Chapter 42

Chern-Simons Forms

Chern-Simons forms arise from the Chern classes by a procedure called **transgression**, which involves pulling a characteristic form C on the base manifold M of a bundle $\pi : E \to M$ back to the total space E by π^*, and thereby making the pullback form an exact form on E:

$$\pi^* C = d(CS(C)) \; .$$

The Chern-Simons form $CS(C)$ so obtained has generated numerous important applications in physics, for example, in topological quantum field theory and condensed matter physics (fractional statistics and the quantum Hall effect).

We will begin with an informal discussion on transgression. Following Chern (see Chern 1990) we use the example of the two-dimensional Gauss-Bonnet formula [c.f. (41.123), Theorem 41.6]. Let M be a two-dimensional oriented Riemannian manifold and $\pi : P \to M$ be the orthonormal frame bundle of M. A point in the total space P is then written $(x; e_1 e_2)$, where (e_1, e_2) is an orthonormal frame with origin at $x \in M$, and the projection π sends $(x; e_1 e_2)$ to its origin x. Choose a frame field (e_1^0, e_2^0) with dual coframe field $((\omega^0)^1, (\omega^0)^2)$. Then we have the local expressions

$$e_1 = (\cos\tau) e_1^0 + (\sin\tau) e_2^0 \; , \quad e_2 = (-\sin\tau) e_1^0 + (\cos\tau) e_2^0 \; ; \tag{42.1}$$

and

$$\omega^1 = (\cos\tau)(\omega^0)^1 + (\sin\tau)(\omega^0)^2 \; , \quad \omega^2 = (-\sin\tau)(\omega^0)^1 + (\cos\tau)(\omega^0)^2 \; , \tag{42.2}$$

where (ω^1, ω^2) is the dual coframe of (e_1, e_2), and τ, the angle between e_1^0 and e_1, is the local fiber coordinate. Eqs. (42.2) show that ω^1 and ω^2 are globally well-defined one-forms on P. Exteriorly differentiating ω^1, we get

$$d\omega^1 = -\sin\tau d\tau \wedge (\omega^0)^1 + \cos\tau d\tau \wedge (\omega^0)^2 + \cos\tau d(\omega^0)^1 + \sin\tau d(\omega^0)^2 \; . \tag{42.3}$$

We can always write

$$d(\omega^0)^1 = a\,(\omega^0)^1 \wedge (\omega^0)^2 \; , \qquad d(\omega^0)^2 = b\,(\omega^0)^1 \wedge (\omega^0)^2 \; , \tag{42.4}$$

where a and b only depend on base manifold coordinates. By (42.2), we then have

$$d(\omega^0)^1 = a'\,\omega^1 \wedge \omega^2 \,, \qquad d(\omega^0)^2 = b'\,\omega^1 \wedge \omega^2 \,, \tag{42.5}$$

where a' and b' now are quantities depending on both base manifold coordinates and the fiber coordinate τ, that is, functions on P. Substituting (42.5) in (42.3), and performing a similar calculation for $d\omega^2$, we obtain

$$d\omega^1 = d\tau \wedge \omega^2 + A\,\omega^1 \wedge \omega^2 \,, \quad d\omega^2 = \omega^1 \wedge d\tau + B\,\omega^1 \wedge \omega^2 \,, \tag{42.6}$$

where A and B are functions on P. Now define the following one-form on P,

$$\omega_1^2 \equiv d\tau + A\omega^1 + B\omega^2 = -\omega_2^1 \,. \tag{42.7}$$

Then,

$$d\omega^1 = \omega_1^2 \wedge \omega^2 = \omega^2 \wedge \omega_2^1 \,, \tag{42.8}$$

$$d\omega^2 = \omega^1 \wedge \omega_1^2 \,. \tag{42.9}$$

These are precisely the structure equations given by (31.33): they completely determine the 1-form ω_1^2, which, in fact, is the connection 1-form. The structure equations (42.8) and (42.9) express the fact that the connection is torsion-free [c.f. (39.123)]. Taking the exterior derivatives of both sides of (42.8) we find

$$0 = d\omega_1^2 \wedge \omega^2 - \omega_1^2 \wedge d\omega^2 \,. \tag{42.10}$$

But, from (42.6) and (42.7),

$$\begin{aligned} \omega_1^2 \wedge d\omega^2 &= (d\tau + A\omega^1 + B\omega^2) \wedge (\omega^1 \wedge d\tau + B\,\omega^1 \wedge \omega^2) \\ &= B d\tau \wedge \omega^1 \wedge \omega^2 + B\,\omega^2 \wedge \omega^1 \wedge d\tau = 0 \,. \end{aligned} \tag{42.11}$$

Hence,

$$d\omega_1^2 \wedge \omega^2 = 0 \,. \tag{42.12}$$

Similarly, exterior differentiation of (42.9) yields

$$d\omega_1^2 \wedge \omega^1 = 0 \,. \tag{42.13}$$

The above two equations imply the following basic result of transgression:

$$\boxed{d\omega_1^2 = -K\,\omega^1 \wedge \omega^2} \,. \tag{42.14}$$

It may be suspected that K is in general a function on P, but it is in fact just a function on M, as can be shown easily as follows. Indeed, exterior differentiation of (42.14) gives

$$\begin{aligned} 0 &= -dK \wedge \omega^1 \wedge \omega^2 - K d(\omega^1 \wedge \omega^2) \\ &= dK \wedge \omega^1 \wedge \omega^2 + K\,(d\omega^1 \wedge \omega^2 - \omega^1 \wedge d\omega^2) = dK \wedge \omega^1 \wedge \omega^2 \,, \end{aligned} \tag{42.15}$$

where the last equality follows from the expressions of $d\omega^1$ and $d\omega^2$ given by (42.6). Now (42.7) implies that

$$\omega^1 \wedge \omega^2 \wedge \omega_1^2 = \omega^1 \wedge \omega^2 \wedge d\tau \neq 0 \; . \tag{42.16}$$

Since ω^1, ω^2 and hence ω_1^2 are all globally defined, the above equation indicates that P has a coframe field ω^1, ω^2, ω_1^2, and is thus **parallelizable**. If K did depend on the fiber coordinate τ, dK would have a term proportional to $d\tau$ and (42.16) would then imply that $dK \wedge \omega^1 \wedge \omega^2 \neq 0$, which contradicts (42.15). Thus K is a function on the base manifold M. We observe that $\omega^1 \wedge \omega^2$ is the area element on M and K is the Gaussian curvature of M. [c.f. (39.166) and the results of Exercises 39.14 to 39.17].

The transgression result (42.14) contains the essence of Chern's elegant proof of the Gauss-Bonnet theorem. Its importance lies in the following observation. By (42.16) and the fact that K is a function on M, $-K\,\omega^1 \wedge \omega^2$ is a 2-form on M, and as such, is not exact. However, when pulled back to P, the LHS of (42.14) indicates that it becomes the exterior derivative of a 1-form on P (the connection 1-form ω_1^2) and is thus exact on P. By the Gauss-Bonnet Theorem the 2-form $\dfrac{1}{2\pi} K\,\omega^1 \wedge \omega^2$ is in fact the Euler class.

