A Quillen model structure for 2-categories

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Abstract

We describe a cofibrantly generated Quillen model structure on the locally finitely presentable category **2-Cat** of (small) 2-categories and 2-functors; the weak equivalences are the biequivalences, and the homotopy relation on 2-functors is just pseudonatural equivalence. The model structure is proper, and is compatible with the monoidal structure given by the Gray tensor product. It is not compatible with the cartesian closed structure, in which the tensor product is the product.

The model structure restricts to a model structure on the full subcategory **PsGpd** of **2-Cat**, consisting of those 2-categories in which every arrow is an equivalence and every 2-cell is invertible. The model structure on **PsGpd** is once again proper, and compatible with the monoidal structure given by the Gray tensor product.

A Quillen model structure on a category allows one to do abstract homotopy theory within that category; it gives rise, for example, to a homotopy relation on maps, a notion of homotopy equivalence, and path object and mapping cylinder constructions. The notion was defined by Quillen in [22, 23] under the name of a closed model category; today the word "closed" is usually dropped (originally there was also a weaker structure called a model category, but this has proved to be of lesser interest) and one speaks simply of a model category, or model structure on a category. The structure consists of three classes of morphisms in the category, called the cofibrations, the weak equivalences, and the fibrations; it is the weak equivalences which are the abstraction of the notion of homotopy equivalence. These classes of morphisms are required to satisfy various conditions, recalled in Section 1 below.

In this paper we describe a Quillen model structure on the category **2-Cat** of small 2-categories and 2-functors. It is closely related to the model structure of Moerdijk and Svensson [20] on the category **2-Gpd** of small 2-groupoids and 2-functors. In particular, the notion of fibration and weak equivalence introduced below generalize those of [20]. The cofibrations, however, look slightly different.

Classical examples of categories with a model structure include the category **SSet** of simplicial sets, the category **Top** of topological spaces, the category **DGAb** of differential graded abelian groups, and the category **Cat** of (small) categories. In the case of **Cat** there are at least two different model structures. One of these, due to Thomason [25] (but see also [8]), is derived from

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the model structure on **SSet**, and a functor is defined to be a weak equivalence if and only if the induced map between the nerves of the categories is a homotopy equivalence of simplicial sets. In the other more "categorical" model structure, the weak equivalences are the equivalences of categories, the cofibrations are the functors which are injective on objects, and the fibrations are the functors $f : A \to B$ with the property that for each object a of A and for each isomorphism $\beta : b \cong fa$ in B, there is an isomorphism $\alpha : a' \cong a$ in A with $fa' = b$ and $f\alpha = \beta$. It is easy to see that the weak equivalences which are also fibrations are precisely the equivalences of categories which are surjective on objects; in [5] these were called the *retract equivalences*, since they are precisely the functors $f : A \to B$ for which there exists a functor $g : B \to A$ with $fg = 1_B$ and $gf \cong 1_A$. This model structure seems to have existed as a folklore definition for a long time. Its first appearance in print known to the author is in [17], where it is shown that an analogously defined model structure exists for the category $\mathbf{Cat}(\mathscr{E})$ of categories internal to a topos $\mathscr{E}$; the case of ordinary categories is then the case $\mathscr{E} = \mathbf{Set}$. It is this "categorical" model structure on **Cat** (henceforth called simply *the* model structure on **Cat**) which is most similar to the model structure we define on **2-Cat**.

For an introduction to 2-categories, one might consult [19]; certain facts about 2-categories are also reviewed in Section 2. In the meantime, we recall that 2-categories have not just objects and morphisms between objects, but also 2-cells between morphisms; and that these 2-cells can themselves be composed in various ways, satisfying strict associativity and unit conditions. Given 2-categories $\mathscr{A}$ and $\mathscr{B}$, one often considers *pseudofunctors* from $\mathscr{A}$ to $\mathscr{B}$ — which preserve composition only up to coherent isomorphism — but **2-Cat** contains only the *2-functors*, which preserve all composites in the strict sense. In fact the model structure on **2-Cat** will allow us to say certain things about the relationship between 2-functors and pseudofunctors.

Although we are considering **2-Cat** only as a category, it does have further structure. For example, it is cartesian closed, and so one can talk about categories enriched in **2-Cat** with respect to this cartesian closed structure: these are precisely the 3-categories. Then **2-Cat** itself has a canonical 3-category structure, involving the 2-functors, 2-natural transformations, and modifications. There is also another symmetric monoidal closed structure on **2-Cat**: the monoidal part of the structure is given by the "Gray tensor product" [12] and the internal hom $[\mathscr{A}, \mathscr{B}]$ is the 2-category whose objects are the 2-functors from $\mathscr{A}$ to $\mathscr{B}$, and whose morphisms and 2-cells are the pseudonatural transformations and the modifications. A category enriched in **2-Cat** with respect to this second monoidal structure is called a **Gray**-category [12, 11]; it is like a 3-category except that the "middle four interchange" holds only up to coherent isomorphism. The canonical **Gray**-category structure on **2-Cat** involves the 2-functors, pseudonatural transformations, and modifications. We shall see that it is only the second monoidal structure which is compatible with the model structure on **2-Cat**.

It is also worth commenting on the fact that we allow only small 2-categories in **2-Cat**. This allows us to employ algebraic techniques in **2-Cat**, and means that **2-Cat** is a locally finitely presentable category. It may, however, seem unnatural: the paradigmatic example of a 2-category is the 2-category **Cat** of small categories, and **Cat** is not small. One can deal with this problem by supposing a hierarchy of universes $\mathscr{U}_0 \in \mathscr{U}_1 \in \mathscr{U}_2$. Then one considers sets, groups, spaces and other "everyday" mathematical objects lying in $\mathscr{U}_0$; categories lying in $\mathscr{U}_1$, and 2-categories in $\mathscr{U}_2$. From this point of view, we would define **2-Cat** to be the category of all 2-categories which are small with respect to $\mathscr{U}_2$. Such precise foundational questions will be ignored in this paper, and we work with a single universe $\mathscr{U}$.

1 Model categories

In this section we recall the definition of a model category, and the special case of a cofibrantly-generated model category; in fact we are interested in the further special case where the category is *locally finitely presentable*, in the sense of Gabriel and Ulmer: see [10], or [2] for a more recent account. A category $\mathscr{K}$ is locally finitely presentable if it is cocomplete and has a small full subcategory $\mathscr{G}$ consisting of finitely presentable objects, with every object of $\mathscr{K}$ a filtered colimit of objects of $\mathscr{G}$. Equivalently, $\mathscr{K}$ is locally finitely presentable if and only if it is equivalent to the category of finite-limit-preserving functors from $\mathscr{C}$ to **Set**, for some small category $\mathscr{C}$ with finite limits. Every locally finitely presentable category is not just cocomplete but also complete.

As was stated in the introduction, a model structure on a category $\mathscr{K}$ consists of three classes of morphisms, called the cofibrations, the weak equivalences, and the fibrations, and denoted $\mathscr{C}$, $\mathscr{W}$, and $\mathscr{F}$ respectively. The category $\mathscr{K}$ is required to have finite limits and finite colimits, and the classes of morphisms are required to satisfy the following conditions involving the notion of weak factorization system, recalled below. We say that a class of morphisms in $\mathscr{K}$ is *closed under retracts* if it is so as a full subcategory of the arrow category $\mathscr{K}^{\mathbf{2}}$.

(1) (2-for-3) If f, g, and h are morphisms in $\mathscr{K}$ with $h = gf$, then if any two of f, g, and h is in $\mathscr{W}$, the third is also in $\mathscr{W}$;

(2) the weak equivalences are closed under retracts;

(3) there is a weak factorization system $(\mathscr{C}, \mathscr{W} \cap \mathscr{F})$ on $\mathscr{K}$;

(4) there is a weak factorization system $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ on $\mathscr{K}$.

Fibrations which are also weak equivalences are called *trivial fibrations*, while cofibrations which are also weak equivalences are called *trivial cofibrations*.

If $\mathscr{I}$ and $\mathscr{P}$ are classes of morphism in a category $\mathscr{K}$ we say that $(\mathscr{I}, \mathscr{P})$ is a weak (or Bousfield) factorization system [6] if the following conditions are satisfied:

(1) (retracts) The classes $\mathscr{I}$ and $\mathscr{P}$, when seen as full subcategories of the arrow-category $\mathscr{K}^{\mathbf{2}}$, are closed under retracts;

(2) (liftings) If in the diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow[p]{} & D \end{array}$$

$i \in \mathscr{I}$ and $p \in \mathscr{P}$ then there exists a "fill-in" $w : B \to C$ with $wi = u$ and $pw = v$;

(3) (factorizations) Any map $f : A \to B$ can be factorized as $f = pi$ where $p : E \to B$ is in $\mathscr{P}$ and $i : A \to E$ is in $\mathscr{I}$.

More generally, if i and p satisfy the lifting condition in the definition of weak factorization system, we say that i has the left lifting property with respect to p, and that p has the right lifting property with respect to i. It turns out that if $(\mathscr{I}, \mathscr{P})$ is a weak factorization system then $\mathscr{I}$

consists of precisely those maps with the left lifting property with respect to $\mathscr{P}$, and $\mathscr{P}$ consists of precisely those maps with the right lifting property with respect to $\mathscr{I}$.

For a modern introduction to model categories one might consult any of the excellent books [9, 14, 15]. In these sources a slightly stronger definition is used, in which the factorizations can be chosen functorially; but this will always be the case in the cofibrantly-generated context that we shall be dealing with. In fact we only deal with cofibrantly generated model categories which are locally finitely presentable, and this makes things particularly simple: see the discussion in [3].

We say that a weak factorization system $(\mathscr{I}, \mathscr{P})$ on a locally presentable category $\mathscr{K}$ is *cofibrantly generated* if there is a small set $\mathscr{G}$ of morphisms in $\mathscr{K}$ with the property that $\mathscr{P}$ consists precisely of those morphisms with the right lifting property with respect to (the maps in) $\mathscr{G}$. Conversely, given a small set $\mathscr{G}$ of maps in a locally presentable category $\mathscr{K}$, there is a weak factorization system $(\mathscr{I}, \mathscr{P})$ in which $\mathscr{P}$ consists of the maps with the right lifting property with respect to $\mathscr{G}$, and $\mathscr{I}$ consists of the maps with the left lifting property with respect to $\mathscr{P}$. The proof is essentially due to Quillen, and is called the "small object argument". It was first formulated for weak factorization systems in [1]. It is possible to give a more explicit description of the maps in $\mathscr{I}$. We say that $f : A \to B$ is a *relative $\mathscr{G}$-cell complex* if it is a transfinite composite of pushouts of maps in $\mathscr{G}$. Then $f : A \to B$ is in $\mathscr{I}$ if and only if there are maps $m : B \to C$ and $r : C \to B$ with $rm = 1_B$ for which mf is a relative $\mathscr{G}$-cell complex. Furthermore, for any f one can choose the factorization $f = pi$ with $p \in \mathscr{P}$ and $i \in \mathscr{I}$ in such a way that i is actually a relative $\mathscr{G}$-cell complex.

A model structure $(\mathscr{C}, \mathscr{W}, \mathscr{P})$ on a locally presentable category $\mathscr{K}$ is said to be cofibrantly generated if each of the weak factorization systems $(\mathscr{C} \cap \mathscr{W}, \mathscr{P})$ and $(\mathscr{C}, \mathscr{W} \cap \mathscr{P})$ is cofibrantly generated; thus the fibrations are the maps with the right lifting property with respect to a small set of "generating trivial cofibrations", and the trivial fibrations are the maps with the right lifting property with respect to a small set of "generating cofibrations".

Example 1.1 The model category **Cat** is cofibrantly generated. A functor is surjective on objects if and only if it has the right lifting property with respect to the unique functor $i : 0 \to 1$ from the initial (empty) category to the terminal category. It is full if and only if it has the right lifting property with respect to the inclusion functor $i' : 2 \to \mathbb{2}$ from the category with two objects and no non-identity arrows to the category with two objects and a single non-identity arrow between them. A functor is faithful if and only if it has the right lifting property with respect to the unique bijective-on-objects functor $i'' : C \to \mathbb{2}$, where C is the category with two objects and a parallel pair of non-identity arrows. Thus we may take i, i', and i'' as the generating cofibrations. Write I for the "free-living isomorphism" consisting of objects x, and y; and non-identity arrows $s : x \to y$ and $s^{-1} : y \to x$ satisfying $s^{-1}s = 1_x$ and $ss^{-1} = 1_y$. Then the fibrations for the model structure on **Cat** are the functors with the right lifting property with respect to the functor $j : 1 \to I$ picking out the object x, thus we may take j as the single generating trivial cofibration.

2 Facts about 2-categories

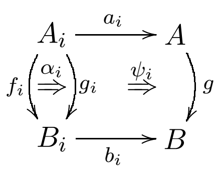
In this section we gather together some basic definitions and facts about 2-categories; see [19] for more details.

Every 2-category $\mathscr{A}$ has an *underlying category* $\mathscr{A}_0$ obtained by forgetting the 2-cells. Similarly every 2-functor $F : \mathscr{A} \to \mathscr{B}$ has an *underlying functor* $F_0 : \mathscr{A}_0 \to \mathscr{B}_0$. These constructions define a functor $U : \mathbf{2\text{-}Cat} \to \mathbf{Cat}$ which preserves limits and colimits. In fact U has both adjoints: the left adjoint sends a category A to the 2-category DA with $(DA)_0 = A$ and with no non-identity 2-cells; while the right adjoint sends A to CA, where $(CA)_0 = A$ and there is a unique 2-cell in CA between every parallel pair of arrows. The 2-categories of the form DA are called *locally discrete* and those of the form CA are called *locally chaotic* or *locally indiscrete*.

We shall say that an arrow $f : A \to B$ in a 2-category $\mathscr{K}$ is an *equivalence* if there exist an arrow $g : B \to A$ and isomorphisms $gf \cong 1$ and $fg \cong 1$.

Given some property of functors, we say that a 2-functor $F : \mathscr{K} \to \mathscr{L}$ has the property *locally* if each $F : \mathscr{K}(A, B) \to \mathscr{L}(FA, FB)$ has the property. For example, a 2-functor is *fully faithful* if and only if it is locally an isomorphism: this implies that the functor between the underlying categories is fully faithful, but also that the 2-functor is "fully faithful on 2-cells" — that is, locally fully faithful.

