# A Quillen model structure for bicategories

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**Abstract**

A cofibrantly generated Quillen model structure on the category $\mathbf{Bicat_s}$ of bicategories and strict homomorphisms is constructed. Another such structure on the category **2-Cat** of 2-categories and 2-functors is described, correcting the construction given in an earlier paper. The fully faithful inclusion of **2-Cat** in $\mathbf{Bicat_s}$ is shown to be the right adjoint part of a Quillen equivalence.

In an earlier paper [5] I tried to describe a Quillen model structure on the category **2-Cat** of 2-categories and 2-functors. Unfortunately, as pointed out to me by André Joyal, this contained an error. The purpose of this note is to correct the error, and moreover to show that the Quillen model structure extends to one on the category $\mathbf{Bicat_s}$ of bicategories and strict homomorphisms of bicategories. The fully faithful inclusion of **2-Cat** in $\mathbf{Bicat_s}$ has a left adjoint, and this adjunction turns out to be a Quillen equivalence.

The problem with the earlier definition is that the generating trivial cofibration $j_1 : 1 \to E$, defined on page 179, is not a weak equivalence. The solution, also suggested by André, is to replace the "free-living equivalence $E$", by the "free-living adjoint equivalence", defined below. This results in a change in the definition of fibration and trivial cofibration, but not in the definition of weak equivalences, trivial fibrations, or cofibrations. All other results in the paper remain valid.

A bicategory [2], like a 2-category, contains objects, 1-cells (or morphisms or arrows) between objects, and 2-cells between 1-cells, and these can be composed as in a 2-category, except that the composition of 1-cells is associative and unital only up to coherent isomorphism. The 2-categories can be seen as those bicategories for which these isomorphisms are in fact identities. A *homomorphism of bicategories* [2] — that is, a homomorphism between bicategories — does not preserve composition or identities strictly, but only up to coherent isomorphism. The *strict homomorphisms*, in which these isomorphisms are identities, and so the structure *is* strictly preserved, are rare in practice, but of theoretical importance, and it is these which are considered here. A strict homomorphism between bicategories which are actually 2-categories is precisely a 2-functor; while a homomorphism between 2-categories is also called a pseudofunctor. The reasons for working with 2-functors rather than pseudofunctors were discussed in [5, Section 4.1], and the reasons for working with strict homomorphisms are precisely the same.

A homomorphism of bicategories (strict or otherwise) $M : \mathscr{A} \to \mathscr{B}$ is said to be a *biequivalence* if the functors $M : \mathscr{A}(A, A') \to \mathscr{B}(MA, MA')$ are equivalences for all objects $A$ and $A'$ of $\mathscr{A}$; and if moreover for every object $B$ of $\mathscr{B}$ there is an object $A$ of $\mathscr{A}$ and an equivalence $MA \simeq B$ in $\mathscr{B}$; the notion of equivalence in a bicategory is recalled in Section 2 below. In [5], the weak equivalences in **2-Cat** were defined to be the biequivalences — that is, those 2-functors which are biequivalences; similarly, the weak equivalences in $\mathbf{Bicat_s}$ will be the (strict homomorphisms which are) biequivalences. Thus a 2-functor will be a weak equivalence in **2-Cat** if and only if it is one in $\mathbf{Bicat_s}$.

The reader is referred to the early sections and references of [5] for more information about model categories and 2-categories.

# 1 Constructions involving bicategories

In this section we describe various adjunctions involving the category $\mathbf{Bicat_s}$.

The structure of bicategory is *essentially algebraic* in the sense of Freyd, and so the category $\mathbf{Bicat_s}$ is both complete and cocomplete, and in fact locally finitely presentable. Similarly, the structure of 2-category is essentially algebraic and the category **2-Cat** is locally finitely presentable. Moreover the fully faithful inclusion of $I : \mathbf{2\text{-}Cat} \to \mathbf{Bicat_s}$ is given by forgetting certain essentially algebraic structure (actually the *property* that certain isomorphisms are identities), and so this inclusion has a left adjoint $L$. We shall describe the left adjoint below.

A (directed) graph consists of a collection of vertices $X$, $Y$, $Z$, ..., and for each pair $(X, Y)$ of vertices a set $\mathscr{G}(X, Y)$ of edges from $X$ to $Y$. A **Cat**-graph [7] consists of a collection of vertices, as above, and for each pair $(X, Y)$ of vertices a *category* $\mathscr{G}(X, Y)$. The **Cat**-graphs form the objects of an evident category **Cat-Graph**, with an evident forgetful functor $U : \mathbf{Bicat_s} \to \mathbf{Cat\text{-}Graph}$. Once again the structure of **Cat**-graph is essentially algebraic, so **Cat-Graph** is locally finitely presentable, and once again the forgetful functor $U$ is given by forgetting certain essentially algebraic structure, and so it has a left adjoint $F$. In fact $\mathbf{Bicat_s}$ is the category of algebras for a 2-operad on **Cat-Graph** — see [1, Section 10] for the details. (Of course **2-Cat** is also the category of algebras for a 2-operad on **Cat-Graph**.)