The transgression procedure described above rests on a basic fact: P is always parallelizable, equivalently, *a global frame field always exists on P, even though it may not exist on the base manifold M*. This fact is easily seen as follows. Consider an arbitrary principal G-bundle $\pi : P \to M$. Using the projection operator π we can construct the pullback bundle $\pi^*(P)$ with base manifold P, called the **square bundle** of P. This bundle admits the "diagonal" global section $p \mapsto (p, p)$, and hence, by Theorem 37.2, is always trivial. In this sense, the total space P is always simpler than the base manifold M.

Let us now apply the transgression procedure to the Chern classes by pulling them back to the frame bundle $\pi' : P \to M$ associated with a complex vector bundle $\pi : E \to M$. Suppose in a local coordinate neighborhood of M we have a local expression ω for a given connection with respect to a choice of a local frame field $\{e_i\}$ on E. Under a local change of gauge, or a local change of frame field,

$$(e')_i = g_i^j\, e_j \; , \quad (g_i^j) \in G \; , \tag{42.17}$$

we have [c.f. (35.17)] the following expression for the connection matrix:

$$\varphi = (dg)\, g^{-1} + g\, \omega\, g^{-1} \; . \tag{42.18}$$

If the elements g_i^j of the matrix g are considered as local fiber coordinates, then φ becomes the pullback of ω by $(\pi')^*$, that is,

$$\varphi = (\pi')^*\,(\omega) \; , \tag{42.19}$$

and is a well-defined connection matrix of one-forms on P. The corresponding curvature matrix of 2-forms

$$\Phi = d\varphi - \varphi \wedge \varphi = g\, \Omega\, g^{-1} \; , \tag{42.20}$$

where $\Omega = d\omega - \omega\wedge\omega$, is also well-defined on P.

Consider the first Chern class [(41.76)]

$$C_1(\Phi) = \frac{i}{2\pi}\, Tr\,\Phi\ . \tag{42.21}$$

By (42.20), $Tr\,\Phi$ is given by

$$Tr\,\Phi = Tr\,d\varphi - Tr\,(\varphi\wedge\varphi) = Tr\,(d\varphi) = d(Tr\,\varphi)\ , \tag{42.22}$$

since

$$Tr\,(\varphi\wedge\varphi) = 0\ , \tag{42.23}$$

by virtue of the fact that

$$Tr\,(\varphi\wedge\varphi) = \varphi^j_i\wedge\varphi^i_j = -\varphi^i_j\wedge\varphi^j_i = -Tr\,(\varphi\wedge\varphi)\ . \tag{42.24}$$

Thus,

$$\boxed{C_1(\Phi) = \frac{i}{2\pi}\, d(Tr\,\varphi)}\quad . \tag{42.25}$$

In other words, the first Chern class can be written explicitly as an exact form on P.

Now consider the second Chern class [(41.77)]

$$C_2(\Phi) = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 [\, Tr\,\Phi\wedge Tr\,\Phi - Tr\,(\Phi\wedge\Phi)\,]\ . \tag{42.26}$$

For the curvature form Φ on P we have the Bianchi identity [c.f. (35.36)]

$$d\Phi = \varphi\wedge\Phi - \Phi\wedge\varphi\ . \tag{42.27}$$

From (42.22),

$$Tr\,\Phi\wedge Tr\,\Phi = d(Tr\,\varphi)\wedge d(Tr\,\varphi) = d(Tr\,\varphi\wedge d(Tr\,\varphi))\ . \tag{42.28}$$

To show that the term $Tr\,(\Phi\wedge\Phi)$ in $C_2(\Phi)$ is also exact on P, we perform the following calculations. First we have, using (42.20) for Φ and the Bianchi identity (42.27),

$$\begin{aligned} &d(Tr\,(\varphi\wedge\Phi)) = Tr\,(d(\varphi\wedge\Phi)) = Tr\,(d\varphi\wedge\Phi - \varphi\wedge d\Phi)\\ &= Tr\,(d\varphi\wedge\Phi) - Tr\,(\varphi\wedge d\Phi)\\ &= Tr\,((\Phi+\varphi\wedge\varphi)\wedge\Phi) - Tr\,(\varphi\wedge(\varphi\wedge\Phi - \Phi\wedge\varphi))\\ &= Tr\,(\Phi\wedge\Phi) + Tr\,(\varphi\wedge\Phi\wedge\varphi) = Tr\,(\Phi\wedge\Phi) - Tr\,(\varphi\wedge\varphi\wedge\Phi)\ , \end{aligned} \tag{42.29}$$

where in the last equality we have used the fact that

$$Tr\,(\varphi\wedge\Phi\wedge\varphi) = -Tr\,(\varphi\wedge\varphi\wedge\Phi)\ , \tag{42.30}$$

which can be demonstrated as follows:

$$\begin{aligned} &Tr\,(\varphi\wedge\Phi\wedge\varphi) = \varphi^j_i\wedge\Phi^k_j\wedge\varphi^i_k = \varphi^j_i\wedge\varphi^i_k\wedge\Phi^k_j \\ &= -\varphi^i_k\wedge\varphi^j_i\wedge\Phi^k_j = -Tr\,(\varphi\wedge\varphi\wedge\Phi)\ . \end{aligned} \tag{42.31}$$

We also have, again from (42.20) and (42.27),

$$\begin{aligned} &d[\,Tr\,(\varphi\wedge\varphi\wedge\varphi)\,] = Tr\,[\,d(\varphi\wedge\varphi\wedge\varphi)\,] = Tr\,[\,d(\varphi\wedge(d\varphi-\Phi))\,] \\ &= Tr\,[\,d(\varphi\wedge d\varphi) - d(\varphi\wedge\Phi)\,] = Tr\,[\,d\varphi\wedge d\varphi - (d\varphi\wedge\Phi - \varphi\wedge d\Phi)\,] \\ &= Tr\,[\,(\Phi+\varphi\wedge\varphi)\wedge(\Phi+\varphi\wedge\varphi) - (\Phi+\varphi\wedge\varphi)\wedge\Phi + \varphi\wedge(\varphi\wedge\Phi - \Phi\wedge\varphi)\,] \\ &= Tr\,[\,\Phi\wedge\varphi\wedge\varphi + \varphi\wedge\varphi\wedge\Phi - \varphi\wedge\Phi\wedge\varphi + \varphi\wedge\varphi\wedge\varphi\wedge\varphi\,] \\ &= 3\,Tr\,(\varphi\wedge\varphi\wedge\Phi)\ . \end{aligned} \tag{42.32}$$

In the last equality, we have used (42.30) and the facts that

$$Tr\,(\Phi\wedge\varphi\wedge\varphi) = Tr\,(\varphi\wedge\varphi\wedge\Phi)\ , \tag{42.33}$$

$$Tr\,(\varphi\wedge\varphi\wedge\varphi\wedge\varphi) = 0\ . \tag{42.34}$$

Exercise 42.1 Prove (42.33) and (42.34). Consult the derivation of (42.30) given by (42.31).