We denote composition in a category or 2-category by juxtaposition, provided that no ambiguity will result; and use periods as necessary for punctuation. Thus we might write gf for the composite of 1-cells $f : A \to B$ and $g : B \to C$, or $\beta\alpha$ for the composite of 2-cells $\alpha : f \to f'$ and $\beta : f' \to f''$, or then again $g\alpha h : gfh \to gf'h$ for the composite of a 1-cell $h : D \to A$, a 2-cell $\alpha : f \to f'$ with $f, f' : A \to B$, and a 1-cell $g : B \to C$. Given 1-cells $f, f' : A \to B$ and $g, g' : B \to C$; and 2-cells $\alpha : f \to f'$ and $\beta : g \to g'$ we can therefore form 2-cells $g\alpha : gf \to gf'$ and $\beta f' : gf' \to g'f'$, and *their* composite would then be written $\beta f'.g\alpha : gf \to g'f'$. Alternatively we could form $\beta f : gf \to g'f$ and $g'\alpha : g'f \to g'f'$, and then $g'\alpha.\beta f : gf \to g'f'$; one of the 2-category axioms states that $g'\alpha.\beta f = \beta f'.g\alpha$. Sometimes we use "pasting diagrams" to define composite 2-cells, as in the diagram



which appears in the proof of Theorem 5.1 below. In this case there are 1-cells $f_i, g_i : A_i \to B_i$, $a_i : A_i \to A$, $b_i : B_i \to B$, and $g : A \to B$; 2-cells $\alpha_i : f_i \to g_i$ and $\psi_i : b_i g_i \to g\alpha_i$; and we form the 2-cell $b_i\alpha_i : b_i f_i \to b_i g_i$ and then the composite 2-cell $\psi_i.b_i\alpha_i : b_i f_i \to ga_i$.

As well as 2-categories, we shall occasionally wish to discuss the more general *bicategories* [4]. These also have objects, 1-cells, and 2-cells, which compose as for 2-categories, except that the associative and identity laws for composition of 1-cells hold only up to coherent isomorphism. Similarly, *homomorphisms* of bicategories are required to preserve identity 1-cells and composition of 1-cells only up to coherent isomorphism. A homomorphism between bicategories which are actually 2-categories is also called a *pseudofunctor*. We shall write $\mathbf{2\text{-}Cat}_{ps}$ for the category of 2-categories and pseudofunctors, and **Bicat** for the category of bicategories and homomorphisms of bicategories. There are natural inclusion functors

$$\mathbf{2\text{-}Cat} \longrightarrow \mathbf{2\text{-}Cat}_{ps} \longrightarrow \mathbf{Bicat}$$

the first is bijective on objects; the second is fully faithful.

The morphisms between homomorphisms of bicategories which are most often considered are the pseudonatural transformations: here we shall recall the definition only in the simplest case of 2-functors between 2-categories. If $F, G : \mathscr{A} \to \mathscr{B}$ are 2-functors, then a pseudonatural transformation $\alpha : F \to G$ consists of a 1-cell $\alpha_A : FA \to GA$ in $\mathscr{B}$ for each object A of $\mathscr{A}$, and an invertible

2-cell α_f as in

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle\alpha_A}\Big\downarrow & \Downarrow\alpha_f & \Big\downarrow{\scriptstyle\alpha_B} \\ GA & \xrightarrow[Gf]{} & GB \end{array}$$

for each 1-cell $f : A \to B$ in $\mathscr{A}$. These 2-cells α_f are required to satisfy the following conditions:

$$\begin{array}{ccccc} FA & \xrightarrow{Ff} & FB & \xrightarrow{Fg} & FC \\ {\scriptstyle\alpha_A}\Big\downarrow & \Downarrow\alpha_f & {\scriptstyle\alpha_B}\Big\downarrow & \Downarrow\alpha_g & \Big\downarrow{\scriptstyle\alpha_C} \\ GA & \xrightarrow[Gf]{} & GB & \xrightarrow[Gg]{} & GC \end{array} \quad = \quad \begin{array}{ccc} FA & \xrightarrow{F(gf)} & FC \\ {\scriptstyle\alpha_A}\Big\downarrow & \Downarrow\alpha_{gf} & \Big\downarrow{\scriptstyle\alpha_C} \\ GA & \xrightarrow[G(gf)]{} & GC \end{array}$$

for 1-cells $f : A \to B$ and $g : B \to C$ in $\mathscr{A}$;

$$\begin{array}{ccc} FA & \overset{Ff}{\underset{Fg}{\rightrightarrows}}\ {\scriptstyle\Downarrow F\rho} & FB \\ {\scriptstyle\alpha_A}\Big\downarrow & \Downarrow\alpha_g & \Big\downarrow{\scriptstyle\alpha_B} \\ GA & \xrightarrow[Gg]{} & GB \end{array} \quad = \quad \begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle\alpha_A}\Big\downarrow & \Downarrow\alpha_f & \Big\downarrow{\scriptstyle\alpha_B} \\ GA & \overset{Gf}{\underset{Gg}{\rightrightarrows}}\ {\scriptstyle\Downarrow \rho} & GB \end{array}$$

for each 2-cell $\rho : f \to g$ in $\mathscr{A}$; and finally α_{1_A} is required to be the identity for each object A. A pseudonatural transformation α for which each α_A is an equivalence will be called a pseudonatural equivalence.

If α and β are pseudonatural transformations from F to G, then a modification from α to β consists of a 2-cell $k_A : \alpha_A \to \beta_A$ for each object A, subject to a coherence condition relating the k_A, α_f, and β_f.

Recall from the introduction the notation $[\mathscr{A}, \mathscr{B}]$ for the 2-category of 2-functors, pseudonatural transformations, and modifications, from $\mathscr{A}$ to $\mathscr{B}$. We consider the special case where $\mathscr{A}$ is the 2-category $\mathbb{2}$ with two objects, a single non-identity arrow between them, and no non-identity 2-cells. An object of $[\mathbb{2}, \mathscr{B}]$ is therefore a 1-cell in $\mathscr{B}$; we write $[\mathbb{2}, \mathscr{B}]_e$ for the full sub-2-category of $[\mathbb{2}, \mathscr{B}]$ consisting of the equivalences in $\mathscr{B}$. The constructions $[\mathbb{2}, \mathscr{B}]$ and $[\mathbb{2}, \mathscr{B}]_e$ are both functorial in $\mathscr{B}$, and there is a natural bijection between 2-functors $\mathscr{C} \to [\mathbb{2}, \mathscr{B}]_e$ and pairs of 2-functors $\mathscr{C} \to \mathscr{B}$ with a pseudonatural equivalence between them. There is an evident 2-functor $\Delta : \mathscr{B} \to [\mathbb{2}, \mathscr{B}]_e$ sending an object B of $\mathscr{B}$ to the constant 2-functor $\mathbb{2} \to \mathscr{B}$ with value B. A 2-functor $H : \mathscr{C} \to [\mathbb{2}, \mathscr{B}]_e$ factorizes as ΔG for some 2-functor G if and only if the corresponding pseudonatural equivalence is in fact the identity on G.

Finally, we describe briefly the "Gray tensor product" of 2-categories; for more detail see [12] or [11]. Just like the tensor product of abelian groups, it can be defined in terms of the corresponding internal hom: $\mathscr{A} \otimes \mathscr{B}$ is the value at $\mathscr{A}$ of the functor $- \otimes \mathscr{B}$: **2-Cat** $\to$ **2-Cat** which is left adjoint to the functor $[\mathscr{B}, -]$: **2-Cat** $\to$ **2-Cat** sending $\mathscr{C}$ to $[\mathscr{B}, \mathscr{C}]$. Like the tensor product of abelian groups once again, there is also a more explicit description. An object of $\mathscr{A} \otimes \mathscr{B}$ is just a pair (A, B) consisting of an object A of $\mathscr{A}$ and an object B of $\mathscr{B}$, but usually we write $A \otimes B$ rather than (A,B). The underlying category of $\mathscr{A}\otimes\mathscr{B}$ is generated by morphisms $A\otimes b : A\otimes B \to A\otimes B'$ for each $b : B \to B'$ and $a\otimes B : A\otimes B \to A'\otimes B$ for each $a : A \to A'$; subject to relations $(A\otimes b')(A\otimes b) = A\otimes b'b$, $A\otimes 1_B = 1_{A\otimes B}$, $(a'\otimes B)(a\otimes B) = a'a\otimes B$, and $1_A\otimes B = 1_{A\otimes B}$. The 2-cells can most efficiently be described by saying that there is a locally fully faithful 2-functor $Q : \mathscr{A}\otimes\mathscr{B} \to \mathscr{A}\times\mathscr{B}$ sending $A\otimes B$ to (A,B), sending $A\otimes b$ to $(1,b) : (A,B) \to (A,B')$, and sending $a\otimes B$ to $(a,1) : (A,B) \to (A',B)$. In fact Q is clearly bijective on objects and surjective on arrows; thus it will turn out to be a trivial fibration for the model structure on **2-Cat** described in the following section.

3 The model structure

In this section we describe a cofibrantly-generated model structure on **2-Cat**. The weak equivalences will be the biequivalences: these are the 2-functors $F : \mathscr{A} \to \mathscr{B}$ which are biessentially surjective on objects and local equivalences. Here we say that F is biessentially surjective on objects if for each object B of $\mathscr{B}$ there is an object A of $\mathscr{A}$ and an equivalence $FA \simeq B$ in $\mathscr{B}$; recall also that F is locally an equivalence if for all objects A and A' of $\mathscr{A}$, the induced functor between hom-categories $F : \mathscr{A}(A,A') \to \mathscr{B}(FA,FA')$ is an equivalence of categories. One easily verifies that the biequivalences satisfy the 2-for-3 property and are closed under retracts.

If $F : \mathscr{A} \to \mathscr{B}$ is a biequivalence then there is a pseudofunctor $G : \mathscr{B} \to \mathscr{A}$ with pseudonatural equivalences $GF \simeq 1_{\mathscr{A}}$ and $FG \simeq 1_{\mathscr{B}}$. It is an important observation that G cannot in general be chosen to be a 2-functor, as Example 3.1 below shows. Thus we cannot internalize the notion of biequivalence to (the **Gray**-category) **2-Cat** in the same way that equivalences are internalized to (the 2-category) **Cat**.

Example 3.1 Let $\mathbb{Z}_2$ be the two-element group, thought of as a one-object 2-category with two arrows and no non-identity 2-cells. Let $\mathbb{Z}$ be the group of integers, thought of as a one-object category, and let A be the 2-category whose underlying category is $\mathbb{Z}$, with a 2-cell from m to n if and only if $m-n$ is even. There is an obvious 2-functor $F : A \to \mathbb{Z}_2$, which is a biequivalence of 2-categories. On the other hand, a 2-functor $G : \mathbb{Z}_2 \to A$ must send the non-identity arrow of $\mathbb{Z}_2$ to an arrow in A whose square is the identity, and clearly the identity itself is the only such arrow in A. Thus there is a unique 2-functor $G : \mathbb{Z}_2 \to A$, and $FG : \mathbb{Z}_2 \to \mathbb{Z}_2$ is the 2-functor sending both maps to the identity. Clearly FG is not equivalent to $1_{\mathbb{Z}_2}$.

The fibrations will be the 2-functors $F : \mathscr{A} \to \mathscr{B}$ with the property that (i) for every object A of $\mathscr{A}$ and every equivalence $B \simeq FA$ in $\mathscr{B}$ consisting of $b : B \to FA$, $b' : FA \to B$, $\beta_1 : bb' \cong 1$ and $\beta_2 : b'b \cong 1$, there exist $a : A' \to A$, $a' : A \to A'$, $\alpha_1 : aa' \cong 1$ and $\alpha_2 : a'a \cong 1$ with $FA' = B$, $Fa = b$, $Fa' = b'$, $F\alpha_1 = \beta_1$, and $F\alpha_2 = \beta_2$; and (ii) F is locally a fibration: that is, for every arrow $a : A \to A'$ in $\mathscr{A}$ and every invertible 2-cell $\beta : b \to Fa$ in $\mathscr{B}$ there is an arrow $a' : A \to A'$ in $\mathscr{A}$ with $Fa' = b$ and an invertible 2-cell $\alpha : a' \to a$ with $F\alpha = \beta$. We shall sometimes call such a 2-functor an *equiv-fibration*, since it is defined in terms of lifting properties not with respect to all arrows but only with respect to equivalences.

The trivial fibrations are those 2-functors which are both fibrations and weak equivalences; it's not hard to see that these are the 2-functors $F : \mathscr{A} \to \mathscr{B}$ which are surjective on objects and for which each $F : \mathscr{A}(A,A') \to \mathscr{B}(FA,FA')$ is a retract equivalence.

We need only two generating trivial cofibrations in order to specify the fibrations. The first is the inclusion 2-functor $j_1 : 1 \to E$, where E is the "free-living equivalence", generated by objects

x and y, arrows $s : x \to y$ and $t : y \to x$, and isomorphisms $\alpha : ts \to 1$ and $\beta : st \to 1$; and where 1 is the 2-category with a single object x and no non-identity arrows or 2-cells. A 2-functor has the right lifting property with respect to j_1 if and only if it satisfies condition (i) in the definition of fibration. As for the second, consider the 2-category $\mathbb{2}$ with objects x and y, a single non-identity morphism $s : x \to y$, and no non-identity 2-cells; and the 2-category D with objects x and y, non-identity morphisms $s, s' : x \to y$ and an invertible 2-cell $\sigma : s \to s'$. A 2-functor has the right lifting property with respect to the inclusion $j_2 : \mathbb{2} \to D$ precisely when it satisfies condition (ii). Thus the fibrations are indeed those arrows with the right lifting property with respect to the generating trivial cofibrations.

We now turn to the trivial fibrations and the generating cofibrations. To say that $F : \mathscr{A} \to \mathscr{B}$ is surjective on objects is just to say that it has the right lifting property with respect to the unique 2-functor $i_1 : 0 \to 1$ from the empty 2-category 0 to the terminal 2-category 1 defined above. To say that each $F : \mathscr{A}(A, A') \to \mathscr{B}(FA, FA')$ is surjective on objects is to say that F has the right lifting property with respect to the 2-functor $i_2 : 2 \to \mathbb{2}$, where 2 is the 2-category with objects x and y and no non-identity arrows or 2-cells, and i_2 is the inclusion. It remains to express the condition that each $F : \mathscr{A}(A, A') \to \mathscr{B}(FA, FA')$ be fully faithful; this requires two further generating cofibrations.

Write C_2 for the 2-category with objects x and y, non-identity 1-cells $s, s' : x \to y$, and non-identity 2-cells $\sigma_1, \sigma_2 : s \to s'$. Write C_1 for the sub-2-category containing σ_1 but not σ_2; and write C_0 for the sub-2-category containing all the 1-cells but no non-identity 2-cells. There is an inclusion 2-functor $i_3 : C_0 \to C_1$ and a 2-functor $i_4 : C_2 \to C_1$ sending both σ_1 and σ_2 to σ_1. To say that $F : \mathscr{A} \to \mathscr{B}$ has the right lifting property with respect to i_3 is to say that each $F : \mathscr{A}(A, A') \to \mathscr{B}(FA, FA')$ is full, while to say that $F : \mathscr{A} \to \mathscr{B}$ has the right lifting property with respect to i_4 is to say that each $F : \mathscr{A}(A, A') \to \mathscr{B}(FA, FA')$ is faithful. Thus the trivial fibrations are precisely those arrows which have the right lifting property with respect to i_1, i_2, i_3, and i_4.

