The legacy of Steve Schanuel!

My friend and collaborator Steve Schanuel died on July 21, 2014.

Steve was a mathematicians' Mathematician. He loved the many facets of mathematics and loved to solve problems that colleagues and students presented to him. He was generous and patient with young students and happiest when he could solve interesting problems that made him sparkle with joy. The students loved him, and so did we, his colleagues.

A free man with no wish for fame or fortune, unencumbered by politics, history, society, gossip, he did not get distracted by philosophy. He gave his time and energy to the problems that presented themselves, he loved to discuss them and spin further solutions. He did not like to write, and seemed to be happy when scribbling and thinking. But when a real mathematical problem presented itself, he was the most serious and hardworking scientist. That applied in particular to the real problem of passing knowledge on to young people.

We both shared the passion of teaching and the belief that large numbers of students could benefit from some explicit knowledge of conceptual methods. It had originally been proposed to us to write a text book on 'Discrete mathematics', to which Steve immediately replied 'no, we will emphasize methods that are applicable to both the continuous and the discrete'.

In the mathematical world he is best known for Schanuel's Lemma, and for Schanuel's Conjecture. Steve discovered the Lemma when he was still a graduate student at Chicago; it became a key instrument for those who participated in the development of Grothendieck's linear 'Klassentheorie' (K-Theory). The brilliant Schanuel's Conjecture, concerning transcendental number theory, has given rise to several advances due to the efforts of Steve himself and of dedicated logicians and number theorists, but it has still not been proved.

But there are further contributions, of relevance to all branches of mathematics, that bear the stamp of elementary clarity so characteristic of Steve's work. For example, his 'What is the Length of a Potato?' presents original contributions in the process of a supremely elementary exposition of the classical subject of geometric measure theory.

When I first met Steve in 1974, he explained to me a way of presenting the theory of affine-linear spaces in terms of the category of vector spaces. We developed that idea for 20 years, during which Steve proved several new mathematical results that I explain in my 1994 contribution to the historical analysis of the work of the great geometer Hermann Grassmann.

Partly in response to some remarks in Federico Gaeta's notes on Grothendieck's 1973 Buffalo course, and partly as a necessary basis for his 1990 theory of Negative Sets, Steve devised the notion of extensive category as a natural relativization of the notion of distributive category. His insight was that the spaces in such categories have both Euler characteristic and dimension, that both of these quantities can be derived from a single 'rig' , and that moreover the two quantities alone sometimes determine the space up to isomorphism. This remarkable non-linear Klassentheorie became a key thread in what we came to call 'Objective Number Theory'. Some of the results, which Steve derived from his theory of rigs, later turned out to be important in the study of O-minimality, in particular in the work of his student Adam Strzebonski on algebraic groups.

In retrospect, it may seem astonishing that the term 'rig' had not been proposed decades earlier: we constantly come across examples of commutative algebraic systems with two constants 0, 1, and two binary operations, $+$, $\times$, which do not necessarily have negatives and hence become rings only upon tensoring with $\mathbb{Z}$; thus omitting the 'n' for negatives, such algebras seem to deserve the name 'rigs'. The previously available name 'Commutative semi-rings with 1' is unwieldy and even carries a faint suggestion that these objects are only half-legitimate. We were amused when we finally revealed to each other that we had each independently come up with the term 'rig'. Thorough algebraist that he was, Steve went on to determine the simple objects in the category of rigs and to develop part of the needed theory of finite presentation; naturally, he also investigated modules over rigs, projective and otherwise.

Steve's basic construction, passing from a distributive category to its rig of isomorphism classes, and then tensoring with standard rigs, gave crucial additional information in some basic examples. Tensoring with $\mathbb{Z}$ to obtain a ring provided Euler characteristics for the spaces in the category. But tensoring instead with the rig '2' (in which $1{+}1 = 1$) measures the dimension of the spaces. In some crucial cases, such as his 'negative sets' (where $X = 1 + 2X$) those two invariants are sufficient to determine the space up to isomorphism. The 'number theory of objects' turns out to be more subtle than either the theory of infinite cardinals or the classification of recursive sets (both of the latter have the same resulting rig, consisting of natural numbers, together with one infinite element satisfying too many equations, of course the theory of recursive SUB-sets is by contrast very rich).

While teaching Conceptual Mathematics, we had noted that in the category of directed graphs, the arrow satisfies the quadratic equation $X^2 = X + 2D$, and hence reasoned that algebraic equations have further uses in combinatorics. Noting that Steve's equation for a negative set is a special case of the equation describing the data type 'lists in the alphabet A', namely $X = 1 + AX$, we tried simple substitution in order to get a description of 'lists of lists', namely $X = 1 + X^2$, which is also known as the binary tree equation for data types. This led easily to the conclusion that the tree data type has the primitive 6th root of unity as its Euler characteristic, and that in fact the more precise form $X^7 = X$ of this conclusion gives an interesting object of dimensions. This led to the conjecture that in some combinatorial categories there are no isomorphisms other than those which are rig-theoretic consequences of the defining equations, in other words, that they objectify the rig presented; that simply means that the proof of entailments is just high school algebra, except that negatives and cancellation are not used; in such calculations the expressions may become longer and longer under repeated substitutions, but then suddenly collapse due to the use of the defining equations. Our friend Andreas Blass dubbed this case of the conjecture 'Seven Trees in One', and proved it in a brilliant paper motivated by the idea of the classifying topos for the equation in question.

Subsequently, several other cases were proved by Robbie Gates. It became clear that equations of the fixed-point kind were appropriate candidates for objectification, at least from the data type point of view, with the right-hand side of the polynomial equation involving a signature in the sense of universal algebra; a fixed-point bijection holds for free algebras over free theories (also known as 'Peano algebras'). Matias Menni's recent work has succeeded to remove restrictive conditions on the signature. Extensions to fixed-point equations in several variables (corresponding to signatures for multi-sorted universal algebra), had also been proposed by Steve.

For many winter vacations Steve accompanied us to Oaxaca, Mexico, where we worked on extending and recording the results of Objective Number Theory. We studied deep into the nights; sometimes Fatima heard us giggle, because we had discovered how simply some results could be proved. In the morning she typed the notes that I had left on her table. Whenever unfinished trains of thought occurred in the manuscript, we wrote SHOULD in large letters.

We felt that our work received an additional inspiration from the ancient Zapotec city that could be glimpsed from our roof top.

Although Steve is gone, his work and his guiding spirit live on.

F. William Lawvere
July 21, 2015

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