ON THE BURNSIDE RING AND
STABLE COHOMOTOPY OF A FINITE GROUP

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0. Introduction.

In this paper we study the connection between permutation representations of finite groups \( G \) and stable cohomotopy of the classifying space \( BG \), analogous to the connection between the character ring of \( G \) and the \( K \)-theory of \( BG \).

Suppose \( S \) is a finite \( G \)-set. Each ordering of \( S \) gives a homomorphism \( \varrho: G \to \Sigma_{|S|} \), where \( \Sigma_n \) denotes the permutation group in \( n \) letters and \( |S| \) is the cardinality of \( S \). Different orderings give conjugate maps, as do isomorphic \( G \)-sets. Hence the homotopy class of \( B\varrho: BG \to B\Sigma_{|S|} \) only depends on the isomorphism class of \( S \).

The disjoint union \( \bigsqcup_{n \geq 0} B\Sigma_n \) is a monoid and its group completion \( \Omega B(\bigsqcup_{n \geq 0} B\Sigma_n) \) is homotopy equivalent (as an \( H \)-space) to the space \( QS^0 = \lim_{n \to \infty} Q^n S^n \) of stable self maps of spheres. Let \( i: \bigsqcup_{n \geq 0} B\Sigma_n \to QS^0 \) be the resulting \( H \)-map and form the composition

\[
\alpha_G(S): BG \to B\Sigma_{|S|} \to \bigsqcup_{n \geq 0} B\Sigma_n \xrightarrow{i} QS^0.
\]

Disjoint union and Cartesian product turn the equivalence classes of \( G \)-sets into a semiring, whose associated ring is the Burnside ring \( A(G) \) ([17], [7]), and the correspondence \( S \to \alpha_G(S) \) defines an additive map

\[
\alpha_G: A(G) \to [BG, QS^0].
\]

[\( BG, QS^0 \)] is by definition the stable cohomotopy \( \pi_S^G(BG) \).

The space \( QS^0 \) admits besides the loop addition the smash product, homotopic to the product given by composition of maps. If \( \bigsqcup_{n \geq 0} B\Sigma_n \) is equipped with the monoid structure induced from the homomorphisms \( \Sigma_n \times \Sigma_m \to \Sigma_{nm} \), then \( i \) respects both structures and \( \alpha_G \) is a ring homomorphism.

The space \( QS^0 \) splits into a disjoint union of homotopy equivalent spaces \( Q_n S^0, n \in \mathbb{Z} \), where \( Q_n S^0 \) denotes the subspace of degree \( n \) maps. Thus we have

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an augmentation \([BG, QS^0] \rightarrow \mathbb{Z}\). Taking cardinality of \(G\)-sets defines an augmentation of \(A(G)\), and \(\alpha_G\) is clearly augmentation preserving. The \((n+1)\)-fold products become trivial on the \(n\)-skeleton \(B_n G\), so \(\alpha_G\) factors as

\[\alpha_G: A(G)/I^{n+1}(G) \rightarrow [B_n G, QS^0].\]

Passing to the limit we get a map from the \(I(G)\)-adic completion

\[\hat{\alpha}_G: \hat{A}(G) \rightarrow \lim_{\leftarrow} [B_n G, QS^0] = [BG, QS^0]\]

(the last isomorphism follows from the finiteness of \([B_n G, QoS^0]\)).

The map \(\hat{\alpha}_G\) is analogous to the isomorphism from the completed representation ring \(\hat{R}(G)\) to \(K(BG)\) [1], and some time ago \(G\). Segal made the

CONJECTURE. \(\hat{\alpha}_G: \hat{A}(G) \rightarrow [BG, QS^0]\) is an isomorphism.

The full conjecture seems very hard and is probably out of reach at the moment. In this paper we study the injectivity of \(\hat{\alpha}_G\).

First we reduce the problem to \(p\)-groups by showing

**Theorem A.** If \(\hat{\alpha}_{G_p}\) is injective for the Sylow subgroups \(G_p\) of \(G\), then \(\hat{\alpha}_G\) is injective.

This is proved by showing that \(\hat{A}(G)\) embeds into \(\bigoplus_p \hat{A}(G_p)\) via the restriction maps, and compatibly with \(BG_p \rightarrow BG\).

For cyclic groups the natural map \(\hat{A}(G) \rightarrow \hat{R}(G)\) is injective, and using Atiyah’s result, \(\hat{R}(G) \cong K^\ast(BG)\), we deduce

**Theorem B.** \(\hat{\alpha}_G\) is injective for cyclic groups \(G\).

One cannot hope to detect the maps \(\alpha_G(x): BG \rightarrow QoS^0\) for \(x \in \text{Ker } (\hat{A}(G) \rightarrow \hat{R}(G))\) by \(K\)-theory, since the space \(QoS^0\) splits as \(J \times \text{cok } J\) with \(\text{cok } J\) a \(K\)-theory point, and at least for groups \(G\) of odd order \([BG, J]\) embeds into \([BG, BU \times Z] = \hat{R}(G)\). By studying induced maps in homology we prove

**Theorem C.** \(\hat{\alpha}_G\) is injective for elementary abelian groups \(G = (\mathbb{Z}/p)^n\).

For Theorem C we need an induction machine, which tells that if \(\alpha_H\) is injective for all genuine subgroups \(H\) of a \(p\)-group \(G\) and furthermore \(\alpha_G\) is injective on a specific summand \(\mathbb{Z}x \subset A(G)\), then \(\hat{\alpha}_G\) is injective. It is the maps \(\alpha_G(nx)\) that induce nontrivial maps in \(\mathbb{Z}/p\)-homology. We show even more: one gets a host of homologically distinct elements \(B((\mathbb{Z}/p)^n) \rightarrow \text{cok } J_p\) for \(n \geq 2\).
THEOREM D. If $G = (\mathbb{Z}/p)^n$, and $\hat{A}_0(G) = \text{Ker} (\hat{A}(G) \to \hat{R}(G))$, then $\hat{a}_G$ maps $\hat{A}_0(G)$ ($2\hat{A}_0(G)$ if $p = 2$) injectively into $[BG, \text{cok } J_p]$.

We note that Theorems A and B combine to show that $\hat{a}_G$ is injective for the groups with cyclic Sylow subgroups. They are all metacyclic. Theorem C enlarges the class of groups for which $\hat{a}_G$ is injective to include e.g. $A_5$, the alternating group on 5 letters.

The smallest groups we cannot settle with our method are $\mathbb{Z}/4 \times \mathbb{Z}/2$, the dihedral D8 and the quaternionic Q8 of order 8. For these groups the maps $\alpha_G(nx)$ induce zero both in homology and K-theory. One wonders if connective K-theory or unitary bordism theory could settle these cases.

The representation ring $R(G)$ and K-theory $K(X)$ admit the structure of a $\lambda$-ring. Atiyah, Tall and Segal [3], [4] have explored the algebraic nature of such rings showing that one gets exponential isomorphisms $\varrho_k: \hat{I}(G) \xrightarrow{\sim} 1 + \hat{I}(G)$ for any $p$-group $G$, and $KSO(X)^{\wedge} \xrightarrow{\sim} (1 + KSO(X))^{\wedge}$ for any finite complex $X$, where $\wedge$ denotes the $p$-adic completion.

Now $A(G)$ has also $\lambda$-operations, yielding $\lambda$-operations on stable cohomotopy $\pi^0_S(X) = [X, QS^0]$, see [19]. Unfortunately the $\lambda$-ring $A(G)$ is not "special", and this breaks down the algebraic program above. However, it is interesting to identify the maps $\varrho_k: Q_0S^0 \to SG_p$. We give a character argument to show

THEOREM E. $\varrho_k: Q_0S^0 \to SG_p$ is the composition $Q_0S^0 \xrightarrow{\varphi} J_p \xrightarrow{S_p} SG_p$.

The paper is divided into 4 sections. The first contains generalities on the Burnside ring: its characters, functorial properties and topology. The main theorem is 1.15 which shows that $\hat{A}(G)$ is detected by $p$-groups.

In section 2 we study the $\lambda$-ring structure of $A(G)$ and note that the natural map $A(G) \to R(G)$ is a $\lambda$-homomorphism. We describe the characters of the associated operations $\lambda^n$, $\psi^n$ and $\varrho_k$, we show they induce operations in the (zero degree) stable cohomotopy $\pi^0_S$ (2.12) and prove Theorem E (2.20). The characters of $\lambda^n$ and $\psi^n$ were obtained independently by C. Siebeneicher [20].

The third section is devoted to the study of $\hat{a}_G$ and contains the proofs of theorems A and B (3.2 and 3.3). We set up the induction machinery needed to prove theorems C and D in section 4 (4.15, 4.22 and 4.23).

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1. The Burnside ring.

Let $G$ be a finite group. A $G$-action on a finite set $S$ is a homomorphism from
$G$ to $\Sigma_S$, the permutation group of $S$. A $G$-set is a finite set with a $G$-action. Two $G$-sets $S$ and $T$ are isomorphic if there exists a $G$-equivariant bijection $f$: $S \to T$, or in other words if the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\Sigma_S} & \Sigma_T \\
\downarrow & & \downarrow \\
 & \Sigma_f & \\
\end{array}
\]

commutes. The equivalence classes of $G$-sets form a commutative semiring $A^+(G)$ under disjoint union and cartesian product. The associated ring $A(G)$ is called the Burnside ring of $G$.

The additive structure of $A(G)$ is easily described. Breaking $G$-sets into $G$-orbits one sees that $A(G)$ is a free abelian group with basis consisting of cosets $G/H$, one for each conjugacy class of subgroups $H$ of $G$. We fix a set $C(G)$ of representatives.

For the multiplicative structure we introduce characters, following W. Burnside (who called them marks [5 p. 236]). Let $H$ be a subgroup of $G$ and $S$ be a $G$-set. Then we set

\[\chi_H(S) = |S^H|,\]

the number of elements in $S$ fixed by $H$. The character $\chi_H$ extends to give a ring homomorphism

\[\chi_H: A(G) \to \mathbb{Z}.\]

On the additive generators of $A(G)$ we have

\[\chi_{H_1}(G/H_2) = \{|gH_2 \mid H_1gH_2 = gH_2\} = \{|gH_2 \mid g^{-1}H_1g \subseteq H_2\} \]

or

\[\chi_{H_1}(G/H_2) = \{|gH_2 \mid H_1 \subseteq H_2\}.\]

This shows that $\chi_H$ depends only on the conjugacy class of $H$.

The homomorphisms $\chi_H$ define together a homomorphism

\[\chi: A(G) \to \bigoplus_{C(G)} \mathbb{Z}.\]

The first statement of the following theorem is due to W. Burnside [5] and the second one to A. Dress (unpublished, cf. [7]).

**Theorem 1.3.** The homomorphism $\chi: A(G) \to \bigoplus_{C(G)} \mathbb{Z}$ is an embedding with
finite cokernel. Its image consists of families \((d_H)_{H \in \mathcal{C}(G)}\) satisfying the congruences

\[
d_H \equiv - \sum_{H \leq K \leq G \atop K/H \text{ cyclic} + 1} \varphi(|K/H|) d_K \pmod{|N_G(H)/H|}
\]

where we set \(d_K = d_{K'}\) for \(K \sim K' \in \mathcal{C}(G)\) and \(\varphi\) is the Euler function.

**Proof.** To prove the injectivity, it is enough to show that non-isomorphic \(G\)-sets \(S\) and \(T\) cannot have the same characters. Write \(S = \sum m_H G/H\), \(T = \sum n_H G/H\) with \(H\) running over \(\mathcal{C}(G)\) and let \(H_0 \in \mathcal{C}(G)\) be maximal with respect to \(m_{H_0} \neq n_{H_0}\). Note from (1.2) that \(\chi_{H_0}(G/H_2)\) is non-zero if and only if \(H_1\) is conjugate to a subgroup of \(H_2\), denoted by \(H_1 \leq H_2\). Then

\[
\chi_{H_0}(S) = m_{H_0} \chi_{H_0}(G/H_0) + \sum_{H_0 \leq H, H + H_0} m_H \chi_{H_0}(G/H)
\]

and

\[
\chi_{H_0}(T) = n_{H_0} \chi_{H_0}(G/H_0) + \sum_{H_0 \leq H, H + H_0} n_H \chi_{H_0}(G/H)
\]

are different as the sum terms coincide but \(m_{H_0} \neq n_{H_0}\).

Let \(S\) be a \(G\)-set. We want to show that the numbers \(\chi_H(S)\) satisfy the congruences for each subgroup \(H\). If \(H < K\) then the \(K\)-fixed points of \(S\) are contained in the \(N_G(H)/H\)-set \(S^H\) so that we are reduced to the case \(H = e\). By a theorem of W. Burnside [5 p. 191] \(|G|^{-1} \sum_{g \in G} |S^g|\) is an integer, namely the number of \(G\)-orbits in \(S\). This implies the congruence

\[
|S| = \chi_e(S) \equiv - \sum_{g + 1} \chi_{(g)}(S) = - \sum_{K \leq G \atop \text{cyclic} + 1} \varphi(|K|) \chi_K(S) \pmod{|G|}
\]

where \(\varphi(|K|)\) is the number of generators in \(K\).