The category **2-Cat** is locally finitely presentable. There are many ways to see this; for instance one can show that it is the category of algebras for a finitary monad on a presheaf topos. Another possibility is to use the results of [18], where it is shown that the category $\mathscr{V}$**-Cat** of (small) $\mathscr{V}$-categories is locally finitely presentable, if $\mathscr{V}$ is a symmetric monoidal closed category which is itself locally finitely presentable. Then one observes that **2-Cat** is just $\mathscr{V}$**-Cat**, where $\mathscr{V}$ is the locally finitely presentable category **Cat**. (The fact that **Cat** is locally finitely presentable is well-known, but also follows from the same result, for **Cat** itself has the form $\mathscr{V}$**-Cat** where $\mathscr{V}$ is now the locally finitely presentable category **Set**.) The practical importance of the fact that **2-Cat** is locally finitely presentable is that it guarantees that the domains and codomains of the generating cofibrations and generating trivial cofibrations will be α-presentable for some regular cardinal α (since in a locally presentable category *every* object is so) but one can also verify directly that these objects are finitely presentable.

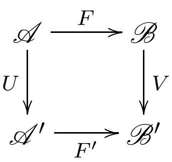
Since **2-Cat** is locally finitely presentable, it is complete and cocomplete (so in particular has finite limits and colimits). We have defined the weak equivalences to be the biequivalences, and shown that they satisfy the 2-for-3 property. We have defined the trivial fibrations, and shown that they are the morphisms with the right lifting property with respect to i_1, i_2, i_3, and i_4. By the small object argument there is a weak factorization system $(\mathscr{C}, \{\text{trivial fibrations}\})$, where $\mathscr{C}$ is the class of all morphisms with the left lifting property with respect to the trivial fibrations; we define $\mathscr{C}$ to be the class of cofibrations. We have defined the fibrations, and shown that they are the morphisms with the right lifting property with respect to j_1 and j_2. By the small object argument once again, there is a weak factorization system $(\mathscr{D}, \{\text{fibrations}\})$, where $\mathscr{D}$ consists of all morphisms with the left lifting property with respect to the fibrations. By [15, Theorem 2.1.19] it will now suffice to show that every relative $\mathscr{G}$-cell complex is a weak equivalence, where $\mathscr{G}$ denotes the set $\{j_1, j_2\}$.

What we actually prove is slightly stronger. Say that a 2-functor $F : \mathscr{A} \to \mathscr{B}$ is a *biequivalence section* if there exist a 2-functor G and a pseudonatural equivalence $\varepsilon : FG \to 1$ for which $GF = 1$ and εF is the identity transformation of F; such a 2-functor is clearly a biequivalence.

Lemma 3.2 *Every relative $\mathscr{G}$-cell complex is a biequivalence section.*

Proof: We shall show that the biequivalence sections are closed under pushout and transfinite composition, and that the generating trivial cofibrations are biequivalence sections.

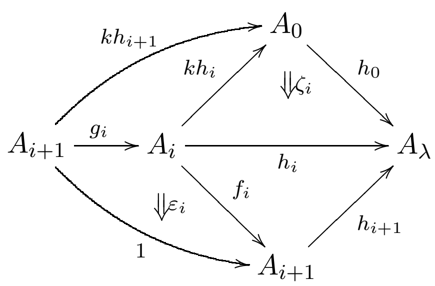
Suppose that



is a pushout in **2-Cat**, and that F is a biequivalence section. Since $UGF = U$ the universal property of the pushout $\mathscr{B}'$ gives a unique 2-functor $G' : \mathscr{B}' \to \mathscr{A}'$ satisfying $G'F' = 1$ and $G'V = UG$. If we can find a pseudonatural equivalence $\varepsilon' : F'G' \to 1$ for which $\varepsilon' F'$ is the identity on F', we shall have proved that F' is a biequivalence section.

Now the pseudonatural equivalence $\varepsilon : FG \to 1$ induces a functor $H : \mathscr{B} \to [\mathbf{2}, \mathscr{B}]_e$, and since εF is the identity, $HF = \Delta F$. We therefore have $[\mathbf{2}, V]_e HF = [\mathbf{2}, V]_e \Delta F = \Delta VF = \Delta F'U$, and so the universal property of the pushout $\mathscr{B}'$ gives a unique $H' : \mathscr{B}' \to [\mathbf{2}, \mathscr{B}']_e$ satisfying $H'F' = \Delta F'$ and $H'V = [\mathbf{2}, V]_e H$. Now H' corresponds to a pseudonatural equivalence $\varepsilon' : F'G' \to 1$, and the fact that $H'F' = \Delta F'$ means that $\varepsilon' F'$ is the identity.

This proves that the biequivalence sections are stable under pushout. We shall now show that they are closed under transfinite composition. Suppose then that λ is an ordinal, regarded as a category, and $A : \lambda \to$ **2-Cat** is a colimit-preserving functor, sending $i < \lambda$ to a 2-category A_i and sending the morphism $i < i+1$ to a biequivalence section $f_i : A_i \to A_{i+1}$. Let A_λ be the colimit of A and $h_i : A_i \to A_\lambda$ the legs of the colimit cocone. Since each f_i is a biequivalence section, there is a $g_i : A_{i+1} \to A_i$ satisfying $g_i f_i = 1$, and an equivalence $\varepsilon_i : f_i g_i \to 1$ with $\varepsilon_i f_i$ the identity. We must show that h_0 is a biequivalence section. Since $h_i g_i f_i = h_i$, there is a unique map $k : A_\lambda \to A_0$ satisfying $kh_0 = 1$ and $kh_{i+1} = kh_i g_i$. We shall construct an equivalence $\zeta : h_0 k \to 1$ for which ζh_0 is an identity; we shall do this by constructing the corresponding map $l : A_\lambda \to [\mathbf{2}, A_\lambda]_e$. We let $lh_0 = \Delta h_0$, and let lh_{i+1} be the map corresponding to the pseudonatural equivalence



where ζ_i is the pseudonatural equivalence corresponding to lh_i. Since $\varepsilon_i f_i$ is the identity, $lh_{i+1}f_i = lh_i$, so there is a unique induced l satisfying the given relations, and we write $\zeta : h_0 k \to 1$ for the corresponding pseudonatural equivalence. Since $lh_0 = \Delta h_0$, the pseudonatural equivalence ζh_0 is the identity on h_0. This proves that h_0 is a biequivalence section.

Finally we should check that $j_1 : 1 \to E$ and $j_2 : \mathbb{2} \to D$ are indeed biequivalence sections. There is a unique 2-functor $k_1 : E \to 1$, and $k_1 j_1 = 1$; there is also an obvious pseudonatural equivalence $\kappa_1 : j_1 k_1 \to 1$ with $\kappa_1 j_1$ the identity. There is a unique 2-functor $k_2 : D \to \mathbb{2}$ which is the identity on objects; $k_2 j_2 = 1$ and there is an evident pseudonatural equivalence $\kappa_2 : j_2 k_2 \to 1$ with $\kappa_2 j_2$ the identity. □

Theorem 3.3 *There is a Quillen model structure on the category* **2-Cat** *of small 2-categories for which the weak equivalences are the biequivalences and the fibrations are the equiv-fibrations.*

Corollary 3.4 *All trivial cofibrations are biequivalence sections.*

PROOF: If $f : A \to B$ is a trivial cofibration, then there exist $m : B \to C$ and $r : C \to B$ with $rm = 1$ and $mf : A \to C$ a relative $\mathscr{G}$-cell complex. Since mf is a biequivalence section, there exists a $g : C \to A$ with $gmf = 1_A$ and a pseudonatural equivalence $\zeta : mfg \to 1$ with ζmf the identity. Now $gm : B \to A$ has the property that $gmf = 1_A$, and $r\zeta m : fgm = rmfgm \to rm = 1$ is a pseudonatural equivalence with the property that $r\zeta mf$ is the identity. □

Remark 3.5 The converse is false. Let E' be the 2-category with objects x and y, generated by arrows $s : x \to y$ and $t : y \to x$ satisfying $ts = 1$, and an isomorphism $\sigma : st \cong 1$ for which $t\sigma$ and σs are identities. The inclusion $j : 1 = \{x\} \to E'$ is a biequivalence section, but not a cofibration. To see this, observe that 1 is cofibrant (the map $0 \to 1$ is after all a generating cofibration) but that E' is not, by Remark 4.9 below.

Among the 2-categories are those in which every arrow is an equivalence and every 2-cell is invertible. We shall call such a 2-category a *pseudogroupoid*, and write **PsGpd** for the full subcategory of **2-Cat** consisting of the pseudogroupoids. Pseudogroupoids were considered in [7] under the name of *2-groupoids*; we shall follow [20], however, in reserving the latter name for those pseudogroupoids in which every arrow is not just an equivalence but in fact an isomorphism. The inclusion of **PsGpd** in **2-Cat** has a right adjoint, which takes a 2-category to its sub-2-category containing only those arrows which are equivalences and only those 2-cells which are invertible. It follows that **PsGpd** is complete and cocomplete: colimits in **PsGpd** are formed as in **2-Cat**, while to form a limit in **PsGpd**, one first forms the limit in **2-Cat**, and then applies the coreflection into **PsGpd**.

The following result is obvious, but will be used several times:

Lemma 3.6 *A 2-category which is biequivalent to a pseudogroupoid is itself a pseudogroupoid.*

An immediate consequence is:

Theorem 3.7 *There is a model structure on* **PsGpd** *for which an arrow is a cofibration, weak equivalence, or fibration if and only if it is one in* **2-Cat**.

Remark 3.8 There is also a model structure on **2-Gpd**, as was proved in [20], with the same notions of weak equivalence and fibration. The situation of 2-groupoids is technically somewhat easier than for general 2-categories; one reason for this is that an invertible arrow in a 2-category has a unique inverse, while an equivalence can have many (isomorphic) equivalence-inverses. The model structure of [20] was used to give a classification of equivariant homotopy 2-types.

4 Cofibrations, cofibrant objects, and pseudofunctors

We recall that an object A of a model category is said to be *fibrant* if the unique map $A \to 1$ from A to the terminal object is a fibration, and that A is said to be cofibrant if the unique map $0 \to A$ from the initial object to A is a cofibration. In the case of **2-Cat**, every object is clearly fibrant, but the situation for cofibrant objects is more interesting. In this section we shall analyze the notions of cofibration and cofibrant object.

The first result says, among other things, that the question of whether a 2-functor is a cofibration depends only on its underlying functor. Recall from Section 2 the functor $U : \mathbf{2\text{-}Cat} \to \mathbf{Cat}$ sending a 2-category to its underlying category, and its left adjoint D and right adjoint C. We shall say that a functor is *surjective* if it is surjective on objects and full.

Lemma 4.1 *A 2-functor $F : \mathscr{A} \to \mathscr{B}$ is a cofibration if and only if its underlying functor has the left lifting property with respect to the surjective functors.*

PROOF: Suppose that F is a cofibration and P a surjective functor. Then UF has the left lifting property with respect to P if and only if F has the left lifting property with respect to CP. Now CP is surjective on objects and locally surjective on objects since P is surjective; while CP is locally fully faithful by construction, thus CP is a trivial fibration and so F does have the left lifting property with respect to CP, and UF does have the left lifting property with respect to P.

Suppose conversely that UF has the left lifting property with respect to the surjective functors, and that P is a trivial fibration. Then UP is a surjective functor, so UF has the left lifting property with respect to UP, and so DUF has the left lifting property with respect to P. If X and Y are 2-functors satisfying $PX = YF$, we have the following diagram of 2-functors:

$$\begin{array}{ccccc} DU\mathscr{A} & \xrightarrow{I} & \mathscr{A} & \xrightarrow{X} & \mathscr{C} \\ {\scriptstyle DUF}\downarrow & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle P} \\ DU\mathscr{B} & \xrightarrow[J]{} & \mathscr{B} & \xrightarrow[Y]{} & \mathscr{D} \end{array}$$

in which I and J denote the canonical inclusions. Since DUF has the left lifting property with respect to P, there is a 2-functor $Z : DU\mathscr{B} \to \mathscr{C}$ with $PZ = YJ$ and $Z.DUF = XI$. Now J is bijective on objects and bijective on arrows, while P is locally fully faithful; from the equation $PZ = YJ$ it therefore follows for general reasons that there is a unique 2-functor W satisfying $WJ = Z$ and $PW = Y$. Finally $PWF = YF = PX$ and $WFI = WJ.DUF = Z.DUF = XI$, and so $WF = X$. Thus W provides the desired fill-in satisfying $WF = X$ and $PW = Y$. □

In particular, a 2-category $\mathscr{A}$ is cofibrant if and only if its underlying category is projective in **Cat** with respect to the surjective functors.

Before going any further we need to look more closely at the relationship between 2-functors and pseudofunctors.

4.1 Pseudofunctors, bicategories, and homomorphisms

The following result is well-known to 2-category theorists, but seems not to exist in the literature in this form:

Proposition 4.2 *The inclusions* **2-Cat** $\to$ **2-Cat**$_{ps}$ *and* **2-Cat** $\to$ **Bicat** *have left adjoints.*

Proof: Since **2-Cat**$_{ps}$ is a full subcategory of **Bicat**, it will suffice to construct the left adjoint to **2-Cat** $\to$ **Bicat**. This we do as follows. Given a bicategory $\mathscr{B}$, we define a 2-category $\mathscr{B}'$ with the same objects as $\mathscr{B}$, whose morphisms are the paths in $\mathscr{B}$ — that is, a morphism in $\mathscr{B}'$ from A to B is a sequence $f_n \centerdot \ldots \centerdot f_2 \centerdot f_1$ of morphisms in $\mathscr{B}$, with the domain of f_{i+1} equal to the codomain of f_i for each i, with the domain of f_1 equal to A, and the codomain of f_n equal to B. The 2-cells in $\mathscr{B}'$ from $f_n \centerdot \ldots \centerdot f_2 \centerdot f_1$ to $g_m \centerdot \ldots \centerdot g_2 \centerdot g_1$ are defined to be the 2-cells in $\mathscr{A}$ from the composite $f_n \ldots f_2 f_1$ to $g_m \ldots g_2 g_1$. Under the evident compositions, this becomes a 2-category. There is an evident homomorphism $P : \mathscr{B} \to \mathscr{B}'$ which is the identity on objects and sends a morphism in $\mathscr{B}$ to the corresponding path of length 1. In fact P is a "biequivalence of bicategories": we shall use this fact in Theorem 4.6 below. Given any homomorphism $F : \mathscr{B} \to \mathscr{C}$ where $\mathscr{C}$ is a 2-category, there is a unique 2-functor $G : \mathscr{B}' \to \mathscr{C}$ with $GP = F$. This now proves the existence of the required adjoint, which sends $\mathscr{B}$ to $\mathscr{B}'$. □

In particular, if $\mathscr{A}$ is a 2-category, then the component at $\mathscr{A}$ of the counit of the adjunction(s) is the evident 2-functor $Q : \mathscr{A}' \to \mathscr{A}$ which is the identity on objects, and sends a path $f_n \centerdot \ldots \centerdot f_2 \centerdot f_1$ to its composite. In fact Q is locally a retract equivalence, and is the identity on objects, so it is a trivial fibration in **2-Cat**.

