THE EULER CHARACTERISTIC, POINCARE-HOPF THEOREM, AND APPLICATIONS

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Abstract. In this paper, we introduce tools from differential topology to analyze functions between manifolds, and how functions on manifolds determine their structure in the first place. As such, Morse theory and the Euler characteristic are discussed, with the central result being a proof of the Poincare-Hopf theorem, which states that the sum of the indices of a smooth vector field is equal to the Euler characteristic. Though a variety of sources are consulted and several of the arguments are original, the strategy of proof is based primarily on that of Milnor's *Topology from the Differentiable Viewpoint*. Knowledge of basic point-set topology and multivariable calculus is assumed.

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1. Introduction

We start with the following question:

Question 1.1. *To what extent does the structure of a shape (for example, its number of holes) limit the ability of certain functions (i.e. smooth ones) to be defined on it?*

For example, we can consider assigning a tangent vector to each point on a sphere, doing so in a way such that the vectors vary smoothly. How many different ways can you assign these vectors? This might sound strange because there are two distinct ideas at work which we don't usually think of as too related. There are the geometric properties of the sphere–it is round, has no holes, etc.–and properties of vector fields, such as smoothness conditions.

As an idea for something that might unite these two ideas, consider the following definition:

Definition 1.2. Take a triangulation of a manifold M; that is, triangles for a two dimensional manifold, tetrahedra for a 3 dimensional manifold, and so on. The *Euler characteristic* $\chi(M)$ is the alternating sum vertices minus edges plus faces (minus number of tetrahedra, and so on).

One can show that the Euler characteristic is independent of the triangulation and is invariant up to homeomorphism. At first glance, this definition seems like it has nothing to do with any sort of smoothness condition we would impose on vector fields. Indeed, since the stereographic projection of the sphere onto $\mathbb{R}^2$ allows us to talk about the sphere and $\mathbb{R}^2$ in a similar manner (it is a smooth function), you might be tempted to say that drawing arrows smoothly on a sphere is like drawing arrows on $\mathbb{R}^2$ using the stereographic projection, or an open disk, since $\mathbb{R}^2$ is homeomorphic to an open ball. Noting that the Euler characteristic on a sphere is not equal to the Euler characteristic on an open disk would prove the whole discussion to be absurd.

Such an argument is flawed, however. As we show in this paper, the geometric structure of a manifold–in particular the Euler characteristic–can impose significant restrictions on which smooth vector fields are possible. Why is this the case? Intuitively, we can think of a manifold as being put together from shapes of different dimensions, where shapes of smaller dimensions guide the construction to higher dimensions. If it is possible to define a differentiable function on our manifold, then using Morse theory, it turns out that an appropriate function actually *entirely* determines how the manifold is to be put together. At this point, we can compare any vector field to the vector fields determined by the differentiable function (in a way that will be explained in greater detail below) to show that they must all preserve a few basic characteristics which are invariant for smooth vector fields.

We present these ideas as follows. In section 2, we define several terms which are essential to the study of differentiable manifolds; tangent space, orientation, etc. In section 3, we introduce the notion of degree and index, and prove some basic theorems, such as the degree's invariance. In section 4, we give some equivalent formulations of the Euler characteristic which require the introduction of homology theory. In section 5, we discuss Morse theory and indicate how it can be used to identify a smooth vector field with the Euler characteristic. Section 6 quickly proves the Poincare-Hopf theorem, tying everything together to easily demonstrate the point made above, that the geometric structure (in this instance the Euler characteristic) of a manifold places restrictions on a smooth vector field on a closed manifold. Section 7 concludes with a few interesting consequences.

2. Preliminaries

We start off with a few basic and fundamental definitions, which arise naturally and immediately in the study of differential topology. First we precisely define manifolds, as discussed in the introduction.

Definition 2.1. Let $M \subset \mathbb{R}^n$. We say that M is a *k-dimensional smooth manifold* if, for every $x \in M$, there exists some open set $U \subset \mathbb{R}^n$ together with a function $f : U \to \mathbb{R}^k$ such that $f : U \to \mathbb{R}^k$ is smooth, $f\mid_{U \cap M} : U \cap M \to \mathbb{R}^k$ is bijective and $f^{-1} : f(U \cap M) \to \mathbb{R}^n$ is smooth. Since $f(U \cap M) = V \subset \mathbb{R}^k$ would necessarily be an open set, we say $U \cap M$ *is diffeomorphic to* V. We call $f\mid_{U \cap M}$ a *chart*, and we call a collection of charts an *atlas*.