Conversely, we construct \(x \in A(G)\) with given characters \((d_H)\) by adding increasing orbits. Start with \(d_G\) points with trivial \(G\)-action. Choose a total order on \(\mathcal{C}(G)\) extending \(\leq\). Assume we have \(y \in A(G)\) such that \(\chi_H(y) = d_H\) for \(H\) greater than \(H_0\). Since the characters of \(y\) and the numbers \((d_H)\) both fulfill the congruences, we have

\[
\chi_{H_0}(y) \equiv d_{H_0} \pmod{|N_G(H_0)/H_0|},
\]

say \(d_{H_0} = \chi_{H_0}(y) + n|N_G(H_0)/H_0|\). We add \(n\) copies of \(G/H_0\) to \(y\). This does not change the earlier adjusted characters, but

\[
\chi_{H_0}(y + nG/H_0) = \chi_{H_0}(y) + n|N_G(H_0)/H_0| = d_{H_0}
\]

by (1.2). This completes the induction.
Finally \( \chi \) has finite cokernel since \( \oplus |G|Z \subset \text{Im} \chi \) by the congruences. The theorem is proved.

**Remark.** In [17] G. Segal defined a ring \( \omega^0_G \) in terms of equivariant stable homotopy. It coincides with \( A(G) \) as both are characterized by theorem 1.3. For a proof and generalization to compact Lie groups, see [16, Theorem 3].

If \( f: H \to G \) is a homomorphism of finite groups, then the pull-back \( f^*S \) of a \( G \)-set \( S \) has the same underlying set with \( H \)-action

\[
H \overset{f^*}{\to} G \to \Sigma_S.
\]

The induced maps \( f^*: A(G) \to A(H) \) make \( A \) into a contravariant functor. The characters of \( f^*x \) are

\[
\chi_U(f^*x) = \chi_{f(U)}(x), \quad U \subseteq H, \ x \in A(G).
\]

In the special case of a subgroup \( i: H \to G \) we call \( i^* \) the restriction homomorphism, and denote it by \( \text{Res}^G_H \).

There is also a covariant induction homomorphism \( \text{Ind}^G_H \) or \( f_* \) for inclusions of subgroups \( f: H \to G \). On the coset basis it is given by

\[
\text{Ind}^G_H(H/H_1) = G/H_1, \quad H_1 \subseteq H \subseteq G
\]

It is easily checked from (1.2) that the characters of \( f_*y \) are

\[
\chi_U(f_*y) = \sum_{U^* \subseteq H} \chi_{U^*}(y), \quad U \subseteq G, \ y \in A(H)
\]

where \( g \) runs through representatives of \( G/H \).

The homomorphisms \( \text{Res} \) and \( \text{Ind} \) are related in the same fashion as the restriction and induction maps in representation theory or cohomology of finite groups. If \( H \) is a subgroup of \( G \), then the Frobenius reciprocity

\[
\text{Ind}^G_H(y \cdot \text{Res}^G_H(x)) = \text{Ind}^H_G(y) \cdot x, \quad y \in A(H), \ x \in A(G)
\]

holds. Further, if \( K \) is another subgroup of \( G \), let \( Kg_1H, \ldots, Kg_rH \subseteq G \) be the double cosets of \( G \) mod \( (K,H) \). Then we have

\[
\text{Res}^G_K \text{Ind}^G_H(x) = \sum_{i=1}^r \text{Ind}^K_{g_iHg_i^{-1}}(c_i^* \text{Res}^{g_i^{-1}Kg_i \cap H}_K(x)), \quad x \in A(H)
\]

where \( c_i \) is conjugation by \( g_i \). The proofs of (1.6) and (1.7) are analogous to the corresponding formulas of representation theory.

Each \( G \)-set \( S \) can be considered as a linear representation of \( G \) over a field \( k \) by extending the \( G \)-action on the canonical basis of \( k^S \) by linearity. As the trace of a permutation matrix is equal to the number of 1's along the diagonal, the
linear character of \( k^S \) can be read off from the characters \( \chi_H \) of (1.1) for \( H \) cyclic:

\[
\chi_{k^S}(g) = \chi_{\langle g \rangle}(S).
\]

If \( k \) has characteristic 0, then the elements in \( R_k(G) \) are detected by their linear characters. We conclude from theorem 1.3

**Lemma 1.8.** Let \( k \) be a field of characteristic 0. Then the kernel of the natural map \( A(G) \to R_k(G) \) coincides with the kernel of the restriction map

\[
\text{Res} : A(G) \to \bigoplus_{C \subseteq G} A(C)
\]

to the cyclic subgroups of \( G \).

In Segal's conjecture the Burnside ring \( A(G) \) is compared with the stable cohomotopy ring \( \pi^S_0(BG) \). As the latter is complete (see section 3), we study here the algebraic process of completing \( A(G) \).

The character \( \chi_e \) just counts the number of points in a \( G \)-set and defines an augmentation \( \varepsilon : A(G) \to \mathbb{Z} \). This is a split surjection so \( A(G) = \mathbb{Z} \oplus I(G) \) where \( I(G) \) is the augmentation ideal \( \varepsilon^{-1}(0) \).

We give the ring \( A(G) \) the usual \( I(G) \)-adic topology, letting the powers \( I(G)^n \) be a neighbourhood basis of 0. The completion of \( A(G) \) is defined as the inverse limit

\[
\hat{A}(G) = \lim_{\longrightarrow} A(G)/I(G)^n.
\]

We shall study the kernel of \( A(G) \to \hat{A}(G) \). First we recall a result from commutative algebra (see e.g. [22 p. 262, Corollary to Theorem 8]). Let \( A \) be a Noetherian ring with no nilpotents and \( \mathfrak{m} \subset A \) a prime ideal. Then the kernel of the natural map from \( A \) to the \( \mathfrak{m} \)-adic completion \( \hat{A} = \lim A/m^n \) is

\[
\bigcap_{n=0}^{\infty} m^n = \bigcap_{p_j + \mathfrak{m} \neq A} p_j
\]

where \( p_j \) runs over such minimal prime ideals of \( A \) that \( p_j + \mathfrak{m} \neq A \).

The ring \( A(G) \) is Noetherian as a finitely generated abelian group. A. Dress determined in [7] the prime ideal structure of \( A(G) \). There are two types of prime ideals in \( A(G) \): the minimal ones

\[
\mathfrak{p}_{U,0} = \{x \in A(G) \mid \chi_U(x) = 0\}
\]

for \( U \leq G \), and the maximal ones

\[
\mathfrak{p}_{U,p} = \{x \in A(G) \mid \chi_U(x) \equiv 0 \pmod{p}\}
\]

for \( U \leq G \) and \( p \) a prime. Furthermore,
\[ p_{U,0} = p_{V,0} \quad \text{if and only if} \quad U \sim V, \]

\[ p_{U,p} = p_{V,q} \quad \text{if and only if} \quad p = q \text{ and } U^p \sim V^p \]

where \( U^p \) is the smallest normal subgroup of \( U \) with \( U/U^p \) a \( p \)-group, and \( p_{U,0} \subset p_{U,p} \) together with (1.9) accounts for all inclusions between prime ideals in \( A(G) \).

**Proposition 1.10.** The kernel of \( A(G) \to \hat{A}(G) \) coincides with the kernel of the restriction map

\[ \text{Res: } A(G) \to \bigoplus_{G_p \subseteq G} A(G_p) \]

to the Sylow subgroups \( G_p \) of \( G \).

**Proof.** It follows from the above that the kernel is

\[ \bigcap_{n=0}^{\infty} I(G)^n = \bigcap_{p_{U,0} \subset I(G) + A(G)} p_{U,0}. \]

Now \( I(G) = p_{e,0} \) and if the ideal \( p_{U,0} + p_{e,0} \) is proper then it is contained in a maximal ideal \( p_{V,p} \). By (1.9) this implies that \( p_{U,p} = p_{e,p} = p_{V,p} \), hence \( U^p = e = V^p \) and \( U \) is a \( p \)-group. Conversely, if \( U \) is a \( p \)-group then \( p_{U,0} + p_{e,0} \subset p_{U,p} \). Thus

\[ \text{Ker} \ (A(G) \to \hat{A}(G)) = \bigcap_{U \subseteq G \text{ } p \text{-group}} \text{Ker} \chi_U, \]

and the claim follows from theorem 1.3.

**Corollary 1.11.** If \( G \) is a \( p \)-group then \( A(G) \to \hat{A}(G) \) is a monomorphism.

In the case of a \( p \)-group \( G \) the \( I(G) \)-adic completion is the familiar \( p \)-adic one:

\[ \hat{A}(G) = \mathbb{Z} \bigoplus (\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} I(G)) \]

where \( \hat{\mathbb{Z}}_p = \lim_{\leftarrow n} \mathbb{Z}/p^n \) denotes the \( p \)-adic integers:

**Proposition 1.12.** If \( G \) is a \( p \)-group then the \( I(G) \)-adic topology of \( A(G) \) is the same as its \( p \)-adic topology.

**Proof.** We have to prove that for each \( m \) there are integers \( n_1, n_2 \) such that

(1) \[ p^{n_1}I(G) \subset I(G)^m \]

(2) \[ I(G)^{n_2} \subset p^mI(G). \]
The first relation follows from Atiyah's

**Lemma 1.13.** For any group $G$, $|G|I(G)^p \subset I(G)^{p+1}$.

(This is a consequence of the reciprocity formula 1.6, see [1p. 269, Proposition 6.13]).

To get the inclusion (2), we note that for any $H \leq G$ and $U \leq G$

$$\chi_U(G/H - |G/H|) \equiv 0 \pmod{p}$$

since the complement of $(G/H)^U$ consists of non-trivial $U$-orbits, hence $\chi_U(I(G)) \subset p\mathbb{Z}$. As $\chi = \oplus_{U \leq G} \chi_U$ is a ring homomorphism we have $\chi(I(G)^p) \subset \oplus_{e \star U \leq G} p^n\mathbb{Z}$ and it is enough to prove

$$\bigoplus_{e \star U \leq G} |G|\mathbb{Z} \subset \chi(I(G))$$

This follows immediately from the congruences 1.3. The proof of 1.12 is complete.

The completion $\hat{A}(G_p)$ is now described for $p$-groups $G_p$. Next we shall show that if $G$ is an arbitrary finite group, then $\hat{A}(G)$ embeds into the sum $\bigoplus \hat{A}(G_p)$, taken over the Sylow subgroups $G_p$ of $G$. This is done by completing the map of proposition 1.10.

Let $A$ be a Noetherian ring and $m \subset A$ an ideal. The $m$-adic completion of a finitely generated $A$-module $M$ is defined to be $\hat{M} = \varprojlim_n M/m^nM$. It is a basic fact that Noetherian completion is an exact functor [1 p. 258, Proposition 3.16].

If $H \leq G$, then $A(H)$ is an $A(G)$-module via the restriction homomorphism $\varrho = \text{Res}_H^G: A(G) \to A(H)$. In the following proof we distinguish the prime ideals $p_{U,p}$ of $A(H)$ and $A(G)$ by upper indices, so that $p_{U,p}^H \subset A(H)$ and $p_{U,p}^G \subset A(G)$.

**Proposition 1.14.** Let $H$ be a subgroup of $G$. Then the $I(H)$-adic topology of $A(H)$ is the same as its $I(G)$-adic topology.

**Proof.** It is enough to show that the radicals of the ideals $q(I(G))$ and $I(H)$ coincide [22 p. 256]. This means that each prime ideal $p \subset A(H)$ either contains the both ideals or none. Since $q(I(G)) \subset I(H)$, one way is trivial. Let $p$ be a prime ideal of $A(H)$ with $q(I(G)) \subset p$. We claim that $I(H) \subset p$. We know that $p$ is of the form $p_{U,0}^H$ or $p_{U,p}^H$ with some subgroup $U \leq H$ and some prime $p$. If $p = p_{U,0}^H$, then

$$p_{e,0}^G = I(G) \subset q^{-1}(p) = p_{U,0}^G$$
which implies \( U = e \) and \( p = p_{e,0}^H = I(H) \) by (1.9). Similarly, if \( p = p_{U,p}^H \), then
\[
p_{e,0}^G = I(G) \subset g^{-1}(p) = p_{U,p}^G
\]
and \( U \) must be a \( p \)-group by (1.9), whence \( p = p_{U,p}^H = p_{e,0}^H \). In both cases \( I(H) \subset p \), are claimed.