Proposition 4.3 *If a 2-functor $F : \mathscr{A} \to \mathscr{B}$ is a trivial fibration then there is a pseudofunctor $G : \mathscr{B} \to \mathscr{A}$ with $FG = 1$.*

Proof: If F is a trivial fibration, then it is surjective on objects, so one can choose for each object B of $\mathscr{B}$ an object GB of $\mathscr{A}$ with $FGB = B$. For all objects B and C of $\mathscr{B}$, the functor $F : \mathscr{A}(GB, GC) \to \mathscr{B}(B, C)$ is a retract equivalence, so one can choose a functor $G : \mathscr{B}(B, C) \to \mathscr{A}(GB, GC)$ whose composite with $F : \mathscr{A}(GB, GC) \to \mathscr{B}(B, C)$ is the identity on $\mathscr{B}(B, C)$. Given $f : A \to B$ and $g : B \to C$ in $\mathscr{B}$, we have $F(G(gf)) = F(G(g)G(f))$, since F is a 2-functor and $FG = 1$; thus there is a unique invertible 2-cell $\psi_{g,f} : G(g)G(f) \to G(gf)$ with $F\psi_{g,f}$ equal to the identity. Similarly $FG(1_B) = F(1_{GB})$, so that there is a unique invertible 2-cell $\psi_B : 1_{GB} \to G(1_B)$ with $G\psi_B$ equal to the identity. This makes G into a pseudofunctor with $FG = 1$. □

Remark 4.4 One cannot in general choose G to be a 2-functor: see Example 3.1 once again, and observe that the biequivalence F constructed there is in fact a trivial fibration.

In many respects pseudofunctors are more natural than 2-functors: Proposition 4.3 and Remark 4.4 illustrate just one way in which this is the case. There are two reasons we have chosen to work with 2-functors rather than the more general and more natural pseudofunctors. The first is that they are formally simpler, the second that they are much better behaved. An example of the poor behaviour of the pseudofunctors is that **2-Cat**$_{ps}$, although it has products and coproducts, has neither equalizers nor coequalizers, as the following example shows.

Example 4.5 Consider the ordered set $\mathbb{3} = \{0 < 1 < 2\}$ considered as a 2-category with no non-identity 2-cells.

Consider a 2-category $\mathscr{K}$ with designated arrows $f : A \to B$, $g : B \to C$ and $h : A \to C$, and an invertible 2-cell $\varphi : gf \to h$. These determine a pseudofunctor $F : \mathbb{3} \to \mathscr{K}$ sending 0, 1, and 2 to A, B, and C; sending $0 < 1$ to f, $1 < 2$ to g and $0 < 2$ to h; strictly preserving the identities, and with φ the pseudofunctoriality constraint. Now suppose that there is a different $h' : A \to C$ and $\varphi' : gf \to h$, and write $F' : \mathbb{3} \to \mathscr{K}$ for the induced pseudofunctor. Were F and F' to have an equalizer, it would have to contain the objects 0, 1, and 2, and the arrows $0 < 1$ and $1 < 2$, but not the arrow $0 < 2$. This is clearly impossible; thus $\mathbf{2\text{-}Cat}_{ps}$ does not have equalizers.

On the other hand, the inclusions $\{1\} \to \{0 < 1\}$ and $\{1\} \to \{1 < 2\}$ have no pushout in $\mathbf{2\text{-}Cat}_{ps}$, as the reader will easily verify.

It is nonetheless the case that the homotopy theory of $\mathbf{2\text{-}Cat}_{ps}$ is very similar to that of **2-Cat**. There are various ways we could formalize this vague statement; here is one. Say that a homomorphism of bicategories is a biequivalence if it is biessentially surjective on objects and locally an equivalence; this extends our earlier definition of 2-functors which are biequivalences. Let $\mathscr{W}$, as above, consist of the biequivalences in **2-Cat**, let $\mathscr{W}_{ps}$ consist of the biequivalences in $\mathbf{2\text{-}Cat}_{ps}$, and $\mathscr{W}_{hom}$ the biequivalences in **Bicat**.

Theorem 4.6 *The inclusion functors* $\mathbf{2\text{-}Cat} \to \mathbf{2\text{-}Cat}_{ps}$ *and* $\mathbf{2\text{-}Cat} \to \mathbf{Bicat}$ *induce equivalences* $\mathbf{2\text{-}Cat}[\mathscr{W}^{-1}] \to \mathbf{2\text{-}Cat}_{ps}[\mathscr{W}_{ps}{}^{-1}]$ *and* $\mathbf{2\text{-}Cat}[\mathscr{W}^{-1}] \to \mathbf{Bicat}[\mathscr{W}_{hom}{}^{-1}]$ *of categories.*

PROOF: The two cases are entirely similar; we shall treat only the second. Since the inclusion $I : \mathbf{2\text{-}Cat} \to \mathbf{Bicat}$ maps $\mathscr{W}$ into $\mathscr{W}_{hom}$, it induces a functor $I_{\mathscr{W}} : \mathbf{2\text{-}Cat}[\mathscr{W}^{-1}] \to \mathbf{Bicat}[\mathscr{W}_{hom}{}^{-1}]$. Likewise the left adjoint L to I maps biequivalences to biequivalences, and so induces a left adjoint $L_{\mathscr{W}}$ to $I_{\mathscr{W}}$. Finally the unit and counit of the adjunction $L \dashv I$ lie in $\mathscr{W}_{hom}$ and $\mathscr{W}$ respectively, so that the induced adjunction $L_{\mathscr{W}} \dashv I_{\mathscr{W}}$ is in fact an equivalence. □

4.2 Cofibrant objects

In this section we shall characterize the cofibrant objects in **2-Cat**, in the next we shall consider the more general problem of characterizing the cofibrations.

First we need a little lemma about free categories:

Lemma 4.7 *Any retract of a free category on a graph is itself free.*

PROOF: Say that an arrow $f : A \to B$ in a category $\mathscr{C}$ is *indecomposable* if (i) f is not an identity and (ii) whenever f is a composite $f = f_2 f_1$ either f_1 or f_2 is an identity. Define a *decomposition* of an arrow f to be a sequence $f_1, \ldots, f_n$ of indecomposable arrows whose composite $f_n \ldots f_1$ is f. (We deem an identity arrow to be the composite of the empty sequence.) A category is clearly free if and only if every arrow has a unique decomposition.

Suppose that $I : \mathscr{A} \to \mathscr{B}$ and $R : \mathscr{B} \to \mathscr{A}$ are functors with $RI = 1$ and $\mathscr{B}$ free; identify $\mathscr{A}$ with its image under I. Any arrow f in $\mathscr{A}$ has a unique decomposition $f = g_n \ldots g_1$ in $\mathscr{B}$; we write $\pi(f)$ for n. If there is an arrow f admitting no decomposition in $\mathscr{A}$, then there is such an f for which $\pi(f)$ is minimal. Certainly f is neither an identity nor indecomposable, so we can write $f = f_2 f_1$ where f_1 and f_2 are non-identity arrows in $\mathscr{A}$. Then the decompositions in $\mathscr{B}$ of f_1 and f_2

must be $f_1 = g_k \dots g_1$ and $f_2 = g_n \dots g_{k+1}$, where $1 \leq k < n$. But if f_1 and f_2 have decompositions in $\mathscr{A}$, then so would f, thus either f_1 or f_2 has no decomposition in $\mathscr{A}$; since $\pi(f_1) < \pi(f)$ and $\pi(f_2) < \pi(f)$ this contradicts the minimality of $\pi(f)$.

It remains to prove that decompositions are unique. Suppose that f has decompositions $f = f_n \dots f_1$ and $f = f'_m \dots f'_1$ in $\mathscr{A}$, and decomposition $f = g_p \dots g_1$ in $\mathscr{B}$. We can choose an n_i for each i with $1 \leq i \leq n$ in such a way that f_i has decomposition $f_i = g_{n_i} \dots g_{n_{i-1}+1}$ in $\mathscr{B}$, and so $f_i = R(f_i) = R(g_{n_i}) \dots R(g_{n_{i-1}+1})$. Now f_i is indecomposable in $\mathscr{A}$, so there is exactly one non-identity arrow in the sequence $R(g_{n_{i-1}+1}), \dots, R(g_{n_i})$. Similarly we can choose an m_i for each i with $1 \leq i \leq m$ in such a way that f'_i has decomposition $f'_i = g_{m_i} \dots g_{m_{i-1}+1}$ in $\mathscr{B}$, and so $f'_i = R(f'_i) = R(g_{m_i}) \dots R(g_{m_{i-1}+1})$, with exactly one non-identity arrow in the sequence $R(g_{m_{i-1}+1}), \dots, R(g_{m_i})$. Now n and m each equal the number of non-identity arrows in the sequence $R(g_j)$ for $1 \leq j \leq p$; while f_i and f'_i each equal the ith non-identity arrow in the sequence, thus $n = m$ and $f_i = f'_i$, and the two decompositions are identical. □

Given an object A of a model category, one can factorize the unique map $0 \to A$ as $0 \to A'$ followed by $p : A' \to A$, where A' is a cofibrant object, called a *cofibrant replacement* for A, and $p : A' \to A$ is a trivial fibration. In the previous section, we constructed for each 2-category $\mathscr{A}$ a 2-category $\mathscr{A}'$ and a trivial fibration $Q : \mathscr{A}' \to \mathscr{A}$. In fact $\mathscr{A}'$ is always cofibrant, as follows from the characterization of cofibrant objects in Theorem 4.8 below, and so $Q : \mathscr{A}' \to \mathscr{A}$ exhibits $\mathscr{A}'$ as a cofibrant replacement for $\mathscr{A}$.

Theorem 4.8 *For a 2-category $\mathscr{A}$ the following conditions are equivalent:*

(i) $\mathscr{A}$ is cofibrant;

(ii) there is a 2-functor $J : \mathscr{A} \to \mathscr{A}'$ with $QJ = 1$;

(iii) $\mathscr{A}$ is a retract of $\mathscr{B}'$ for some 2-category $\mathscr{B}$.

(iv) the underlying category of $\mathscr{A}$ is free.

Proof: The implication $(i) \Rightarrow (ii)$ follows immediately from the fact that Q is a trivial fibration, while the implication $(ii) \Rightarrow (iii)$ is trivial, and $(iii) \Rightarrow (iv)$ follows from Lemma 4.7. Finally, if the underlying category of $\mathscr{A}$ is free, then the unique 2-functor $0 \to \mathscr{A}$ clearly satisfies the condition of Lemma 4.1, and so $\mathscr{A}$ is cofibrant. □

Remark 4.9 This means in particular that a 2-category $\mathscr{A}$ cannot be cofibrant if there exist non-identity arrows $f : A \to B$ and $g : B \to A$ satisfying $gf = 1_A$.

The 2-categories of the form $\mathscr{A}'$ are "free" in the sense that they lie in the image of the left adjoint to the inclusion **2-Cat** $\to$ **2-Cat**$_{ps}$. The theorem shows that the cofibrant objects are precisely the retracts of the free objects. This situation is familiar from homological algebra, where, for example, an R-module is projective if and only if it is a retract of a free module; of course the condition of being cofibrant just says that an object is "projective with respect to the trivial fibrations". From another point of view, the cofibrant objects are precisely the free ones: that is, the 2-categories whose underlying categories are free.

Remark 4.10 The construction of $\mathscr{A}'$ is very closely related to the results of [5], involving a 2-monad T on a 2-category $\mathscr{K}$. One defines algebras for the 2-monad as for ordinary monads, but there are two sorts of morphisms, the "strict" (corresponding in our case to the 2-functors) and the "pseudo" (corresponding in our case to the pseudofunctors). Once again one can construct, for each algebra A, an algebra A' with the property that pseudo maps from A to B are in bijection with strict maps from A' to B. Once again the algebras A with the property that the canonical map $A' \to A$ has a section are important; in [5] they are called the *flexible algebras*. Thus if one defines flexible 2-categories by obvious analogy with [5], one sees that a 2-category is flexible if and only if it is cofibrant. In fact there is a way to view 2-categories within the framework of [5]: for a fixed set X, there is a 2-monad T on the 2-category $\mathbf{Cat}^{X\times X}$ whose algebras are the 2-categories with object-set X, whose strict morphisms are the identity-on-object 2-functors, and whose pseudo morphisms are the identity-on-object pseudofunctors — for the details see [21] — and then our construction $\mathscr{A}'$ agrees with that of [5].

4.3 Cofibrations

In the previous section we saw how to characterize the cofibrant objects; when it comes to the cofibrations in general, we have relatively little to add to Lemma 4.1. We do have:

Lemma 4.11 *There is a cofibrantly generated weak factorization system $(\mathscr{I}, \mathscr{P})$ on* **Cat** *with $\mathscr{P}$ consisting of the surjective functors. One can choose the set $\mathscr{G}$ of generating cofibrations to consist of the functors $i : 0 \to 1$ and $i' : 2 \to \mathbb{2}$ of Example 1.1.*

PROOF: It suffices to repeat the observation of Example 1.1 that a functor is surjective on objects if and only if it has the right lifting property with respect to $0 \to 1$, and full if and only if it has the right lifting property with respect to $2 \to \mathbb{2}$. ☐

Corollary 4.12 *A functor $F : A \to B$ lies in $\mathscr{I}$ if and only if it is injective on objects, faithful, and there are functors $I : B \to C$ and $R : C \to B$ satisfying $RI = 1$ such that C is obtained from the image of A by freely adjoining objects and then arrows between specified objects.*

We saw in the last section that for functors with domain the initial category the class of relative $\mathscr{G}$-cell complexes is closed under retracts, but in general this is no longer the case, as the following example shows:

Example 4.13 Let Q be the monoid of non-negative rationals under addition. We think of Q as a one-object category with object X, and we write the arrows as x^a with a a non-negative rational. Let A be the monoid obtained from Q by adjoining an element y satisfying $y^2 = x^0 y = y = yx^a$ for all a, and $x^a y = x^a$ for all $a > 0$. The only new arrow in A is therefore y itself. Let C be the category obtained from A by adjoining an object Z and an arrow $g : Z \to X$. Then the inclusion $i : A \to C$ is a relative $\mathscr{G}$-cell complex.

The subcategory B of C containing every arrow except g is a retract via the functor sending g to yg. Thus the inclusion $j : A \to B$ is a retract of a relative $\mathscr{G}$-cell complex $i : A \to C$. Suppose now that j were a relative $\mathscr{G}$-cell complex. The arrow yg is not freely adjoined, since it satisfies $y(yg) = yg$. On the other hand if a and b are distinct positive rationals, then $x^a g$ and $x^b g$ cannot both be freely adjoined, since $y(x^a g) = y(x^b g)$. Thus the only possibility is that B is obtained from A by freely adjoining a single arrow $x^a g$. This, however, would imply that the monoid Q was freely generated by a, which is of course impossible.

Thus $j : A \to B$ is a retract of a relative $\mathscr{G}$-cell complex but not itself a relative $\mathscr{G}$-cell complex.

