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EQUIVARIANT BUNDLES WITH ABELIAN STRUCTURAL GROUP

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Let G and A be compact Lie groups and recall that a principal (G,A)-bundle is a principal A-bundle $p: D \to X$ such that X and D are G-spaces, p is a G-map, and the actions of G and A on D commute. For a G-space X of the homotopy type of a G-CW complex, define $\mathcal{B}(G,A)(X)$ to be the set of equivalence classes of principal (G,A)-bundles over X. For a space Y of the homotopy type of a CW-complex, define $\mathcal{B}(A)(Y)$ to be the set of equivalence classes of principal A-bundles over Y. Let $X_G = EG \times_G X$, where EG is a contractible and G-free G-CW complex. Define a natural transformation

$$\Phi: \mathcal{B}(G,A)(X) \longrightarrow \mathcal{B}(A)(X_G)$$

by sending a (G,A)-bundle $p: D \to X$ to the A-bundle $p_G: D_G \to X_G$. We shall give an elementary proof of the following result.

<u>Theorem A.</u> If A is Abelian, then Φ is an isomorphism.

A choice of basepoint in EG determines a natural injection $\iota: X \to X_G$ and thus a natural transformation

$$\iota^*: \mathcal{B}(A)(X_G) \to \mathcal{B}(A)(X).$$

If $\pi: EG \times X \to X_G$ is the quotient map and $\varepsilon: EG \times X \to X$ is the projection, then $\iota \circ \varepsilon \simeq \pi$ and thus ι^* agrees with the composite

$$\mathcal{B}(A)(X_G) \xrightarrow{\pi^*} \mathcal{B}(A)(EG \times X) \xrightarrow{(\varepsilon^*)^{-1}} \mathcal{B}(A)(X).$$

The composite $\iota^*\Phi\colon \mathcal{B}(G,A)(X) \to \mathcal{B}(A)(X)$ coincides with the forgetful transformation Ψ from (G,A)-bundles to A-bundles. Its image consists of those A-bundles over X which admit a structure of (G,A)-bundle, that is, for which the action of G on X lifts appropriately to the total space.

<u>Corollary B.</u> If A is Abelian, then the image of ι^* is the set of A-bundles over X which admit a structure of (G,A)-bundle.

When A is a torus and X is connected and locally finite, this is the main theorem of Hattori and Yoshida [1,1.1].

Since the product and inverse maps of an Abelian group are homomorphisms, they induce natural internal operations which make $\mathcal{B}(G,A)(X)$ and $\mathcal{B}(A)(Y)$ Abelian groups. When $A = S^1$, the product may be viewed as the tensor product of complex line bundles and these are known as Picard groups. Clearly Φ and ι^* are homomorphisms.

<u>Corollary C.</u> If A is Abelian, then the (G,A)-bundle structures (if any) on a given A-bundle over X are in bijective correspondence with the elements of the kernel of ι^*.

Again, when A is a torus and X is connected and locally finite, essentially this enumeration was given by Hattori and Yoshida [1,4.1].

Of course, ι is the inclusion of a fibre in the natural bundle $\gamma\colon X_G \to BG$ and we can use the Serre spectral sequence of γ to compute ι^*. We assume that X is connected throughout the following discussion.

Since A is isomorphic to the product of a torus T^n and a finite Abelian group F, each (G,A)-bundle decomposes uniquely into the Whitney sum of a (G,T^n)-bundle and a (G,F)-bundle. Thus we can discuss these cases separately.

Of course, we have

$$\mathcal{B}(F)(X) = [X,BF] = H^1(X;F),$$

an F-bundle ξ being given by an F-characteristic class $f_1(\xi)$. An immediate calculation gives that the bottom row is exact in the commutative diagram

$$\begin{array}{ccccccccc} & & \mathcal{B}(G,F)(pt) & \xrightarrow{\varepsilon^*} & \mathcal{B}(G,F)(X) & \xrightarrow{\Psi} & \mathcal{B}(F)(X) & & \\ & & \Phi\downarrow\cong & & \cong\downarrow\Phi & & \cup & & \\ 0 & \longrightarrow & H^1(BG;F) & \xrightarrow{\gamma^*} & H^1(X_G;F) & \xrightarrow{\iota^*} & H^1(X;F)^G & \xrightarrow{d_2} & H^2(BG;F). \end{array}$$

Thus ξ lifts to a (G,F)-bundle if and only if $f_1(\xi)$ is G-invariant and annihilated by d_2, and there is then one lift for each element of $H^1(BG;F)$.

