



WHAT IS . . .

an Oka Manifold?

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The prototypical complex manifold is the complex plane \mathbb{C} . In three cases out of four we find something interesting by considering the class of complex manifolds X with “many” or “few” holomorphic maps $X \rightarrow \mathbb{C}$ or $\mathbb{C} \rightarrow X$. The trick, of course, is to come up with a fruitful interpretation of the words “many” and “few”.

As undergraduates, most of us take a course in complex analysis on domains in \mathbb{C} . Many of the theorems proved in such a course extend to a class of manifolds called *Stein manifolds*. Stein manifolds play a fundamental role in higher-dimensional complex analysis and complex geometry, similar to affine varieties in algebraic geometry.

One of the many equivalent definitions of a Stein manifold X says, roughly speaking, that there are many holomorphic maps $X \rightarrow \mathbb{C}$, enough in fact to embed X as a closed complex submanifold of \mathbb{C}^m for some m . Another is the famous Theorem B of H. Cartan that for every coherent analytic sheaf \mathcal{F} on X , the cohomology groups $H^k(X, \mathcal{F})$ vanish for all $k \geq 1$. A third is a convexity property: there is a proper smooth function $X \rightarrow [0, \infty)$ which is strictly plurisubharmonic. Plurisubharmonicity is ordinary convexity weakened just enough to make it biholomorphically invariant. The equivalence of any two of these definitions is a deep theorem.

While it is nontrivial to interpret the word “many”, the word “few” has a straightforward interpretation as “no nonconstant”. A complex manifold X is *Brody hyperbolic* if every holomorphic map $\mathbb{C} \rightarrow X$ is constant. It turns out that the notion of Kobayashi hyperbolicity, equivalent to Brody hyperbolicity for compact manifolds but stronger

in general, is more important. A complex manifold X is *Kobayashi hyperbolic* if there is a metric (a nondegenerate distance function) d on X such that $d(f(z), f(w)) \leq \delta(z, w)$ for all holomorphic maps f from the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to X , and all $z, w \in \mathbb{D}$. Here δ denotes the Poincaré distance on \mathbb{D} . Picard’s little theorem says that the twice-punctured plane $\mathbb{C} \setminus \{0, 1\}$ is Brody hyperbolic; it is in fact Kobayashi hyperbolic.

Hyperbolicity problems in higher-dimensional complex geometry have been intensively studied in recent years. Many deep problems remain unsolved, some to do with a mysterious connection with arithmetic. S. Lang conjectured that a smooth complex projective variety defined over a number field K is Kobayashi hyperbolic if and only if it has only finitely many rational points over each finite extension of K . In the one-dimensional case, this is a celebrated theorem of G. Faltings.

It is only recently that a good notion of a complex manifold X having “many” holomorphic maps $\mathbb{C} \rightarrow X$ has emerged. The new notion has its origins in a seminal paper of M. Gromov, the 2009 Abel laureate, published in 1989 [2]. Gromov’s ideas and results have been developed further over the past ten years, primarily by F. Forstnerič, partly in joint work with J. Prezelj. Forstnerič has proved the equivalence of over a dozen properties, saying, in one way or another, that a complex manifold is the target of many holomorphic maps from \mathbb{C} [1]. He has named such manifolds *Oka manifolds*, after K. Oka, a pioneer in several complex variables. In the remainder of this article, we will motivate the definition of an Oka manifold, sketch what is known about them, and mention two major applications of the ambient theory.

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(What about the fourth class, of complex manifolds with “few” holomorphic maps to \mathbb{C} ? Even if we interpret “few” as “no nonconstant”, this class seems too big to be of interest. It contains all compact manifolds and a whole lot more.)

Runge and Weierstrass. The story begins with two well-known theorems of nineteenth-century complex analysis concerning a domain Ω in \mathbb{C} . The *Runge approximation theorem* says that if K is a compact subset of Ω with no holes in Ω , then every holomorphic map $K \rightarrow \mathbb{C}$ can be approximated, uniformly on K , by holomorphic maps $\Omega \rightarrow \mathbb{C}$. (By a holomorphic map $K \rightarrow \mathbb{C}$ we mean a holomorphic function on some open neighborhood of K .) The *Weierstrass theorem* says that if T is a discrete subset of Ω , then every map $T \rightarrow \mathbb{C}$ extends to a holomorphic map $\Omega \rightarrow \mathbb{C}$.

In the formative years of modern complex analysis, in the mid-twentieth century, these theorems were extended to higher dimensions, generalizing Ω to a Stein manifold S . The *Oka-Weil approximation theorem* replaces the topological condition that K have no holes in S with the subtle, nontopological condition that K be *holomorphically convex* in S . This means that for every $x \in S \setminus K$, there is a holomorphic function f on S with $|f(x)| > \sup_K |f|$. The *Cartan extension theorem*, on the other hand, generalises T to a closed complex subvariety of S and says that every holomorphic map $T \rightarrow \mathbb{C}$ extends to a holomorphic map $S \rightarrow \mathbb{C}$.