Combining Lemma 4.1 and Corollary 4.12 we have:

Proposition 4.14 *A 2-functor $F : \mathscr{A} \to \mathscr{B}$ is a cofibration if and only if its underlying ordinary functor satisfies the conditions of Corollary 4.12; in particular, F must be injective on objects and 1-cells.*

5 The homotopy relation

We now explore the homotopy relation. In a general model category there are several different ways of defining the homotopy relation, but they all coincide and are well-behaved when one restricts oneself to objects which are both fibrant and cofibrant. In the case of **2-Cat**, we have seen that all objects are fibrant, but that relatively few are cofibrant.

In a model category $\mathscr{C}$, a *path object* for an object X consists of a factorization

$$\begin{array}{ccccc} & & PX & & \\ & {}^{r}\nearrow & & \searrow^{\binom{p_1}{p_2}} & \\ X & & \xrightarrow[\Delta]{} & & X \times X \end{array}$$

of the diagonal, where r is a weak equivalence, and $\binom{p_1}{p_2}$ is a fibration. We know that such a factorization is possible, and that r may be chosen to be a trivial cofibration, but here r is assumed only to be a weak equivalence. A *right homotopy* from $f : B \to X$ to $g : B \to X$ consists of a path object on X, as above, and a morphism $h : B \to PX$ with $p_1 h = f$ and $p_2 h = g$. If there exists a right homotopy from f to g we say that f and g are right homotopic.

If, as in our case, every object is fibrant, the general theory tells us that right homotopy is an equivalence relation on $\mathscr{C}(B, X)$, respected by composition on either side, and that right homotopy implies left homotopy.

The main result of this section is:

Theorem 5.1 *If $F, G : \mathscr{B} \to \mathscr{A}$ are 2-functors, then they are right homotopic if and only if there is a pseudonatural equivalence from F to G.*

Proof: If

$$\mathscr{A} \xrightarrow{R} \mathscr{P}\mathscr{A} \xrightarrow{\binom{P_1}{P_2}} \mathscr{A} \times \mathscr{A}$$

is a path object for $\mathscr{A}$, then $P_1 R = 1_{\mathscr{A}}$ and R is a biequivalence. It follows that P_1 is also a biequivalence, and that there exists a pseudonatural equivalence $\Theta : RP_1 \simeq 1_{\mathscr{P}\mathscr{A}}$. If now $H : \mathscr{B} \to \mathscr{P}\mathscr{A}$ is a right homotopy from F to G, then we have

$$F = P_1 H = P_2 R P_1 H \xrightarrow[\simeq]{P_2 \Theta H} P_2 H = G$$

and so F is pseudonaturally equivalent to G.

On the other hand, for every 2-category $\mathscr{A}$, we have the following canonical choice of path object $\mathscr{P}\mathscr{A}$. An object of $\mathscr{P}\mathscr{A}$ consists of three objects A, A_1, and A_2 of $\mathscr{A}$ and two equivalences

$a_1 : A_1 \to A$ and $a_2 : A_2 \to A$ in $\mathscr{A}$. A morphism in $\mathscr{P}\mathscr{A}$ consists of three morphisms $f : A \to B$, $f_1 : A_1 \to B_1$, and $f_2 : A_2 \to B_2$ and invertible 2-cells as in:

$$\begin{array}{ccccc} A_1 & \xrightarrow{a_1} & A & \xleftarrow{a_2} & A_2 \\ {\scriptstyle f_1}\downarrow & \overset{\varphi_1}{\Rightarrow} & \downarrow{\scriptstyle f} & \overset{\varphi_2}{\Leftarrow} & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow[b_1]{} & B & \xleftarrow[b_2]{} & B_2 \end{array}$$

A 2-cell in $\mathscr{P}\mathscr{A}$ consists of 2-cells $\alpha : f \to g$, $\alpha_1 : f_1 \to g_1$, and $\alpha_2 : f_2 \to g_2$ in $\mathscr{A}$ satisfying:

$$\begin{array}{ccc} A_i & \xrightarrow{a_i} & A \\ {\scriptstyle f_i}\Big(\overset{\alpha_i}{\Rightarrow}\Big){\scriptstyle g_i} & \overset{\psi_i}{\Rightarrow} & \Big\downarrow{\scriptstyle g} \\ B_i & \xrightarrow[b_i]{} & B \end{array} \quad = \quad \begin{array}{ccc} A_i & \xrightarrow{a_i} & A \\ {\scriptstyle f_i}\Big\downarrow & \overset{\varphi_i}{\Rightarrow} & {\scriptstyle f}\Big(\overset{\alpha}{\Rightarrow}\Big){\scriptstyle g} \\ B_i & \xrightarrow[b_i]{} & B \end{array}$$

for $i = 1$ and $i = 2$. With the obvious compositions one obtains a 2-category, equipped with 2-functors $P_1 : \mathscr{P}\mathscr{A} \to \mathscr{A}$ and $P_2 : \mathscr{P}\mathscr{A} \to \mathscr{A}$ for which the induced 2-functor $\binom{P_1}{P_2} : \mathscr{P}\mathscr{A} \to \mathscr{A} \times \mathscr{A}$ is a fibration in **2-Cat**.

There is is an evident factorization $\Delta = \binom{P_1}{P_2}R$, where $R : \mathscr{A} \to \mathscr{P}\mathscr{A}$ sends an object A to

$$A \xrightarrow{1} A \xleftarrow{1} A$$

and is defined similarly on 1-cells and 2-cells. Now $R : \mathscr{A} \to \mathscr{P}\mathscr{A}$ is a biequivalence of 2-categories, and so R, P_1, and P_2 exhibit $\mathscr{P}\mathscr{A}$ as a path object for $\mathscr{A}$. Let $F, G : \mathscr{B} \to \mathscr{A}$ be 2-functors, and let $\Phi : F \to G$ be a pseudonatural equivalence. Then we can define a right homotopy $H : \mathscr{B} \to \mathscr{P}\mathscr{A}$ from F to G as follows. On objects H sends B to

$$FB \xrightarrow{\Phi B} GB \xleftarrow{1} GB$$

and on morphisms H sends $f : B \to C$ to

$$\begin{array}{ccccc} FB & \xrightarrow{\Phi B} & GB & \xleftarrow{1} & GB \\ {\scriptstyle Ff}\downarrow & \overset{\Phi f}{\Rightarrow} & \downarrow{\scriptstyle Gf} & & \downarrow{\scriptstyle Gf} \\ FC & \xrightarrow[\Phi C]{} & GC & \xleftarrow[1]{} & GC \end{array}$$

where Φf is the isomorphism expressing the pseudonaturality of Φ. Finally on 2-cells, H sends $\alpha : f \to g$ to $(F\alpha, G\alpha, G\alpha)$. This is clearly a 2-functor with $P_1 H = F$ and $P_2 H = G$. $\square$

6 Stability of biequivalences

A model category is said to be *right proper* if the pullback of a weak equivalence by a fibration is a weak equivalence, and *left proper* if the pushout of a weak equivalence by a cofibration is a weak equivalence. It is said to be *proper* if it is both right proper and left proper. A model category in which every object is fibrant is automatically right proper; see [14, Corollary 11.1.3].

The model category **Cat** is proper: every object is fibrant, and so it is right proper, while the fact that it is left proper was proved in [16].

Example 6.1 It is not true that equivalences in **Cat** are stable under pullback by an arbitrary functor. For example, consider the functor $j : 1 \to I$ of Example 1.1; recall that I is the "free-living isomorphism". Let $j' : 1 \to I$ be the other functor. Then j and j' are both equivalences, but their pullback is the initial category 0, and the inclusion $0 \to 1$ is not an equivalence.

Similarly, equivalences in **Cat** are not stable under pushout by an arbitrary functor. To see this, let B be the group $\mathbb{Z}$ of integers, thought of as a one-object category; and let A be the category with two objects x_1 and x_2, with $A(x_i, x_j) = \mathbb{Z}$ for $i, j \in \{1, 2\}$, and with composition given by addition. There is an equivalence $P : A \to B$ sending each $n : x_i \to x_j$ to n, and an equivalence $Q : A \to B$ sending $n : x_i \to x_j$ to $n + j - i$. The pushout of P along Q is the terminal category 1, and the unique functor $B \to 1$ is not an equivalence.

We now turn to the case of **2-Cat**. Once again, every object is fibrant, and so the model structure is immediately seen to be right proper, but the fact that it is left proper is more involved. Throughout the proof we shall refer to the following pushout diagram:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{I} & \mathscr{C} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathscr{B} & \xrightarrow[J]{} & \mathscr{D} \end{array} \qquad (*)$$

The main step is:

Lemma 6.2 *If in the pushout $(*)$, P is a trivial fibration and I is the pushout of a generating cofibration, then Q is a trivial fibration.*

PROOF: Suppose in $(*)$ that P is a trivial fibration, and I is the pushout of a generating cofibration. We consider in turn the four possible generating cofibrations: $i_1 : 0 \to 1$, $i_2 : 2 \to \mathbf{2}$, $i_3 : C_0 \to C_1$, and $i_4 : C_2 \to C_1$.

If I is a pushout of i_1, then $\mathscr{C}$ is the coproduct $\mathscr{A} + 1$ and I the inclusion. It follows that $\mathscr{D} = \mathscr{B} + 1$ and $Q = P + 1$, which is clearly a trivial fibration if (and only if) P is one.

If I is a pushout of i_2, then $\mathscr{C}$ is obtained from $\mathscr{A}$ by freely adjoining an arrow $c : A \to B$ between given objects A and B in $\mathscr{A}$. It follows that $\mathscr{D}$ is obtained from $\mathscr{B}$ by freely adjoining an arrow $d : PA \to PB$, and that Q is the unique map extending P which sends c to d. More explicitly, $\mathscr{C}$ is the 2-category with the same objects as $\mathscr{A}$, whose arrows are the well-formed words of the form $a_n \centerdot c \centerdot a_{n-1} \centerdot \ldots \centerdot c \centerdot a_1$ where each a_i is an arrow in $\mathscr{A}$, and whose 2-cells are the well-formed words of the form $\alpha_n \centerdot c \centerdot \alpha_{n-1} \centerdot \ldots \centerdot c \centerdot \alpha_1$ where each α_i is a 2-cell in $\mathscr{A}$. There is a similar description for $\mathscr{D}$, and Q acts letter-by-letter on a word, sending an arrow a in $\mathscr{A}$ to Pa, a 2-cell α in $\mathscr{A}$ to $P\alpha$, and c to d. It is clear from this description that Q inherits from P each of the properties of being surjective on objects, full, and locally fully faithful.

The case of i_3 is the hardest; first we treat the easier case of i_4. If I is a pushout of i_4, then $\mathscr{C}$ is obtained from $\mathscr{A}$ by imposing the relation $\alpha = \alpha'$ on a pair of 2-cells $\alpha, \alpha' : f \to g : A \to B$ in $\mathscr{A}$. Then $\mathscr{D}$ is obtained from $\mathscr{B}$ by imposing the relation $P\alpha = P\alpha'$ on $\mathscr{B}$, and Q is the 2-functor obtained from P by passing to the quotient. This process has no effect on objects and arrows, so Q will, like P, be surjective on objects and full. More explicitly, the hom-category $\mathscr{C}(C, D)$ is obtained from the corresponding hom-category $\mathscr{A}(C, D)$ by imposing the relation $h\alpha k = h\alpha' k$ for all $h : B \to D$ and $k : C \to A$. Similarly $\mathscr{D}(PC, PD)$ is obtained from $\mathscr{B}(PC, PD)$ by imposing the relation $m.P\alpha.n = m.P\alpha'.n$ for all $m : PB \to PD$ and $n : PC \to PA$. But in this case, since P is full, we can write $m = Ph$ and $n = Pk$ for some $h : B \to D$ and some $k : C \to A$, so that $\mathscr{D}(QC, QD)$ is obtained from $\mathscr{B}(PC, PD)$ by imposing the relations $Ph.P\alpha.Pk = Ph.P\alpha'.Pk$ for all $h : B \to D$ and all $k : C \to A$. This means that

$$\begin{array}{ccc} \mathscr{A}(C,D) & \xrightarrow{I} & \mathscr{C}(C,D) \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathscr{B}(PC,PD) & \longrightarrow & \mathscr{D}(QC,QD) \end{array}$$

is in fact a pushout in **Cat**. Since $P : \mathscr{A}(C, D) \to \mathscr{B}(PC, PD)$ is a retract equivalence and $I : \mathscr{A}(C, D) \to \mathscr{C}(C, D)$ is injective on objects, it follows that $Q : \mathscr{C}(C, D) \to \mathscr{D}(QC, QD)$ is also a retract equivalence. Since we have already observed that $Q : \mathscr{C} \to \mathscr{D}$ is surjective on objects, it now follows that it is a trivial fibration.

Finally, if I is a pushout of i_3, then $\mathscr{C}$ is obtained from $\mathscr{A}$ by freely adjoining a 2-cell $\gamma : f \to g : A \to B$ between given arrows $f, g : A \to B$ of $\mathscr{A}$. Then $\mathscr{D}$ is obtained from $\mathscr{B}$ by freely adjoining a 2-cell δ from Pf to Pg, and $Q : \mathscr{C} \to \mathscr{D}$ is the unique 2-functor extending P which sends γ to δ. This process has no effect on objects and arrows, so Q will, like P, be surjective on objects and full. To go any further, we need a more explicit description of $\mathscr{B}$ and $\mathscr{D}$.

If C and D are objects of $\mathscr{A}$, write $\mathscr{E}(C, D)$ for the category obtained from $\mathscr{A}(C, D)$ by freely adjoining an arrow $k\alpha h : kfh \to kgh$ for all $k : B \to D$ and $h : C \to A$, subject to relations of the form

$$\begin{array}{ccc} lfkfh & \xrightarrow{l\alpha kfh} & lgkfh \\ {\scriptstyle lfk\alpha h}\downarrow & & \downarrow{\scriptstyle lgk\alpha h} \\ lfkgh & \xrightarrow[l\alpha kgh]{} & lgkgh \end{array} \qquad \begin{array}{ccc} kfh & \xrightarrow{\kappa fh} & k'fh \\ {\scriptstyle k\alpha h}\downarrow & & \downarrow{\scriptstyle k'\alpha h} \\ kgh & \xrightarrow[\kappa gh]{} & k'gh \end{array} \qquad \begin{array}{ccc} kfh & \xrightarrow{kf\lambda} & kfh' \\ {\scriptstyle k\alpha h}\downarrow & & \downarrow{\scriptstyle k\alpha h'} \\ kgh & \xrightarrow[kg\lambda]{} & kgh'. \end{array}$$

We shall try to define a composition making $\mathscr{E}$ into a 2-category with the required universal property of $\mathscr{B}$.