It follows from (42.29) and (42.32) that

$$\begin{aligned} &Tr\,(\Phi\wedge\Phi) = d(Tr\,(\varphi\wedge\Phi)) + Tr\,(\varphi\wedge\varphi\wedge\Phi) \\ &= d(Tr\,(\varphi\wedge\Phi)) + \frac{1}{3}\,d(Tr\,(\varphi\wedge\varphi\wedge\varphi)) = d\,[\,Tr\,(\varphi\wedge\Phi) + \frac{1}{3}\,Tr\,(\varphi\wedge\varphi\wedge\varphi)\,]\ ; \end{aligned} \tag{42.35}$$

and hence

$$\boxed{C_2(\Phi) = d\left[\frac{1}{2}\left(\frac{i}{2\pi}\right)^2\{Tr\,\varphi\wedge d(Tr\,\varphi) - CS(\varphi)\}\right]}\ , \tag{42.36}$$

where $CS(\varphi)$, known as the **Chern-Simons 3-form**, is given by

$$\boxed{CS(\varphi) \equiv Tr\,(\varphi\wedge\Phi) + \frac{1}{3}\,Tr\,(\varphi\wedge\varphi\wedge\varphi)}\ . \tag{42.37}$$

Thus, as for $C_1(\Phi)$, the second Chern class $C_2(\Phi)$ is also exact on P. Using (42.20), the Chern-Simons 3-form can also be written as

$$\boxed{CS(\varphi) = Tr\,(\varphi\wedge d\varphi - \frac{2}{3}\,\varphi\wedge\varphi\wedge\varphi)}\ . \tag{42.38}$$

There is a very simple reason why the pullback forms $C_i(\Phi)$ on P must be exact. First, by naturality of the Chern classes,

$$C_i(\Phi) = C_i((\pi')^* \, \Omega) = (\pi')^* \, (C_i(\Omega)) \; . \tag{42.39}$$

Thus $C_i(\Phi)$ is the pullback of the Chern class $C_i(\Omega)$, where the latter is defined on the frame bundle $\pi' : P \to M$. In other words $C_i(\Phi)$ is a Chern class of the square bundle $(\pi')^* \, (P)$ (with base space P), which, as we have seen, is trivial. Since $C_i(\Phi) \in H^{2i}(P)$, and all characteristic classes of trivial bundles are trivial [by Corollary 41.1], $C_i(\Phi)$ must be exact (as a $2i$-form on P). One can then always write

$$C_i(\Phi) = (\pi')^* \, (C_i(\Omega)) = d(TC_i(\varphi)) \; , \tag{42.40}$$

where $TC_i(\varphi)$ denotes the transgression of the i-th Chern class. $TC_i(\varphi)$ are also referred to as the **Chern-Simons forms**. Note that they are not necessarily characteristic classes, since $TC_i(\varphi)$ are not necessarily closed $[d(TC_i(\varphi)) = C_i(\Phi)]$.

The Chern-Simons forms can be given explicit general expressions by the following theorem.

Theorem 42.1. *Let $P_j(\Omega)$ be an ad-invariant polynomial of order j [c.f. (41.43)], where Ω is the curvature form of a $\mathcal{G}$-valued connection ω on a principal G-bundle $\pi' : P \to M$ (c.f. Defs. 37.7 and 37.11). Let $\Phi = (\pi')^* \, (\Omega)$ and set*

$$\Phi_t \equiv t\Phi + \frac{1}{2} \, (t^2 - t) \, [\, \varphi, \varphi \,]_{\mathcal{G}} \; , \tag{42.41}$$

where $[\, , \,]_{\mathcal{G}}$ is in the sense of (37.90) and $\varphi = (\pi')^* \, (\omega)$. Define the $(2j-1)$-form

$$TP_j(\, \varphi) \equiv j \int_0^1 P_j(\varphi, \underbrace{\Phi_t, \dots, \Phi_t}_{(j-1) \; times}) dt \; . \tag{42.42}$$

Then

$$d(TP_j(\varphi)) = (\pi')^* \, (P_j(\Omega)) = P_j(\Phi) \; . \tag{42.43}$$

Proof. Let $f(t) \equiv P_j(\Phi_t)$. Then $f(0) = 0$ and $f(1) = P_j(\Phi)$, so that

$$P_j(\Phi) = \int_0^1 \frac{dP_j(\Phi_t)}{dt} dt = \int_0^1 f'(t) dt \; .$$

We need to show that

$$f'(t) = j \, dP_j(\varphi, \overbrace{\Phi_t, \dots, \Phi_t}^{(j-1) \text{ times}}) \; . \tag{42.44}$$

By the definition of $f(t)$,

$$f'(t) = \frac{dP_j(\Phi_t)}{dt} = j \, P_j(\frac{d\Phi_t}{dt}, \overbrace{\Phi_t, \dots, \Phi_t}^{(j-1) \text{ times}}) \; . \tag{42.45}$$

From (42.41),

$$\frac{d\Phi_t}{dt} = \Phi + (t - 1/2)\,[\,\varphi,\,\varphi\,]_{\mathcal{G}}\ . \tag{42.46}$$

Thus

$$f'(t) = j\,P_j(\Phi, \Phi_t, \ldots, \Phi_t) + j\,(t - 1/2)\,P_j([\,\varphi, \varphi\,]_{\mathcal{G}}, \Phi_t, \ldots, \Phi_t\,)\ . \tag{42.47}$$

On the other hand, recalling the proof of (41.31),

$$\begin{aligned} j\,dP_j(\varphi, \Phi_t, \ldots, \Phi_t) &= j\,P_j(d\varphi, \Phi_t, \ldots, \Phi_t) \\ &\quad - j(j-1)\,P_j(\varphi, d\Phi_t, \underbrace{\Phi_t, \ldots, \Phi_t}_{(j-2)\text{ times}})\ . \end{aligned} \tag{42.48}$$

Since, by the structural equation (37.116) of the connection φ,

$$\Phi = d\varphi + \frac{1}{2}\,[\,\varphi,\,\varphi\,]_{\mathcal{G}}\ , \tag{42.49}$$

(42.48) implies that

$$\begin{aligned} j\,dP_j\,(\varphi, \Phi_t, \ldots, \Phi_t\,) &= j\,P_j(\Phi, \Phi_t, \ldots, \Phi_t) \\ - \frac{1}{2}\,j\,P_j(\,[\,\varphi,\,\varphi\,]_{\mathcal{G}}, \Phi_t, \ldots, \Phi_t\,) &- j(j-1)\,P_j(\varphi, d\Phi_t, \underbrace{\Phi_t, \ldots, \Phi_t}_{(j-2)\text{ times}})\ . \end{aligned} \tag{42.50}$$