The left adjoint $L : \mathbf{Bicat_s} \to \mathbf{2\text{-}Cat}$ may be described as follows. Given a bicategory $\mathscr{B}$, consider the underlying **Cat**-graph $V\mathscr{B}$ of $\mathscr{B}$, and the free 2-category $GV\mathscr{B}$ on $V\mathscr{B}$. (The free 2-category functor $G : \mathbf{Cat\text{-}Graph} \to \mathbf{2\text{-}Cat}$ is of course the composite $LF$, but it may be described more simply than either $L$ or $F$ using the idea of *paths* in a **Cat**-graph — see [7] for the details.) The 2-category $L\mathscr{B}$ is obtained as a quotient of $GV\mathscr{B}$; specifically, by the universal 2-functor $GV\mathscr{B} \to L\mathscr{B}$ for which the composite $\mathscr{B} \to IGV\mathscr{B} \to IL\mathscr{B}$ is a strict homomorphism of bicategories. Clearly this $\mathscr{B} \to IL\mathscr{B}$ is the identity on objects and surjective on 1-cells. Moreover, since it only identifies 1-cells which are isomorphic, this quotienting process does not result in the creation of new 2-cells, and so the functors $\mathscr{B}(B, B') \to IL\mathscr{B}(B, B')$ are full. They need not be faithful, as the following example shows. (A bicategory is said to have *strict identities* if the identity law for composition of 1-cells holds, and the "identity isomorphisms" $f1_A \cong f \cong 1_B f$ are identities, for any 1-cell $f : A \to B$.)

**Example 1** Consider the bicategory $\mathscr{B}$ with strict identities, with objects $A$, $B$, $C$, $D$; with non-identity 1-cells $j : A \to B$, $f, g : B \to C$, $q : C \to D$, $fj = gj = k : A \to C$, $qf = qg = r : B \to D$, and $rj, qk : A \to D$; and with the only non-identity 2-cells being between $rj$ and $qk$, these being freely generated by $\alpha : rj \cong qk$ and $\beta : rj \cong qk$. The only non-trivial associativity isomorphisms are $(qf)j = rj \cong qk = q(fj)$, given by $\alpha$, and $(qg)j = rj \cong qk = q(gj)$, given by $\beta$. The monoid of 2-cells from $rj$ to $rj$ is clearly non-trivial. In the 2-category $L\mathscr{B}$, however, there is only one 1-cell $(qf)j = q(fj) = q(gj) = (qg)j$ from $A$ to $D$ and no non-identity 2-cells.

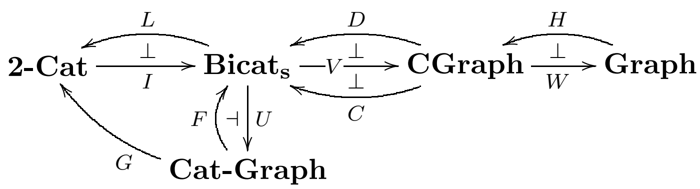
We summarize our results so far as:

**Proposition 2** *The strict homomorphism $\mathscr{B} \to IL\mathscr{B}$ is bijective on objects, surjective on arrows, and locally full; it is not necessarily locally faithful, but if it is locally faithful then it is a biequivalence.*

Next we describe a different sort of structure underlying a bicategory. We define a *compositional graph* to be a (directed) graph, with a chosen identity $1_X : X \to X$ for each object $X$, and a chosen composite $gf : X \to Z$ for each composable pair $(f : X \to Y, g : Y \to Z)$, but with neither the associative law nor the identity laws assumed to hold. The compositional graphs are the object of an evident category **CGraph**, which once again is locally finitely presentable, and once again the evident forgetful functor $V : \mathbf{Bicat_s} \to \mathbf{CGraph}$ has a left adjoint $D$. This time, however, the forgetful functor also has a *right* adjoint, which sends a compositional graph $G$ to the (unique) bicategory with the same underlying compositional graph, and with a single (invertible) 2-cell between any parallel pair of arrows. The counit of the adjunction $V \dashv C$ is clearly invertible, so $C$ is fully faithful; it follows that $D$ is also fully faithful, and that the unit of the adjunction $D \dashv V$ is invertible.

Finally observe that there is a further forgetful functor $W : \mathbf{CGraph} \to \mathbf{Graph}$ from the category of compositional graphs to the category of (directed) graphs, and that once again **Graph** is locally finitely presentable, and $W$ has a left adjoint $H$.