To make $\mathscr{E}$ into a 2-category, we should define a functor $\mathscr{E}(c, d) : \mathscr{E}(C, D) \to \mathscr{E}(C', D')$ for all arrows $d : D \to D'$ and $c : C \to C'$. The composite functor

$$\mathscr{A}(C,D) \xrightarrow{\mathscr{A}(c,d)} \mathscr{A}(C',D') \longrightarrow \mathscr{E}(C',D')$$

sends kfh to $dkfhc$ and kgh to $dkghc$; there is now a unique functor $\mathscr{E}(c, d) : \mathscr{E}(C, D) \to \mathscr{E}(C', D')$ extending $\mathscr{A}(c, d)$ and sending $k\alpha h$ to $dk\alpha hc$. The functors of the form $\mathscr{E}(c, d)$ satisfy $\mathscr{E}(c', d')\mathscr{E}(c, d) = \mathscr{E}(cc', d'd)$ where $d' : D' \to D''$ and $c' : C' \to C''$; and also $\mathscr{E}(1_C, 1_D) = 1_{\mathscr{E}(C,D)}$. This much makes $\mathscr{E}$ into a sesquicategory; it will be a 2-category if it also satisfies the "middle four interchange" law, which states that if $\varphi : f \to f' : A \to B$ and $\psi : g \to g' : B \to C$ are 2-cells in $\mathscr{E}$, then $(\psi f')(g\varphi) = (g'\varphi)(\psi f)$. This follows from the relations imposed on the $\mathscr{E}(C, D)$. The verification of the universal property is straightforward; and so $\mathscr{E}$ is just $\mathscr{B}$. Summarizing, $\mathscr{B}(C, D)$ is obtained from $\mathscr{A}(C, D)$ by freely adjoining arrows of the form $k\alpha h : kfh \to kgh$ subject to three classes of relations.

Similarly, $\mathscr{D}(QC, QD)$ is obtained from $\mathscr{B}(PC, PD)$ by freely adjoining arrows of the form $m.P\alpha.n : m.Pf.n \to m.Pg.n$ subject to corresponding relations. Now P is full, so that m and n

have the form Pk and Ph for some $k : B \to D$ and $h : C \to A$. Next we show that if $k_1, k_2 : B \to D$ and $h_1, h_2 : C \to A$ satisfy $Pk_1 = Pk_2 = f$ and $Ph_1 = Ph_2 = g$, then $P(k_1\alpha h_1) = P(k_2\alpha h_2)$. Since $Pk_1 = Pk_2$ there is a unique invertible 2-cell $\kappa : k_1 \to k_2$ which is mapped to the identity 2-cell on f; similarly there is a unique invertible $\lambda : h_1 \to h_2$ mapped to the identity on g. Now the relations

$$\begin{array}{ccc} k_1fh_1 & \xrightarrow{\kappa fh_1} & k_2fh_1 \\ {\scriptstyle k_1\alpha h_1}\downarrow & & \downarrow{\scriptstyle k_2\alpha h_1} \\ k_1gh_1 & \xrightarrow[\kappa gh_1]{} & k_2gh_1 \end{array} \qquad \begin{array}{ccc} k_2fh_1 & \xrightarrow{k_2f\lambda} & k_2fh_2 \\ {\scriptstyle k_2\alpha h_1}\downarrow & & \downarrow{\scriptstyle k_2\alpha h_2} \\ k_2gh_1 & \xrightarrow[k_2g\lambda]{} & k_2gh_2 \end{array}$$

imply, respectively, that $P(k_1\alpha h_1) = P(k_2\alpha h_1)$ and $P(k_2\alpha h_1) = P(k_2\alpha h_2)$; thus $P(k_1\alpha h_1) = P(k_2\alpha h_2)$ as claimed. Thus $\mathscr{D}(QC, QD)$ can be obtained from $\mathscr{B}(PC, PD)$ by freely adjoining arrows of the form $Pk.P\alpha.Ph$ subject to three classes of relations. The third class, for example, consists of relations of the form

$$\begin{array}{ccc} Pk.Pf.Ph & \xrightarrow{Pk.Pf.\nu} & Pk.Pf.Ph' \\ {\scriptstyle Pk.P\alpha.Ph}\downarrow & & \downarrow{\scriptstyle Pk.P\alpha.Ph'} \\ Pk.Pg.Ph & \xrightarrow[Pk.Pg.\nu]{} & Pk.Pg.Ph' \end{array}$$

but since P is locally fully faithful, $\nu = P\lambda$ for a unique $\lambda : h \to h'$. The other classes of relations are similar, and so $\mathscr{D}(QC, QD)$ can be obtained from $\mathscr{B}(PC, PD)$ by freely adjoining arrows of the form $Pk.P\alpha.Ph$ subject to relations of the form

$$\begin{array}{ccc} Pl.Pf.Pk.Pf.Ph & \xrightarrow{Pl.P\alpha.Pk.Pf.Ph} & Pl.Pg.Pk.Pf.Ph \\ {\scriptstyle Pl.Pf.Pk.P\alpha.Ph}\downarrow & & \downarrow{\scriptstyle Pl.Pg.Pk.P\alpha.Ph} \\ Pl.Pf.Pk.Pg.Ph & \xrightarrow[Pl.P\alpha.Pk.Pg.Ph]{} & Pl.Pg.Pk.Pg.Ph \end{array}$$

$$\begin{array}{ccc} Pk.Pf.Ph & \xrightarrow{P\kappa.Pf.Ph} & Pk'.Pf.Ph \\ {\scriptstyle Pk.P\alpha.Ph}\downarrow & & \downarrow{\scriptstyle Pk'.P\alpha.Ph} \\ Pk.Pg.Ph & \xrightarrow[P\kappa.Pg.Ph]{} & Pk'.Pg.Ph \end{array} \qquad \begin{array}{ccc} Pk.Pf.Ph & \xrightarrow{Pk.Pf.\lambda} & Pk.Pf.Ph' \\ {\scriptstyle Pk.P\alpha.Ph}\downarrow & & \downarrow{\scriptstyle Pk.P\alpha.Ph'} \\ Pk.Pg.Ph & \xrightarrow[Pk.Pg.P\lambda]{} & Pk.Pg.Ph'. \end{array}$$

This now proves that the diagram

$$\begin{array}{ccc} \mathscr{A}(C, D) & \xrightarrow{I} & \mathscr{C}(C, D) \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathscr{B}(PC, PD) & \longrightarrow & \mathscr{D}(QC, QD) \end{array}$$

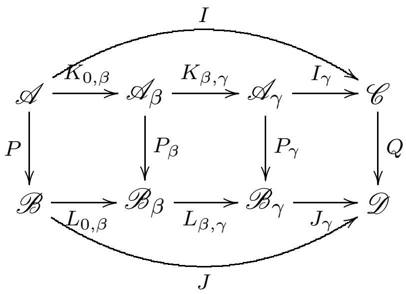
is a pushout in **Cat**, in which I is injective on objects and P is a retract equivalence. It follows that $Q : \mathscr{C}(C, D) \to \mathscr{D}(QC, QD)$ is a retract equivalence, and so that $Q : \mathscr{C} \to \mathscr{D}$ is a trivial fibration. □

Theorem 6.3 *The model category* **2-Cat** *is proper.*

Proof: We have already seen that **2-Cat** is right proper; the other half will involve three steps. The first of these uses the fact that the generating cofibrations and generating trivial cofibrations have domains which are finitely presentable objects of **2-Cat**; the latter two steps are actually general facts about model categories.

Step 1: If P is a trivial fibration and I is a relative $\mathscr{G}$-cell complex then Q is a trivial fibration.

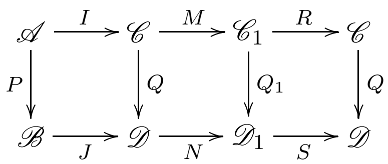
We know that I can be written as the transfinite composite of a chain of 2-functors which are pushouts of generating cofibrations. We prove the result by transfinite induction on the length of the chain. Lemma 6.2 takes care of all the steps involving successor ordinals; thus it will suffice to prove the result for a limit ordinal α, given that it holds for all lesser ordinals. Thus we have a chain of pushouts



in which the P_β are trivial fibrations, and the horizontal arrows are cofibrations. The fact that Q is a trivial fibration now follows by [15, Lemma 7.4.1].

Step 2: If P is a trivial fibration and I is a cofibration then Q is a trivial fibration.

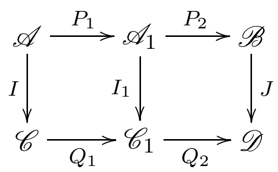
We may form pushouts



where $RM = 1$, $SN = 1$, and MI is a relative $\mathscr{G}$-cell complex. Now Q_1 is a trivial fibration by Step 1, but Q is a retract of Q_1, so it too is a trivial fibration.

Step 3: If P is a weak equivalence and I is a cofibration then Q is a weak equivalence.

We may form pushouts



where $P = P_2P_1$, $Q = Q_2Q_1$, P_1 is a trivial cofibration, and P_2 is a trivial fibration. Trivial cofibrations and are stable under pushout, so Q_1 is a trivial cofibration; and cofibrations are stable under pushout, so I_1 is a cofibration. Now Q_2 is a trivial fibration by Step 2, and so Q is the composite of a trivial cofibration and a trivial fibration, hence a weak equivalence. □

Corollary 6.4 *The model category* **PsGpd** *is proper.*

PROOF: Since pushouts in **PsGpd** are formed as in **2-Cat**, the fact that **PsGpd** is left proper is immediate from the fact that **2-Cat** is left proper. On the other hand, **PsGpd** is right proper since every object is fibrant. □

7 Monoidal structure on 2-Cat

The category **2-Cat** is cartesian closed, but it also has a symmetric monoidal closed structure for which the tensor product is the "Gray tensor product", described in Section 2. The internal hom of the closed structure is given by what we have been writing $[\mathscr{A}, \mathscr{B}]$: the 2-category of 2-functors, pseudonatural transformations, and modifications, from $\mathscr{A}$ to $\mathscr{B}$. In each case the unit for the tensor product is the terminal 2-category 1.

In this section we investigate the relationship between these monoidal structures and the model structure. In particular, one might hope that these structures make **2-Cat** into a *monoidal model category* [15] and that it further satisfies the "monoid axiom" of [24]. If $\mathscr{V}$ is a monoidal model category, then the monoidal closed structure passes to the homotopy category. Furthermore, the category A-**Mod** of modules for a monoid A in $\mathscr{V}$ has a canonical model structure, provided that A is cofibrant as an object of $\mathscr{V}$. If $\mathscr{V}$ also satisfies the monoid axiom, then the assumption that A be cofibrant is no longer required; moreover, the category of monoids in $\mathscr{V}$ has a canonical model structure.

A monoidal model category is defined to be a category with a monoidal closed structure and a model structure which satisfy a kind of "internalized lifting condition" (essentially Quillen's axiom SM7) and a technical condition which is automatically satisfied if the unit for the tensor product is cofibrant. Since the terminal 2-category 1 is cofibrant, we shall not bother to spell out the technical condition. The "internalized lifting condition" involves the internal hom $[-,-]$ coming from the closed structure. The condition is that if $p : C \to D$ is a fibration, $i : A \to B$ a cofibration, and

$$\begin{array}{ccc} P & \longrightarrow & [B,D] \\ \downarrow & & \downarrow{\scriptstyle [i,D]} \\ [A,C] & \underset{[A,p]}{\longrightarrow} & [A,D] \end{array}$$

a pullback then the induced map $[B,C] \to P$ is a fibration, trivial if either the fibration p or the cofibration i is so.

It is also possible to use the adjunction connecting the tensor product and internal hom to reformulate this condition. Given arrows $i : A \to B$ and $j : C \to D$, form the pushout

$$\begin{array}{ccc} A \otimes D & \longrightarrow & A \otimes D +_{A\otimes C} B \otimes C \\ {\scriptstyle A\otimes j}\uparrow & & \uparrow \\ A \otimes C & \underset{i\otimes C}{\longrightarrow} & B \otimes C \end{array}$$

and write $i\Box j$ for the induced map $A \otimes D +_{A\otimes C} B \otimes C \to B \otimes D$. Then the category satisfies the internalized lifting condition if and only if (a) $i\Box j$ is a cofibration whenever i and j are cofibrations, and (b) $i\Box j$ is a trivial cofibration whenever one of i and j is a cofibration and the other a trivial cofibration. Furthermore, if the model structure is cofibrantly generated, then it suffices to check (a) and (b) when i and j are *generating* cofibrations or *generating* trivial cofibrations; see [15].

Example 7.1 **Cat** is a monoidal model category via the cartesian closed structure. If $p : C \to D$ is a fibration and $i : A \to B$ a cofibration, as above, we must show that the induced functor $w : [B, C] \to P$ is a fibration, trivial if either p or i is so.

The fact that w is a fibration amounts to the fact that if $f : A \to C$, $u : B \to C$, and $g : B \to D$ are functors satisfying $pf = gi$; and $\varphi : f \to ui$ and $\psi : g \to pu$ natural isomorphisms satisfying $p\varphi = \psi i$; then there exist a functor $v : B \to C$ satisfying $pv = g$ and $vi = f$ and a natural isomorphism $\theta : v \to u$ satisfying $p\theta = \psi$ and $\theta i = \varphi$.

To see this, we define for each object b of B an object vb of C and an isomorphism $\theta_b : vb \to ub$ as follows. If $b = ia$, let $vb = fa$ and $\theta_b = \varphi_a$; otherwise, use the fact that p is a fibration to choose an arbitrary $\theta_b : vb \to ub$ with the property that $pvb = gb$ and $p\theta_b = \psi_b$. There is now a unique way to make v into a functor so that $\theta : v \to u$ is a natural isomorphism; and the resulting v and θ satisfy all the requirements.

It remains to show that if i or p is trivial, then so is w. If i is trivial, then $[i, D]$ is a trivial fibration, so its pullback $P \to [A, C]$ is a trivial fibration; since also $[i, C]$ is a trivial fibration, the fibration w is trivial by the 2-for-3 property. The case where i is trivial is entirely analogous.

Example 7.2 **2-Cat** is not a monoidal model category via the cartesian closed structure. To see this, consider the generating cofibration $i_2 : 2 \to \mathbb{2}$; we shall show that $i_2\Box i_2$ is not a cofibration. The pushout $2\times\mathbb{2}+_{2\times 2}\mathbb{2}\times 2$ is the "non-commuting square": it has four objects $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$; arrows generated by $(0, 0) \to (0, 1)$, $(0, 0) \to (1, 0)$, $(0, 1) \to (1, 1)$, and $(1, 0) \to (1, 1)$, but the two paths $(0, 0) \to (1, 1)$ are not equal. On the other hand, they become equal after one applies $i_2\Box i_2$. Thus $i_2\Box i_2$ is not injective on arrows, so cannot be a cofibration.

We now turn to the case of the Gray tensor product.

Lemma 7.3 *If i and j are cofibrations then so is $i\Box j$. In particular, if A is cofibrant and j is a cofibration then $A \otimes j$ is a cofibration.*

Proof: It suffices to consider the case where $i, j \in \{i_1, i_2, i_3, i_4\}$. For any C we have isomorphisms $0 \otimes C \cong 0$ and $1 \otimes C \cong C$; thus if $i : A \to B$ is i_1 then $i\Box j$ is just j, so is a cofibration. Similarly, $i\Box j = i$ if $j = i_1$.