Besides (37.116) we have the Bianchi identity [c.f. (37.125)]:

$$d\Phi = [\,\Phi,\,\varphi\,]_{\mathcal{G}}\ . \tag{42.51}$$

From these two equations

$$\begin{aligned} d\Phi_t &= d\left(t\Phi + \frac{1}{2}\,(t^2 - t)\,[\,\varphi,\,\varphi\,]_{\mathcal{G}}\right) = td\Phi + \frac{1}{2}\,(t^2 - t)\,d[\varphi,\,\varphi\,]_{\mathcal{G}} \\ &= t\,[\,\Phi, \varphi\,]_{\mathcal{G}} + \frac{1}{2}(t^2 - t)\,\{\,[\,d\varphi,\,\varphi\,]_{\mathcal{G}} - [\,\varphi,\,d\varphi\,]_{\mathcal{G}}\} \\ &= t\,[\,\Phi,\,\varphi\,]_{\mathcal{G}} + \frac{1}{2}\,(t^2 - t)\,\left\{\,[\,\Phi - \frac{1}{2}\,[\,\varphi, \varphi\,]_{\mathcal{G}},\,\varphi\,]_{\mathcal{G}} - [\,\varphi,\,\Phi - \frac{1}{2}\,[\,\varphi,\,\varphi\,]_{\mathcal{G}}\,]_{\mathcal{G}}\right\} \\ &= t\,[\,\Phi,\,\varphi\,]_{\mathcal{G}} + \frac{1}{2}\,(t^2 - t)\,\{\,[\,\Phi, \varphi\,]_{\mathcal{G}} - [\,\varphi,\,\Phi\,]_{\mathcal{G}}\,\} \\ &= t\,[\,\Phi, \varphi\,]_{\mathcal{G}} + (t^2 - t)\,[\,\Phi,\,\varphi\,]_{\mathcal{G}} = t^2[\,\Phi, \varphi\,]_{\mathcal{G}} = t\,[\,\Phi_t,\,\varphi\,]_{\mathcal{G}}\ , \end{aligned} \tag{42.52}$$

where in the third equality we have used (37.97) for the calculation of $d\,[\,\varphi,\,\varphi\,]_{\mathcal{G}}$, and in the fourth and the last equality we have used the result $[\,[\,\varphi,\,\varphi\,],\,\varphi\,]_{\mathcal{G}} = 0$, which in turn follows from (37.96). Using (42.52) in (42.50) we have

$$\begin{aligned} j\,dP_j\,(\varphi, \Phi_t, \ldots, \Phi_t) &= j\,P_j(\Phi, \Phi_t, \ldots, \Phi_t) - \frac{1}{2}\,jP_j(\,[\,\varphi,\,\varphi\,]_{\mathcal{G}}, \Phi_t, \ldots, \Phi_t\,) \\ &\quad - j(j-1)\,tP_j(\,\varphi,\,[\,\Phi_t,\,\varphi\,]_{\mathcal{G}}, \underbrace{\Phi_t, \ldots, \Phi_t}_{(j-2)\text{ times}})\ . \end{aligned} \tag{42.53}$$

We will next use the result (41.29), written in a slightly different form as

$$\sum_{i=1}^{j}(-1)^{d_1+\ldots+d_i}\, P_j(A_1,\ldots,[\,A_i,\,\theta\,]_{\mathcal{G}},\ldots,A_j)=0\;. \tag{42.54}$$

With $A_1=\theta=\varphi$, $A_2=\cdots=A_j=\Phi_t$, it yields

$$-P_j([\,\varphi,\,\varphi\,]_{\mathcal{G}},\Phi_t,\ldots,\Phi_t)-(j-1)P_j(\varphi,[\,\Phi_t,\,\varphi\,]_{\mathcal{G}},\Phi_t,\ldots,\Phi_t)=0\;. \tag{42.55}$$

Applying this result to the last term on the RHS of (42.53) we finally have

$$\begin{aligned} j\,dP_j(\varphi,\Phi_t,\ldots,\Phi_t) &= j\,P_j(\Phi,\Phi_t,\ldots,\Phi_t)-\frac{1}{2}\,jP_j(\,[\,\varphi,\varphi\,]_{\mathcal{G}},\Phi_t,\ldots,\Phi_t\,)\\ &\quad + jtP_j(\,[\,\varphi,\,\varphi\,]_{\mathcal{G}},\Phi_t,\ldots,\Phi_t\,)=f'(t)\;, \end{aligned} \tag{42.56}$$

where the last equality follows from (42.47) □

As an example let us consider $P_2(\Phi)=Tr\,(\Phi\wedge\Phi)$. Formula (42.42) for the transgression of $P_2(\Phi)$ gives

$$TP_2(\varphi)=2\int_0^1 P_2(\varphi,\Phi_t)dt=2\int_0^1 dt\,Tr\,(\varphi\wedge\Phi_t)\;. \tag{42.57}$$

Eq. (42.41) gives

$$\begin{aligned} \Phi_t &= t\Phi-\frac{1}{2}\,(t^2-t)\,2\varphi\wedge\varphi\\ &= t(d\varphi-\varphi\wedge\varphi)-(t^2-t)\,\varphi\wedge\varphi=td\varphi-t^2\,\varphi\wedge\varphi\;, \end{aligned} \tag{42.58}$$

where we have used (37.135). Thus

$$\begin{aligned} TP_2(\varphi) &= Tr\int_0^1 dt\,(2t\,\varphi\wedge d\varphi-2t^2\,\varphi\wedge\varphi\wedge\varphi)\\ &= Tr\,\left(\varphi\wedge d\varphi-\frac{2}{3}\,\varphi\wedge\varphi\wedge\varphi\right)=CS(\varphi)\;, \end{aligned} \tag{42.59}$$

which is the same result as (42.38).

We note the following properties of transgression forms without proof.

Theorem 42.2. *Let P_l be an $ad(G)$-invariant symmetric polynomial of degree l and Q_s be the same of degree s. Then*

$$i)\qquad P_lQ_s(\,\overbrace{\Phi,\ldots,\Phi}^{(l+s)\ times}\,)=P_l(\overbrace{\Phi,\ldots,\Phi}^{l\ times})\wedge Q_s(\overbrace{\Phi,\ldots,\Phi}^{s\ times})\;. \tag{42.60}$$

$$\begin{aligned} ii)\qquad T(P_lQ_s\,(\varphi)) &= TP_l(\varphi)\wedge Q_s(\Phi,\ldots,\Phi)+\ \textit{exact form}\\ &= TQ_s(\varphi)\wedge P_l(\Phi,\ldots,\Phi)+\ \textit{exact form}\;. \end{aligned} \tag{42.61}$$

Theorem 42.3. *Let $\omega(t)$ be a smooth one-parameter family of connections on a principal G-bundle $\pi : P \to M$ with $t \in [0,1]$. Let $\varphi(t) = \pi^*(\omega(t))$. Set $\varphi(0) = \varphi$ and*

$$\phi' \equiv \frac{d}{dt}\varphi(t)\bigg|_{t=0} . \tag{42.62}$$

If P_j is an $ad(G)$-invariant symmetric polynomial of degree j, then

$$\frac{d}{dt}(TP_j(\varphi(t)))\bigg|_{t=0} = j\,P_j(\,\varphi',\,\overbrace{\Phi,\ldots,\Phi}^{(j-1)\ times}\,) + \ exact\ form\,, \tag{42.63}$$

where $\Phi = d\varphi - \varphi\wedge\varphi$ is the curvature form corresponding to φ.