In the torus case, we have

$$\mathcal{B}(T^n)(X) = [X,BT^n] = H^2(X;Z^n),$$

a T^n-bundle ξ being given by a Z^n-characteristic class $c_1(\xi)$. We consider $E_2^{p,q} = H^p(BG;H^q(X;Z^n))$. The corollaries concern

$$\iota^*: H^2(X_G;Z^n) \to E_\infty^{0,2} \subset E_2^{0,2} = H^2(X;Z^n)^G \subset H^2(X;Z^n),$$

where $E_\infty^{0,2} = \mathrm{Ker}(d_3:E_3^{0,2} \to E_3^{3,0}) \subset \mathrm{Ker}(d_2:E_2^{0,2} \to E_2^{2,1})$, and we have the short exact sequence

$$(\alpha) \qquad 0 \longrightarrow E_\infty^{2,0} \longrightarrow \mathrm{Ker}\ \iota^* \longrightarrow E_\infty^{1,1} \longrightarrow 0,$$

where $E_\infty^{2,0} = \mathrm{Coker}(d_2:E_2^{0,1} \to E_2^{2,0})$ and $E_\infty^{1,1} = \mathrm{Ker}(d_2:E_2^{1,1} \to E_2^{3,0})$. We conclude that <u>$\xi$ lifts to a (G,T^n)-bundle if and only if $c_1(\xi)$ is G-invariant and killed by d_2 and d_3, and the exact sequence (α) then determines the number of liftings.</u> For example, if G is simply connected, then BG is 3-connected and every ξ lifts uniquely, this being a result of Stewart [8]. Now assume that $H^1(X;Z) = 0$. Then the bottom row is exact in the commutative diagram

$$\begin{array}{ccccccccc}
 & & \mathcal{B}(G,T^n)(pt) & \xrightarrow{\varepsilon^*} & \mathcal{B}(G,T^n)(X) & \xrightarrow{\Psi} & \mathcal{B}(T^n)(X) & & \\
 & & \Phi\big\downarrow\cong & & \cong\big\downarrow\Phi & & \bigcup & & \\
0 & \longrightarrow & H^1(BG;Z^n) & \xrightarrow{\gamma^*} & H^2(X_G;Z^n) & \xrightarrow{\iota^*} & H^2(X;Z^n)^G & \xrightarrow{d_3} & H^3(BG;Z^n).
\end{array}$$

Note that $H^3(BG;Z^n) = 0$ if G is Abelian and $H^2(X;Z^n)^G = H^2(X;Z^n)$ if G is connected. Thus every ξ lifts uniquely if G is a torus, this being a result of Su [9]. If $n = 1$, the top row is the exact sequence of Picard groups discussed by Liulevicius [3,Thm 2].

To prove Theorem A, we first interpret Φ on the classifying space level; A need not be Abelian for this part. There is a classifying G-space $B(G,A)$ such that

$$\mathcal{B}(G,A)(X) = [X,B(G,A)]_G,$$

where $[X,X']_G$ denotes the set of homotopy classes of G-maps $X \to X'$. We may take $B(G,A)$ to be a G-CW complex. Of course, we also have

$$\mathcal{B}(A)(Y) = [Y,BA] = [Y,BA]_G,$$

where Y and BA are regarded as G-trivial G-spaces. As a nonequivariant space, $B(G,A)$ is itself a classifying space for A-bundles. Moreover, there is a map $\zeta\colon BA \to B(G,A)$ which takes values in $B(G,A)^G$ (hence may be regarded as a G-map) and is a nonequivariant homotopy equivalence. On the level of represented functors on spaces, ζ corresponds to the transformation which sends an A-bundle to the same A-bundle regarded as a G-trivial (G,A)-bundle. The following diagram of functors clearly commutes.

$$\text{(I)}\qquad \begin{array}{ccc}
\mathcal{B}(G,A)(X) & \xrightarrow{\Phi} & \mathcal{B}(A)(X_G) \\
\big\downarrow \varepsilon^* & & \big\downarrow \zeta_* \\
\mathcal{B}(G,A)(EG \times X) & \xleftarrow{\pi^*} & \mathcal{B}(G,A)(X_G)
\end{array}$$

We also have the obvious commutative diagram

$$\text{(II)}\qquad \begin{array}{ccccc} [EG \times X, BA]_G & \xleftarrow{\pi^*} & [X_G, BA]_G & = & [X_G, BA] \\ \Big\downarrow{\scriptstyle \zeta_*} & & \Big\downarrow{\scriptstyle \zeta_*} & & \\ [EG \times X, B(G,A)]_G & \xleftarrow{\pi^*} & [X_G, B(G,A)]_G & = & [X_G, B(G,A)^G] \end{array} .$$