Next observe that the underlying functors of i_3 and i_4 are isomorphisms of categories, and that the "underlying category functor" U : **2-Cat** $\to$ **Cat** preserves pushouts, while the underlying category of the **Gray** tensor product $C \otimes D$ only depends on the underlying categories of C and D. It follows that $i\Box j$ is an isomorphism if either i or j is either i_3 or i_4.

This leaves only the case $i = j = i_2 : 2 \to \mathbb{2}$, which we treat by direct calculation. An object of $\mathbb{2} \otimes \mathbb{2}$ is a pair (x, y) where $x, y \in \{0, 1\}$. The arrows of $\mathbb{2} \otimes \mathbb{2}$ are freely generated by arrows $(1, \eta) : (x, y) \to (x, y')$ and $(\xi, 1) : (x, y) \to (x', y)$, where $\xi : x \to x'$ and $\eta : y \to y'$ are arrows in $\mathbb{2}$, subject to the "Gray relations" $(1, \eta')(1, \eta) = (1, \eta'\eta)$, $(\xi', 1)(\xi, 1) = (\xi'\xi, 1)$, and $(1_x, 1_y) = 1_{(x,y)}$. Thus the underlying category of $\mathbb{2} \otimes \mathbb{2}$ is the non-commuting square of Example 7.2. On the other hand, since the category 2 is discrete, the canonical trivial fibrations $C \otimes 2 \to C \times 2$ and $2 \otimes C \to 2 \times C$ are isomorphisms, and so $2 \otimes \mathbb{2} +_{2 \otimes 2} \mathbb{2} \otimes 2$ is just $2 \times \mathbb{2} +_{2 \times 2} \mathbb{2} \times 2$, which we saw in Example 7.2 to be the non-commuting square. Thus $i_2 \square i_2$ is an isomorphism at the level of underlying categories, and is therefore a cofibration. $\square$

The next result is of independent interest:

Lemma 7.4 *If $F : \mathscr{A} \to \mathscr{B}$ is a biequivalence and $\mathscr{C}$ is any 2-category, then $\mathscr{C} \times F : \mathscr{C} \times \mathscr{A} \to \mathscr{C} \times \mathscr{B}$ and $\mathscr{C} \otimes F : \mathscr{C} \otimes \mathscr{A} \to \mathscr{C} \otimes \mathscr{B}$ are biequivalences.*

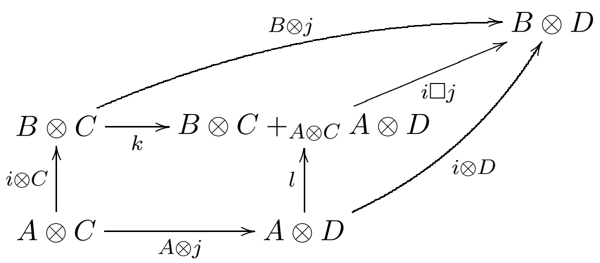
PROOF: An object of $\mathscr{C} \times \mathscr{B}$ is a pair (C, B) consisting of an object of $\mathscr{C}$ and an object of $\mathscr{B}$. Since F is biessentially surjective on objects, there is an object A of $\mathscr{A}$ with an equivalence $FA \simeq B$; thus $(\mathscr{C} \times F)(C, A) \simeq (C, B)$ and $\mathscr{C} \times F$ is biessentially surjective on objects. Similarly, the fact that $\mathscr{C} \times F$ is locally essentially surjective on objects and locally fully faithful follows easily from the corresponding facts for F. Thus $\mathscr{C} \times F$ is a biequivalence.

On the other hand $\mathscr{C} \otimes F$ is a biequivalence since $Q'(\mathscr{C} \otimes F) = (\mathscr{C} \times F)Q$, where $Q : \mathscr{C} \otimes \mathscr{A} \to \mathscr{C} \times \mathscr{A}$ and $Q' : \mathscr{C} \otimes \mathscr{B} \to \mathscr{C} \times \mathscr{B}$ are the canonical trivial fibrations. $\square$

Theorem 7.5 **2-Cat** *is a monoidal model category with respect to the monoidal structure given by the Gray tensor product.*

PROOF: We have seen that the unit for the tensor is cofibrant, and that $i \square j$ is a cofibration if i or j is one. It remains to show that if i and j are cofibrations, one of which is trivial, then $i \square j$ is a weak equivalences.

Suppose that $i : A \to B$ is a cofibration with A cofibrant, and that $j : C \to D$ is a trivial cofibration. In the diagram



$A \otimes j$ is a cofibration by Lemma 7.3, and a trivial one by Lemma 7.4; thus k is a trivial cofibration since it is a pushout of $A \otimes j$. On the other hand $B \otimes j$ is a weak equivalence by Lemma 7.4 once again; thus $i \square j$ is a weak equivalence by the 2-for-3 property. Since all the generating cofibrations have cofibrant domains, it follows that $i \square j$ is a weak equivalence whenever i is a cofibration and j a trivial cofibration.

The case where i is a trivial cofibration and j a cofibration follows by symmetry. $\square$

The symmetric monoidal closed structure on **2-Cat** given by the Gray tensor product restricts to the full subcategory **PsGpd**, since $[\mathscr{A}, \mathscr{B}]$ and $\mathscr{A} \otimes \mathscr{B}$ are pseudogroupoids if $\mathscr{A}$ and $\mathscr{B}$ are: the case of $[\mathscr{A}, \mathscr{B}]$ is obvious, while $\mathscr{A} \otimes \mathscr{B}$ is biequivalent to $\mathscr{A} \times \mathscr{B}$, which once again is obviously a pseudogroupoid.

Theorem 7.6 **PsGpd** *is a monoidal model category with respect to the monoidal structure given by the Gray tensor product.*

PROOF: Consider the formulation of the internal lifting condition which involves tensor products and pushouts. These are both formed in **PsGpd** as in **2-Cat**, so the condition for **PsGpd** follows immediately from the corresponding condition for **2-Cat**. □

We now turn to the monoid axiom of [24]; for convenience we follow [24] in stating it only for symmetric monoidal categories. The axiom says that any transfinite composite of pushouts of maps of the form $j \otimes C$, where j is a trivial cofibration and C is arbitrary, is a weak equivalence. If, as in the case of **2-Cat**, the model category is cofibrantly generated, then it suffices to consider $j \otimes C$ with j a generating trivial cofibration.

Theorem 7.7 *The monoidal model categories* **2-Cat** *and* **PsGpd** *satisfy the monoid axiom.*

PROOF: The case of **PsGpd** clearly follows from that of **2-Cat**. Suppose that $F : \mathscr{A} \to \mathscr{B}$ is a biequivalence section, with $GF = 1$ and $\varepsilon : FG \to 1$. Then $(\mathscr{C} \otimes G)(\mathscr{C} \otimes F) = \mathscr{C} \otimes (GF) = \mathscr{C} \otimes 1 = 1$. On the other hand, $(\mathscr{C} \otimes F)(\mathscr{C} \otimes G) = \mathscr{C} \otimes (FG)$, and so $\mathscr{C} \otimes \varepsilon$ is a pseudonatural equivalence from $(\mathscr{C} \otimes F)(\mathscr{C} \otimes G)$ to 1, with $(\mathscr{C} \otimes \varepsilon)(\mathscr{C} \otimes F) = \mathscr{C} \otimes (\varepsilon F)$ equal to the identity. Thus $\mathscr{C} \otimes F$ is a biequivalence section.

We saw in the proof of Lemma 3.2 that the biequivalence sections are closed under pushout and transfinite composition, and that they contain the generating trivial cofibrations. Thus any transfinite composite of pushouts of maps of the form $j \otimes C$ with j a generating trivial cofibration must be a biequivalence section, and therefore a weak equivalence. □

8 Relationships between homotopy categories

This section is devoted to relationships between the homotopy categories of various model categories studied in the rest of the paper. Most of these relationships will be expressed in terms of the notion of Quillen adjunction, recalled below, but first we look directly at various categories equivalent to the homotopy category of **2-Cat**. We have already seen one such relationship: the categories $\mathbf{2\text{-}Cat}[\mathscr{W}^{-1}]$, $\mathbf{2\text{-}Cat}_{ps}[\mathscr{W}_{ps}{}^{-1}]$, and $\mathbf{Bicat}[\mathscr{W}_{hom}{}^{-1}]$ are all equivalent, by Theorem 4.6.

If $F, G : \mathscr{A} \to \mathscr{B}$ are homomorphisms of bicategories, write $F \sim G$ if there is a pseudonatural equivalence from F to G. This defines a congruence on **Bicat**, and we may form the quotient $Q : \mathbf{Bicat} \to \mathbf{Bicat}/\sim$ which is the identity on objects, but identifies arrows F and G if (and only if) $F \sim G$. Similarly, we may form $Q : \mathbf{2\text{-}Cat}_{ps} \to \mathbf{2\text{-}Cat}_{ps}/\sim$. If $F : \mathscr{A} \to \mathscr{B}$ is a biequivalence of bicategories, then there is a homomorphism $G : \mathscr{B} \to \mathscr{A}$ with $GF \sim 1$ and $FG \sim 1$; thus $Q : \mathbf{Bicat} \to \mathbf{Bicat}/\sim$ inverts the biequivalences. On the other hand, if $F, G : \mathscr{A} \to \mathscr{B}$ are pseudonaturally equivalent, we may form a "path object" $\mathscr{P}\mathscr{B}$ with biequivalences $P_1, P_2 : \mathscr{P}\mathscr{B} \to \mathscr{B}$ and $R : \mathscr{B} \to \mathscr{P}\mathscr{B}$ exactly as in the proof of Theorem 5.1, and obtain a homomorphism $H : \mathscr{A} \to \mathscr{P}\mathscr{B}$ with $P_1 H = F$ and $P_2 H = G$. Any functor which inverts the biequivalences must identify P_1 and P_2, and so also identify F and G. The case of pseudofunctors between 2-categories is entirely analogous, and we deduce:

Theorem 8.1 *The categories* $\mathbf{Bicat}[\mathscr{W}_{hom}{}^{-1}]$ *and* $\mathbf{Bicat}/\sim$ *are canonically equivalent, as are the categories* $\mathbf{2\text{-}Cat}_{ps}[\mathscr{W}_{ps}{}^{-1}]$ *and* $\mathbf{2\text{-}Cat}_{ps}/\sim$.

An adjunction between model categories is called a *Quillen adjunction* if the left adjoint sends cofibrations to cofibrations and trivial cofibrations to trivial cofibrations; this is equivalent to asking that the right adjoint send fibrations to fibrations and trivial fibrations to trivial fibrations. A Quillen adjunction induces an adjunction (the "derived adjunction") between the homotopy categories of the model categories.

In more detail, if $\mathscr{A}$ is a model category, write $\mathrm{Ho}\mathscr{A}$ for the homotopy category, write $Z : \mathscr{A} \to \mathscr{A}$ for the cofibrant replacement functor, and $\zeta_A : ZA \to A$ for the associated trivial fibration; and write $X : \mathscr{A} \to \mathscr{A}$ for the fibrant replacement functor, and $\xi_A : A \to XA$ for the associated trivial cofibration. Suppose that $\mathscr{B}$ is also a model category, and $U : \mathscr{B} \to \mathscr{A}$ is part of a Quillen adjunction $F \dashv U$ with unit $\eta : 1 \to UF$ and counit $\varepsilon : FU \to 1$. The composite $FZ : \mathscr{A} \to \mathscr{B}$ sends weak equivalences to weak equivalences, and so induces a functor $F' : \mathrm{Ho}\mathscr{A} \to \mathrm{Ho}\mathscr{B}$, called the *total left derived functor* of F. Similarly, the composite $UX : \mathscr{B} \to \mathscr{A}$ sends weak equivalences to weak equivalences, and so induces a functor $U' : \mathrm{Ho}\mathscr{B} \to \mathrm{Ho}\mathscr{A}$ called the *total right derived functor* of U. The unit $\eta' : 1 \to U'F'$ of the derived adjunction has components

$$A \xrightarrow{\zeta_A^{-1}} ZA \xrightarrow{\eta_{ZA}} UFZA \xrightarrow{U\xi FZA} URFZA$$

while the counit $\varepsilon' : F'U' \to 1$ has components

$$FZUXB \xrightarrow{F\zeta_{UXB}} FUXB \xrightarrow{\varepsilon_{XB}} XB \xrightarrow{\xi_B^{-1}} B.$$

Thus η' is invertible if and only if $U\xi FA.\eta_A$ is a weak equivalence for all cofibrant objects A, while ε' is invertible if and only if $\varepsilon_B.F\zeta_{UB}$ is a weak equivalence for all fibrant objects B. If both η' and ε' are invertible, then the adjunction $F' \dashv U'$ is an equivalence, and we say that $F \dashv U$ is a Quillen equivalence. (One warning is appropriate here: we have used the "prime" notation to denote both total right derived functors and total left derived functors; if as well as the Quillen adjunction $F \dashv U$ there were a Quillen adjunction $G \dashv F$ this would create ambiguity in the notation F'. This ambiguity will not arise in the examples considered below.)

We saw in Section 2 two adjunctions between **Cat** and **2-Cat**, namely $D \dashv U$ and $U \dashv C$. The first of these fails to be a Quillen adjunction since D fails to preserve cofibrant objects: every category A is cofibrant, but DA is cofibrant in **2-Cat** if and only if A is free. The second adjunction, $U \dashv C$, fails to be a Quillen adjunction because C fails to preserve fibrations. Let A be the category with two objects x and y, and two non-identity arrows $s, s' : x \to y$; and let $p : \mathbb{2} \to A$ be the inclusion. Then p is a fibration, but Cp is not, since there is an isomorphism $s \cong s'$ in CA which does not lift through Cp.

On the other hand, the functor $D : \mathbf{Cat} \to \mathbf{2\text{-}Cat}$ does preserve fibrations and trivial fibrations, so would be part of a Quillen adjunction if it had a *left* adjoint, which is in fact the case. The left adjoint P to D sends a 2-category $\mathscr{A}$ to the category $P\mathscr{A}$ with the same objects as $\mathscr{A}$ and with hom-sets $(P\mathscr{A})(A,B) = \pi_0(\mathscr{A}(A,B))$, where $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ is the functor sending a category to its set of connected components.

Theorem 8.2 *The adjunction $P \dashv D : \mathbf{Cat} \to \mathbf{2\text{-}Cat}$ is a Quillen adjunction. The counit of the derived adjunction is invertible but the unit is not.*

Proof: The discussion preceding the theorem contained the proof that $P \dashv D$ is a Quillen adjunction. The counit $\varepsilon' : P'D' \to 1$ is invertible if and only if $\varepsilon_B.P\zeta_{DB} : PZDB \to B$ is a weak equivalence for every fibrant category B — that is, for *every* category. Now ε_B is an isomorphism, so we need only show that $P\zeta_{DB} : PZDB \to PDB$ is a weak equivalence. The 1-cells of ZDB are the paths in the 1-cells in B; all 2-cells in ZDB are invertible, and there is a (unique) 2-cell between two paths if and only if their composite in B is the same. Taking connected components on the hom-categories of ZDB therefore simply identifies such paths with the same composite in B; thus $P\zeta_{DB}$ is also invertible. This proves that ε' is invertible.