The various adjunctions described in this section are summarized in the following diagram:




## 2 The model structure for bicategories

As described in the introduction, we use the same notion of weak equivalence in $\mathbf{Bicat_s}$ as was used in **2-Cat**. Similarly, we use the same notion of trivial fibration: a strict homomorphism $F : \mathscr{A} \to \mathscr{B}$ is a *trivial fibration* if it is surjective on objects, and each functor $F : \mathscr{A}(A, A') \to \mathscr{B}(FA, FA')$ is surjective on objects and an equivalence. Thus a 2-functor is a weak equivalence or trivial fibration in **2-Cat** if and only if it is one in $\mathbf{Bicat_s}$.

Recall that an arrow $b : B' \to B$ in a bicategory is an *equivalence* if there exist an arrow $b^* : B \to B'$ and invertible 2-cells $bb^* \cong 1_B$ and $1_{B'} \cong b^*b$. A strict homomorphism $F : \mathscr{A} \to \mathscr{B}$ is said to be a *fibration* if it satisfies the following conditions:

($i$) For every object $A$ in $\mathscr{A}$ and every equivalence $b : B' \to FA$ in $\mathscr{B}$ there is an equivalence $a : A' \to A$ in $\mathscr{A}$ with $FA' = B'$ and $Fa = b$;

($ii$) For every 1-cell $a : A' \to A$ in $\mathscr{A}$ and every invertible 2-cell $\beta : b' \to Fa$ in $\mathscr{B}$, there is an invertible 2-cell $\alpha : a' \to a$ in $\mathscr{A}$ with $Fa' = b'$ and $F\alpha = \beta$.

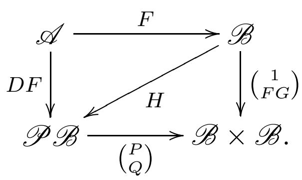
We also say that $F$ has the *equivalence lifting property*; note that this refers both to the lifting of *1-cells* which are equivalences, as in $(i)$, and to the lifting of *2-cells* which are equivalences (actually isomorphisms), as in $(ii)$.

We say that a 2-functor is a fibration if it has the equivalence lifting property; *this is not the same as the definition in* [5], which involved the lifting not just of the equivalence $b$, but also of the $b^*$ and the invertible 2-cells. We shall see in Proposition 6 below how the current definition allows one to lift these other data as well, provided that one works with *adjoint equivalences*.

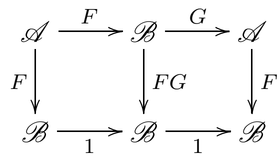
It is straightforward to check that the biequivalences are closed under retracts and satisfy the 2-out-of-3 property, and that the trivial fibrations are precisely the biequivalences with the equivalence lifting property. We define the cofibrations to be the arrows with the left lifting property with respect to the trivial fibrations, and we define the trivial cofibrations to be the arrows with the left lifting property with respect to the fibrations.

Next we show that the trivial cofibrations are weak equivalences, using the existence of *path objects* in $\mathbf{Bicat_s}$. Every bicategory $\mathscr{B}$ is clearly fibrant; moreover we can factorize the diagonal map $\mathscr{B} \to \mathscr{B} \times \mathscr{B}$ as a weak equivalence $D : \mathscr{B} \to \mathscr{P}\mathscr{B}$ followed by a fibration $\binom{P}{Q} : \mathscr{P}\mathscr{B} \to \mathscr{B} \times \mathscr{B}$ (such a $\mathscr{P}\mathscr{B}$ is called a path object for $\mathscr{B}$). Explicitly, an object of $\mathscr{P}\mathscr{B}$ is an equivalence $b : B' \to B$ in $\mathscr{B}$; a morphism from $b : B' \to B$ to $c : C' \to C$ consists of arrows $g : B \to C$ and $g' : B' \to C'$ and an invertible 2-cell $\psi : cg' \cong gb$; a 2-cell from $(g, \psi, g')$ to $(f, \varphi, f')$ consists of 2-cells $\gamma : g \to f$ and $\gamma' : g' \to f'$ satisfying the obvious condition expressing compatibility with $\psi$ and $\varphi$. These objects, arrows, and 2-cells form an evident bicategory $\mathscr{P}\mathscr{B}$, with strict homomorphisms $P, Q : \mathscr{P}\mathscr{B} \to \mathscr{B}$ sending $b : B' \to B$ to $B'$ and to $B$, and with a strict homomorphism $D : \mathscr{B} \to \mathscr{P}\mathscr{B}$ sending an object $B$ to the identity $1_B : B \to B$. We omit the routine details, and the verification that $D$ is a weak equivalence and $\binom{P}{Q} : \mathscr{P}\mathscr{B} \to \mathscr{B} \times \mathscr{B}$ is a fibration.

The fact that trivial cofibrations are weak equivalences now follows by a standard argument: if $F : \mathscr{A} \to \mathscr{B}$ is a trivial cofibration, then since $\mathscr{A}$ is fibrant there is a map $G : \mathscr{B} \to \mathscr{A}$ with $GF = 1$; since $\binom{P}{Q}$ is a fibration there is now a fill-in $H$ as in




Now $PD = QD = 1$ and $D$ is a weak equivalence, so $P$ and $Q$ are weak equivalences; thus $H$ is a weak equivalence since $PH = 1$, and $FG$ is a weak equivalence since $QH = FG$. Finally the diagram




exhibits $F$ as a retract of $FG$, and so $F$ is a weak equivalence as required.