**Theorem 1.15.** Let \( G \) be a finite group and \( \{ G_p \} \) its Sylow subgroups. Then the completion of the restriction maps \( \text{Res}_{G_p}^G \) gives an injective homomorphism
\[
0 \rightarrow \hat{A}(G) \rightarrow \bigoplus_p \hat{A}(G_p).
\]

**Proof.** By 1.10 \( A(G)/\bigcap_{n=0}^{\infty} I^n(G) \) maps injectively into \( \bigoplus_p A(G_p) \). Both modules have \( I(G) \)-adic topology by 1.14. The claim follows since Noetherian completion is an exact functor.

We close the chapter with some examples. The first two are abelian \( p \)-groups. The last one illustrates the restrictions to Sylow subgroups and completion.

**Example 1.16.** The cyclic group \( \mathbb{Z}/p^n \). It has a unique subgroup of order \( p^{n-m} \), \( 0 \leq m \leq n \). Let \( \eta_m \) be the quotient \( (\eta_0 = 1) \). We have additively
\[
A(\mathbb{Z}/p^n) = \mathbb{Z} \oplus \mathbb{Z} \eta_1 \oplus \ldots \oplus \mathbb{Z} \eta_n.
\]
From the characters
\[
\chi_{\mathbb{Z}/p^i}(\eta_m) = \begin{cases} p^m, & i \leq n-m \\ 0, & i > n-m \end{cases}
\]
one gets the multiplication \( \eta_1 \cdot \eta_m = p^l \eta_m \) for \( l \leq m \).

**Example 1.17.** The elementary abelian group \( (\mathbb{Z}/p)^n \). It can be interpreted geometrically as a vector space over the finite field \( F_p \), with subgroups corresponding to linear subspaces. The number of \( m \)-dimensional planes is
\[
G(m, n) = \frac{(p^n - 1)(p^n - p) \ldots (p^n - p^{m-1})}{(p^m - 1)(p^m - p) \ldots (p^m - p^{m-1})}
\]
\[
= \frac{(p^n - 1)(p^{n-1} - 1) \ldots (p^{n-m+1} - 1)}{(p^m - 1)(p^{m-1} - 1) \ldots (p - 1)}
\]
\((G \text{ stands for Grassmann}). \ A((\mathbb{Z}/p)^n) \) is additively generated by the \( m\)-
dimensional quotient planes \( \eta^i_{m}, 0 \leq m \leq n, 1 \leq i \leq G(n - m, n) = G(m, n) \), and

\[
F^n_p / V_1 \times F^n_p / V_2 = |F^n_p / V_1 + V_2| F^n_p / V_1 \cap V_2.
\]

**Example 1.18.** The alternating group \( A_4 \). The diagram of subgroups is

\[
\begin{align*}
A_4 & \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
\mathbb{Z}/3 & \sim \mathbb{Z}/3 \sim \mathbb{Z}/3 \sim \mathbb{Z}/3 \\
\mathbb{Z}/2 & \sim \mathbb{Z}/2 \sim \mathbb{Z}/2 \\
e &
\end{align*}
\]

and the character table is given in table 1.19 where the small letters \( 1, a, b, c, \) and \( d \) denote the cosets \( A_4 / H \) in the given order.

<table>
<thead>
<tr>
<th>( A_4 / H )</th>
<th>( A_4 )</th>
<th>( \mathbb{Z}/2 \oplus \mathbb{Z}/2 )</th>
<th>( \mathbb{Z}/3 )</th>
<th>( \mathbb{Z}/2 )</th>
<th>( e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( a )</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( b )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>( c )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>( d )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>12</td>
</tr>
</tbody>
</table>

The Burnside rings of the Sylow subgroups of \( A_4 \) are described in the preceding examples:

\[
A(\mathbb{Z}/2 \oplus \mathbb{Z}/2) = \mathbb{Z} \oplus \mathbb{Z} \eta_1^1 \oplus \mathbb{Z} \eta_1^2 \oplus \mathbb{Z} \eta_1^3 \oplus \mathbb{Z} \eta_2
\]

\[
A(\mathbb{Z}/3) = \mathbb{Z} \oplus \mathbb{Z} \xi.
\]

The restriction map \( A(G) \rightarrow A(G_2) \oplus A(G_3) \) is read from the character table using (1.4) \( \chi_H(i^*x) = \chi_H(x) \). The result is

\[
\begin{align*}
1 & \rightarrow \ (1, 1) \\
a & \rightarrow \ (3, \xi) & a_1 & \rightarrow \ (0, \xi) \\
b & \rightarrow \ (\eta_2, 1 + \xi), \quad \text{or} & b_1 & \rightarrow \ (\eta_2, 0) \\
c & \rightarrow \ (\eta_1^1 + \eta_1^2 + \eta_3^2, 2\xi) & c_1 & \rightarrow \ (\eta_1^1 + \eta_1^2 + \eta_3^3, 0) \\
d & \rightarrow \ (3\eta_2, 4\xi) & d_1 & \rightarrow \ (0, 0)
\end{align*}
\]
in the basis $a_1 = a - 3$, $b_1 = b - a - 1$, $c_1 = c - 2a$, $d_1 = d - 3b - a + 3$ for $I(A_4)$. Here $x$ denotes $x - \varepsilon(x) \in I(G_p)$. This shows that the image of $A(G)$ in $A(G_2) \oplus A(G_3)$ consists precisely of the stable elements. These are the pairs $(x_2, x_3)$ with

\[(1.20) \quad (1) \; \varepsilon(x_2) = \varepsilon(x_2) \]
\[(2) \; \chi_{H_1}(x_p) = \chi_{H_2}(x_p), \text{ if } H_1 < G_p \text{ and } H_2 < G_p \text{ are conjugate in } G.\]

The condition (2) rules out $\eta_1^1, \eta_1^2$ and $\eta_1^3$ since they have different characters on the three $Z/2 < Z/2 \oplus Z/2$ which are conjugate in $A_4$.

It might be interesting to know whether (1.20) characterizes the image of $A(G)$ in $\bigoplus_p A(G_p)$ in general. If the Segal conjecture $\tilde{A}(G) = \pi^0_3(BG)$ is true, then this holds at least on the completion level by general properties of cohomology theories on $BG$ (see [11, 1.7]).

Finally, the multiplication table for $I(A_4)$

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$b_1$</th>
<th>$c_1$</th>
<th>$d_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$-3a_1$</td>
<td>$d_1$</td>
<td>0</td>
<td>$-3d_1$</td>
</tr>
<tr>
<td>$b_1$</td>
<td>$-4b_1 - d_1$</td>
<td>$-4c_1$</td>
<td>$-d_1$</td>
<td></td>
</tr>
<tr>
<td>$c_1$</td>
<td>$6b_1 - 10c_1 + 2d_1$</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d_1$</td>
<td></td>
<td></td>
<td></td>
<td>$3d_1$</td>
</tr>
</tbody>
</table>

shows that $\tilde{A}(G) = Z \oplus \hat{Z}_3 a_1 \oplus \hat{Z}_2 b_1 \oplus \hat{Z}_2 c_1$ and $I(G)^\infty = Z d_1$.

2. $\lambda$-Operations on the Burnside ring.

Let $G$ be a finite group. If $k$ is a field and $V$ is a representation of $G$ over $k$ then the exterior powers $\lambda^n V$ are also $G$-representations. We want to construct operations in $A(G)$ which under the natural map $A(G) \to R_k(G)$ correspond to the exterior powers. As it is not clear how to make sense of the relation $x \wedge y = -y \wedge x$ in a $G$-set, we consider first the symmetric powers $s^n V$ where no signs are needed.

Let $S = \{s_1, \ldots, s_l\}$ be a $G$-set. The vector space $s^n(k^S)$ has a basis consisting of monomials of degree $n$ in $s_i \in s^1(k^S)$, considered as elements of the symmetric algebra $s(k^S)$. We define the $n$th symmetric power of $S$ as

\[s^n(S) = S^n / \Sigma_n\]

with the diagonal $G$-action. It is clear that

\[(2.1) \quad s^1(S) = S\]
\[(2.2) \quad s^n(S \cup T) = \sum_{i=0}^n s^i(S)s^{n-i}(T).\]
We assign to $S$ the formal power series

\begin{equation}
(2.3) \quad s_t(S) = 1 + \sum_{n \geq 1} s^n(S)t^n \in A(G)[[t]].
\end{equation}

It is invertible as the leading coefficient is 1, and (2.2) shows that $s_t(S \cup T) = s_t(S) \cdot s_t(T)$. The homomorphism $s_t$ is uniquely extended to $A(G)$ by $s_t(S - T) = s_t(S) \cdot s_t(T)^{-1}$.

In the representation ring $R_k(G)$ the symmetric powers are connected to the exterior powers by means of the identity

\[ \lambda_t(V)s_{-t}(V) = 1. \]

Thus we are lead to

**Definition 2.4.** The $n$th exterior power of $x \in A(G)$, denoted by $\lambda^n(x)$, is the coefficient of $t^n$ in the series $\lambda_t(x) = s_{-t}(x)^{-1}$.

The formulae (2.1)-(2.3) translate to give

\begin{equation}
(2.5) \quad (i) \quad \lambda^0(x) = 1 \quad (ii) \quad \lambda^1(x) = x \quad (iii) \quad \lambda^n(x + y) = \sum_{i=0}^{n} \lambda^i(x)\lambda^{n-i}(y). \end{equation}

This is summarised in saying that the operations $\lambda^n$, $n \geq 1$, give $A(G)$ the structure of a $\lambda$-ring [4]. By construction they are natural with respect to induced maps, and $A(G) \to R_k(G)$ is a $\lambda$-homomorphism.

We shall calculate the character of $\lambda^n(x)$, $\chi_U(\lambda^n(x))$. By (2.5) and naturality it is enough to consider a single $G$-orbit $x = G/H$. A point $(s_1, \ldots, s_n) \in (G/H)^n/\Sigma_n$ is fixed under $G$ only if it can be split up to $G$-orbits $G/H$. This implies that $n$ is a multiple of $|G/H|$, and

\begin{equation}
(2.6) \quad \chi_G(s^n(G/H)) = \begin{cases} 0, & n \not\equiv 0 \ (|G/H|) \\ 1, & n \equiv 0 \ (|G/H|). \end{cases}
\end{equation}

Thus $\chi_G(s_t(G/H)) = (1 - t^{|G/H|})^{-1}$. For a subgroup $U$ of $G$ we can break up $S$ into $U$-orbits $\cup S_i$ so

\begin{equation}
(2.7) \quad \chi_U(\lambda_t(S)) = \prod_{S_i \in U\text{-orbits}} (1 - (-t)^{|S_i|}).
\end{equation}

In particular the degree of $\chi_U(\lambda_t(S))$ is equal to $|S|$, hence $\lambda^n(S) = 0$ if $n > |S|$. Also $\varepsilon(\lambda_t(s)) = \chi_\varepsilon(\lambda_t(s)) = (1 + t)^{|S|}$. Thus $A(G)$ is a finite-dimensional augmented $\lambda$-ring.
We define the Adams operations $\psi^n: A(G) \to A(G)$, $n \geq 1$, by

$$\psi^{-i}(x) = -t^{\lambda^{-i}(x)} \frac{\lambda^{-i}(x)}{\lambda^{-i}(x)}$$

where $\psi_i(x) = \sum_{n \geq 1} \psi^n(x)t^n$.

Then (2.5) implies that $\psi^n$ is additive, $\psi^n(x+y) = \psi^n(x) + \psi^n(y)$. As to the characters, the logarithmic differentiation of (2.7) yields

$$\chi_U(\psi_i(X)) = \sum_{S, s \text{ U-orbits}} |S| t^{l^S} (1 - t^{l^S})^{-1}.$$ 

This proves

**Proposition 2.8.** $\chi_U(\psi^n(S)) = \sum_{\text{S U-orbits}} |S|_n |S_i|$, where $S = \bigcup S_i$ the decomposition into U-orbits.

**Corollary 2.9.** The Adams operations are periodic of period dividing the order of $G$.

(Indeed, the length of each U-orbit $U/H$ is a divisor of $|G|$).

If $G$ is a $p$-group and $(n, p) = 1$, then the only orbits occuring in 2.8 are the U fixed points. This proves

**Corollary 2.10.** If $G$ is a $p$-group, then $\psi^n = \text{id}$ for $n$ relatively prime to $p$.

The operations $\lambda^n$ have geometrical significance: they induce natural transformations of $\pi_0^\infty$, the zeroth stable cohomotopy functor.