On the other hand, $\eta' : 1 \to D'P'$ is invertible if and only if $D\xi_{PA}.\eta_A : A \to DXPA$ is a weak equivalence for all cofibrant 2-categories A. Now $\xi_P A$ is an equivalence of categories, and so $D\xi_{PA}$ is a biequivalence (in fact an equivalence of 2-categories), so that η' will be invertible if and only if η_A is a weak equivalence for all cofibrant 2-categories A. Let A be the 2-category with two objects x and y, non-identity arrows $s, s' : x \to y$, and non-identity 2-cell $\sigma : s \to s'$; since the underlying category of A is free, A is cofibrant. Then PA is just $\mathbb{2}$, and η_A identifies the non-isomorphic arrows s and s', so fails to be a biequivalence. □

Next we consider the inclusion $I : \mathbf{PsGpd} \to \mathbf{2\text{-}Cat}$. This preserves all of the model structure, and so any adjunction involving it will be a Quillen adjunction. Now I does have a right adjoint R, which sends a 2-category to the sub-2-category containing all the objects, but only those 1-cells which are equivalences, and only those 2-cells which are invertible. On the other hand I does not have a left adjoint, since it does not preserve equalizers. To see this, let E denote the "free-living equivalence", generated by objects x and y, 1-cells $s : x \to y$ and $t : y \to x$, and invertible 2-cells $\alpha : ts \to 1$ and $\beta : st \to 1$; let E' be generated by objects x and y, 1-cells $s : x \to y$ and $t, t' : y \to x$, and invertible 2-cells $\alpha : ts \to 1$, $\alpha' : t's \to 1$, $\beta : st \to 1$, and $\beta' : st' \to 1$. As well as the inclusion $i : E \to E'$, there is another 2-functor $i' : E \to E'$ with the same effect on objects, but sending t to t'. The equalizer in **2-Cat** of i and i' is $\mathbb{2}$, which does not lie in **PsGpd**. Thus $I : \mathbf{PsGpd} \to \mathbf{2\text{-}Cat}$ does not preserve limits and so does not have a left adjoint.

Theorem 8.3 *The adjunction $I \dashv R : \mathbf{2\text{-}Cat} \to \mathbf{PsGpd}$ is a Quillen adjunction. The unit of the derived adjunction is invertible but the counit is not.*

Proof: The discussion preceding the theorem contained the proof that $I \dashv R$ is a Quillen adjunction. The unit $\eta' : 1 \to R'I'$ is invertible if and only if $R\xi_{IA}.\eta_A : A \to RXIA$ is a weak equivalence for every cofibrant pseudogroupoid A. Now η_A is invertible for all A, so it will suffice to show that $R\xi_{IA}$ is a weak equivalence. But $\xi_{IA} : XIA \to IA$ is a weak equivalence and IA is a pseudogroupoid, so XIA is a pseudogroupoid, so $RXIA = XIA$ and $R\xi_{IA} = \xi_{IA}$. Thus η' is invertible.

On the other hand, $\varepsilon' : I'R' \to 1$ is invertible if and only if $\varepsilon_B.I\zeta_{RB} : IZRB \to B$ is a weak equivalence for every fibrant 2-category B — that is, for *every* 2-category. Now I preserves weak equivalences, so this is equivalent to asking that $\varepsilon_B : IRB \to B$ be a weak equivalence; but if ε_B were a weak equivalence, then B would be biequivalent to the pseudogroupoid IRB and so would itself be a pseudogroupoid. The 2-category $\mathbb{2}$ is not a pseudogroupoid, so ε' is not invertible. □

When we turn to the inclusion $I : \mathbf{2\text{-}Gpd} \to \mathbf{2\text{-}Cat}$ the situation is somewhat different: I has a right adjoint R sending a 2-category to its sub-2-category of invertible arrows and 2-cells, but it also has a left adjoint L which freely adjoins inverses to all 1-cells and 2-cells. We have already seen that a non-trivial 2-groupoid can never be cofibrant in **2-Cat**, so I fails to preserve cofibrant objects, and $I \dashv R$ is not a Quillen adjunction. On the other hand, I does preserve fibrations and trivial fibrations.

Theorem 8.4 *The adjunction $L \dashv I$: **2-Gpd** $\to$ **2-Cat** is a Quillen adjunction. The counit of the derived adjunction is invertible but the unit is not.*

Proof: Once again, it remains only to prove the invertibility of $\varepsilon' : L'I' \to 1$ and non-invertibility of $\eta' : 1 \to I'L'$.

Recall from Section 4.1 that the cofibrant replacement functor Z can be constructed so that there is a pseudofunctor $\rho_A : A \to ZA$ inducing a natural bijection between pseudofunctors $A \to C$ and 2-functors $ZA \to C$. (In Section 4.1 we used the notation $\mathscr{A}'$ for what we are here calling ZA.) Thus if B and G are 2-groupoids, the composite pseudofunctor $\eta_{ZIB}\rho_{IB} : IB \to ILZIB$ induces a natural bijection between pseudofunctors $B \to G$ and 2-functors $LZIB \to G$. Then $\varepsilon_B.L\zeta_{IB}$ is the unique 2-functor $\pi_B : LZIB \to LIB$ satisfying $I\pi_B.\eta_{ZIB}\rho_{IB} = 1$.

But we can construct a 2-groupoid $LZIB$ satisfying this universal property analogously to the construction of ZA in Section 4.1: the underlying category of $LZIB$ is the free groupoid on the underlying graph of IB, and the 2-cells are chosen so that the induced 2-functor $\pi_B : LZIB \to B$ is locally fully faithful. Thus since π_B is bijective on objects and surjective on 1-cells, it is a biequivalence. This proves that ε' is invertible.

As for the non-invertibility of η', since I preserves weak equivalence, $I\xi_{LA}$ is a weak equivalence for any 2-category A, and so $I\xi_{LA}.\eta_A : A \to IXLA$ is a weak equivalence if and only if $\eta_A : A \to ILA$ is one. Now ILA is a 2-groupoid, and so in particular a pseudogroupoid; thus if η_A were to be a weak equivalence, A would have to be a pseudogroupoid. Once again, there are cofibrant 2-categories, such as $\mathbb{2}$, which are not pseudogroupoids. $\square$

Theorem 8.5 *The Quillen adjunction of Theorem 8.4 restricts to a Quillen equivalence between* **2-Gpd** *and* **PsGpd**.

Proof: Since the fully faithful inclusion **PsGpd** $\to$ **2-Cat** preserves all of the model structure, we deduce from Theorem 8.4 a Quillen adjunction $L \dashv I$: **2-Gpd** $\to$ **PsGpd** for which the counit of the derived adjunction is invertible. It remains to show that the unit is so, which will be the case provided that $I\xi_{LA}.\eta_A : A \to IXLA$ is a weak equivalence whenever A is a cofibrant pseudogroupoid. Once again, I preserves weak equivalences, which leaves us to prove that $\eta_A : A \to ILA$ is a weak equivalence.

We shall prove that η_A is a biequivalence for any pseudogroupoid A; first, however, we need a more explicit description of ILA. Make a definite choice, for each $f : a \to b$ in A, of an arrow $f' : b \to a$ and (invertible) 2-cells $\beta : ff' \to 1_b$ and $\alpha : f'f \to 1_a$. An object of ILA is an object of A, while the arrows of ILA are generated by the arrows f in A and their formal inverses f^{-1} subject to the relations $ff^{-1} = 1$ and $f^{-1}f = 1$. Thus every arrow can be represented as a composite $f_n f_{n-1}^{-1} \dots f_2 f_1^{-1}$; for each arrow, make a definite choice of such a representation. We now define a 2-cell in ILA from $f_n f_{n-1}^{-1} \dots f_2 f_1^{-1}$ to $g_m g_{m-1}^{-1} \dots g_2 g_1^{-1}$ to be a 2-cell in A from $f_n f'_{n-1} \dots f_2 f'_1$ to $g_m g'_{m-1} \dots g_2 g'_1$. Under the obvious definition of composition this makes ILA into a groupoid, and the 2-functor $\eta_A : A \to ILA$ sending an arrow $f : A \to B$ to $f1_A^{-1}$ clearly has the required universal property. Moreover, η_A is a biequivalence: it is straightforward to verify that is bijective on objects and locally an equivalence, but one can also construct a pseudofunctor $ILA \to A$ which is the identity on objects and sends an arrow $f_n f_{n-1}^{-1} \dots f_2 f_1^{-1}$ to $f_n f'_{n-1} \dots f_2 f'_1$, and this pseudofunctor is a biequivalence-inverse to η_A. $\square$

Corollary 8.6 *The inclusion of* **2-Gpd** *in* **2-Cat** *induces a fully faithful functor* Ho **2-Gpd** $\to$ Ho **2-Cat** *with both adjoints.*

Proof: We saw in Theorem 8.4 that the inclusion **2-Gpd** $\to$ **2-Cat** induces a functor J : Ho**2-Gpd** $\to$ Ho**2-Cat** with a left adjoint for which the counit is invertible; the latter condition means that the functor is fully faithful.

We saw in Theorem 8.5 that the inclusion **2-Gpd** $\to$ **PsGpd** induces an equivalence of categories J_1 : Ho**2-Gpd** $\to$ Ho**PsGpd**; and we saw in Theorem 8.3 that the cofibrant replacement functor Z : **PsGpd** $\to$ **PsGpd** followed by the inclusion **PsGpd** $\to$ **2-Cat** induces a functor J_2 : Ho**PsGpd** $\to$ Ho**2-Cat** with a right adjoint. Thus the composite J_2J_1 has a right adjoint; but this composite is induced by the functor **2-Gpd** $\to$ **2-Cat** sending a 2-groupoid to its cofibrant replacement "as a pseudogroupoid", and since the resulting 2-category is weak equivalent to the original 2-groupoid, J_2J_1 is just J. □

Finally, we turn to the classifying space of a 2-category. The nerve functor N : **2-Cat** $\to$ **SSet** has a left adjoint W. It was proved in [20] that the composite adjunction $LW \dashv NI$: **2-Gpd** $\to$ **SSet** is a Quillen adjunction. The counit of the derived adjunction is invertible, and so $(NI)'$: Ho**2-Gpd** $\to$ Ho**SSet** is fully faithful; its image consists of the homotopy 2-types.

The functor N : **2-Cat** $\to$ **SSet** does not preserve fibrations: recall that for a 2-category $\mathscr{A}$ the 0-simplices of $N\mathscr{A}$ are the objects of $\mathscr{A}$, and the 1-simplices are the arrows. The two 2-functors $1 \to \mathbb{2}$ are fibrations of 2-categories, but are mapped by N to the two maps $\Delta[0] \to \Delta[1]$, which are not Kan fibrations. Thus the adjunction $W \dashv N$ is not a Quillen adjunction.

In [13], a class of 2-functors called *2-fibrations* is defined. It seems plausible that N will map 2-fibrations to Kan fibrations, but it seems unlikely that the 2-fibrations will form part of a model structure, so we have not pursued this possibility.

References

[1] J. Adámek, H. Herrlich, J. Rosický, and W. Tholen, On a generalized small-object argument for the injective subcategory problem, preprint, 2000.

[2] J. Adámek and J. Rosický, *Locally presentable categories and accessible categories*, LMS Lecture Notes Series 189, Cambridge University Press, Cambridge, 1994.

[3] T. Beke, Sheafifiable homotopy model categories, *Math. Proc. Cambridge Philos. Soc.* 129: 447–475, 2000.

[4] Jean Bénabou, Introduction to bicategories, *Reports of the Midwest Category Seminar*, Lecture Notes in Mathematics 47:1–77, Springer-Verlag, 1967.

[5] R. Blackwell, G. M. Kelly, and A. J. Power, Two-dimensional monad theory, *J. Pure Appl. Algebra* 59:1–41, 1989.

[6] A.K. Bousfield, Constructions of factorization systems in categories. *J. Pure Appl. Algebra* 9:207–220, 1976.

[7] L. Breen, On the classification of 2-gerbes and 2-stacks, *Astérisque* 225, 1994.

[8] Denis-Charles Cisinski, La classe des morphismes de Dwyer n'est pas stable par retractes *Cahiers Topologie Gom. Diffrentielle Catg.* 40:227–231, 1999.

[9] W.G. Dwyer, P.S. Hirschhorn, and D.M. Kan, *Model categories and general abstract homotopy theory*

[10] P. Gabriel and F. Ulmer, *Lokal-Präsentierbare Kategorien*, Lecture Notes in Mathematics 221, Springer-Verlag, 1971.

[11] R. Gordon, A. J. Power, and R.H. Street, *Coherence for Tricategories*, AMS Memoirs 558, American Mathematical Society, Providence, 1995.

[12] J. Gray, *Formal Category Theory: Adjointness for 2-categories* Lecture Notes in Mathematics 391, Springer-Verlag, 1974.

[13] Claudio Hermida, Some properties of **Fib** as a fibred 2-category, *J. Pure Appl. Alg.*, 134:83–109, 1999.

[14] Philip S. Hirschhorn, *Localization of model categories*, unpublished book manuscript, available electronically from `http://www-math.mit.edu/~psh`

[15] Mark Hovey, *Model Categories,* Mathematical Surveys and Monographs 63, Amer. Math. Soc., 1998.

[16] André Joyal and Ross Street, Pullbacks equivalent to pseudopullbacks, *Cahiers Topologie Gom. Diffrentielle Catgoriques* 34:153–156, 1993.

[17] A. Joyal and M. Tierney, Strong stacks and classifying spaces, *Category Theory, Proceedings of Como 1990,* Lecture Notes in Mathematics 1488, Springer-Verlag, 213–236, 1991.

[18] G.M. Kelly and Stephen Lack, $\mathscr{V}$**-Cat** is locally presentable or locally bounded if $\mathscr{V}$ is so, *Theory Appl. Categ.* 8:555–575, 2001.

[19] G.M. Kelly and R.H. Street, Review of the elements of 2-categories, *Sydney Category Seminar,* Lecture Notes in Mathematics 420, 75–103, Springer-Verlag, 1974.

[20] Ieke Moerdijk and Jan-Alve Svensson, Algebraic classification of equivariant homotopy 2-types I, *J. Pure Appl. Algebra* 89:187–216, 1993.

[21] A.J. Power, A general coherence result, *J. Pure Appl. Alg.* 57:165–173, 1989.

[22] Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics 43, Springer-Verlag, 1967.

[23] Daniel Quillen, Rational homotopy theory, *Ann. Math.* 90:205–295, 1969.

[24] S. Schwede and B. Shipley, Algebras and modules in monoidal model categories, *Proc. London Math. Soc.* 80:491–511, 2000.

[25] R. Thomason, Cat as a closed model category, *Cahiers Topologie Gom. Diffrentielle* 21:305-324, 1980.