In Section 6, we shall describe small (in fact finite) sets $I$ and $J$ of generating cofibrations and generating trivial cofibrations, so that the trivial fibrations are the arrows with the right lifting property with respect to the generating cofibrations, and the fibrations are the arrows with the right lifting property with the generating trivial cofibrations. Since $\mathbf{Bicat_s}$ is locally finitely presentable, and so every object is small, this guarantees the existence of the required factorizations, and by [4, Theorem 2.1.19] we have:

**Theorem 3** *There is a cofibrantly generated model structure on the category* $\mathbf{Bicat_s}$ *of bicategories and strict homomorphisms, for which the weak equivalences are the biequivalences, and the fibrations are the strict homomorphisms with the equivalence lifting property.*

Notice that $\mathscr{P}\mathscr{B}$ is a 2-category if $\mathscr{B}$ is one, and so everything in this section works with **2-Cat** in place of $\mathbf{Bicat_s}$. The generating cofibrations and generating trivial cofibrations for **2-Cat** are obtained by applying $L : \mathbf{Bicat_s} \to \mathbf{2\text{-}Cat}$ to those for $\mathbf{Bicat_s}$; this is also described in Section 6. We then deduce:

**Theorem 4** *There is a cofibrantly generated model structure on the category* **2-Cat** *of 2-categories and 2-functors, for which the weak equivalences are the biequivalences, and the fibrations are the 2-functors with the equivalence lifting property.*

This corrects the faulty definition in [5]. Note that everything in [5] becomes correct if we replace the generating trivial cofibration $j_1 : 1 \to E$ given on page 179 of [5] by the new generating trivial cofibration $j_1' : 1 \to E'$, described in Section 6 below, where in place of the free-living equivalence $E$ we have the free-living adjoint equivalence $E'$. The later sections of [5], concerning cofibrations, the homotopy relation, properness, relations with monoidal structure, and so on, can all be read unchanged, and we now have two different proofs of the model category axioms for **2-Cat**.

# 3 Fibrations

In this section we look at an alternative characterization of the fibrations. We defined a 1-cell $b : B' \to B$ in a bicategory $\mathscr{B}$ to be an equivalence if there exist a 1-cell $b^* : B \to B'$, and invertible 2-cells $\beta_1 : 1_{B'} \to b^*b$ and $\beta_2 : bb^* \to 1_B$. We say that $(b, b^*, \beta_1, \beta_2)$ is an *adjoint equivalence from* $B'$ *to* $B$ if moreover the triangle equations are satisfied: these assert the commutativity of the diagrams

$$\begin{array}{ccccc} & & b1_{B'} & \xrightarrow{b\beta_1} & b(b^*b) \\ & \nearrow & & & \downarrow \\ b & & & & \\ & \searrow & & & \\ & & 1_B b & \xleftarrow[\beta_2 b]{} & (bb^*)b \end{array} \qquad \begin{array}{ccccc} & & 1_{B'}b^* & \xrightarrow{\beta_1 b^*} & (b^*b)b^* \\ & \nearrow & & & \downarrow \\ b^* & & & & \\ & \searrow & & & \\ & & b^*1_B & \xleftarrow[b^*\beta_2]{} & b^*(bb^*) \end{array}$$

of 2-cells in $\mathscr{B}$ in which the unnamed arrows are associativity and unit isomorphisms in the bicategory.

Every equivalence $b : B' \to B$ is part of an adjoint equivalence; in fact we have the following well-known result about adjoint equivalences:

**Lemma 5** *Let* $b : B' \to B$ *and* $b^* : B \to B'$ *be 1-cells in a bicategory, and let* $\beta_2 : bb^* \to 1_{B'}$ *be an invertible 2-cell. If* $b^*b \cong 1$*, then there is a unique invertible 2-cell* $\beta_1 : 1 \to b^*b$ *for which* $(b, b^*, \beta_1, \beta_2)$ *is an adjoint equivalence.*

The promised charaterization of fibrations is:

**Proposition 6** *A strict homomorphism $F : \mathscr{A} \to \mathscr{B}$ of bicategories is a fibration if and only if it satisfies condition (ii) in the definition of fibration, and also:*

*(i′) for any object $A$ of $\mathscr{A}$ and any adjoint equivalence $(b, b^*, \beta_1, \beta_2)$ from $B'$ to $FA$ in $\mathscr{B}$, there exists an adjoint equivalence $(a, a^*, \alpha_1, \alpha_2)$ from $A'$ to $A$ in $\mathscr{A}$, with $FA' = B'$, $Fa = b$, $Fa^* = b^*$, $F\alpha_1 = \beta_1$, and $F\alpha_2 = \beta_2$.*
*Similarly, a 2-functor $F : \mathscr{A} \to \mathscr{B}$ is a fibration if and only if it satisfies conditions (i′) and (ii).*

Proof: We prove the statement involving strict homomorphisms; the case of 2-functors is an immediate consequence.

