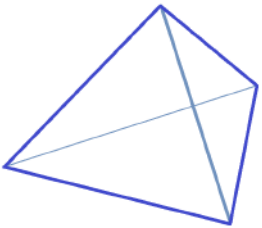

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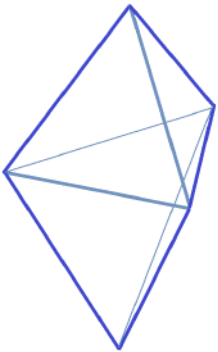

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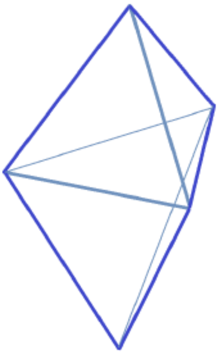

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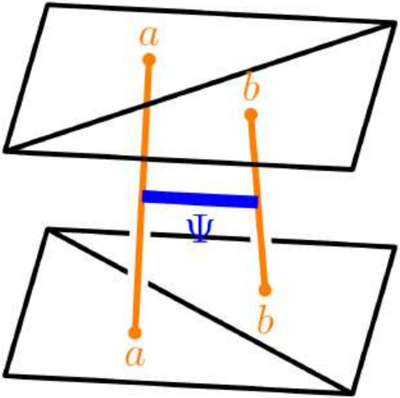
Outline

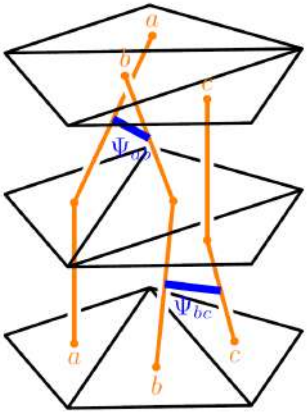
BPS states and holomorphic disks

- Consider double cover of C and M
- BPS states $\sim$ holomorphic disks (M2 branes) stretched between branching loci



Example: two disks

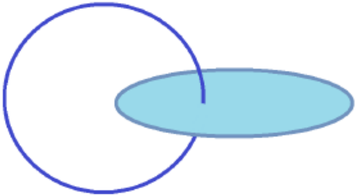
- Branching loci and disks in $C \sim$ pentagon, $M \sim$ bipiramid



[Cecotti-Cordova-Vafa]

Linking

- When we close M_0 by $M_{\pm}$, we can define linking
- We count intersections between one disk and the boundary of the other



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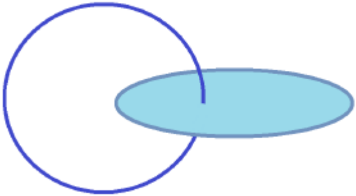
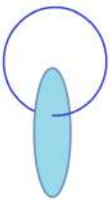


Linked holomorphic disks and symmetric quivers

- Knot-quiver correspondence $\Rightarrow$ linked holomorphic disks can be associated to symmetric quivers
  - holomorphic disks $\sim$ quiver nodes
  - linking $\sim$ quiver arrows
- We can apply this idea also to disks that do not come from knots!



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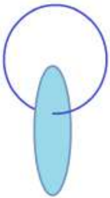


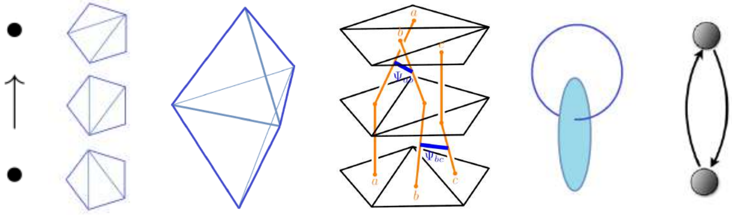
[Ekholm-P.K.-Longhi]