First recall the Barratt–Quillen theorem. The group completion map $i: \coprod_{n \geq 1} B\Sigma_n \to QS^0$ gives a natural transformation of monoid-valued functors

$$\left[ X, \bigcoprod_{n \geq 1} B\Sigma_n \right] \to [X, QS^0] .$$

Here $A(X) = [X, \coprod_{n \geq 1} B\Sigma_n]$ is the set of isomorphism classes of finite coverings of $X$ organized to a semiring under disjoint union and fibrewise cartesian product of the total spaces, and $\pi_0^\infty(X) = [X, QS^0]$ is the stable cohomotopy of $X$ (in degree 0), an abelian group with respect to loop sum. The group completion theorem states [18, Proposition 4.1]

**Theorem 2.11.** The transformation $A \to \pi_0^\infty$ is universal among transformations $\theta: A \to F$, where $F$ is a representable abelian-group-valued homotopy functor on compact spaces, and $\theta$ is a transformation of monoid-valued functors.

We deduce from this the existence of $\lambda$-operations on $\pi_0^\infty$ along the lines of Segal [19].
THEOREM 2.12. There are natural transformation \( \lambda^n : \pi_3^0 \to \pi_3^0 \) for \( n \geq 0 \), such that

(i) \( \lambda^0(x) = 1 \)
(ii) \( \lambda^1(x) = x \)
(iii) \( \lambda^n(x + y) = \sum_{i=0}^{n} \lambda^i(x) \lambda^{n-i}(y) \).

PROOF. We define a transformation \( \lambda^n : A(X) \to \pi_3^0(X) \). Assume \( X \) is connected. An \( m \)-fold covering \( Y \downarrow X \) can be written as \( P \times \Sigma_m[m] \downarrow X \), where \( P \) is the principal \( \Sigma_m \)-bundle associated to \( Y \) consisting of mappings of \( [m] = (1, 2, \ldots, m) \) onto the fibres of \( Y \), and \( [m] \) has the usual \( \Sigma_m \)-action. Let \( \lambda^n([m]) = S - T \in A(\Sigma_m) \). We associate to \( Y \downarrow X \) the difference

\[
\lambda^n(Y) = P \times S - P \times T \in \pi_3^0(X)
\]

where we have used \( A \to \pi_3^0 \) from 2.11.

Let us form the mapping

\[
\lambda_t = \sum_{n \geq 0} \lambda^n t^n : A(X) \to 1 + \pi_3^0(X)[[t]]^+ = \prod_{n \geq 1} \pi_3^0(X).
\]

It is a monoid homomorphism, when we use multiplication of power series on the right. As \( 1 + \pi_3^0(X)[[t]]^+ \) is a representable abelian-group-valued functor, \( \lambda_t \) extends by theorem 2.11 to a group homomorphism

\[
\lambda_t : \pi_3^0(X) \to 1 + \pi_3^0(X)[[t]]^+.
\]

This completes the proof of theorem 2.12.

In the articles [3] and [4] Atiyah, Tall and Segal showed that special \( p \)-adic \( \lambda \)-rings possess certain canonical exponential isomorphisms between the additive group \( \hat{I}(G) \) and the multiplicative group \( 1 + \hat{I}(G) \). Unfortunately the Burnside ring \( A(G) \) is special only if \( G \) is cyclic: The Adams operations \( \psi^n \) are ring homomorphisms in special \( \lambda \)-rings, but Siebencher showed that this is not true in \( A(G) \) for any non-cyclic \( G \) [20, p. 232]. On the other hand, \( A(G) \) embeds as a sub-\( \lambda \)-ring of \( R(G) \) if \( G \) is cyclic.

However, it is interesting to study the exponential map \( e_k \). We first do the algebra and then identify the resulting geometric map \( Q_0 S^0 \to SG[1/k] \) with the composition

\[
Q_0 S^0 \to \text{Im} J_p \xrightarrow{\tau} SG \left[ \frac{1}{k} \right].
\]

Philosophically this is a negative result: the \( \lambda \)-operations on \( A(G) \) do not give any information on the fibre of \( e \), the space usually denoted \( \text{cok} J_p \).
Let $G$ be a finite group. We shall encounter series of the form $\sum \alpha^n \lambda^n(x)$. To show their convergence in $\hat{A}(G)$ we introduce a new topology on $A(G)$. Define the Grothendieck operations by

$$\gamma^n(x) = \lambda^n(x+n-1).$$

If $\gamma_t(x) = 1 + \sum_{n \geq 1} \gamma^n(x) e^n$, then

$$\gamma_t(x) = 1 + \sum_{n \geq 1} \gamma^n(x) e^n,$$

$$\gamma_t(x + y) = \gamma_t(x) \gamma_t(y).$$

The $\gamma$-operations are convenient on the augmentation ideal $I(G)$ as the generators $G/H - \varepsilon(G/H)$ have finite $\gamma$-dimension but infinite $\lambda$-dimension. In fact, (2.7) and (2.14) imply

$$\chi_U(\gamma_t(S - \varepsilon(S))) = \prod_{S_i \subseteq S \text{ U-orbit}} [(1 - t)^{|S_i|} - (-t)^{|S_i|}]$$

for any $G$-set $S$.

Define the $\gamma$-filtration by

$$I_n \text{ is the group generated additively by } \gamma^{n_1}(x_1) \ldots \gamma^{n_r}(x_r) \text{ with } x_i \in I(G), \sum n_i \geq n.$$ 

Then $I_m \cdot I_n \subseteq I_{m+n} I_0 = A(G)$ and $I_1 = I(G)$. Thus the filtration $(I_n)_{n \geq 0}$ defines a topology on $A(G)$, the $\gamma$-topology.

**Proposition 2.16.** If $G$ is a $p$-group, then the $p$-adic, $I(G)$-adic and $\gamma$-topologies on $A(G)$ are equivalent.

**Proof.** We proved in 1.1.2 that the first two topologies coincide. Atiyah [1, Corollary 12.3] shows that the $\gamma$-topology is equivalent to the $I(G)$-adic if $I(G)$ has a finite number of generators, each of finite $\gamma$-dimension.

Let $G$ be a $p$-group $x \in I(G)$ and $x \in \hat{Z}_p$. Then the series $\sum_{n \geq 1} \alpha^n \gamma^n(x)$ converges in the $\gamma$-topology, hence also in $\hat{A}(G)$. More generally, if $x \in A$ where $A$ is a finitely generated $\hat{Z}_p$-algebra, then $\gamma_n(x)$ exists in $1 + \hat{I}(G) \otimes \hat{Z}_p A$. We fix a prime $k$ different from $p$ and apply this to $A = \hat{Z}_p[\xi]$, where $\xi$ is a primitive $k$th root of 1.

**Definition 2.17.** $q_k(x) = \prod_{u \neq 1} \lambda_{-u}(x), \ x \in I(G)$.

A priori $q_k(x)$ belongs to $1 + \hat{I}(G) \otimes \hat{Z}_p[\xi]$. But it is invariant under the action of the Galois group of $\hat{Q}_p(\xi)/\hat{Q}_p$ so actually $q_k(x) \in 1 + \hat{I}(G)$.

We compute the character of $q_k(x)$. A substitution in (2.7) yields
\[ \chi_U(\varrho_k(S - \varepsilon(S))) = \prod_{u^1 \neq 1} \left( \prod_{S_i \subset S} (1 - u^{|S_i|}) \right) (1 - u)^{-|S|}. \]

As the sizes of the \( U \)-orbits \( S_i \) are 1 or multiples of \( p \) and \((k, p) = 1\), \( 1 - u^{|S_i|} \) runs through the same values as \( 1 - u \), when \( S_i \) is fixed. Noting that \( \prod_{u^1 \neq 1} (1 - u) = k \)
we get

\[ (2.18) \]
\[ \chi_U(\varrho_k(S - \varepsilon(S))) = k^{o_U(S) - \kappa(S)} \]

where \( o_U(S) \) is the number of \( U \)-orbits in \( S \).

Next we show that \( \varrho_k \) can be obtained by a direct operation on \( G \)-sets.

**Proposition 2.19.** For a \( G \)-set \( S \) let \( \theta_k(S) \) be the underlying set of the vector space \( F^S_k \) with the linear \( G \)-action extending the permutation of the basis. Then \( \theta_k \) satisfies

(i) \[ \theta_k(S + T) = \theta_k(S)\theta_k(T) \]

(ii) \[ \varepsilon \theta_k(S) = k^{\kappa(S)} \]

(iii) \[ \theta_k \text{ is natural} \]

(iv) \[ \theta_k(S) = \prod_{u^1 \neq 1} \lambda_{-u}(S) \]

on \( A^+(G) \).

**Proof.** Properties (i)–(iii) are obvious. To prove (iv) it is enough to check \( \chi_U(\theta_k(S)) \) for \( U = G \) by naturality and for a transitive \( G \)-set \( S \) by (i). A point \( \sum_{x \in S} a_x x, a_x \in F_k \), is fixed under \( G \) only if it is of the form \( a \sum_{x \in S} x \), thus

\[ \chi_G(\theta_k(S)) = |F_k| = k \]

but \( \chi_G(\prod \lambda_{-u}(S)) = k^{o_G(S)} = k \).

**Remark.** There is no problem about the convergence of \( \lambda_{-u}(S) \) in (iv), since \( \lambda_{-u}(S) \) is a polynomial.

We return now to the stable cohomotopy interpretation. Let \( p \) and \( k \) be different primes. The operation \( \theta_k : A^+(\Sigma_n) \to A^+(\Sigma_n) \) induces a natural transformation \( \theta_k : A \to A \) as in theorem 2.12: if \( Y \downarrow X \) is an \( n \)-fold covering, write it as \( Y = P \times \Sigma_r[n] \) with some principal \( \Sigma_r \)-bundle \( P \) and set \( \theta_k(Y) = P \times \Sigma_r \theta_k[n] \). By (2.19) (i) and (ii) the composite

\[ A \xrightarrow{\theta_k} A \to \pi^0_S \]
is exponential and maps n-fold coverings of \(X\) to the component \([X, Q_k S_0^0]\).

In order to apply theorem 2.11 the elements \(\theta_k(\alpha) \in \pi_3^0(X)\) have to be invertible in the composition product, in particular the maps in \(Q_k S_0^0\) must have an inverse of degree \(k^{-n}\). This can be accomplished by forming the localization \(\lambda: Q S_0^0 \rightarrow Q S_0^0_p\) of the space \(Q S_0^0\) at \(p\) [21, sections 2 and 4]. Denote the 1-component of \(Q S_0^0_p\) by \(SG_p\). Then the transformation

\[
\theta_k: A(X) \rightarrow [X, SG_p]
\]

which takes an n-fold covering \(Y \downarrow X\) to

\[
X \xrightarrow{\theta_k(Y)} Q_k S_0^0 \xrightarrow{\lambda} Q_k S_0^0 \xrightarrow{\cdot k^{-n}} SG
\]

extends to a homomorphism

\[
(*) \quad \theta_k: \pi_3^0(X) \rightarrow [X, SG_p]
\]

by theorem 2.11.

The restriction of (*) to \(\pi_3^0\) corresponds to an \(H\)-map \(\theta_k: Q_0 S_0^0 \rightarrow SG_p\), defined up to homotopy. The space \(SG_p\) splits as a product \(J_p \times \text{cok} J_p\) (this will be discussed in section 4), and we point out here

**Theorem 2.20.** The map \(\theta_k: Q_0 S_0^0 \rightarrow SG_p\) factors \(Q_0 S_0^0 \xrightarrow{e} J_p \xrightarrow{s} SG_p\).

**Proof.** Compare proposition 2.19 to [14, p. 236].

The natural homomorphism \(A(G) \rightarrow R(G)\) is a \(\lambda\)-ring homomorphism. For the elementary abelian groups its kernel is large. We evaluate the Adams operations on \(A((\mathbb{Z}/p)^n)\) in the concluding example.