Suppose $dim(M) = m$ (where M is the base manifold of $\pi' : P \to M$). If $2j > m$, then $P_j(\Omega) = 0$ and so

$$P_j(\Phi) = P_j((\pi')^*\Omega) = (\pi')^*(P_j(\Omega)) = 0\,.$$

Thus by (42.43), $TP_j(\varphi)$ is closed, and so defines an element $(TP_j(\varphi)) \in H^{2j-1}(P)$, that is, a cohomology class in P. We have the following basic theorem concerning the transgression forms $TP_j(\varphi)$, which will be stated without proof.

Theorem 42.4. *Let $\pi' : P \to M$ be a principal G-bundle with connection ω and corresponding curvature Ω; $\varphi = (\pi')^*(\omega)$ and $\Phi = (\pi')^*(\Omega)$ be the pullback connection and curvature on P, respectively. Suppose $P_j(\Omega)$ is an ad-invariant polynomial of order j, and $dim(M) = m$. Then*

i) If $2j-1 > m$, then $TP_j(\varphi)$ is closed and $[\,TP_j(\varphi)\,] \in H^{2j-1}(P,\mathbb{R})$ is independent of the choice of the connection φ.

ii) If $2j-1 = m$, then $TP_j(\varphi)$ is closed and $[\,TP_j(\varphi)\,] \in H^m(P,\mathbb{R})$ depends on the connection φ.

For the case $2j-1 > m$, then, the transgression form $TP_j(\varphi)$ defines a characteristic class called a **secondary characteristic class**. For example, for $dim(M) = m = 3$, the Chern-Simons form $CS(\varphi) = TP_2(\varphi)$ $(j = 2)$ is closed but depends on the choice of the connection φ.

Suppose now that $G = GL(m,\mathbb{R})$ and $\pi : P \to M$ is the (principal) frame bundle of M, with a $\mathcal{GL}(m,\mathbb{R})$-valued connection φ. This connection can be restricted to a subbundle of orthonormal frames, $\pi' : P' \to M$, with structure group $O(m)$. We then have the following theorem (stated without proof).

Theorem 42.5. *Let $\pi : P \to M$ be the frame bundle on M $(dim(M) = m)$ with connection φ. Suppose φ restricts to a connection on an $O(m)$ subbundle of P, and let the curvature form on this subbundle be Φ. Let*

$$Q_j(A_1,\ldots,A_j) = \frac{1}{j!}\sum_{\sigma\in\mathcal{S}_j} Tr\,(\,A_{\sigma(1)}A_{\sigma(2)}\ldots A_{\sigma(j)}\,) \tag{42.64}$$

be the **symmetrized trace** *of the* $m \times m$ *matrices* $A_1, \ldots, A_j$. *Then*

$$\text{i)} \qquad Q_{2j+1}(\overbrace{\Phi, \ldots, \Phi}^{(2j+1)\ times}) = 0\,, \tag{42.65}$$

$$\text{ii)} \qquad TQ_{2j+1}(\varphi) \ \textit{is exact}\,. \tag{42.66}$$

i) and ii) imply that $(TQ_{2j+1}(\varphi))$ *is the trivial cohomology class on the subbundle of orthonormal frames on* M.

We will now specialize to the case where M is a Riemannian manifold with a Riemannian metric g, and ω the corresponding Levi-Civita connection on TM. Consider the frame bundle $\pi : P \to M$ and let $\varphi = \pi^*(\omega)$ be the pullback of ω on P. φ restricts to a connection (still denoted by φ) on the $O(m)$-subbundle of orthonormal frames $\pi' : P' \to M$. We say that two Riemannian metrics g and $\hat{g}$ are **conformally related** if

$$\hat{g} = e^h\, g\,, \tag{42.67}$$

where $h \in C^\infty(M)$ (a smooth function on M). The transgression forms of an ad-invariant, symmetric polynomial P_j then exhibit the following fundamental result, stated without proof.

Theorem 42.6. *Let* g *and* $\hat{g}$ *be conformally related Riemannian metrics on a Riemannian manifold* M, *and let* $\varphi, \Phi, \hat{\varphi}, \hat{\Phi}$ *denote the corresponding connection and curvature forms on the principal* $O(m)$*-bundle of orthonormal frames on* M, $\pi' : P' \to M$, *where* $dim(M) = m$. *Then, for any ad-invariant, symmetric polynomial* P_j *of* $m \times m$ *matrices,*

$$\text{i)} \qquad TP_j(\hat{\varphi}) - TP_j(\varphi) \ \textit{is exact}\,. \tag{42.68}$$

$$\text{ii)} \qquad P_j(\overbrace{\hat{\Phi}, \ldots, \hat{\Phi}}^{j\ times}) = P_j(\overbrace{\Phi, \ldots, \Phi}^{j\ times})\,, \tag{42.69}$$

iii) *If* $P_j(\Phi, \ldots, \Phi) = 0$, *then the cohomology class* $(TP_j(\varphi)) \in H^{2j-1}(P', \mathbb{R})$ *is a* ***conformal invariant***.

Note that iii) follows immediately from i), ii) and Theorem 42.1, while ii) follows immediately from i) and Theorem 42.1.

Let us consider some applications of the Chern-Simons form $CS(\varphi)$ in physics. Instead of φ we will write A, the connection symbol usually used for gauge potentials in physics. It turns out, for instance, that the Chern-Simons form $CS(A)$ is of considerable interest in a quantum field theory in 3 dimensions (Witten 1989). Consider the **Chern-Simons action** of the gauge potential A on a vector bundle $\pi : E \to M$ associated with a principal G-bundle $\pi' : P \to M$:

$$\boxed{S_{CS}(A) = \int_M Tr\,(A \wedge dA - \frac{2}{3}\, A \wedge A \wedge A)}\,, \tag{42.70}$$

where M is a compact, oriented 3-dimensional manifold without boundary. The Chern-Simons form, and hence the corresponding field theory given by (42.70), is defined without reference to a metric in M.

The Chern-Simons action has the following interesting property

Theorem 42.7. *The Chern-Simons action S_{CS} [given by (42.70)] is "almost" gauge-invariant: it is not invariant under a general gauge transformation of A, but is so under a gauge transformation $g \in G$ connected to the identity in G.*

Proof. Let $g_t \in G$, $t \in [0,1]$ be a family of gauge transformations such that $g_0 = 1$ (the identity in G) and $g_1 = g$. In other words, g is connected to the identity. Starting with a connection A, with corresponding curvature 2-form F, the gauge-transformed connection

$$A_t \equiv g_t\, d(g_t)^{-1} + g_t\, A\, (g_t)^{-1} \tag{42.71}$$

is well-defined on the total space P. We need to prove that

$$\frac{d}{dt} S_{CS}(A_t)\bigg|_{t=0} = 0\,. \tag{42.72}$$

Let

$$T \equiv \frac{d}{dt}\, g_t\bigg|_{t=0}\,. \tag{42.73}$$

Then, from

$$0 = \frac{d}{dt}\, g_t (g_t)^{-1} = \left(\frac{dg_t}{dt}\right)(g_t)^{-1} + g_t \left(\frac{d(g_t)^{-1}}{dt}\right)\,, \tag{42.74}$$