Here the upper map π^* is clearly a bijection and will be regarded as an identification. We shall see in a moment that the left map ζ_* is also a bijection. This implies that $\pi^*\zeta_*$ is a bijection in both diagrams and that Φ may be regarded as the composite

$$[X,B(G,A)]_G \xrightarrow{\varepsilon^*} [EG \times X, B(G,A)]_G \xrightarrow{\zeta_*^{-1}} [EG \times X, BA]_G.$$

For G-spaces X and X', let $M(X,X')$ denote the function G-space of continuous maps $X \to X'$, with G acting by conjugation. Then ε^* and ζ_* are obtained by application of the functor $[X,?]_G$ to the G-maps

$$(*)\qquad B(G,A) = M(pt, B(G,A)) \xrightarrow{\varepsilon^*} M(EG, B(G,A)) \xleftarrow{\zeta_*} M(EG, BA).$$

Recall that a G-map $f\colon D \to E$ is said to be a weak G-equivalence if its fixed point map $f^H\colon D^H \to E^H$ is an ordinary weak equivalence for each closed subgroup H of G. By the G-Whitehead theorem [5,10],

$$f_*\colon [X,D]_G \to [X,E]_G$$

is then a bijection for any G-space X of the homotopy type of a G-CW complex. Since we have restricted ourselves to such X, we may as well regard classifying G-spaces as defined only up to weak G-homotopy type. The point is that such function G-spaces as $M(EG,BA)$ will generally fail to have the homotopy types of G-CW complexes. Our assertion above that ζ_* is a bijection was proven by obstruction theory in [2,1.4], but we give the following simple argument to illustrate the convenience of using function G-spaces in this context.

Lemma 1 ζ_*: M(EG,BA) $\to$ M(EG,B(G,A)) is a weak G-equivalence.

Proof: Via $f \longleftrightarrow \sigma$ if $\sigma(x) = (x,f(x))$, $M(EG,B(G,A))^H$ may be identified with the space of sections of the natural fibration

$$EG \times_H B(G,A) \longrightarrow EG/H = BH.$$

Similarly, $M(EG,BA)^H = M(BH,BA)$ is the space of sections of

$$EG \times_H BA = BH \times BA \longrightarrow BH.$$

Since $1 \times \zeta$: EG $\times$ BA $\to$ EG $\times$ B(G,A) is a G-map and a nonequivariant homotopy equivalence between free G-spaces of the homotopy type of G-CW complexes, it is a G-homotopy equivalence by the G-Whitehead theorem. Therefore $1 \times_H \zeta$ is a homotopy equivalence over BH and thus a fibre homotopy equivalence (by a standard elementary argument). The induced homotopy equivalence between the respective spaces of sections coincides with the fixed point map $(\zeta_*)^H$.

Henceforward, we assume that A is Abelian. It is clear from the discussion above that Theorem A is an immediate consequence of the following result, which implies that (*) displays a weak G-equivalence between B(G,A) and M(EG,BA).

Theorem 2. ε^*: B(G,A) $\to$ M(EG,B(G,A)) is a weak G-equivalence when A is Abelian.

For the proof, we note first that ε^* and ζ_* in (*) are Hopf G-maps between Hopf G-spaces. Indeed, our G-spaces have equivariant sums which make them Abelian topological G-groups up to homotopy. This is clear for M(EG,BA), which inherits a structure of Abelian topological G-group from the structure of Abelian topological group on BA. For B(G,A) and M(EG,B(G,A)), it follows from the fact that, up to G-homotopy, B(G,A) is a product-preserving functor of A. The zero of $B(G,A)$ is the image of the point $B(G,\{0\})$. We shall use the following triviality.

Lemma 3. Let Y be a homotopy associative and commutative Hopf space such that $\pi_0 Y$ is a group and let Y_0 be the basepoint component of Y. Then Y is naturally equivalent as a Hopf space to $Y_0 \times \pi_0 Y$.

Proof: Choose a point a in each component Y_a, writing a^{-1} for the chosen point in the inverse component. Define $\alpha: Y \to Y_0 \times \pi_0 Y$ by $\alpha(y) = (a^{-1} \cdot y, Y_a)$ for $y \in Y_a$ and define $\beta: Y_0 \times \pi_0 Y \to Y$ by $\beta(z, Y_a) = a \cdot z$ for $z \in Y_o$. Homotopy associativity ensures that α and β are inverse equivalences; homotopy commutativity ensures that they are Hopf maps.

Thus to prove Theorem 2 it suffices to show that $(\varepsilon^*)^H$ restricts to an equivalence on basepoint components and induces an isomorphism on π_0 for each $H \subset G$.

