QUANTIFIERS AND SHEAVES

by F. W. LAWVERE

The unity of opposites in the title is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. At the same time, in the present joint work with Myles Tierney there are important influences in the other direction: a Grothendieck " topology " appears most naturally as a modal operator, of the nature " it is locally the case that ", the usual logical opera- tors such as ∀, ∃, ⇒ have natural analogues which apply to families of geometrical objects rather than to propositional functions, and an important technique is to lift constructions first understood for " the " category $\underline{S}$ of abstract sets to an arbitrary topos. We first sum up the principal contradictions of the Grothendieck-Giraud- Verdier theory of topos in terms of four or five adjoint functors, significantly generaliz- ing the theory to free it of reliance on an *external* notion of infinite limit (in particular enabling one to claim that in a sense logic is a special case of geometry). The method thus developing is then applied to intrinsically define the concept of Boolean-valued model for $\underline{S}$ ($\underline{BVM}/\underline{S}$) and to prove the independence of the continuum hypothesis free of any use of transfinite induction. The second application of the method outlined here is an intrinsic geometric construction of the Chevalley-Hakim global spectrum of a ringed topos free of any choice of a " site of definition ".

When the main contradictions of a thing have been found, the scientific procedure is to summarize them in slogans which one then constantly uses as an ideological weapon for the further development and transformation of the thing. Doing this for " set theory " requires taking account of the experience that the main pairs of oppos- ing tendencies in mathematics take the form of adjoint functors, and frees us of the mathematically irrelevant traces (∈) left behind by the process of accumulating (∪) the power set (P) at each stage of a metaphysical " construction ". Further, experience with sheaves, permutation representations, algebraic spaces, etc., shows that a " set theory " for geometry should apply not only to *abstract* sets divorced from time, space, ring of definition, etc., but also to more general sets which do in fact develop along such parameters. For such sets, usually logic is " intuitionistic " (in its formal pro- perties) usually the axiom of choice is false, and usually a set is not determined by its points defined over 1 only.

1. By a topos we mean a category $\underline{E}$ which has finite limits and finite colimits, which is (a) cartesian closed and which (b) has a subobject classifier T. That is (a) on the one hand there is for each object A an internal hom functor $(\ )^A$ right adjoint to cartesian product $(\ ) \times A$, and (b) on the other hand there is a single map true: $1 \to T$ such that any monomorphism $X' \rightarrowtail X$ in $\underline{E}$ is the pullback of true along a unique characteristic map $X \to T$. This is the principal struggle in the internal theory of an arbitrary topos, and leads to very rapid development. The " set " T of

" truth-values " for $\underline{E}$ is shown to be a Heyting-algebra object which is complete in the sense that for any map $f : X \rightarrow Y$ in $\underline{E}$ there is a left adjoint $\exists\limits_{f}$ to the induced map T^f and also a right adjoint

$$\forall\limits_{f} : T^X \rightarrow T^Y$$

to T^f. Usually T is not a Boolean-algebra; for example if $\underline{E} = $ all $\underline{S}$-valued sheaves on a topological space, T is that sheaf whose sections over any U is the set of open subsets of U, while if $\underline{E} = \hat{\underline{C}} = \underline{C}^{\mathrm{op}}\underline{S}$ is set-valued functors on a small category $\underline{C}$, then $T(C) = $ all cribles of C. For any $\varphi : X \rightarrow T$, we denote by $\{ X/\varphi \}$ the corresponding subobject, correctly suggesting that to appropriate formulas of higher-order logic, a corresponding actual subobject exists.