Since every equivalence is part of an adjoint equivalence, $(i')$ certainly implies $(i)$. Suppose conversely that $F : \mathscr{A} \to \mathscr{B}$ satisfies $(i)$ and $(ii)$, and let $A$ be an object of $\mathscr{A}$, and $(b, b^*, \beta_1, \beta_2)$ an adjoint equivalence in $\mathscr{B}$ from $B'$ to $FA$. By $(i)$, there is an equivalence $a : A' \to A$ in $\mathscr{A}$ with $FA' = B'$ and $Fa = b$. By Lemma 5 there is an adjoint equivalence $(a, a', \alpha_1', \alpha_2')$ in $\mathscr{A}$ from $A'$ to $A$. Since $b$ is an equivalence and $F\alpha_2'$ is invertible, there is a unique invertible 2-cell $\zeta : b^* \to Fa'$ in $\mathscr{B}$ for which the composite

$$bb^* \xrightarrow{b\zeta} b.Fa' = F(aa') \xrightarrow{F\alpha_2'} 1_B$$

is equal to $\beta_2$. By condition $(ii)$ there is an invertible 2-cell $\xi : a^* \to a'$ with $Fa^* = b^*$ and $F\xi = \zeta$. Let $\alpha_2 : aa^* \to 1$ be the (invertible) composite

$$aa^* \xrightarrow{a\xi} aa' \xrightarrow{\alpha_2'} 1$$

in $\mathscr{A}$, so that $F\alpha_2 = \beta_2$. Since $a^*a \cong a'a \cong 1$, there is by Lemma 5 a unique invertible 2-cell $\alpha_1$ with $(a, a^*, \alpha_1, \alpha_2)$ an adjoint equivalence in $\mathscr{A}$. Since $Fa = b$, $Fa^* = b^*$, and $F\alpha_2 = \beta_2$, by the uniqueness aspect of Lemma 5 also $F\alpha_1 = \beta_1$. □

# 4 Cofibrations

In this section we study the cofibrations and cofibrant objects in $\mathbf{Bicat_s}$ using the adjunctions $D \dashv V \dashv C$ and $H \dashv W$.

We shall say that a morphism $F : \mathscr{G} \to \mathscr{H}$ of compositional graphs is *full* if for all objects $X$ and $Y$ the map $F : \mathscr{G}(X, Y) \to \mathscr{H}(FX, FY)$ is surjective, and say that $F$ is *surjective* if it is surjective on objects and full. Finally we say that a compositional graph is *projective* if it is projective with respect to the surjections in **CGraph**.

**Proposition 7** *A strict homomorphism of bicategories $M : \mathscr{A} \to \mathscr{B}$ is a cofibration if and only if $VM : V\mathscr{A} \to V\mathscr{B}$ has the left lifting property with respect to the surjections in* **CGraph**.

Proof: If $M$ is a cofibration and

$$\begin{array}{ccc} V\mathscr{A} & \xrightarrow{VM} & V\mathscr{B} \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle T} \\ G & \xrightarrow[P]{} & H \end{array}$$

is a diagram in **CGraph** with $P$ surjective, then the adjunction $V \dashv C$ induces a diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{M} & \mathscr{B} \\ {\scriptstyle S'}\downarrow & & \downarrow{\scriptstyle T'} \\ CG & \xrightarrow[CP]{} & CH \end{array}$$

in $\mathbf{Bicat_s}$ with $CP$ a trivial fibration. Since $M$ is a cofibration, there exists a fill-in $R' : \mathscr{B} \to CG$ for the latter square, and so a fill-in $R : V\mathscr{B} \to G$ for the former square. This proves that $VM$ has the left lifting property with respect to the surjections in **CGraph**.

Suppose conversely that $VM$ has the left lifting property with respect to the surjections in **CGraph** and that

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{M} & \mathscr{B} \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle T} \\ \mathscr{E} & \xrightarrow[P]{} & \mathscr{D} \end{array}$$

is a diagram in $\mathbf{Bicat_s}$ with $P$ a trivial fibration. Applying $V$ gives a diagram in **CGraph** with $VP$ a surjection, and so a fill-in $R_0$ as in

$$\begin{array}{ccc} V\mathscr{A} & \xrightarrow{VM} & V\mathscr{B} \\ {\scriptstyle VS}\downarrow & \swarrow{\scriptstyle R_0} & \downarrow{\scriptstyle VT} \\ V\mathscr{E} & \xrightarrow[VP]{} & V\mathscr{D}. \end{array}$$

Since $DV\mathscr{B}$ has the same objects, arrows, composition, and identities as $\mathscr{B}$, while $P : \mathscr{E} \to \mathscr{D}$ is locally fully faithful, there is a unique strict homomorphism $R : \mathscr{B} \to \mathscr{E}$ with $VR = R_0$ and $PR = T$. Moreover, $V(RM) = R_0.VM = VS$, so that $RM$ agrees with $S$ on objects and arrows, while $PRM = TM = PS$ ensures that $RM$ agrees with $S$ on 2-cells; thus $RM = S$, and so $R$ provides the desired fill-in to exhibit $M$ as a cofibration. $\square$

We shall say that a compositional graph is *free* if it is in the image of the left adjoint $H$ : **Graph** $\to$ **CGraph**.