Remarks 2.2. The fact that $U \cap M$ will often not be open in $\mathbb{R}^n$ prevents us from outright saying that "M is a k-dimensional smooth manifold if, around every $x \in M$ there is a neighborhood U such that $U \cap M$ is diffeomorphic to an open subset of $\mathbb{R}^k$." But, essentially, this is what we mean, and it is formalized in the above definition.
We sometimes use a superscript to emphasize the dimension of M, and refer to our manifold as $M^k \subset \mathbb{R}^n$. In general, k and l will be dimensions of manifolds and n and m will be dimensions of ambient spaces.

Definition 2.3. Let $M \subset \mathbb{R}^n$. We say that M is a *k-dimensional smooth manifold with boundary* if, for any $x \in M$, there exists some open set U such that $U \cap M$ is diffeomorphic to some open subset of $\mathbb{H}^k = \mathbb{R}^{k-1} \times \{x \in \mathbb{R} \mid x \geq 0\}$. Other terms are defined analogously as above. The subset of M which maps to $\mathbb{R}^{k-1} \times \{0\}$ in a parameterization is called the *boundary.* Note that the boundary of M does not depend on the chart.

Definition 2.4. We say that N is a *submanifold* of M if N is a manifold and $N \subset M$. Note that N and M do not necessarily have the same dimension.

Examples of manifolds abound. A few useful ones are listed below.

Examples 2.5. *(i)* Trivially, $\mathbb{R}^n$ is an n dimensional manifold, with parameterization given by the identity map. Additionally, the unit interval $[0, 1]$ is a 1-dimensional manifold with boundary, since $[0, 1)$ is open in $\{x \in \mathbb{R} \mid x \geq 0\}$, and $[0, 1)$ is diffeomorphic to $(0, 1]$.
(ii) The unit circle in $\mathbb{R}^2$, henceforth referred to as S^1, is a 1-dimensional manifold. Let $f_1, f_2, f_3, f_4 : (0, 1) \to S^1$ be defined by $f_1(x) = (x, \sqrt{1-x})$, $f_2(x) = (x, -\sqrt{1-x})$, $f_3(x) = (\sqrt{1-x}, x)$ and $f_4(x) = (-\sqrt{1-x}, x)$. Then each f_i is a diffeomorphism between $\mathbb{R}$ and $U \cap S^1$ for some $U \subset \mathbb{R}^2$, and also such a U exists for any $x \in S^1$. Hence S^1 is a differentiable manifold. A few slight modifications of this argument show that, for any $n \geq 2$, the set $\{x \in \mathbb{R}^n \mid ||x|| = 1\}$ is an $n-1$ dimensional manifold.
(iii) The Cartesian product of two manifolds, as it is usually defined, is a manifold. To see this, suppose X and Y are manifolds without boundary. Consider $x \in X \times Y$. Then there exist diffeomorphisms $f : \mathbb{R}^k \to U \cap X$ and $g : \mathbb{R}^p \to V \cap Y$. Under the product topology, $U \times V$ is open in $X \times Y$. Additionally, define a function $h : \mathbb{R}^{p+k} \to U \times V \cap X \times Y$ by $h(x, y) = (f(x), g(y))$. Then h is a diffeomorphism of $\mathbb{R}^{p+k}$ and an intersection between an open set and $X \times Y$. Since this can be done for every $x \in X \times Y$, we have $X \times Y$ is a manifold.
(iv) The product of a manifold with boundary and a manifold (without boundary) is a manifold with boundary. The proof is nearly identical to the case of the product of two manifolds. In particular, if X is a manifold with boundary and Y is a manifold, then the boundary of $X \times Y$ is the boundary of X.
(v) By the above, $\mathbb{T}^n = \prod_{i=1}^n S^1$, that is, the n-torus, is an n dimensional manifold.
(vi) Also by the above, the cylinder, $S^1 \times [0, 1]$ is a manifold with boundary equal to $S^1 \times \{0\} \cup S^1 \times \{1\}$.