**Example 2.21.** Elementary abelian groups \((\mathbb{Z}/p)^n\).

Let \(G = (\mathbb{Z}/p)^n\). Each generator \(G/H\) of \(A(G)\) is the image of the regular representation under \(\pi^\#: A(G/H) \rightarrow A(G)\). By naturality it is thus enough to find \(\psi^k(\eta^n)\), where we denote \(\eta^n = G/e\) (see 1.17). From 2.8 we get

\[
\chi_H(\psi^k(\eta^n)) = \begin{cases} 0, & k \equiv 0 \pmod{|H|} \\ p^n, & k \equiv 0 \pmod{|H|} \end{cases} \quad H \subseteq (\mathbb{Z}/p)^n
\]

which depends only on the size of \(H\). This suggests that we begin with the sum of all cosets of cardinality \(p^m\),

\[
\eta^\text{tot} = \sum_{|H| = p^m} G/H
\]

which has the characters \(\chi_H(\eta^\text{tot}) = 0\) if \(|H| > p^{n-m}\), \(\chi_{\mathbb{Z}/p^{n-m}}(\eta^\text{tot}) = p^m\), and then correct \(\chi_H\) for smaller \(H\) by adding linear combinations of \(\eta^\text{tot}_{m+k}, k > 0\). An inductive calculation shows that the element.
\[ a_m = \sum_{k=0}^{n-m} (-1)^k p^{k(k-1)/2} G(k, m+k) \frac{\eta^{\text{tot}}_{m+k}}{p^{m+k}} \in A(G) \left[ \frac{1}{p} \right], \]

where \( G(k, m+k) \) is defined in example 1.17, has characters

\[ \chi_{(\mathbb{Z}/p)^r}(a_m) = \begin{cases} 1, & k = n-m \\ 0, & k \neq n-m \end{cases}. \]

By the first formula we then get

\[ \psi^{p^h}(\eta_n) = \sum_{i=n-m}^n p^i a_i = p^m \sum_{k=0}^m (-1)^k p^{(k)} G(k, n-m+k-1) \eta^{\text{tot}}_{n-m+k} \]

if \( (p, h) = 1 \), \( 0 \leq m < n \) and

\[ \psi^{p^h}(\eta_n) = p^n. \]

3. The map \( \alpha_G \).

In this section we study the injectivity of the map \( \hat{\alpha}_G : \hat{A}(G) \to \pi^0_S(BG) \) from the completion of the Burnside ring of a finite group \( G \) to the stable cohomotopy of its classifying space \( BG \).

Recall the definition of \( \alpha_G \). Each \( G \)-set \( S \) with \( G \)-action \( \varrho : G \to \Sigma_{|S|} \) gives rise to a map \( \alpha_G(S) : BG \to QS^0 \) by

\[ BG \xrightarrow{BG_{\varrho}} B\Sigma_{|S|} \xrightarrow{\prod_{n \geq 0} B\Sigma_n} QS^0, \]

where \( i \) is the group completion map. The homotopy class of \( \alpha_G(S) \) depends only on the class of \( S \) in \( A(G) \), and the correspondence \( S \mapsto \alpha_G(S) \) extends to a ring homomorphism

\[ \alpha_G : A(G) \to [BG, QS^0] \]

(by definition, \( \pi^0_S(BG) = [BG, QS^0] \)).

Alternatively, we can define \( \alpha_G(S) \) as the image of the covering \( EG \times S \downarrow BG \) in \( \pi^0_S(BG) \) (cf. 2.11). This is quite analogous to the homomorphism

\[ \alpha : R(G) \to K^*(BG) \]

studied by Atiyah in [1]: if \( \varrho : G \to \text{Gl}(n, \mathbb{C}) \) is a complex representation of \( G \), then \( \alpha(\varrho) \) is the class of the vector bundle \( EG \times (\mathbb{C}^n, \varrho) \) in \( K^0(BG) \). It is no surprise that \( \alpha_G \) and \( \alpha \) are connected via the natural map \( A(G) \to R(G) \):

**Proposition 3.1.** Let \( G \) be a finite group. Then the diagram

\[ A(G) \xrightarrow{\alpha_G} \pi^0_S(BG) \]

\[ \downarrow \quad \downarrow \epsilon_* \]

\[ R(G) \xrightarrow{\alpha} K^*(BG) \]
commutes, where \( e: \mathcal{Q} \mathcal{S}^0 \to BU \times \mathcal{Z} \) is induced from a unit of the unitary spectrum.

**Proof.** Let \( S \) be a \( G \)-set of cardinality \( n \), and let \( \varphi: G \to \Sigma_n \) be its \( G \)-action. If \( P: \Sigma_n \to U_n \) is the permutation representation, then \( \alpha_G(S) \) and \( \alpha(CS) \) are represented by the upper and lower horizontal arrows in

\[
\begin{array}{ccc}
BG & \xrightarrow{BG} & BS_n \\
\downarrow \downarrow & & \downarrow \downarrow \\
BP & \xrightarrow{BP} & BU_n \\
\end{array}
\]

But the right hand squares commute [12, Corollary 5.31].

**Remark.** We shall give in section 4 a closer description of the map \( e \) (see 4.18).

In section 1 we considered the \( I(G) \)-adic topology on \( A(G) \). If \( X \) is a CW-complex with \( n \)-skeleton \( X^n \), we filter \( \pi_S^0(X) \) by

\[
(F) \quad F^n\pi_S^0(X) = \ker \left( \pi_S^0(X) \to \pi_S^0(X^{n-1}) \right).
\]

Then \( F^n \cdot F^m \subseteq F^{n+m} \) by diagonal approximation. J. W. Milnor’s original construction of \( BG \) gave a CW-complex with finite skeletons \( B_nG \). As the stable homotopy groups \( \pi_S^0(S^n) \) are finite, so are the groups \( \pi_S^0(B_nG) \). Hence \( \pi_S^0(BG) = \lim_{\to} \pi_S^0(B_nG) \), and \( \pi_S^0(BG) \) is complete in the filtration topology. It follows from the definition that \( \alpha_G(I(G)) \subseteq [BG, Q_o S^0] = F^1\pi_S^0(BG) \), and since \( \alpha_G \) is a ring homomorphism, \( \alpha_G(I(G)^n) \subseteq F^n\pi_S^0(BG) \). Thus \( \alpha_G \) is continuous and induces a homomorphism

\[
\hat{\alpha}_G: \hat{A}(G) \to \pi_S^0(BG)
\]

between the completions.

All the maps of 3.1 are continuous homomorphisms, when \( R(G) \) is equipped with augmentation ideal topology and \( K^*(BG) \) with a filtration topology similar to (F). Passing to completions we have

\[
\begin{array}{ccc}
\hat{A}(G) & \xrightarrow{\hat{\alpha}_G} & \pi_S^0(BG) \\
\downarrow \downarrow & & \downarrow \downarrow \\
\hat{R}(G) & \xrightarrow{\hat{\alpha}_G} & K^*(BG)
\end{array}
\]

The main result of Atiyah [1] states that \( \hat{\alpha}_G \) is an isomorphism. Therefore we can conclude the injectivity of \( \hat{\alpha}_G \) if \( \hat{A}(G) \) embeds into \( \hat{R}(G) \). If \( G \) is cyclic then \( A(G) \to R(G) \) is injective by Lemma 1.8. If moreover the order of \( G \) is a prime
power $p^n$, then the augmentation ideals of both rings have the $p$-adic topology by Proposition 1.12 and [4, p. 277]. But then $\hat{A}(G) \to \hat{R}(G)$ is injective, since the $p$-adic completion is an exact functor. We have proved

**Theorem 3.2.** Let $G$ be a cyclic group of prime power order. Then $\hat{\alpha}_G$ is injective.

We can express theorem 3.2 by saying that the maps $\alpha_G(x)$ for cyclic $G$ are detected by $K$-theory. Indeed, the proof of 3.1 shows that $\alpha(x): BG \to BU \times \mathbb{Z}$ factors as

$$BG \xrightarrow{\alpha_G(x)} QS^0 \xrightarrow{\epsilon} BU \times \mathbb{Z}.$$  

Since the map induced by $\alpha(x)$ in $K$-theory is non-trivial, if $x \neq 0$, so must be the one induced by $\alpha_G(x)$, too.

Next we invoke theorem 1.15 to show that the injectivity of $\hat{\alpha}_G$ can be deduced from that of $\hat{\alpha}_{G_p}$, for all Sylow subgroups $G_p$ of $G$. Consider the commutative diagram

$$
\begin{array}{c}
\hat{A}(G) \\
\oplus_{p} \hat{A}(G_p)
\end{array} \xrightarrow{\oplus_{G_p} \hat{\alpha}_{G_p}} \begin{array}{c}
\pi^0_{S}(BG) \\
\oplus_{p} \pi^0_{S}(BG_p)
\end{array}
$$

By Theorem 1.15, Res is injective. If the maps $\hat{\alpha}_{G_p}$ are injective for all $G_p \leq G$, then $\hat{\alpha}_G$ must be injective. Hence

**Theorem 3.3.** Let $G$ be a finite group and $\{G_p\}$ its Sylow subgroups. If $\hat{\alpha}_{G_p}$ is injective for all $G_p \leq G$, then $\hat{\alpha}_G$ is injective.

Theorem 3.3 reduces the study of $\hat{\alpha}_G$ to $p$-groups $G$. First, we note

**Lemma 3.4.** Let $G$ be a $p$-group. Then $\hat{\alpha}_G$ is injective if and only if $\alpha_G$ is injective.

(Indeed, as $A(G)$ embeds into $\hat{A}(G)$ by Corollary 1.11, one way is trivial and the converse follows since the $p$-adic completion is an exact functor and $I(G)$ and $\hat{\pi}^0_S(BG)$ both have $p$-adic topology, $I(G)$ by Proposition 1.12 and $\hat{\pi}^0_S(BG)$ being profinite with $BG$ $p$-local [21].)

The smallest non-trivial $p$-group is the cyclic $\mathbb{Z}/p$, where we can apply Theorem 3.2. Suppose inductively that $\alpha_H$ is injective for all genuine subgroups
$H$ of $G$. By naturality of $\alpha$ an element in $A(G)$ which has a non-zero restriction to some $H < G$, cannot lie in the kernel of $\alpha_G$. Applying Theorem 1.3 we get

**Lemma 3.5.** Let $G$ be a $p$-group. Suppose $\alpha_H$ is injective for all genuine subgroups $H < G$. Then $\alpha_G$ is injective on $\ker \chi_G$.

To handle the rest, we have

**Proposition 3.6.** Let $G$ be a $p$-group. There exists an element $x \in A(G)$ with $\chi_G(x) = p$ and $\chi_H(x) = 0$ for $H < G$. It is induced from an epimorphism $G \to (\mathbb{Z}/p)^d$.

**Proof.** The existence of $x$ follows from the congruences of 1.3. However, we prefer to construct it directly.

Let $\Phi(G)$ be the Frattini subgroup of $G$, that is, the intersection of all maximal subgroups of $G$. We recall some elementary facts about $\Phi(G)$ [8, III § 3]:

1) $\Phi(G) = G^p[G, G]$,
2) $\Phi(G) \triangleleft G$ and $G/\Phi(G)$ is a maximal elementary abelian quotient of $G$, say $(\mathbb{Z}/p)^d$, and
3) the elements of $\Phi(G)$ are redundant in any set of generators for $G$.

One can also characterize the quotient $G/\Phi(G)$ as $H_1(G; \mathbb{Z}/p)$. In $A(\mathbb{Z}/p)^d$ we write down the element

$$y = p - \eta_1^{\text{tot}} + \eta_2^{\text{tot}} - \ldots + (-1)^d \eta_1^{\text{tot}} - \ldots - \eta_d$$

(See example 2.21, $y = p a_0$ in (2.22)). If $\pi: G \to G/\Phi(G)$ denotes the projection, then $\pi^*(y)$ has the required properties. Clearly $\chi_G(\pi^*(y)) = (\chi_G(\pi_1^{(1)}(y))) = p$. If $H < G$, then $\pi(H) < (\mathbb{Z}/p)^d$ and $\chi_H(\pi^*(y)) = \chi_{\pi(H)}(y) = 0$. Indeed, if $\pi(H) = (\mathbb{Z}/p)^d$, then $H \Phi(G) = G$, which implies $H = G$ by 3) above. This completes the proof of proposition 3.6.

**Lemma 3.7.** $\mathcal{Z}x = \bigcap_{H < G} \ker \text{Res}_H^G$ and it is a $\lambda$-ideal of $A(G)$.

**Proof.** The second claim follows from the first, since the maps $\text{Res}_H^G$ are $\lambda$-homomorphisms. By definition $\text{Res}_H^G(x) = 0$ for each $H < G$. For the other containment suppose $\text{Res}_H^G(y) \equiv 0$ for each $H < G$; we must show $\chi_G(y) \equiv 0$ (mod $p$). Let $H < G$ be a subgroup of index $p$. As $H < N(H), H$ must be normal in $G$. The congruences of 1.3 become

$$0 = \chi_H(y) \equiv -\varphi(p)\chi_G(y) = -(p-1)\chi_G(y) \pmod{p}$$
thus $\chi_G(y) \equiv 0 \pmod{p}$.

We are ready to state the final result.

**Theorem 3.8.** Let $G$ be a $p$-group. Suppose that

1) $\alpha_H$ is injective for all $H < G$
2) $\alpha_G$ is injective for the $\lambda$-ideal $\mathbb{Z}x$ described in Proposition 3.6.

Then $\hat{\alpha}_G$ is injective.

**Proof.** We first show that $\alpha_G$ is injective on $\mathbb{Z}x \oplus \text{Ker } \chi_G$. If $m \in \mathbb{Z}$, $\chi_G(y) = 0$ and $\alpha_G(mx + y) = 0$, then also

$$0 = \alpha_G(mx + y)\alpha_G(y) = \alpha_G(y^2)$$

since $xy = 0$ (all characters are 0). By Lemma 3.5 $y^2 = 0$, so $y = 0$. Thus $\alpha_G(mx) = 0$ and $m = 0$ by assumption 2).