we have

$$\frac{d}{dt}\,(g_t)^{-1} = -T\,. \tag{42.75}$$

Thus

$$\begin{aligned} A' \equiv \frac{d}{dt} A_t\bigg|_{t=0} &= \frac{d}{dt}\left(g_t\, d(g_t)^{-1} + g_t\, A\,(g_t)^{-1}\right)\bigg|_{t=0} \\ = \left(TA(g_t)^{-1} - g_t A T + T\, d(g_t)^{-1} - g_t\, dT\right)\Big|_{t=0} &= [T,\, A]_{\text{gr}} - dT\,, \end{aligned} \tag{42.76}$$

where $[T,\, A]_{\text{gr}}\ (= TA - AT)$ was given by (35.65). From Theorem 42.3 we have

$$\frac{d}{dt} S_{CS}(A_t)\bigg|_{t=0} = \int_M \frac{d}{dt}\, CS(A_t)\bigg|_{t=0} = 2\int_M Tr\,(A' \wedge F) + \int_M d\theta\,, \tag{42.77}$$

where $F = dA - A \wedge A$. By Stokes' Theorem

$$\int_M d\theta = \int_{\partial M} \theta = 0\,,$$

since by assumption, $\partial M = 0$. It follows from (42.76) that

$$\begin{aligned} &\frac{d}{dt}\, S_{CS}(A_t)\bigg|_{t=0} \\ &= 2\int_M Tr\,\{\,(\,[T,\,A]_{\rm gr} - dT\,) \wedge (\,dA - A\wedge A\,)\,\} \\ &= 2\int_M Tr\,\{\,[T,\,A]_{\rm gr}\wedge dA - [T,\,A]_{\rm gr}\wedge A\wedge A + dT\wedge A\wedge A\,\}\ , \end{aligned} \tag{42.78}$$

where the term $dT \wedge dA$ does not contribute since, again by Stokes' theorem,

$$\begin{aligned} &\int_M Tr\,(dT\wedge dA) = \int_M Tr\,d(T\wedge dA) = \int_M d(Tr\,(T\wedge dA)) \\ &= \int_{\partial M} Tr\,(T\wedge dA) = 0\ . \end{aligned} \tag{42.79}$$

Next we note that

$$\int_M Tr\,(\,[T,\,A]_{\rm gr}\wedge A\wedge A) = 0\ , \tag{42.80}$$

since by the **graded cyclic property** of the trace [see (42.82) below],

$$Tr\,(TA\wedge A\wedge A) = Tr\,(AT\wedge A\wedge A)\ . \tag{42.81}$$

Note that by definition, T is a matrix of 0-forms.

Exercise 42.2 Prove the graded cyclic property of the trace, which states that

$$Tr\,(\theta\wedge\phi) = (-1)^{pq}\,Tr\,(\phi\wedge\theta)\ , \tag{42.82}$$

where θ and ϕ are matrices (of the same size) of p- and q-forms, respectively.

Eq. (42.78) then implies

$$\begin{aligned} &\frac{d}{dt}\, S_{CS}(A_t)\bigg|_{t=0} \\ &= 2\int_M Tr\,\{\,[T,\,A]_{\rm gr}\wedge dA + dT\wedge A\wedge A\,\} \\ &= 2\int_M Tr\,(\,T\wedge A\wedge dA - A\wedge T\wedge dA + dT\wedge A\wedge A\,) \\ &= 2\int_M Tr\,d(T\wedge A\wedge A) = 2\int_M d\,\{\,Tr\,(T\wedge A\wedge A)\,\} \\ &= 2\int_{\partial M} Tr\,(T\wedge A\wedge A) = 0\ , \end{aligned} \tag{42.83}$$

where the last two equalities follow from Stokes' Theorem. □

The following fact is especially relevant for the formulation of a generally **covariant quantum field theory** (one without an a priori choice of a metric on M) based on the Chern-Simons action.

Theorem 42.8. *Let A and A' be connection 1-forms on a principal G-bundle $\pi : P \to M$ related by a gauge transformation $g \in G$:*

$$A' = (dg)g^{-1} + g\,A\,g^{-1}\,. \tag{42.84}$$

Then

$$\boxed{S_{CS}(A') - S_{CS}(A) = 8\pi^2 n\,, \quad n \in \mathbb{Z}}\,, \tag{42.85}$$

where the Chern-Simons action S_{CS} is defined in (42.70).

Proof. Let $A(t)$, $t \in [\,0,1\,]$, be a 1-parameter family of connections on $\pi : P \to M$ such that $A(0) = A$ and $A(1) = A'$. Let us consider the 4-dimensional manifold $S^1 \times M$, where M is a compact 3-dimensional manifold without boundary. Defining the projection map $p : S^1 \times M \to M$ by $(t,x) \mapsto x$, $t \in S^1$, $x \in M$, we have the induced bundle $\tilde{P} = p^*(P)$ and the corresponding induced connection $\tilde{A} = p^*(A)$, whose local expression is given by $A(t)$, if S^1 is coordinatized by $t \in [\,0,1\,]$ with the two ends of the interval identified. Let $\tilde{F}$ be the corresponding curvature form. Now, according to the fact that the Chern classes $C_j(\tilde{F})$ [as defined by (41.74)] are integral cohomology classes [c.f. (41.73)], it follows from the expression for C_2 [(41.77)] that

$$\frac{1}{8\pi^2}\int_{S^1\times M} Tr\,(\tilde{F}\wedge\tilde{F}) = n\,, \quad n \in \mathbb{Z}\,. \tag{42.86}$$

On the other hand, by (42.35) and the Stokes Theorem

$$\begin{aligned}\int_{S^1\times M} Tr\,(\tilde{F}\wedge\tilde{F}) &= \int_{[\,0,1\,]\times M} d(CS(\tilde{A}))\\ &= \int_{\partial([\,0,1\,]\times M)} CS(\tilde{A}) = S_{CS}(A') - S_{CS}(A)\,.\end{aligned} \tag{42.87}$$

Eqs. (42.86) and (42.87) then imply the theorem. □

Thus the classical Chern-Simons action is not gauge-invariant. But in a quantum field theory based on the Chern-Simons action formulated in terms of **path integrals**, we can define the **vacuum expectation value** $\langle f \rangle$ of an observable f (considered as a gauge-invariant function of the connection A) as

$$\langle f \rangle = \frac{1}{Z}\int_{A/G} f(A)\,\exp\left\{\frac{ik}{4\pi}\,S_{CS}(A)\right\} DA\,, \quad k \in \mathbb{Z}\,, \tag{42.88}$$

$$Z \equiv \int_{A/G} \exp\left\{\frac{ik}{4\pi}\,S_{CS}(A)\right\} DA\,, \tag{42.89}$$

in which $\exp\left\{\frac{ik}{4\pi} S_{CS}(A)\right\}$ is clearly gauge-invariant, by the result of Theorem 42.8. In analogy to a similar quantity in statistical mechanics, Z is called the **partition function**. The path measure DA, in the space of gauge potentials (up to gauge transformations) has, in fact, not been rigorously defined. But this has not prevented mathematical physicists from working formally with expressions like (42.88) and (42.89) to obtain extremely interesting and useful results.