BPS counts and quivers

Count of BPS states in 4d theory $T[C]$ for directed quiver =

= Count of BPS states in 3d theory $T[M]$ such that $\partial M \sim C$

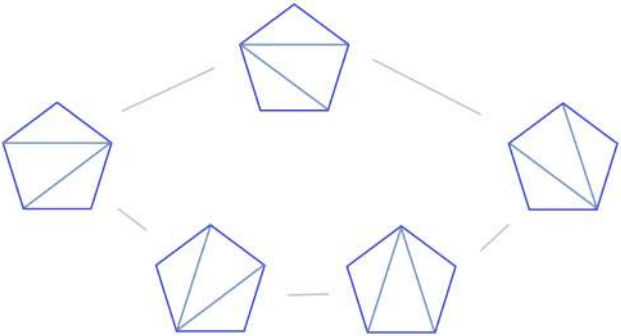
= Generating series of symmetrisation of the initial quiver



Outline

Chambers of 4d theory A_2 – pentagon relation

- We can do another, longer sequence of flips!



- We can also have 3 flips $\Rightarrow$ 3 BPS states
- Another chamber of the same 4d theory

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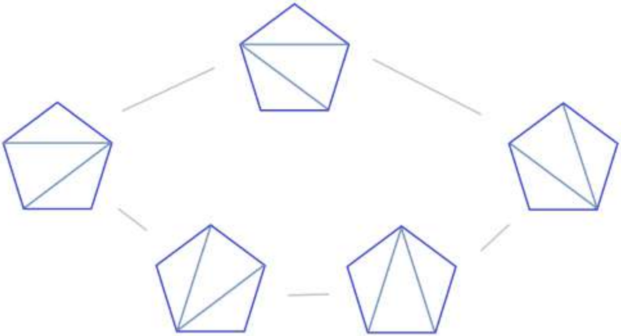

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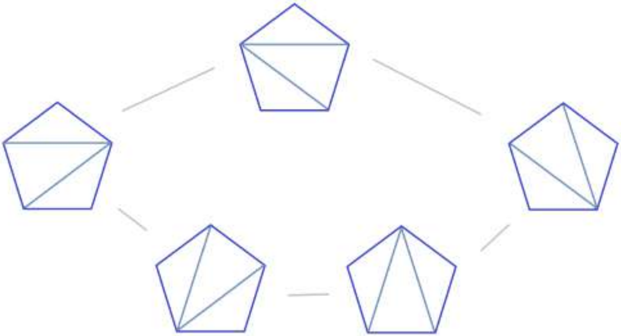

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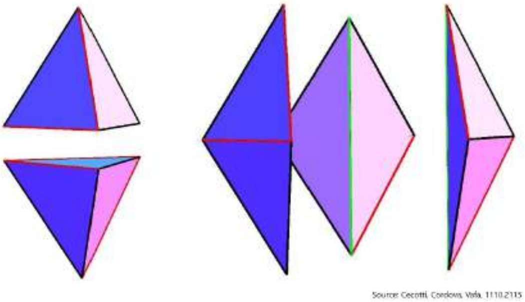
Dual 3d theories – Pachner move

- We can divide the bipiramid in two different ways!



- We can also have 3 tetrahedra ⇒ 3 BPS states
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[Cecotti-Cordova-Vafa, Dimofte-Gaiotto-Gukov, Terashima-Yamazaki]

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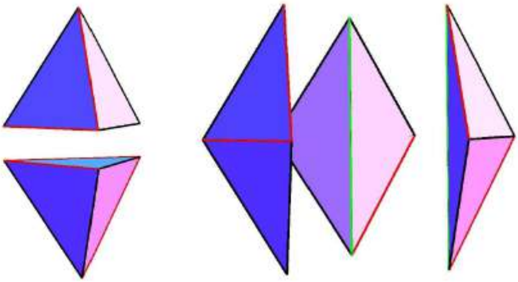

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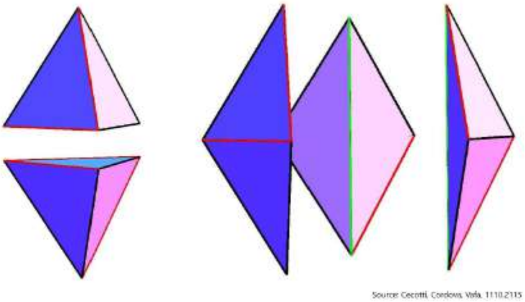