**Lemma 8** *A bicategory is cofibrant if and only if it is a retract of a bicategory whose underlying compositional graph is free.*

PROOF: By the proposition, a bicategory $\mathscr{B}$ will be cofibrant if $V\mathscr{B}$ is projective (with respect to the surjections in **CGraph**). But any free compositional graph will clearly be projective. Since a retract of a cofibrant object is cofibrant this proves one half of the lemma. Suppose conversely that $\mathscr{B}$ is cofibrant, and consider its underlying compositional graph $V\mathscr{B}$, which by Proposition 7 will be projective. The canonical map $HWV\mathscr{B} \to V\mathscr{B}$ is bijective on objects and surjective on arrows, hence the same is true of $DHWV\mathscr{B} \to DV\mathscr{B}$, while the canonical $DV\mathscr{B} \to \mathscr{B}$ is bijective on objects and bijective on arrows. Thus the composite $DHWV\mathscr{B} \to \mathscr{B}$ is bijective on objects and surjective on arrows, and when we factorize this composite as

$$DHWV\mathscr{B} \xrightarrow{E} Q\mathscr{B} \xrightarrow{J} \mathscr{B}$$

where $E$ is bijective on objects and bijective on arrows and $J$ is locally fully faithful, $J$ will also be bijective on objects and surjective on arrows, and so a trivial fibration. Since $\mathscr{B}$ is cofibrant, $J$ will have a section and so $\mathscr{B}$ is a retract of $Q\mathscr{B}$. But $VQ\mathscr{B} = VDHWV\mathscr{B} = HWV\mathscr{B}$, hence the result. $\square$

Our final result for this section will be crucial in the proof that $\mathbf{Bicat_s}$ and **2-Cat** are Quillen equivalent.

**Lemma 9** *If $\mathscr{B}$ is cofibrant, and $M : \mathscr{B} \to \mathscr{C}$ is a homomorphism of bicategories, then there is a strict homomorphism $M' : \mathscr{B} \to \mathscr{C}$ which is equivalent to $M$; in particular, if $M$ is a biequivalence then $M'$ will be one.*

Proof: For an arbitrary bicategory $\mathscr{B}$ we have the strict homomorphism $\varepsilon : FU\mathscr{B} \to \mathscr{B}$, and we factorize this as

$$FU\mathscr{B} \xrightarrow{E} \mathscr{B}' \xrightarrow{J} \mathscr{B}$$

where $E$ and $J$ are strict homomorphisms, with $E$ bijective on objects and arrows, and with $J$ locally fully faithful. Since $\varepsilon$ is bijective on objects and surjective on arrows, the same is true of $J$, and so $J$ is in fact a trivial fibration.

Given a homomorphism of bicategories $M : \mathscr{B} \to \mathscr{C}$, there is an underlying morphism $UM : U\mathscr{B} \to U\mathscr{C}$ of **Cat**-graphs, and so a strict homomorphism $FUM : FU\mathscr{B} \to FU\mathscr{C}$. Consider the following (non-commuting) square:

$$\begin{array}{ccc} FU\mathscr{B} & \xrightarrow{FUM} & FU\mathscr{C} \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ \mathscr{B} & \xrightarrow[M]{} & \mathscr{C}. \end{array}$$

The two paths around the square agree on objects, and on the arrows which are the "generators" of the free bicategory $FU\mathscr{B}$. They agree on all arrows only if the homomorphism $M$ is strict, but they will nonetheless agree on all arrows *up to isomorphism*. In fact we may take these isomorphisms as the pseudonaturality isomorphisms of a pseudonatural equivalence between the two paths around the square, with the components of the pseudonatural transformation being identity arrows in $\mathscr{C}$.

Since $E : FU\mathscr{B} \to \mathscr{B}'$ is bijective on objects and bijective on arrows, and $J : \mathscr{C}' \to \mathscr{C}$ is locally fully faithful, there is now a strict homomorphism $M' : \mathscr{B}' \to \mathscr{C}'$ for which the upper square in the diagram

$$\begin{array}{ccc} FU\mathscr{B} & \xrightarrow{FUM} & FU\mathscr{C} \\ {\scriptstyle E}\downarrow & & \downarrow{\scriptstyle E} \\ \mathscr{B}' & \xrightarrow{M'} & \mathscr{C}' \\ {\scriptstyle J}\downarrow & \simeq & \downarrow{\scriptstyle J} \\ \mathscr{B} & \xrightarrow[M]{} & \mathscr{C} \end{array}$$

commutes and the lower square does so up to (pseudonatural) equivalence: one uses $FUM$ to define $M'$ on objects and arrows, and uses $M$ to define $M'$ on 2-cells. Finally if $\mathscr{B}$ is cofibrant, then there is a strict homomorphism $K : \mathscr{B} \to \mathscr{B}'$ with $JK = 1$, so we may take $F' = JM'K$.