Recall that we have studied one physical example of the integrality of Chern classes already, namely, the quantization of the magnetic charge [c.f. (38.32)]. That result is just a consequence of the fact that the first Chern number

$$c_1 = \int_M C_1(\Omega)$$

is an integer. The **integrality condition**

$$C_j(\Omega) \in H^{2j}(M, \mathbb{Z}) \tag{42.90}$$

in particular implies that

$$c_{\frac{m}{2}} = \int_M C_{\frac{m}{2}}(\Omega)\ , \quad m = dim(M)\ , \tag{42.91}$$

is a topological invariant of bundles with even-dimensional base manifolds. When $m = 4$, these are called **instanton numbers**, and are special examples of so-called **Chern numbers**, or **topological quantum numbers** in physics. Thus the magnetic monopole is described by a $U(1)$-bundle over S^2, and the quantized magnetic charge M is given by

$$M = \left(\frac{\hbar c}{2e}\right) c_1\ , \quad c_1 \in \mathbb{Z}\ . \tag{42.92}$$

As a further example, we will consider **instantons** described by $SU(2)$-bundles over S^4. For a particular connection A with associated curvature F, the instanton number k is defined in terms of the second Chern character by [c.f. (41.83)]

$$-k \equiv \int_{S^4} Ch_2(F) = -\frac{1}{8\pi^2}\int_{S^4} Tr\,(F \wedge F)\ . \tag{42.93}$$

In fact, for instantons, $\star F = \pm F$ (where $\star$ denotes the Hodge star) [recall the discussion following (36.42)]. Thus the Yang-Mills action functional [(36.36)] can be written as

$$S_{YM} = \mp \int_{S^4} Tr\,(F \wedge F)\ , \tag{42.94}$$

where the signs $(-)$ and $(+)$ give the self-dual and anti self-dual instantons, respectively. S^4 can be considered as the one-point compactification of R^4, and

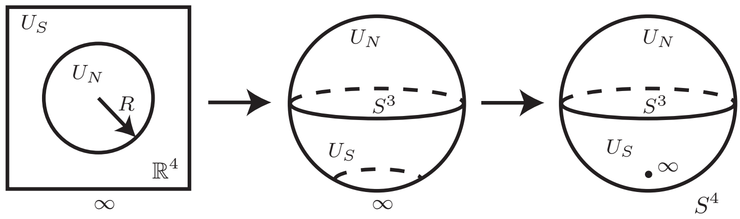

FIGURE 42.1

similar to S^2, can be covered by two coordinate patches, U_N (the "northern hemisphere") and U_S (the "southern hemisphere"), defined by

$$U_N = \{ x \in \mathbb{R}^4 \,|\, |x| \le R + \epsilon \} , \quad U_S = \{ x \in \mathbb{R}^4 \,|\, |x| \ge R - \epsilon \} , \tag{42.95}$$

where $R > 0$ (see Fig. 42.1). The region of overlap (in the limit $\epsilon \to 0$) can be contracted to $S^3 \sim SU(2)$ [recall (11.71)]. Thus the connections A can be classified by homotopy classes of transition functions

$$g_{SN} : S^3 \to SU(2) \sim S^3 ,$$

or the homotopy group

$$\pi_3(S^3) = \mathbb{Z} . \tag{42.96}$$

The integer characterizing the homotopy class is called the **degree** of the map g_{SN}. Without loss of generality we can set

$$A^{(S)} = 0 , \quad x \in U_S . \tag{42.97}$$

Thus

$$A^{(N)} = (dg_{SN})\,(g_{SN})^{-1} , \quad x \in U_N . \tag{42.98}$$

We will denote a transition function of degree n by $g_{SN}^{(n)}$. Analogous to (38.14) for the magnetic monopole, we have

$$g_{SN}^{(1)} : S^3 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SU(2) \quad \text{(the constant map)} , \tag{42.99}$$

$$g_{SN}^{(1)}(x) = (\, x^4 + i\, x^j \sigma_j) , \quad j = 1, 2, 3, \quad \text{(the identity map)} , \tag{42.100}$$

where

$$|x|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1 ,$$

and σ_i, $i = 1, 2, 3$ are the Pauli matrices given by (11.16), and

$$g_{SN}^{(n)} = (x^4 + ix^j \sigma_j)^n \; . \tag{42.101}$$

In the evaluation of the instanton numbers [(42.93)] we can use Stokes' Theorem as follows.

$$\begin{aligned} \int_{S^4} Tr\,(F \wedge F) &= \int_{U_N} Tr\,(F \wedge F) + \int_{U_S} Tr\,(F \wedge F) \\ = \int_{U_N} Tr\,(F \wedge F) &= \int_{U_N} d\{CS(A)\} = \int_{\partial\, U_N} CS(A) = \int_{S^3} CS(A) \; , \end{aligned} \tag{42.102}$$

where the local Chern-Simons form $CS(A)$ is given by [c.f. (42.37)]

$$\boxed{CS(A) = Tr\,(A \wedge F) + \frac{1}{3}\, Tr\,(A \wedge A \wedge A)} \quad . \tag{42.103}$$

Note that in the above equation, as distinct from (42.37), the connection A and curvature F are local forms on S^4 (the base manifold), and $Tr\,(F \wedge F)$, regarded as a 4-form on S^4, is only locally exact. Since in U_S, $A^{(S)} = F^{(S)} = 0$, which implies $F^{(N)} = g_{SN}\, F^{(S)}\, g_{SN}^{-1} = 0$, Eqs. (42.102) and (42.103) imply that

$$\begin{aligned} k = \frac{1}{8\pi^2} \int_{S^4} Tr\,(F \wedge F) = \frac{1}{24\pi^2} &\int_{S^3} Tr\,(\, A^{(N)} \wedge A^{(N)} \wedge A^{(N)}\,) \\ = \frac{1}{24\pi^2} &\int_{S^3} Tr\,(\, dg\, g^{-1} \wedge dg\, g^{-1} \wedge dg\, g^{-1}\,) \; , \end{aligned} \tag{42.104}$$

where we have written g for g_{SN}. We will show that this integral yields precisely the degree of the map $g : S^3 \to SU(2)$ (where g is understood to be $g_{SN}^{(n)}$). Consider the 3-form

$$\alpha = \frac{1}{24\pi^2} Tr\,(\, dg\, g^{-1} \wedge dg\, g^{-1} \wedge dg\, g^{-1}\,) \tag{42.105}$$

on S^3, where both g and dg are expressed in terms of the local coordinates of S^3. It is equal to the pullback of some 3-form β on $SU(2)$:

$$\alpha = g^*\, \beta \; . \tag{42.106}$$

Observe that α is closed on S^3, since $d\alpha$ is a 4-form and S^3 is 3-dimensional. By the integrality of Chern classes, α thus determines a cohomology class $[\,\alpha\,] \in H^3(S^3, \mathbb{Z}) = \mathbb{Z}$. Consider a map $h : S^3 \to SU(2)$ homotopic to g ($h \sim g$). Then we have, by Theorem 38.1

$$[\, g^*\, \beta\,] = [\, h^*\, \beta\,] \; . \tag{42.107}$$

This implies that

$$g^* \beta = h^* \beta + d\phi \; , \tag{42.108}$$

where ϕ is some 2-form on S^3. By Stokes' Theorem,

$$\int_{S^3} \alpha = \int_{S^3} g^*\beta = \int_{S^3} h^*\beta \, . \tag{42.109}$$