All of the usual exactness properties of a topos follow quickly, most of them from the fact that there is for any $f : X \rightarrow Y$ a functor

$$\prod\limits_{f} : \underline{E}/X \rightarrow \underline{E}/Y$$

right adjoint to pulling back families $E \rightarrow Y$ over Y along f to families $E \underset{Y}{\times} X$ indexed by X. This extends to the case where the fibers are being acted upon as follows: If $\underline{C}$ in Cat $(\underline{E})$ is an internal category object in $\underline{E}$ with object-of-objects X, we can consider all actions of $\underline{C}$ on arbitrary families $E \rightarrow X$ of objects internally parameterized by X, obtaining a new *topos* $\hat{\underline{C}} = C^{\mathrm{op}} \underline{E}$ of internal $\underline{E}$-valued presheaves on $\underline{C}$. If $f : \underline{C} \rightarrow \underline{C}'$ is any internal functor, there is a right adjoint $f_* : \hat{\underline{C}} \rightarrow \hat{\underline{C}}'$ to the induced functor (as well as a left adjoint f_0), which means that in a very useful sense, any topos (even if countable) is internally complete.

Let us denote by σ_X the " support " functor which to any family $E \rightarrow X$ assigns the caracteristic map of the image of the structure map. This then allows consideration of particular direct contradictions between logic and geometry of a kind arising in proof theory and reminiscent of virtual vector bundles:

(II)

For every map $f : X \rightarrow Y$,

$$\begin{array}{ccc} \underline{E}/X & \underset{\underset{f}{\prod}}{\rightarrow} & \underline{E}/Y \\ \sigma_X \downarrow & & \downarrow \sigma_Y \\ \underline{E}(X, T) & \underset{f}{\overset{\forall}{\rightarrow}} & \underline{E}(Y, T) \end{array}$$

The above diagram commutes for permutation representations of a group, but not for the category $\underline{2}\,\underline{S}$ of maps in sets. On the other hand, both in intuitionistic logic and algebraic geometry we have to consider the extent to which the internal algebraicly defined operator $\exists$ actually means existence, which is essentially means whether

($\exists$) For every object E, the epi part *splits* in the following diagram

$$E \rightarrow \{ 1 \,|\, \sigma_1(E) \} \rightarrow 1$$

Now the latter condition *fails* for $\underline{G}\,\underline{S}$, $\underline{G}$ a non-trivial group, but *holds* for $\underline{P}\,\underline{S}$ where $\underline{P}$ is any well-ordered set (such as $\underline{2}$). Actually the conjunction of the two conditions (II) and ($\exists$) is equivalent to the condition that every epi splits, which geometrically we would call 0-dimensionality and logically we would call the axiom of choice. If $\underline{E}$ is the category of equivalence classes of formulas in some higher theory, the condi-

tion (∃) is a Skolem condition, but the problem arises also if E is of a geometrical nature since $\exists \varphi = $ true usually means actual existence only locally.

Often in a topos we have to make use of a further adjoint reflecting the contradiction between primitive recursion data and the family of sequences which it defines (T-valued sequences being the case known as mathematical induction), for example in analyzing a coequalizer or forming the free group or free ring object generated by a given object:

(ω) $\underline{E}^{\vec{\gamma}} \to E$ is not an equivalence and has a left adjoint ( ) $\times \omega$. Here $E^{\vec{\gamma}}$ is the usual category of objects-together-with an-endomap. However we did not include this axiom in the definition of topos partly because of the useful generality and partly because it is automatically lifted to any topos $\underline{E}$ " defined over " another one $\underline{E}_0$ in which it is true.

" Defined over " refers to a given *geometrical morphism* of topos, by which we mean a functor having an exact left adjoint. There are also *logical morphisms* of topos, which means a functor preserving up to isomorphism all the structure involved in the concept of topos. The two unite in *local homeomorphism*, which is a geometrical morphism u whose left adjoint part u^* is actually a logical morphism.