□

## 5 The Quillen equivalence

We now compare the model structures on **2-Cat** and $\mathbf{Bicat_s}$, using the adjunction $L \dashv I$. Since $I$ preserves weak equivalences and fibrations, this is a Quillen adjunction. We shall show that it is in fact a Quillen equivalence, so that the derived adjunction between the homotopy categories will be an equivalence.

By [4, Proposition 1.3.13], the adjunction will be a Quillen equivalence provided that the composites

$$\mathscr{B} \xrightarrow{\rho} IL\mathscr{B} \xrightarrow{Ir} IRL\mathscr{B}$$

$$LQI\mathscr{C} \xrightarrow{Lq} LI\mathscr{C} =\!=\!=\!= \mathscr{C}$$

are weak equivalences whenever $\mathscr{B}$ is a cofibrant bicategory and $\mathscr{C}$ is a fibrant 2-category, where $q : QI\mathscr{C} \to I\mathscr{C}$ is a cofibrant replacement in $\mathbf{Bicat_s}$ of $I\mathscr{C}$, and $r : L\mathscr{B} \to RL\mathscr{B}$ is a fibrant replacement of $L\mathscr{B}$ in **2-Cat**. Since every 2-category is fibrant, this amounts to proving that $\rho : \mathscr{B} \to IL\mathscr{B}$ is a weak equivalence if $\mathscr{B}$ is cofibrant, and that $Lq : LQI\mathscr{C} \to LI\mathscr{C}$ is a weak equivalence for any $\mathscr{C}$.

We first prove:

**Lemma 10** *If $\mathscr{B}$ is a cofibrant bicategory then $\rho : \mathscr{B} \to IL\mathscr{B}$ is a trivial fibration, and so in particular a biequivalence of bicategories.*

PROOF: By Proposition 2 it will suffice to show that $\rho : \mathscr{B} \to IL\mathscr{B}$ is locally faithful. By the coherence theorem for bicategories [6], there is a biequivalence $M : \mathscr{B} \to I\mathscr{C}$ for some 2-category $\mathscr{C}$. By Lemma 9, we can replace $M$ by an equivalent 2-functor $M'$ which will still be a biequivalence. By the universal property of $L\mathscr{B}$ there is a 2-functor $N : L\mathscr{B} \to \mathscr{C}$ with $M'$ equal to the composite of $\rho$ and $IN$. Now $M'$ is locally faithful since it is a biequivalence, thus $\rho$ is locally faithful, and so $\rho$ is in fact a trivial fibration. □

We are now ready to show:

**Theorem 11** *Consider the model structures on the categories* $\mathbf{Bicat_s}$*, of bicategories and strict homomorphisms, and* **2-Cat***, of 2-categories and 2-functors. The fully faithful inclusion of* **2-Cat** *into* $\mathbf{Bicat_s}$ *is the right adjoint part of a Quillen equivalence.*

PROOF: It remains to show that $Lq : LQI\mathscr{C} \to LI\mathscr{C}$ is a weak equivalence for any 2-category $\mathscr{C}$. Consider the diagram

$$\begin{array}{ccc} ILQI\mathscr{C} & \xrightarrow{ILq} & ILI\mathscr{C} \\ \uparrow{\scriptstyle\rho} & & \uparrow{\scriptstyle\rho} \\ QI\mathscr{C} & \xrightarrow[q]{} & I\mathscr{C} \end{array}$$

which commutes by naturality of $\rho$. Now $\rho : I\mathscr{C} \to ILI\mathscr{C}$ is invertible, and $q : QI\mathscr{C} \to \mathscr{C}$ is a weak equivalence, while $\rho : QI\mathscr{C} \to ILQI\mathscr{C}$ is a weak equivalence since $QI\mathscr{C}$ is cofibrant. Thus $ILq$ is a weak equivalence, and so finally $Lq$ is one. □

## 6 Generating cofibrations and trivial cofibrations

In this section we describe the generating cofibrations and generating trivial cofibrations for the model structure on $\mathbf{Bicat_s}$, using the forgetful functor $U : \mathbf{Bicat_s} \to \mathbf{Cat\text{-}Graph}$ and its left adjoint $F$.