Now any element in $A(G)$ can be written as $n + z$ with $0 \leq n < p$ and $z \in \mathbb{Z}x \oplus \text{Ker } \chi_G$ since the latter ideal consists of $z$ such that $\chi_G(z) \equiv 0 \pmod{p}$. Suppose $\alpha_G(n + z) = 0$, then

$$0 = \alpha_G(z)\alpha_G(n + z) = \alpha_G(nz + z^2)$$

where $nz + z^2 \in \mathbb{Z}x \oplus \text{Ker } \chi_G$. From the above $nz = -z^2$, and taking characters we get

$$\chi_H(z) = 0 \text{ or } -n \text{ for all } H \leq G.$$ 

As $\chi_G(z) \equiv 0 \pmod{p}$ we must have $\chi_G(z) = 0$. But then $\chi_H(z) = 0$ for all $H < G$: if $H < G$ is a maximal subgroup with $\chi_H(z) = -n$ then by 1.3

$$-n = \chi_H(z) \equiv -\sum \phi(K/H)\chi_K(z) = 0 \pmod{|N(H)/H|}$$

which is impossible, since $|N(H)/H|$ is a positive power of $p$. Thus $z = 0$ and $\alpha_G(n) = 0$ implies $n = \deg \alpha_G(n) = 0$.

This completes the proof of Theorem 3.8.

**4. Homological study of $\alpha_G$.**

In this section we shall study the maps induced by $\alpha_G(x) : BG \rightarrow QS^0$ in homology for elementary abelian groups $G$. As a corollary we get that $\alpha_{(\mathbb{Z}/p)^n}$ is injective. We obtain also information relative to the splitting $Q_0S_p^0 \cong J_p \times \text{cok } J_p$. We suppress the index $G$ and write $\alpha$ for $\alpha_G$.

Consider $\alpha(S)$ for a $(\mathbb{Z}/p)^n$-set $S$. The map $\alpha$ is additive, so we can restrict to transitive sets: $S = (\mathbb{Z}/p)^n/H$ where $H \leq (\mathbb{Z}/p)^n$. Both $H$ and the quotient $(\mathbb{Z}/p)^n = (\mathbb{Z}/p)^n/H$ are elementary abelian, and $S$ is induced from the regular representation $\eta_m = (\mathbb{Z}/p)^n/1$ of $(\mathbb{Z}/p)^n$. Thus $\alpha(S)$ factors
(4.1) \[ B(\mathbb{Z}/p)^n \rightarrow B(\mathbb{Z}/p)^m \xrightarrow{\alpha(\eta_n)} QS^0 \]

Recall that the composition product in \( QS^0 \) corresponds to the product in \( \Pi \Sigma_n \) coming from the homomorphisms
\[ \psi_{n,m} : \Sigma_n \times \Sigma_m \rightarrow \Sigma_{nm} \]
defined as
\[ \psi_{n,m}(g, h)(i, j) = (g(i), h(j)) . \]
Here \( \Sigma_{nm} \) is regarded as the permutation group of pairs \((i, j), 1 \leq i \leq n, 1 \leq j \leq m. \)
This requires a linear ordering of the pairs; we use the lexicographic one.

We can express \( \eta_m \) inductively in terms of \( \psi_{n,m} \). The first \( \eta_1 \) is just the inclusion \( \mathbb{Z}/p \subset \Sigma_p \) as cyclic permutations. Then \( \eta_2 = \psi_{p,p} \circ (\eta_1 \times \eta_1) \), and generally
\[ \eta_n : (\mathbb{Z}/p)^n = \mathbb{Z}/p \times (\mathbb{Z}/p)^{n-1} \xrightarrow{\eta_1 \times \eta_{n-1}} \Sigma_p \times \Sigma_p^{n-1} \xrightarrow{\psi_{p,p}^{n-1}} \Sigma_p^n . \]
Hence \( \alpha(\eta_n) \) can be written as the composition
\[ \alpha(\eta_n) : B(\mathbb{Z}/p)^n = (B\mathbb{Z}/p)^n \xrightarrow{(B\eta)^n} (B\Sigma_p)^n \xrightarrow{\psi_{p,p}^{n-1}} QS^0 . \]

We shall need certain facts about the homology of \( QS^0 \) with \( \mathbb{Z}/p \)-coefficients. General references for this are [10] and [12] for \( p = 2 \) and [6] for \( p > 2 \). Here is a summary.

The space \( QS^0 \) has two products: the loop sum \( * \) and the composition product \( \cdot \). They induce products on \( H_*(QS^0; \mathbb{Z}/p) \), denoted similarly. They are homomorphisms \( Q^b : H_*(QS^0; \mathbb{Z}/p) \rightarrow H_*(QS^0; \mathbb{Z}/p) \) with the following properties [modifications for the case \( p = 2 \) are stated inside square brackets]:

(4.3) Degree: \( Q^b \) raises degree by \( 2b(p-1) \) \( [b] \)

(4.4) Evaluation:
\[ \begin{align*}
Q^b x &= 0 \quad \text{if } 2b < \deg x \quad [b < \deg x] \\
Q^b x &= x \cdot p \quad \text{if } 2b = \deg x \quad [b = \deg x]
\end{align*} \]

(4.5) Cartan formula: \( Q^b(x \ast y) = \sum Q^i x \ast Q^j y \).

(4.6) Adem relations: If \( a > pb \) then
\[ Q^a Q^b x = \sum (-1)^{a+t} \binom{(p-1)(t-b)-1}{pt-a} Q^{a+b-t} Q^t x ; \]
if \( p > 2, a \geq pb \) and \( \beta \) denotes the mod \( p \) Bockstein, then
\[ Q^a \beta Q^b x = \sum (-1)^{a+t} \binom{(p-1)(t-b)}{pt-a} \beta Q^{a+b-t} Q^t x . \]
\[ + \sum (-1)^{a+t} \binom{(p-1)(t-b) - 1}{pt-a - 1} Q^{a+t} \beta Q'^t x. \]

In all cases the summation is over \( t \) such that \( (p+1)t \geq a+b \).

(4.7) Nishida relations: If \( P^*_1 \) is dual to the reduced \( p \)th power \( P^r \) [the square \( \text{Sq}^r \)] then

\[ P^a_x Q^b x = \sum_{t \geq 0} (-1)^{a+t} \binom{(p-1)(b-a)}{a- pt} Q^{b-a+t} P^*_1 x; \]

if \( p > 2 \) then

\[ P^a_x \beta Q^b x = \sum_{t \geq 0} (-1)^{a+t} \binom{(p-1)(b-a) - 1}{a- pt} \beta Q^{b-a+t} P^*_1 x + \sum_{t \geq 0} (-1)^{a+t} \binom{(p-1)(b-a) - 1}{a- pt - 1} Q^{b-a+t} P^*_1 \beta x. \]

Let \([k] \in H_0(\text{QS}^0; \mathbb{Z}/p)\) denote the component of maps of degree \( k \). E. Dyer and R. Lashof showed that the homology ring \( H_*(\text{QS}^0; \mathbb{Z}/p) \) was generated by successive operations of \( Q^*, \beta Q^* \) on \([1]\) as an algebra under *\(^{1}\). To make a precise statement, we introduce the Dyer–Lashof algebra \( R(p) \).

Let \( \mathcal{F} \) be the free graded associative algebra generated by the symbols \( Q^s \), \( s \geq 0 \) and \( \beta Q^s \), \( s > 0 \) with degrees \( 2s(p-1) \) and \( 2s(p-1) - 1 \) respectively [if \( p = 2 \), \( \mathcal{F} \) is generated by \( Q^s, s \geq 0 \), with degree \( s \)]. The monomials in \( \mathcal{F} \) can be written as

\[ \beta^{e_1} Q^{s_1} \ldots \beta^{e_k} Q^{s_k} \]

with \( e_i = 0 \) or 1 and \( s_i \geq e_i \). Denote them by \( Q^I \), where \( I = (e_1, s_1, \ldots, e_k, s_k) \). We say that \( I \) is admissible if \( s_1 \leq ps_2 - e_2, \ldots, s_{k-1} \leq ps_k - e_k \), and we define the length and excess of \( I \) by \( l(I) = k \) and \( e(I) = \sum_{j=2}^k 2s_j(p-1)-e_j \) (\( p > 2 \))

\[ e(I) = s_1 - \sum_{j=2}^k s_j \quad (p = 2) \]

The quotient of \( \mathcal{F} \) by the ideal generated by the Adem relations and by monomials with \( e(I) < 0 \) is the Dyer–Lashof algebra \( R(p) \).

The formulas (4.3)–(4.6) tell that \( R(p) \) acts on \( H_*(\text{QS}^0; \mathbb{Z}/p) \). In fact the set

(4.8) \[ X = \{ Q^I[1] \mid I \text{ admissible}, e(I) + e_1 > 0 \} \]

forms a basis for the *-algebra \( H_*(\text{QS}^0; \mathbb{Z}/p) \) up to component shift. Indeed, let \( \mathbb{Z}/p[\mathbb{Z}] \) be the group ring of \( \mathbb{Z} = \pi_0(\text{QS}^0) \). Then
\[ H_*(QS^0; \mathbb{Z}/2) = PX \otimes \mathbb{Z}/2[\mathbb{Z}] \]
\[ H_*(QS^0; \mathbb{Z}/p) = PX^+ \otimes EX^- \otimes \mathbb{Z}/p[\mathbb{Z}], \quad \text{if } p > 2 \]

where \( P \) and \( E \) denote the polynomial and exterior algebras, respectively, and \( X^+ (X^-) \) is the even (odd) degree part of \( X \).

The composition product is related to the operations \( Q^b \) by May's formula:

\[
Q^b(x) \cdot f = \sum_{t \geq 0} Q^{b+t}(x \cdot P_t^* f) \quad \text{and, if } p > 2
\]
\[
(4.9) \quad \beta Q^b(x) \cdot f = \sum_{t \geq 0} \beta Q^{b+t}(x \cdot P_t^* f) - (-1)^{\deg x} \sum_{t \geq 0} Q^{b+t}(x \cdot P_t^* f)
\]

After these preparations we turn to the evaluation of (4.2) in homology. The map \( i: B\Sigma_p \to QS^0 \) is obtained from the Dyer–Lashof map \( \theta_p \), as the composite

\[ i: B\Sigma_p = E\Sigma_p \times (\ast)^p \rightarrow E\Sigma_p \times (QS^0)^p \xrightarrow{\theta_p} QS^0, \]

where \( \ast \) goes to the identity map in the 1-component \( Q_1S^0 \). By (4.2), \( \alpha(\eta_1) \) is of the form \( BZ/p \xrightarrow{B\eta_1} B\Sigma_p \to QS^0 \). This is precisely the map used in the definition of \( Q^s \) [6, pp. 7–8]; if \( e_m \in H_m(BZ/p; \mathbb{Z}/p) \) denotes the standard generator then

\[
(4.10) \quad \alpha(\eta_1)_*(e_m) = Q_m[1] = \begin{cases} 
(-1)^sQ^s[1] & \text{if } m = 2s(p-1) \\
(-1)^s\beta Q^s[1] & \text{if } m = 2s(p-1) - 1 \quad \text{for } p \text{ odd} \\
0 & \text{otherwise}
\end{cases}
\]

\[
= Q^m[1] \quad \text{for } p = 2.
\]

It follows from (4.2) and (4.10) that \( \alpha(\eta_n)_* \) takes the generators \( e_{i_1} \otimes \ldots \otimes e_{i_n} \) of \( H_*(B(\mathbb{Z}/p)^n; \mathbb{Z}/p) \) to products of the form

\[ \pm \beta^{e_1}Q^1[1] \cdot \ldots \cdot \beta^{e_n}Q^n[1]. \]

We would like to express these elements in the \( \ast \)-product basis (4.8). A two-fold product, for example \( Q^a[1] \cdot Q^b[1] \), becomes

\[
(4.11) \quad Q^a[1] \cdot Q^b[1] = \sum_{t \geq 0} Q^{a+t}(P_t^*Q^b[1])
\]

\[ = \sum_{t \geq 0} (-1)^t \binom{(p-1)(b-t)}{t} Q^{a+t}Q^{b-t}[1] \]

by (4.9) and (4.7). Applying the Adem relations (4.6) it can be written as a linear combination of admissible terms \( Q^sQ^r[1] \). If the excess is negative then \( Q^r[1] = 0 \) by (4.4). Similarly it is shown by induction on the length of the product that
Lemma 4.12.

\[ \beta^{e_1}Q^n[1] \cdots \beta^{e_n}Q^n[1] = \sum \lambda_i Q^i[1] \]

where \( I \) ranges over admissible sequence of length \( n \) and excess \( \geq 0 \).