THEOREM. — Any geometrical morphism $u : \underline{E}' \to \underline{E}''$ of topos can be factored into

$$\underline{E}' \xrightarrow{\;u\;} \underline{E}''$$
$$u' \searrow \qquad \nearrow u''$$
$$\underline{E}$$

Where $\underline{E}$ is also a topos, where u', u'' are geometrical morphisms of topos with the additional properties that $(u'')_*$ is full and faithful $\underline{E} \to \underline{E}''$ while the left adjoint $(u')^* : \underline{E} \to \underline{E}'$ reflects isomorphisms. Further, u'' (hence any full and faithful geometrical morphism) is entirely determined by a single map $j_u : T'' \to T''$ in $\underline{E}''$ of the kind we call a Grothendieck " topology ", in fact as the j_u-sheaves.

Shifting to a topos denoted by $\underline{E}$ (rather than $\underline{E}''$) the conditions which such a modal operator $j : T \to T$ should satisfy are that it is (*a*) idempotent and that it (*b*) commutes with true and with the conjunction map $\wedge : T \times T \to T$. Such induces functorially a closure operator on the set of subobjects of any object (not a Kuratowski closure; for example in presheaves on a topological space the appropriate j assigns to any order ideal of open sets the principal ideal determined by its union). In order to show that j yields a full and faithful geometrical morphism $\underline{E}_j \to \underline{E}$ of topos, we show that the usual condition of being a j-sheaf is equivalent to having a diagonal j-closed in the square (" separated ") and being j-closed in any separated object into which embedded. Then the associated sheaf functor is constructed without any appeal to infinite direct limits by using the following four observations about a Grothendieck " topology " (= modal operator j satisfying axioms (*a*) and (*b*)): 1) The image T_j of j is a j-sheaf. 2) Y^X is a j-sheaf if Y is. 3) For any X, the j-closure in $X \times X$ of the diagonal is an equivalence relation. 4) If $X \to Y$ is any mono of X into any sheaf Y, then the j-closure in Y of X is the associated sheaf of X. (The first step (prior to applying the four observations) is to consider the singleton map $\{ \ \} : X \to T^X$). One then proves that the associated sheaf functor is exact by studying the morphisms which it inverts.

An important example in which we use the above factorization theorem is (lifted

to an arbitrary base topos $\underline{E}$ instead of $\underline{S}$) the Godement construction of sheaves on a topology basis by the method of resolving the contradiction between presheaves and (" discrete ") espace étalé. By a topology basis is meant a triplet consisting of an object X (of " points "), an object A (of " indices for the basis elements "), and a pairing $X \times A \to T$ which satisfies a directness condition so that the induced pair of adjoints

$$\underline{E}/X \xrightarrow[\Gamma]{\overset{\varinjlim}{\longleftarrow}} \underline{A}^{\mathrm{op}}\underline{E}$$

is a geometrical morphism of topos (Here by $\underline{A}$ we mean the poset whose " hom " order relation $F \rightrightarrows A$ on " objects " in A is just thepullback along $A \to T^X$ of the standard order relation on subobjects of X). Then the " image " topos is the usual category of sheaves, describable either using a Grothendieck " topology " in $\underline{A}^{\mathrm{op}}\underline{E}$ or a left exact cotriple (standard construction) in $\underline{E}/X$.

There is a standard Grothendieck topology in any topos, namely double negation, which is more appropriately put into words as " it is cofinally the case that ". The category $\underline{E}_{\neg\neg}$ of double negation sheaves always satisfies the additional condition that the logic is classical:

$$(\neg) \qquad 1 + 1 \;\Rightarrow\; T$$

which is equivalent with the condition that T (e. g. $T_{\neg\neg}$ in $\underline{E}_{\neg\neg}$) is a Boolean algebra object, which again geometrically is equivalent with the condition that every mono $X' \rightarrowtail X$ is part of a (unique) direct sum diagram $X' + (\neg X') \Rightarrow X$.