We usually consider these theorems as results about Stein manifolds, and of course they are, but we can also view them as expressing properties of the target \mathbb{C} . We can then formulate them for a general target. To avoid topological obstructions, which are not relevant here, we restrict ourselves to very special S , K , and T .

CAP and CIP. A complex manifold X satisfies the *convex approximation property* (CAP) if, whenever K is a convex compact subset of \mathbb{C}^m for some m , every holomorphic map $K \rightarrow X$ can be approximated, uniformly on K , by holomorphic maps $\mathbb{C}^m \rightarrow X$. A complex manifold X satisfies the *convex interpolation property* (CIP) if, whenever T is a contractible subvariety of \mathbb{C}^m for some m , every holomorphic map $T \rightarrow X$ extends to a holomorphic map $\mathbb{C}^m \rightarrow X$.

It is rather easy to see that CIP implies CAP. (This is not to say that the Cartan extension theorem implies the Oka-Weil approximation theorem: the proof that CIP implies CAP uses the Oka-Weil theorem.) Forstnerič’s work contains a difficult, roundabout proof of the converse; no simple proof is known.

We define a complex manifold to be *Oka* if it satisfies the equivalent properties CAP and CIP.

Oka Properties. There are more than a dozen other so-called *Oka properties* that are nontrivially equivalent to CAP and CIP. If S is a Stein manifold and X is an Oka manifold, then every continuous

map $f : S \rightarrow X$ can be deformed to a holomorphic map. If f is already holomorphic on a subvariety T of S , then the restriction $f|_T$ may be kept fixed during the deformation. If f is already holomorphic on a holomorphically convex compact subset K of S , then the restriction $f|_K$ may be kept arbitrarily close to being fixed during the deformation. All this can be done parametrically. If we have a family of maps f , depending continuously on a parameter in a compact subset P of \mathbb{R}^k , then the maps can be deformed with continuous dependence on the parameter. If the maps parameterized by a compact subset of P are already holomorphic on S , then they may be kept fixed during the deformation.

It follows that the inclusion $\mathcal{O}(S, X) \hookrightarrow C(S, X)$ is a weak homotopy equivalence. Here, the spaces $\mathcal{O}(S, X)$ of holomorphic maps and $C(S, X)$ of continuous maps $S \rightarrow X$ are endowed with the compact-open topology.

Examples. The “classical” examples of Oka manifolds, by renowned work of H. Grauert from around 1960, are complex Lie groups and their homogeneous spaces. Among other examples are the complement in \mathbb{C}^n of an algebraic or a tame analytic subvariety of codimension at least 2, the complement in complex projective space of a subvariety of codimension at least 2, Hopf manifolds, Hirzebruch surfaces, and the complement of a finite set in a complex torus of dimension at least 2. A Riemann surface is Oka if and only if it is not hyperbolic. Our understanding of the geography of Oka manifolds is poor. For example, it is an open problem to determine which compact complex surfaces are Oka.

Gromov’s Oka Principle. The most important sufficient condition for the Oka property to hold is ellipticity, introduced by Gromov in [2]. It is yet another way to say that a complex manifold X is the target of many holomorphic maps from \mathbb{C} . More precisely, X is *elliptic* if there is a holomorphic map $s : E \rightarrow X$, called a *dominating spray*, defined on the total space of a holomorphic vector bundle E over X , such that $s(0_x) = x$ and $s|_{E_x} \rightarrow X$ is a submersion at 0_x for all $x \in X$. The theorem that ellipticity implies the Oka property is one version of Gromov’s Oka principle.

A Stein manifold is elliptic if and only if it is Oka. There are no known examples of Oka manifolds that are not elliptic. So why focus on the Oka property rather than ellipticity? One reason is that the Oka property has good functorial properties that we cannot at present prove or disprove for ellipticity.

Model categories. There is abstract homotopy theory lurking in the background. The author has shown that the category of complex manifolds can be embedded into a model category in the sense of D. Quillen (roughly speaking, a category in which one can do homotopy theory) in such a way that a



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manifold is cofibrant if and only if it is Stein, and fibrant if and only if it is Oka.

Applications. The fact that the complement in \mathbb{C}^n , $n \geq 2$, of an algebraic subvariety of codimension at least 2 is Oka is a crucial ingredient in the proof of *Forster's conjecture* by Y. Eliashberg and Gromov, and by J. Schürmann. For each $n \geq 2$, Forster's conjecture identifies the smallest $N(n) = n + [n/2] + 1$ such that every n -dimensional Stein manifold embeds into $\mathbb{C}^{N(n)}$.

B. Ivarsson and F. Kutzschebauch have used Gromov's Oka principle, as developed by Forstnerič, to solve the *holomorphic Vaserstein problem* posed by Gromov [3]. They show that the inclusion of the ring of holomorphic functions on a contractible Stein manifold into the ring of continuous functions does not induce an isomorphism of K_1 -groups, whereas by Grauert's Oka principle it does induce an isomorphism of K_0 -groups. Here, amusingly, Gromov's Oka principle reveals a limitation of a more general Oka principle.

References

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