The terms \( Q^i[1] \) with \( e(i) + e_1 = 0 \) decompose as \(*\)-products of shorter \( Q^j[1] \)'s (4.4).

We shall now find the special case of Lemma 4.12 in the lowest degree where we can get an admissible \( Q^i[1] \) of length \( n \) and excess \( > 0 \) involving no Bocksteins \( \beta \). It is clearly \( Q^n Q^{n-1} \cdots Q^1[1] \) in dimension \( 2(p^n+1-1) \) with excess 2 [if \( p = 2 \) then \( d = 2^n+1-1 \) and \( e = 1 \)]. We give the proofs of the next two lemmas only for \( p > 2 \). The (easier) case \( p = 2 \) follows by trivial modifications.

Lemma 4.13. \( Q^n[1] \cdot Q^{n-1}[1] \cdots Q^1[1] = Q^n Q^{n-1} \cdots Q^1[1] \).

Proof. To begin with, \( Q^n[1] \cdot Q^1[1] = Q^n Q^1[1] \) by (4.11). Suppose by induction that the claim holds for \( n \). Since \( x_n = Q^n Q^{n-1} \cdots Q^1[1] \) is primitive, so is also \( P^t_* x_n \). If \( t > 0 \), then according to (4.7) \( P^t_* x_n \) is a linear combination of \( Q^i[1] \)'s of length \( n \) and degree \( < 2(p^{n+1}-1) \), without Bocksteins. By the minimality of \( x_n \), \( P^t_* x_n \) is \(*\)-decomposable. By a general theorem of Hopf algebras [15, Proposition 4.23] \( P^t_* x_n \) must then be a \(* - p\)th power, especially

\[ \deg P^t_* x_n = 2(p^{n+1}-1) - 2t(p-1) \equiv -2 + 2t \equiv 0 \quad (\text{mod } p) \]

so that \( t \equiv 1 \quad (\text{mod } p) \). Now we can apply May's formula (4.9) to get

\[ Q^{n+i}[1] \cdot Q^n[1] \cdots Q^1[1] = Q^n Q^{n+1}[1] \cdot x_n = \sum_{i \geq 0} Q^{n+i+1}(P^i_* x_n) = Q^{n+i} Q^n \cdots Q^1[1] \]

since \( Q^s(x^{*p}) \equiv 0 \) only if \( s \equiv 0 \quad (\text{mod } p) \) in virtue of the Cartan formula (4.5).

Now we can prove that the map \( \alpha: A(\mathbb{Z}/p)^n \to \pi_0^s(B(\mathbb{Z}/p)^n) \) is injective on \( \mathbb{Z} \eta_n \) and thereby on the whole of \( A(\mathbb{Z}/p)^n \).

Proposition 4.14. \( \alpha(m \eta_n) \) is homologically non-trivial for all non-zero integers \( m \).

Proof. Let first \( m > 0 \). Then using the diagonal formula

\[ \psi(e_{2i}) = \sum e_{2i_1} \otimes \cdots \otimes e_{2i_m}, \quad i_1 + \cdots + i_m = i \]

for \( B \mathbb{Z}/p \to (B \mathbb{Z}/p)^m \) and Lemmas 4.12 and 4.13 we obtain

\[ \alpha(m \eta_n)_* (e_{2m(p-1)} \otimes \cdots \otimes e_{2m(p-1)}) = (Q^n Q^{n-1} \cdots Q^1[1])^{*m} + \cdots \]
where the other terms are of the form $Q^l[1]^* \cdots Q^l[1]$ with $l(l_j) = n$ and 
$\deg (l_j) < 2(p^{n+1} - 1)$ for at least one $j$. Since $Q^p Q^{p-1} \cdots Q^1[1]$ is a polynomial generator, they cannot cancel the first term.

If $m < 0$, then apply the loop inverse $\chi_*$, and note that $\chi_*(x) = x [-2 \deg x]$ on primitive elements $x$.

**Theorem 4.15.** $\hat{A}((\mathbb{Z}/p)^n) \to \pi^0_{\text{et}} p(B(\mathbb{Z}/p)^n)$ is injective for all primes $p$.

**Proof.** By Theorems 3.2 and 3.8 we are reduced to showing that $\alpha$ is injective on $\mathbb{Z}x$, where

$$x = p - \eta_1^{\text{tot}} + \cdots + (-1)^n p^{\frac{n-1}{2}} \eta_n.$$ 

The argument of Proposition 4.14 applies also here, since the terms $\eta_1^{\text{tot}}$ contribute in homology only by $*$-products of $Q^l[1]$ with $l(l) < n$ (cf. (4.1)).

This completes the proof of theorem 4.15.

Let $Q_0 S^0$ be the 0-component of $Q S^0$. Let $X_p$ denote the localization of the space $X$ at a prime $p$ [21]. D. Sullivan has showed that the space $Q_0 S^0$ splits locally

$$Q_0 S^0_p \cong J_p \times \text{cok} J_p.$$ 

The space $J_2$ is defined as the fibre of $\psi^2 - 1: BO_2 \to B\text{Spin}_2$. At odd primes $J_p$ is the fibre of $\psi^k - 1: BU_p \to BU_p$, where $k$ is a prime power generating the group of units in $\mathbb{Z}/p^2$. The homotopy groups of $J_p$ are essentially the $p$-primary part of the image of the $J$-homomorphism $O \to G$ in the stable homotopy of spheres. To describe the second factor $\text{cok} J_p$ we recall the discrete models for $J_p$ due to D. Quillen [14, chapter VIII].

First, let $p = 2$. Let $F_3$ denote the finite field with 3 elements. Let $N_n(F_3)$ be the group of orthogonal transformations of the quadratic space $(F_3^n, x_1^2 + \cdots + x_n^2)$ for which the determinant and the spinor norm [14, p. 164] agree. We encounter now a similar situation to the construction of $Q S^0$ from the symmetric groups: there are sum and product maps on the disjoint union

$$\prod_{n \geq 0} BN_n(F_3)$$

coming from direct sum and tensor product of quadratic spaces.

Let $\bar{F}_3$ be an algebraic closure of $F_3$ and choose an embedding $\mu: \bar{F}_3^* \to \mathbb{C}^*$. If $G$ is a finite group and $g: G \to \text{Gl}_n(\bar{F}_3)$ a representation of $G$, then the complex-valued function on $G$ 

$$\chi(g) = \sum_{i=1}^n \mu(\lambda_i)$$
where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( g(g) \), is the character of a unique element in the complex representation ring \( R(G) \). Moreover, Quillen has proved that if \( g \) takes values in \( O_n(F_3) \), then \( \chi \) is the character of an element in the real representation ring \( RO(G) \).

We lift the standard representations of \( N_n(F_3) \) in \( F_3^s \) in the above way to virtual representations in \( RO(N_n(F_3)) \) and apply \( \alpha : RO(G) \rightarrow KO(BG) \) to get maps

\[
v_n : BN_n(F_3) \rightarrow BO \times (n).
\]

They are compatible with the sum and product on (4.17), giving rise to an \( H \)-map

\[
v : \Omega B \left( \coprod_{n \geq 0} BN_n(F_3) \right) \rightarrow BO \times Z
\]

from its group completion. Now the Adams operation \( \psi^3 \) is characterized by its action on the characters \( (\psi^3 \chi)(g) = \chi(g^3) \), so \( \psi^3 \circ v_n = v_n \) as the Frobenius map \( \lambda \rightarrow \lambda^3 \) just permutes the eigenvalues of any representation realizable over \( F_3 \).

Let \( J^s_2 \) denote the zero component of \( \Omega B \left( \coprod_{n \geq 0} BN_n(F_3) \right) \) localized at 2. Then \( v : J^s_2 \rightarrow BO_2 \) lifts to an \( H \)-map \( J^s_2 \rightarrow J_2 \), which can be proved to be a homotopy equivalence e.g. by cohomological methods. From now on we identify \( J^s_2 \) and \( J_2 \).

Let

\[
e : QS^0 \cong \Omega B \left( \coprod_{n \geq 0} B \Sigma_n \right) \rightarrow \Omega B \left( \coprod_{n \geq 0} BN_n(F_3) \right)
\]

be induced from the functor which takes a finite set \( S \) to the vector space \( F_3^S \).

We restrict \( e \) to the zero component and localize to get \( e : Q_0S^0_2 \rightarrow J_2 \). We shall also use the analogous map \( e : Q_0S^0_2 \rightarrow BO_2 \), induced from the functor \( S \mapsto R^S \). Then the triangle

\[
\begin{array}{ccc}
Q_0S^0_2 & \xrightarrow{e} & J_2 \\
\downarrow e & & \downarrow \\
BO_2 & & \\
\end{array}
\]

(4.18)

clearly commutes. The space \( \text{cok} J_2 \) is defined as the fibre in

\[
\text{cok} J_2 \rightarrow Q_0S^0_2 \xrightarrow{e} J_2.
\]

(4.19)

There exists a splitting \( \alpha_2 : J_2 \rightarrow Q_1S^0_2 \), and \( \alpha_2 [*[-1]] \) gives (4.16).

At odd primes \( p \) the model for \( J_p \) is constructed from general linear groups...
over the finite field $F_k$: $J_p$ is equivalent to the zero component of the group completion of

$$\prod_{n \geq 0} B\text{GL}_n(F_k), \text{ } k \text{ a prime power generating } (\mathbb{Z}/p^2)^*.$$

As before defines $e: Q_0 S^0_p \to J_p$ and gets a commutative diagram

$$\begin{array}{ccc}
Q_0 S^0_p & \xrightarrow{e} & J_p \\
\downarrow e & & \downarrow \\
BU_p & & \\
\end{array}$$

and $\text{cok } J_p$ is defined as the fibre in

$$\text{cok } J_p \to Q_0 S^0_p \xrightarrow{e} J_p.$$

Let now $G$ be a $p$-group. We shall in the following consider the abelian groups $[BG, X_p]$, where $X = Q_0 S^0$, $BU$, $U$ for all $p$ and in addition to these, $X = BO$ and $SO$ for $p = 2$. We claim that in all cases

$$[BG, X_p] = [BG, X].$$

If $X = Q_0 S^0$ this holds because $Q_0 S^0$ has finite homotopy groups, so $Q_0 S^0 = \prod_p Q_0 S^0_p$, and $BG$ is $p$-local (even $p$-complete) [21, section 3].

For the other spaces we recall the results of Atiyah [1] and Atiyah–Segal [2]. Consider the representable $K$-theory and the theory $K^* (\_ ; Z_{(p)})$ defined by the unitary spectrum and its localization at $p$. For any finite CW-complex $Y$ we have

$$K^* (Y; Z_{(p)}) \cong K^* (Y) \otimes Z_{(p)}.$$

The formula is valid also for $Y = BG$ since it follows from [1] and [2] that $\lim^1$ of the inverse systems $K^* (B_n G)$ and $K^* (B_n G) \otimes Z_{(p)}$ vanishes, so that $K^* (\widehat{BG}) = \lim K^* (B_n G)$ and $K^* (BG; Z_{(p)}) = \lim K^* (B_n G) \otimes Z_{(p)}$. For any group $G$

$$K^0 (BG) = \hat{K} (G) \text{ and } K^1 (BG) = 0$$

[1, p. 270] and in the case of $p$-groups the completion is the $p$-adic one: $\hat{K}^0 (BG) = I (G) \otimes \hat{Z}_p$ [4 p. 277]. Since these groups are clearly unaffected by $\otimes Z_{(p)}$, we get

$$[BG, BU] = [BG, BU_p] = I (G) \otimes \hat{Z}_p, \text{ } [BG, U_p] = 0$$

where $I (G)$ is the augmentation ideal of $R(G)$.

If $p = 2$, then using the Real $K$-theory $KR^*$ instead of $K^*$ and [2, p. 17] we obtain in the same fashion

$$[BG, BO] = [BG, BO_2] = I (G) \otimes \hat{Z}_2,$$
for 2-groups $G$, where $I(G)$ is the augmentation ideal of $R_R(G)$.

Let $G$ be a $p$-group. After these preliminaries we turn to the question: when does a map $\hat{\alpha}(x): BG \to Q_0S^0$ lift to $\text{cok} J_p$ in the fibration (4.19), (4.19'). In order for $e \circ \hat{\alpha}(x): BG \to Q_0S^0 \to J_p$ to be nullhomotopic, it is necessary in the light of (4.18) and (4.18') that the image of $\hat{\alpha}(x)$ under $e_\ast: \tilde{\pi}_s^0(BG) \to \tilde{K}_s^0(BG)$ is zero. From Proposition 3.1, this is equivalent to

$$x \in \hat{A}_0(G) = \text{Ker} (\hat{A}(G) \to \hat{K}(G)).$$

If $p$ is odd, this condition is also sufficient, since in the mapping sequence of the fibration $J_p \to BU_p \xrightarrow{\psi^*} BU_p$

$$[BG, U_p] \to [BG, J_p] \to [BG, BU_p]$$

the first group is trivial (4.20), so $x \in \hat{A}_0(G)$ maps to zero already in $[BG, J_p]$.