Thus the instanton number only depends on the homotopy class of the map $g : S^3 \to SU(2)$ (which is given by an element of $\pi_3(S^3)$), or the degree of the map g. Now suppose $g = g_{SN}^{(1)}$ (degree 1) and $h = g_{SN}^{(n)}$ (degree n). Let $\alpha^{(1)} = g^*\beta = (g_{SN}^{(1)})^*\beta$ and $\alpha^{(n)} = h^*\beta = (g_{SN}^{(n)})^*\beta$. Since $H^3(S^3, \mathbb{Z}) = \mathbb{Z}$, we see that $[\,\alpha^{(1)}\,] = 1$ and $[\,\alpha^{(n)}\,] = n$ (as cohomology classes). Thus

$$[\,\alpha^{(n)} - n\alpha^{(1)}\,] = [\,\alpha^{(n)}\,] - n[\,\alpha^{(1)}\,] = 0 \in H^3(S^3, \mathbb{Z}) \, . \tag{42.110}$$

In other words, $\alpha^{(n)} - n\alpha^{(1)}$ is the trivial class, and so must be exact. We then have

$$\alpha^{(n)} - n\alpha^{(1)} = d\eta \, , \tag{42.111}$$

for some 2-form η on S^3. Integration over S^3 gives

$$\int_{S^3} \alpha^{(n)} = n \int_{S^3} \alpha^{(1)} \, . \tag{42.112}$$

We will calculate the integral on the RHS,

$$\int_{S^3} \alpha^{(1)} = \frac{1}{24\pi^2} \int_{S^3} Tr\,(\, dg_{SN}^{(1)}(g_{SN}^{(1)})^{-1} \wedge dg_{SN}^{(1)}(g_{SN}^{(1)})^{-1} \wedge dg_{SN}^{(1)}(g_{SN}^{(1)})^{-1}\,) \, , \tag{42.113}$$

explicitly. Using (42.101) we have

$$(g_{SN}^{(1)})^{-1} = x^4 - ix^j\sigma_j \, . \tag{42.114}$$

Exercise 42.3 Write $g_{SN}^{(1)}$ as a 2×2 matrix [$\in SU(2)$]. Then use the explicit expressions for the Pauli matrices σ_j [given by (11.6)] to verify (42.114).

So

$$(dg_{SN}^{(1)})(g_{SN}^{(1)})^{-1} = (dx^4 + i(dx^j)\sigma_j)(x^4 - ix^j\sigma_j) \, . \tag{42.115}$$

The value of the integral (42.113) is not changed if we push the "equator" ($\sim S^3$) in Fig. 42.1 up the "northern hemisphere" towards the north pole (see Fig. 42.2). In each of these retracted boundaries, $dx^4 = 0$. In the limit of vanishing radius of the S^3 boundary of the northern coordinate patch, this boundary approaches the north pole, at which point $x^4 = 1$, $x^1 = x^2 = x^3 = 0$. We then have, at the north pole,

$$(dg_{SN}^{(1)})(g_{SN}^{(1)})^{-1} = i\sigma_j \, dx^j \, , \tag{42.116}$$

and the integrand of (42.113) becomes

$$Tr\,(\, A^{(N)} \wedge A^{(N)} \wedge A^{(N)}\,) = i^3\, Tr\,(\sigma_j\sigma_k\sigma_l)\, dx^j \wedge dx^k \wedge dx^l \, . \tag{42.117}$$

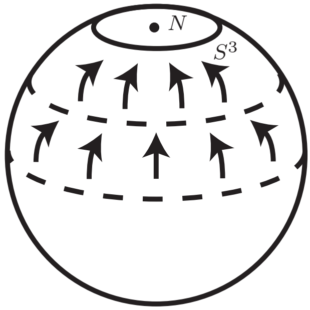

FIGURE 42.2

Using the properties of the σ-matrices (c.f. Exercise 11.5), we have

$$\begin{aligned} &Tr\,(\sigma_j\sigma_k\sigma_l) = Tr\,(\,\sigma_j(\delta_{kl} + i\varepsilon_{kl}{}^m\sigma_m)\,) \\ &= i\varepsilon_{kl}{}^m\,Tr\,(\sigma_j\sigma_m) = i\varepsilon_{kl}{}^m\,Tr\,(\,\delta_{jm} + i\varepsilon_{jm}{}^n\sigma_n\,) \\ &= i\varepsilon_{kl}{}^m\delta_{jm}\,Tr\,(e) = 2i\varepsilon_{klj}\ . \end{aligned} \tag{42.118}$$

Hence

$$Tr\,(A^{(N)}\wedge A^{(N)}\wedge A^{(N)}) = 2\varepsilon_{klj}\,dx^k\wedge dx^l\wedge dx^j = 12\,dx^1\wedge dx^2\wedge dx^3\ , \tag{42.119}$$

where $dx^1\wedge dx^2\wedge dx^3$ is the volume element of S^3. Since the integral in (42.113) cannot depend on the radius of the boundary S^3 [compare the present situation with that of the magnetic monopole given by (38.8)], we can use the volume of the unit S^3, that is,

$$\int_{S^3} dx^1\wedge dx^2\wedge dx^3 = 2\pi^2\ , \tag{42.120}$$

and get, finally,

$$\int_{S^3}\alpha^{(1)} = \frac{1}{24\pi^2}\int_{S^3} Tr\,(A^{(N)}\wedge A^{(N)}\wedge A^{(N)}\,) = \frac{12(2\pi^2)}{24\pi^2} = 1\ . \tag{42.121}$$

This result, together with (42.112), show that the instanton number $k^{(n)}$, corresponding to a gauge potential $A^{(n)}$ characterized by $n\in\pi_3(S^3)$ [the degree of the map $g^{(n)}: S^3\to SU(2)$], is just given by

$$k^{(n)} = n\ . \tag{42.122}$$

In conclusion we mention that a quantum field theory with a Chern-Simons action term, in addition to couplings between fermion fields and a gauge potential describing a magnetic flux tube, has been found to be very useful in describing the physics of the **fractional quantum Hall effect**, and the associated phenomenon of **fractional statistics** (see, for example, E. Fradkin

1991). The geometrical setup, analogous to the case of the integral quantum Hall effect, is a $U(1)$-bundle whose base space is a compact $(2+1)$-dimensional space M (2 spatial and 1 time) without boundary. Since the gauge group $U(1)$ is abelian, the Chern-Simons action is

$$S_{CS} = \theta \int_M Tr\,(A \wedge dA)\;, \tag{42.123}$$

where θ is a "strength" constant. On another front, the study of a topological quantum field theory based on the Chern-Simons action has revealed deep connections between quantum field theory on the one hand, and the topology of three-dimensional spaces on the other, through certain invariants of knot theory (E. Witten 1989, S. Hu, 2001). In short, the study of topological invariants of fiber bundles through the Chern classes and their transgression forms via Chern-Simons theory has brought vast areas of physics and mathematics together in a stunning fashion.