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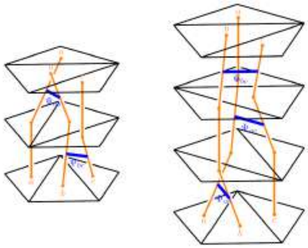
2 and 3 holomorphic disks

- 3 BPS states correspond to 3 holomorphic disks



- Algebraic version for non-commutative operators representing holomorphic disks – 2-3 wall-crossing relation:

$$\Psi_{ab}\Psi_{bc} = \Psi_{bc}\Psi_{ac}\Psi_{ab}$$
$$\Psi_1\Psi_2 = \Psi_2\Psi_{1+2}\Psi_1$$

[Fadeev-Kashaev, Kontsevich-Soibelman]

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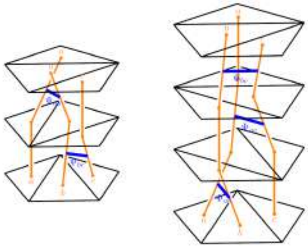

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Equivalent symmetric quivers – unlinking

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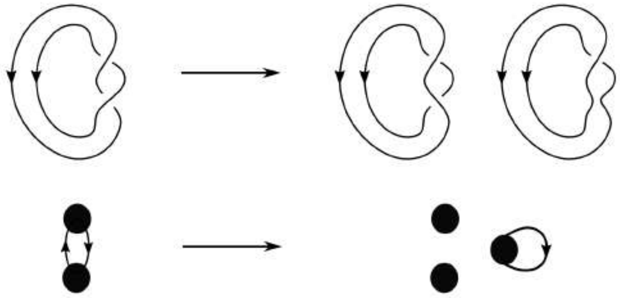

- Two symmetric quivers with the same generating series

[Ekholm-P.K.-Longhi]

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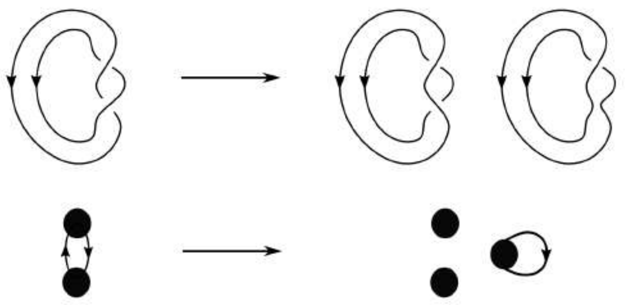

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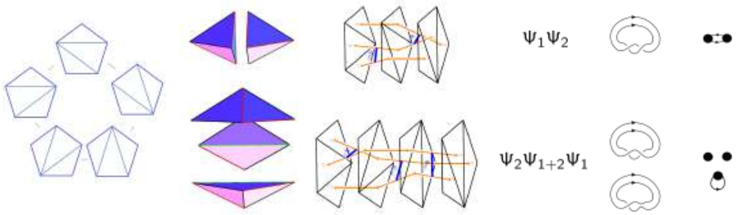
2-3 move

Two different chambers with 2 and 3 BPS states in 4d theory $\sim$

$\sim$ Two dual 3d theories with 2 and 3 BPS states

$\sim$ Two equivalent symmetric quivers with 2 and 3 nodes



Outline

More complicated wall-crossing: choices

- Example: Argyres-Douglas theory associated to A_3 quiver

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet}$$

- We can perform 2-3 move for (12) and (23)
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More complicated wall-crossing: choices