In particular, if $G$ is an elementary abelian group $(\mathbb{Z}/p)^n$ with odd $p$, we know that all the maps $\hat{\alpha}(x): BG \to Q_0S^0, x \in \hat{A}_0(G)$ are homotopically distinct (Theorem 4.15). Thus $\hat{\alpha}$ lifts to a monomorphism $\hat{\alpha}'$

$$\hat{A}_0(G) \xrightarrow{\hat{\alpha}} [BG, cok J_p] \to [BG, Q_0S^0] \to [BG, J_p].$$

**Theorem 4.22.** Let $p$ be an odd prime and $G$ the elementary abelian group $(\mathbb{Z}/p)^n$. Then the ideal

$$\hat{A}_0(G) = \text{Ker} (\hat{A}(G) \to \hat{K}(G))$$

maps monomorphically into $[BG, \text{cok} J_p]$.

Let then $G$ be a 2-group and $x \in \hat{A}_0(G)$. Then the image of $\hat{\alpha}(x)$ is 0 in $[BG, BU_2]$. To see when $e \circ \hat{\alpha}(x): BG \to J_2$ is non-trivial, we consider the maps $J_2 \to BO_2 \xrightarrow{c} BU_2,$ where $c$ is complexification. The map $[BG, BO_2] \to [BG, BU_2]$ corresponds to the completion of $R_R(G) \subset R(G)$ by (4.20), (4.21) and [2, p. 17], which is injective. In the mapping sequence of $J_2 \to BO_2 \xrightarrow{\psi^*} BSpin_2$

$$[BG, Spin_2] \to [BG, J_2] \to [BG, BO_2]$$

the first group is a subgroup of $[BG, SO_2]$ as $SO_2 \cong RP^\infty \times Spin_2$, hence it is a vector space over $\mathbb{Z}/2$. Thus (at least) 2x maps to zero in $[BG, J_2]$. We have proved the first half of
Theorem 4.23. Let $G$ be the elementary abelian 2-group $(\mathbb{Z}/2)^n$. Then the ideal $2\hat{A}_0(G)$ maps monomorphically into $[BG, \text{cok } J_2]$. There are elements in $\hat{A}_0(G)$ which do not lift to cok $J_2$.

Proof. Consider the critical element $x \in A_0(G)$ with $\chi_G(x)=2$ and $\chi_H(x)=0$ for all genuine subgroups $H < G$. We claim that $1-x$ can be written as a product in terms of the $2^n-1$ quotients $\eta_i = G/(\mathbb{Z}/2)^{n-1}$:

$$1-x = \prod_{i=1}^{2^n-1} (\eta_i^i - 1).$$

Indeed, check the characters. First $\chi_H(\eta_1^i)=2$ or 0 according to whether $H \leq (\mathbb{Z}/2)^{n-1}$ or $H \not\leq (\mathbb{Z}/2)^{n-1}$, the hyperplane defining $\eta_1^i$. Therefore we get

$$\chi_G\left(\prod_{i=1}^{2^n-1} (\eta_i^i - 1)\right) = (-1)^{2^n-1} = -1.$$

On the other hand each hyperplane containing $H$ corresponds to a line inside $H^\perp$. If $H < G$ the number of these, $|H|^\perp - 1$, is odd, so

$$\chi_H\left(\prod_{i=1}^{2^n-1} (\eta_i^i - 1)\right) = 1^{\text{odd}} (-1)^{\text{even}} = 1, \quad H < G.$$

Thus $\alpha(1-x)$ is a composition product of maps of the form

$$BG \xrightarrow{B\eta_i^i} B\mathbb{Z}/2 \xrightarrow{i_3} Q_2S^0 \xrightarrow{e[-1]} SG.$$

But the map $i_2 *[1] : B\mathbb{Z}/2 \to SG$ is homotopy equivalent to the composite

$$\mathbb{R}P^\infty \to SO \xrightarrow{J} SG$$

[6, p. 120] so that $\alpha(1-x)$ factors through $J : SO \to SG$. Let $e_1 : SG \to J^\otimes$ be the 1-component of the map $e$ defined just before (4.18). We showed in 4.14 that $\alpha(-x)$, hence $\alpha(1-x)$ induces a non-trivial map in homology. It is well-known that the composite

$$H_*(SO) \xrightarrow{J_*} H_*(SG) \xrightarrow{e_1*} H_*(J)$$

is injective [6, p. 120 and Theorem 12.5 p. 185]. Then $e_1 \circ \alpha(1-x)$, hence $e \circ \alpha (-x)$ must be homologically non-trivial.

This completes the proof of Theorem 4.23.

Remark 4.24. Theorems 4.22 and 4.23 enable us to get hold of elements in $H_*(\text{cok } J_p; \mathbb{Z}/p)$. Let us consider the first case $A_0(\mathbb{Z}/2 \oplus \mathbb{Z}/2) = \mathbb{Z}x$ ($A_0(\mathbb{Z}/2) = 0$). It is most convenient to evaluate $f = \alpha(1-x)$, since from the preceding proof

$$1-x = (\eta_1^1 - 1)(\eta_2^1 - 1)(\eta_3^1 - 1)$$
where \( \eta^i_1 : \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \) are projections to the first and second factor for \( i = 1, 2 \), and \( \eta^i_2 \) takes the quotient modulo the diagonal subgroup \( \Delta \mathbb{Z}/2 \subset \mathbb{Z}/2 \oplus \mathbb{Z}/2 \).

The maps \( f_i = \alpha(\eta^i_1 - 1) : \mathbb{R}P^\infty \times \mathbb{R}P^\infty \to \mathbb{R}P^\infty, \ i = 1, 2, 3 \), have the effect

\[
f_i_*(e_m \otimes e_n) = \delta_{m0} x_m, \ f_2_*(e_m \otimes e_n) = \delta_{m0} x_n
\]

and

\[
f_3_*(e_m \otimes e_n) = \binom{m+n}{m} x_{m+n}
\]

on homology (cf. 4.10). Here \( x_k = Q^k[1] *[ -1] \in H_*(SG; \mathbb{Z}/2) \), and adding up we get

\[
(4.25) \quad f_*(e_m \otimes e_n) = \sum_{i=0}^m \sum_{j=0}^n \binom{m+n-i-j}{m-i} x_i \cdot x_j \cdot x_{m+n-i-j}.
\]

As a special case of this formula \( f_*(e_{2n} \otimes e_1) = p_{2n+1} \), where the polynomial

\[
p_{2n+1} = x_{2n+1} + \sum_{i=1}^n x_i x_{2n+1-i}
\]

is the standard primitive element of degree \( 2n+1 \) in the subalgebra \( E(x_1, x_2, \ldots) \subset H_*(SG) \).

Let \( \tilde{f} \) denote \( \alpha(-x) = f*[ -1] \). We know from theorem 4.23 that \( 2\tilde{f} : (\mathbb{R}P^\infty)^2 \to Q_0 S^0 \) lifts to \( \text{cok} J_2 \). Therefore the elements

\[
C_{4n+2} = (2\tilde{f})_*(e_{2n} \otimes e_2) = \tilde{f}_*(e_{2n} \otimes e_1) * \tilde{f}_*(e_{2n} \otimes e_1)
\]

\[
= p_{2n+1} * p_{2n+1} *[ -2]
\]

\( n \geq 1 \), lie in \( \text{Ker} e_* \). Since they are primitive, the lie in \( H_*(\text{cok} J_2, \mathbb{Z}/2) \). (The elements \( C_{2n-2} \) have a connection with the Arf invariant conjecture: they are spherical if and only if there are stable homotopy classes in \( \pi_{2n-2}^S(S^0) \) of Arf invariant one [9].)

**Remark 4.26.** We succeeded in proving that \( \delta_G \) is injective for elementary abelian \( G \) by evaluating the maps \( \alpha_G(nx) \) in homology. Let us indicate briefly where this program fails for more complicated groups. The smallest ones we have not covered are the following three groups of order 8:

\[
Z/4 \oplus Z/2 = \langle x, y \mid x^4 = y^2 = 1, xy = yx \rangle
\]

\[
D_8 = \langle x, y \mid x^4 = y^2 = 1, y^{-1}xy = x^3 \rangle
\]

\[
Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, y^{-1}xy = x^3 \rangle
\]
(D8 and Q8 are the dihedral and the quaternion groups). In all cases the Frattini subgroup \( \Phi(G)=G^2 \) is \( \mathbb{Z}/2 \) generated by \( x^2 \). The cohomology of \( G \) (with \( \mathbb{Z}/2 \) coefficients) can be computed from the spectral sequence of the central extension

\[ 1 \to \Phi(G) \to G \xrightarrow{\pi} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 1. \]

The \( E^2 \)-term is

\[ H^*(\mathbb{Z}/2) \otimes H^*(\mathbb{Z}/2 \oplus \mathbb{Z}/2) = P(t) \otimes P(t_1, t_2). \]

We choose \( t_1 \) and \( t_2 \) as the generators of the cohomology of \( \langle \pi(x) \rangle \) and \( \langle \pi(y) \rangle \). The differentials are determined by the characteristic class \( d_2(t) \in H^2(\mathbb{Z}/2 \oplus \mathbb{Z}/2) \), which is

\[ t_1^2, t_1^2 + t_1 t_2 \quad \text{and} \quad t_1^2 + t_1 t_2 + t_2^2, \]

respectively.

The critical elements \( BG \to QS^0 \) are compositions of

\[ BG \xrightarrow{B \pi} B(\mathbb{Z}/2 \oplus \mathbb{Z}/2) = (\mathbb{RP}^\infty)^2 \]

with the maps \( \alpha(nx): (\mathbb{RP}^\infty)^2 \to QS^0 \). We evaluated \( \bar{f} = \alpha(-x) \) in the preceding remark. From (4.25) we get

\[ f_\pi^*(e_n \otimes e_m) = f_\pi^*(e_m \otimes e_n), \quad f_\pi^*(e_n \otimes e_0) = 0 \quad (n > 0). \]

Consider now e.g. the cohomology of \( G = D8 \). In its spectral sequence

\[ d_3(t^2) = d_3(\text{Sq}^1 t) = \text{Sq}^1 d_2 t = \text{Sq}^1 (t_1^2 + t_1 t_2) = t_1^2 t_2 + t_1 t_2^2 = t_2 d_2 t = 0 \]

so that \( E^3 = E^\infty \) and

\[ H^*(D8) = P(s) \otimes P(t_1, t_2)/(t_1^2 + t_1 t_2) \]

where \( s \in H^2(D8) \) is any element whose image is \( t^2 \in H^2(\mathbb{Z}/2) \), and \( t_1 \) and \( t_2 \) come from \( H^*(\mathbb{Z}/2 \oplus \mathbb{Z}/2) \). Thus the image of \( B \pi^* \) in \( H^*(D8) \) is generated by the elements \( t_1^n t_2 = t_1^{n-1} t_2, \ldots, t_1^{n-1} t_2 \) and \( t_2^n \). Dually the image of \( B \pi^*_\ast \) in \( H_*(\mathbb{Z}/2 \oplus \mathbb{Z}/2) \) is generated by

\[ e_n \otimes e_0 + e_{n-1} \otimes e_1 + \ldots + e_1 \otimes e_{n-1} \quad \text{and} \quad e_0 \otimes e_n. \]

From (4.27) \( f_\pi^*(\text{Im } B \pi^*_\ast) = 0 \). Hence all maps \( \alpha_{D8}(nx) = (-n\bar{f}) \circ B \pi \) vanish in \( \mathbb{Z}/2 \)-homology.

A similar computation shows that \( \bar{f} \circ B \pi \) induces the zero map \( H^*_\ast(Q8) \to H^*_\ast(\mathbb{Q}S^0) \). In fact here \( \text{Im } B \pi^* = 0 \) from dimension 4 on. Finally for \( G = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \) we get that \( (\bar{f} \circ B \pi)_\ast \) is non-trivial precisely in dimension 3. But then \( (2\bar{f} \circ B \pi)_\ast \) vanishes.

By Proposition 3.1 these maps induce 0 also in \( K \)-theory. We pose the
QUESTION. Are the maps

\[ f_n : BG \to \mathbb{R}P^{\infty} \times \mathbb{R}P^{\infty} \to Q_0 S^0 \]

where \( G = \mathbb{Z}/4 \oplus \mathbb{Z}/2, D8, Q8 \) and \( x = 2 - \eta_1^4 - \eta_2^2 - \eta_1^2 + \eta_2 \in A_0(\mathbb{Z}/2 \oplus \mathbb{Z}/2) \) homotopic to zero?

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