For constructing logical morphisms of topos we need to use geometrical morphisms, but also another construction, a generalized ultraproduct, which does *not* give a geometrical morphism in general and hence leads outside the realm of externally complete (i. e. defined over given $\underline{E}_0$) topos considered up to now in geometry. The data needed for the generalized ultraproduct is a pair consisting of a functor $u_* : \underline{E} \to \underline{E}_0$ between two topos, which may be a geometrical morphism but which in general is only required to preserve finite inverse limits, and of a homomorphism $h : u_*(T) \to T_0$ of Heyting algebra objects of $\underline{E}_0$. A new category $\underline{E}_h$ is then obtained from $\underline{E}$ by formally inverting all monomorphisms $X' \rightarrowtail X$ in $\underline{E}$ whose " universal quantification belongs to the ultrafilter " in the sense that

$$h\big(u_*\big(\sigma_1\big(\prod_{X \to 1} X'\big)\big)\big) = \mathrm{true}_0$$

THEOREM. — $\underline{E}_h$ is a topos and $\underline{E} \to \underline{E}_h$ a logical morphism. $\underline{E}_h$ is defined over $\underline{E}_0$ *in the sense of closed categories* but usually not in the geometrical sense of topos.

The above is needed, for example, to show that a $\underline{BVM}/\underline{S}$ can always be collapsed to a two-valued model, allowing most work on independence results to take place in higher topos without actually choosing h and making the collapse.

2. We can now make more precise what it is usually necessary to assume about " the " category $\underline{S}$ of abstract sets: it can be any topos satisfying conditions (Π), $(\exists)$, (ω), $(\neg)$ above as well as the following " irreducibility of 1 " condition:

$$(\vee) \quad \text{If } \varphi_i : 1 \to T \qquad \text{and} \qquad \varphi_1 \vee \varphi_2 = \mathrm{true},$$

then
$$\varphi_1 = \text{true} \quad \text{or} \quad \varphi_2 = \text{true}.$$

Now conditions ($\neg$) and ($\vee$) together imply that there are only two subobjects of 1, but not conversely as $\underline{M}^{\mathrm{op}}\underline{S}$, for $\underline{M}$ a monoid but not a group, shows. On the other hand ($\exists$) and ($\neg$) together imply that the subobjects of 1 (which form a " complete " Boolean algebra then) also form a *generating* family for the category; a topos satisfying ($\exists$) and ($\neg$) we call " Boolean ", and in such usually write $2 = T$. By a Boolean-valued model $\underline{E}$ of $\underline{S}$ (in symbols $\underline{E} \in \underline{BVM}/\underline{S}$) we mean then simply that $\underline{E}$ is a Boolean topos defined over $\underline{S}$. We can then show that any BVM over $\underline{S}$ actually also satisfies (Π) i. e. the axiom of choice, and indeed that the bi-category $\underline{BVM}/\underline{S}$ is equivalent to the category $\underline{CBA}(\underline{S})$ of $\underline{S}$-complete Boolean algebra objects in $\underline{S}$.

Actually the BVM's can be constructed another way, namely as double negation sheaves $\tilde{\underline{P}} = (\underline{P}\,\underline{S})_{\neg\neg}$ in the category of $\underline{S}$-valued functors on some poset $\underline{P}$ in $\underline{S}$. In this case (as well as others) the terminology of Cohen is suggestive: if $X \in \tilde{\underline{P}}$, $q \geq p$ in $\underline{P}$, $\varphi : X \rightarrow 2$ and x is an element of X defined at p, say that " q forces $\varphi(x)$ " iff $\varphi(x/q) = \text{true}$. Then in $\tilde{P}$, q forces $\varphi(x)$ iff r forces $\varphi(x)$ for a set of r cofinal beyond q.