- Algebraic perspective:
  - operators connected by arrows satisfy 2-3 wall-crossing relation
  - operators which are not connected by arrows commute

$$\Psi_3\Psi_{2+3}\Psi_2\Psi_{1+2+3}\Psi_{1+2}\Psi_1$$

$$\nearrow \qquad\qquad\qquad \nwarrow$$

$$\Psi_2\Psi_3\Psi_{1+2+3}\Psi_{1+2}\Psi_1 \qquad\qquad \Psi_3\Psi_{2+3}\Psi_{1+2+3}\Psi_1\Psi_2$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

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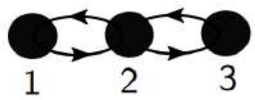
More complicated unlinking: choices

- Example: symmetrisation of the A_3 quiver



- We can unlink (12) and (23)
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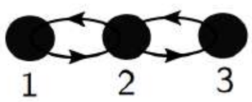

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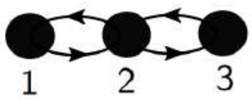

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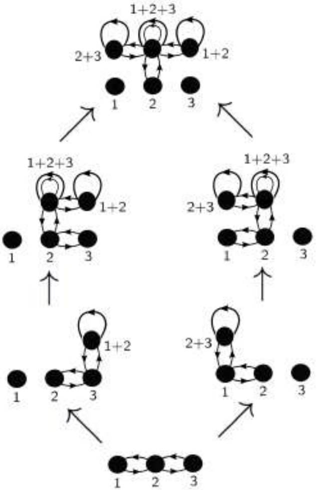

Wall-crossing for A_3 and unlinking for its symmetrisation

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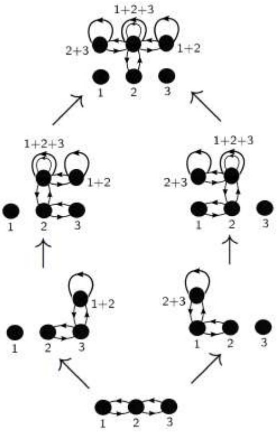
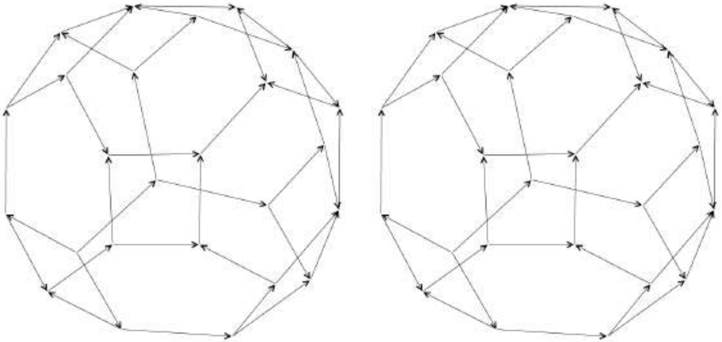


Wall-crossing for A_4 and unlinking for its symmetrisation



- The same pattern works for all A_n quivers and their symmetrisations!

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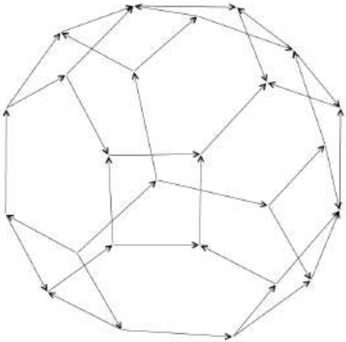

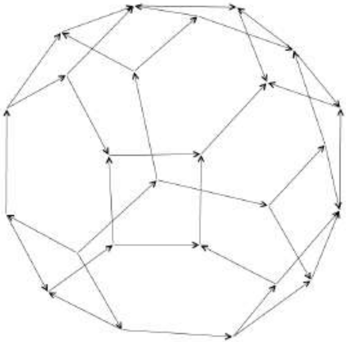

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Summary

- Proper division of theory on M5-brane (6d = 4d+2d = 3d+3d) allows for relating BPS spectra in 4d and 3d theories associated to 2d and 3d manifolds
- This allows to relate directed quivers to their symmetrisations
- This relation maps the structure of wall-crossing to the structure of unlinkings

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Last messages

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Thank you for attention!