We say that a morphism $M : \mathscr{G} \to \mathscr{H}$ of **Cat**-graphs is a trivial fibration if it is surjective on objects and each $M : \mathscr{G}(X, Y) \to \mathscr{H}(FX, FY)$ is a surjective equivalence; clearly a strict homomorphism $M : \mathscr{A} \to \mathscr{B}$ of bicategories is a trivial fibration in $\mathbf{Bicat_s}$ if and only if $UM$ is a trivial fibration in **Cat-Graph**. We shall describe "generating cofibrations" in **Cat-Graph**, and then apply $F : \mathbf{Cat\text{-}Graph} \to \mathbf{Bicat_s}$ to obtain generating cofibrations in $\mathbf{Bicat_s}$.

Let 0 denote the empty **Cat**-graph, and 1 the **Cat**-graph with a single object $*$ and $1(*, *)$ equal to the empty category. To give a morphism $1 \to \mathscr{G}$ is to give an object of $\mathscr{G}$, and so a morphism of **Cat**-graphs is surjective on objects if and only if it has the right lifting property with respect to the unique map $! : 0 \to 1$. If $C$ is a category, write $2_C$ for the **Cat**-graph with objects $X$ and $Y$, and hom-categories $2_C(X, X) = 2_C(Y, Y) = 2_C(Y, X) = 0$ and $2_C(X, Y) = C$; this is obviously functorial in $C$, so that a functor $f : C \to D$ induces a morphism $2_f : 2_C \to 2_D$ of **Cat**-graphs. To give a morphism $2_C \to G$ is to give objects $X$ and $Y$ of $G$, and a functor $C \to G(X, Y)$. A morphism $M : \mathscr{G} \to \mathscr{H}$ of **Cat**-graphs has each $\mathscr{G}(X, Y) \to \mathscr{H}(MX, MY)$ a surjective equivalence if and only if $M$ has the right lifting property with respect to $2_i$, $2_{i'}$, and $2_{i''}$, where $i$, $i'$, and $i''$ are the three generating cofibrations for the model structure on **Cat** [5, Example 1.1]. (Explicitly, $i$ is the unique functor from the empty category to the terminal category, $i'$ is the identity-on-objects functor from the discrete category with two objects to the "arrow category" $\mathbb{2}$, and $i''$ is the identity-on-objects functor from the category with two objects and two parallel non-identity arrows to the arrow-category $\mathbb{2}$.) Thus we have four "generating cofibrations" $! : 0 \to 1$, $2_i$, $2_{i'}$, and $2_{i''}$ in **Cat-Graph**, and now applying the left adjoint $F : \mathbf{Cat\text{-}Graph} \to \mathbf{Bicat_s}$ we obtain the desired generating cofibrations $F!$, $F2_i$, $F2_{i'}$, and $F2_{i''}$ in $\mathbf{Bicat_s}$.

Next we consider the fibrations and generating trivial cofibrations. Condition $(ii)$ in the definition of fibration can once again be expressed using the adjunction $F \dashv U$. If $j$ is the generating trivial cofibration of [5, Example 1.1] (from the terminal category to the "free-living isomorphism"), then a strict homomorphism of bicategories has the right lifting property with respect to $F2_j$ if and only if it satisfies condition $(ii)$. We now turn to condition $(i')$. Here we use the bicategory $\mathscr{E}$ with two objects $x$ and $y$, freely generated by 1-cells $s : x \to y$ and $t : y \to x$, and invertible 2-cells $1 \to ts$ and $st \to 1$ satisfying the triangle equations. An explicit construction is complicated by the fact that composition will not be strictly associative, but all we really need is the existence of $\mathscr{E}$, and about this there is no doubt. To give an adjoint equivalence in $\mathscr{B}$ is now precisely to give a strict homomorphism $\mathscr{E} \to \mathscr{B}$. The **Cat**-graph morphism $1 \to U\mathscr{E}$ picking out the object $x$ induces a strict homomorphism $k : F1 \to \mathscr{E}$, and the right lifting property with respect to $k$ is precisely condition $(i')$. Thus the fibrations are precisely the strict homomorphisms with the right lifting property with respect to the generating trivial cofibrations $F2_j$ and $k$.

This completes the missing step in the proof of Theorem 3.

Finally we observe that applying the left adjoint $L : \mathbf{Bicat_s} \to \mathbf{2\text{-}Cat}$ to the generating cofibrations recovers the generating cofibrations in **2-Cat** given in [5]; and applying $L$ to $F2_j$ recovers the generating trivial cofibration $j_2$ of [5]. Applying $L$ to $k$ gives $j'_1 : 1 \to E'$, which is what "should" have been taken as the final generating trivial cofibration. It is easier to describe concretely than $k$; the domain is the terminal 2-category 1, and the codomain has objects $x$ and $y$, 1-cells given $1_x$, $1_y$, and all "alternating non-empty words in $s$ and $t$", such as $sts$, $tststs$, and so on; there is a unique 2-cell between any parallel pair of 1-cells, and every 2-cell is invertible.

This completes the missing step in the proof of Theorem 4.

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