To refute the continuum hypothesis in some BVM $\tilde{P}$ we also follow Cohen by choosing a set I in $\underline{S}$ with $2^\omega < I$ in the sense that there is a mono but no epi. Then $\underline{P}$ is the poset (ordered by extension) of all partial maps $\omega \times I \rightarrow 2$ with finite domain (definable as an object in any topos). Then in $\tilde{\underline{P}}$

$$\omega < u^*(2^\omega) < 2^\omega$$

where u^* is the " constant sheaf " functor left adjoint to the " global sections " functor $u_* : \tilde{\underline{P}} \rightarrow \underline{S}$. For the proof, one notes that $\underline{P}$ itself is essentially the definition of a map $u^*(I) \rightarrowtail 2^\omega$ *on a covering*, hence for sheaves there is such a map. The main point is then the

LEMMA. — If $\underline{P}$ is any poset in $\underline{S}$ satisfying a suitable " countable chain condition ", X in $\underline{S}$ and Y in $\underline{S}$ with $Y \times \omega \cong Y$, then

$$\text{Epi }(X, Y) = 0 \text{ in } \underline{S} \qquad \text{implies} \qquad \text{Epi }(u^*(X), u^*(Y)) = 0 \text{ in } \tilde{\underline{P}}.$$

Here Epi (X, Y) is an object defined in any topos by pulling back " image " along " true ".

3. A particular sort of topology basis arises if an object A has the structure of a (multiplicative) commutative monoid and one is given a *homomorphism* $u : A \rightarrow T^X$ into the monoid of subobjects of an object X, where multiplication is defined as conjunction (intersection). In this case we have moreover that the order-relation-object $F \rightrightarrows A$ determines a submonoid of the constant functor $\bar{A}$ in $\underline{A}^{\mathrm{op}}\underline{E}$ and that the" membership " relation $P \rightarrowtail X \times A$ induced by the pairing determines a submonoid of the constant family $\dot{A}$ in $\underline{E}/X$. We may then form fractions to obtain new commutative monoid objects $(\dot{A})_P$ in $\underline{E}/X$ and $(\bar{A})_F$ in $\underline{A}^{\mathrm{op}}\underline{E}$ and in particular $\tilde{A}$ (in the intermediate sheaf category) which is the reflection of $(\bar{A})_F$ and which is reflected to $(\dot{A})_P$.

Suppose now that A is actually a commutative *ring* in $\underline{E}$. Because of the intuitionistic nature of logic (already for $\underline{E} = 2\,\underline{S}$) we are forced to define a *prime* x of A

to be, not an ideal, but a subobject of A satisfying rather four conditions of the form

1) $[1 \in x]$ $= \text{true}$
2) $[f.g \in x]$ $= [f \in x] \wedge [g \in x]$
3) $[0 \in x]$ $= \text{false}$
4) $[f + g \in x] \leq [f \in x] \vee [g \in x]$

Note that 2) is an if-and-only-if condition and that the disjunction in the conclusion of the implication in 4) means essentially sup of two subobjects, which in a general topos may mean actual disjunction only locally. We further say that a ring is *local* iff the subobject of units is a prime. By a finite inverse limit, we get $X \rightarrowtail T^A$, " the subobject of T^A consisting of all subobjects of A which are prime ". This gives a topology basis in $\underline{E}$ whose sheaves form the topos Spec (A) known as the global spectrum of $\underline{E}$, A ; in Spec (A), $\tilde{A}$ is a local ring object, and indeed the universal local A-algebra in topos defined over $\underline{E}$. Note that in the process, the membership relation is exactly transformed into its opposite.

4. While the application of our method to algebraic geometry has only begun, other questions also immediately arise. Unpublished work of George Rousseau shows that the semantics often given for intuitionistic logic is simply ordinary (i. e. for abstract sets) semantics done in a suitable topos $\underline{\Lambda} \underline{S}$; a similar statement is true for Läuchli's proof-theoretic interpretation, as was recently shown by Anders Kock. But it would seem also possible to consider parameters designed to be applied to actual materialist time rather than just to stages in an imagined " construction ". In any topos satisfying (ω) each definition of the real numbers yields a definite object, but it is not yet known what theorems of analysis can be proved about it.

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