The basepoint component of $B(G,A)^H$ classifies H-trivial (H,A)-bundles and is thus a copy of BA. Indeed, ζ may be regarded as the inclusion of the basepoint component in $B(G,A)^H$ for any H. In view of Lemma 1 and the obvious commutative diagram

$$\begin{array}{ccc} BA & \xrightarrow{\varepsilon^*} & M(BH,BA) = M(EG,BA)^H \\ \zeta\downarrow & & \downarrow(\zeta_*)^H \\ B(G,A)^H & \xrightarrow{(\varepsilon^*)^H} & M(EG,B(G,A))^H \end{array} ,$$

$(\varepsilon^*)^H$ will be a weak equivalence on basepoint components provided that ε^* is a weak equivalence from BA to the basepoint component $M_0(BH,BA)$ of $M(BH,BA)$, the basepoint being the trivial map. Now A has the form $F \times T^n$, where F is finite, and we have the commutative diagram

$$\begin{array}{ccc} \pi_q BA & = & H^1(S^q;F) \oplus H^2(S^q;Z^n) \\ \downarrow \varepsilon^* & & \downarrow \varepsilon^* \oplus \varepsilon^* \\ \pi_q M_0(BH,BA) & = & H^1(\Sigma^q BH^+;F) \oplus H^2(\Sigma^q BH^+;Z^n), \end{array}$$

where BH^+ is the union of BH and a disjoint basepoint. If $q = 1$, the first summand ε^* is clearly an isomorphism and the second summands are zero since $H^1(BH;Z) = 0$. If $q = 2$, the first summands are clearly zero and the second summand ε^* is clearly an isomorphism. If $q \geq 3$, all groups are zero.

It remains to consider $(\varepsilon^*)^H$ on π_0. For any G-space X, $\pi_0(X^H) = [G/H,X]_G$. By Lemma 1 and diagrams (I) and (II), $(\varepsilon^*)^H$ will induce an isomorphism on π_0 provided that

$$\Phi\colon \mathcal{B}(G,A)(G/H) \longrightarrow \mathcal{B}(A)(BH)$$

is an isomorphism. We claim that Φ here may be identified with the homomorphism

$$B\colon \mathrm{Hom}(H,A) \longrightarrow [BH,BA]$$

given by the classifying space functor, where $\mathrm{Hom}(H,A)$ denotes the Abelian group of continuous homomorphisms $\rho\colon H \to A$. Indeed, we obtain an isomorphism from $\mathrm{Hom}(H,A)$ to $\mathcal{B}(G,A)(G/H)$ by sending ρ to the natural (G,A)-bundle $\xi_\rho\colon G \times_H A_\rho \to G/H$, where A_ρ denotes A regarded as an H-space via ρ, and Φ carries ξ_ρ to the natural A-bundle $EG \times_H A_\rho \to BH$. It is classical bundle theory that the latter is classified by $B\rho$. Thus the following result completes the proof of Theorem 2.

<u>Proposition 4.</u> $B\colon \mathrm{Hom}(G,A) \to [BG,BA]$ is an isomorphism when A is Abelian.

<u>Proof:</u> If A is finite, elementary calculations show that π_0 and π_1 are isomorphisms in the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(G,A) & \xrightarrow{B} & [BG,BA]. \\ \pi_0 \downarrow & & \downarrow \pi_1 \\ \mathrm{Hom}(\pi_0 G, \pi_0 A) & = & \mathrm{Hom}(\pi_1 BG, \pi_1 BA) \end{array}$$

In general, $A = F \times T^n$ where F is finite, hence it suffices to prove the result when A is the circle group S^1. This is very easy if G is finite (by group cohomology [4,IV.5.5]), if G is a torus (by inspection), or if G is connected (by use of a maximal torus). The general case is easily handled by use of the third author's continuous group cohomology theory [7]. For topological G-modules A, there are cohomology groups $H^*(G;A)$ (denoted $R^*\Gamma^G A$ in [7]). We shall only be concerned with trivial G actions. Here $H^*(G;A)$ is the ordinary cohomology $H^*(BG;A)$ if A is discrete [7,3.3], and $H^1(G;A) = \mathrm{Hom}(G,A)$ in general [7,4.3]. If A is contractible, then $H^*(G;A)$ can be calculated by continuous cochains [7,3.1], and it follows from Mostow [6,2.5 and 2.14] that $H^q(G;\mathbf{R}) = 0$ for $q > 0$. Suitable topological short exact sequences in A give rise to long exact sequences of cohomology groups [7,1.3]. In particular, the extension $Z \to \mathbf{R} \to S^1$ gives rise to a connecting isomorphism $\delta: H^1(G;S^1) \to H^2(G;Z)$. A comparison of definitions shows that δ coincides with B.

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