As an aside, one might wonder how many example 1 dimensional manifolds there are. The following classification, which we state without proof, definitively answers that question.

Theorem 2.6 (Classification of 1-Manifolds)**.** *Any smooth, connected 1-dimensional manifold is diffeomorphic to the circle S^1 or an interval in $\mathbb{R}$.*

Theorem 2.7. *Let M be a k-dimensional manifold with boundary. Then the boundary of M is a $k-1$ dimensional submanifold.*

Proof. Consider any $x \in \partial M$. Then there exists some open set $U \subset \mathbb{R}^n$ containing x such that $f : U \times M \to \mathbb{R}^{k-1} \times [0, \infty)$ is a diffeomorphism. By definition of the boundary, $f|_{\partial M} U \cap \partial M \to \mathbb{R}^{k-1} \times \{0\}$ is a diffeomorphism of $U \cap \partial M$ and $V \times \{0\}$ where V is an open subset of $\mathbb{R}^{k-1}$. Forgetting about the last coordinate, we see that it is a diffeomorphism of $U \cap \partial M$ and $V \subset \mathbb{R}^{k-1}$. Doing this for all $x \in \partial M$ demonstrates that ∂M is a $k-1$ dimensional manifold. □

Definition 2.8. Consider a differentiable manifold $M \subset \mathbb{R}^n$ of dimension k, and let $f : U \to M$ be a chart of M. Let $dg_u : \mathbb{R}^m \to \mathbb{R}^n$ be the derivative of g, that is,

$$dg_u(v) = \lim_{t \to 0} \frac{g(u + tv) - g(u)}{t}.$$

Recall from multivariable calculus that the derivative as defined is, indeed, a linear map. We say that the *tangent space* TM_u is the image of $\mathbb{R}^m$ under dg_u, that is, $dg_u(\mathbb{R}^m)$.

Note that the tangent space is defined for every point in our manifold. In fact, as the next definition shows, it is also unique for every point in our manifold.

Theorem 2.9. *The tangent space as defined is independent of choice chart.*

Proof. Suppose that $f : U \to M$ and $g : U \to M$ are two charts of a manifold. We wish to show that the tangent space is the same substituting each of these into the definition. Assume without loss of generality that $f(U) = g(U)$, that is, that f and g parameterize the same part of M. Then let $h = g^{-1} \circ f$. Then h is the composition of diffeomorphisms and is hence a diffeomorphism. But $f = g \circ h$, so by the chain rule, $df_x = dg_x \circ dh_x$. Hence $df_x(\mathbb{R}^k) = dg_x \circ dh_x(\mathbb{R}^k)$, so that $df_x(\mathbb{R}^k) \subset dg_x(\mathbb{R}^k)$. Switching f and g and making the same argument shows that $dg_x(\mathbb{R}^k) \subset df_x(\mathbb{R}^k)$, so that the tangent space defined by each chart is identical. □

Note that we have not yet discussed maps between manifolds. The next definition rectifies this problem.

Definition 2.10. Now let M and N be two manifolds, and $f : M \to N$ be a function between them. For some $x \in M$, pick charts $\phi : U \to M$ and $\psi : V \to N$ such that $x \in \phi(U)$ and $f(x) \in \psi(V)$. Let $g : U \to V$ be defined by $g = \psi^{-1} \circ f \circ \phi$. We define the *derivative of f at x* to be the map $df_x = d\psi_{\psi^{-1}(f(x))} \circ dg_{\phi^{-1}(x)} \circ d\phi_x^{-1}$. Note that $df_x : TM_x \to TN_{f(x)}$ is a linear transformation, since the derivative is.

Remarks 2.11. The trick in this definition is to transform the question of derivatives on manifolds to a question of derivatives between Euclidean spaces and manifolds, and a question of derivatives between different Euclidean spaces. Otherwise, the definition is mostly a formality. That this definition does not depend on choice of chart is almost identical to the proof above. That the chain rule holds is an exercise in drawing commutative diagrams.
On this note, it is useful to further emphasize that charts allow all of our definitions and theorems from multivariable calculus to be used for manifolds. In other words, having a chart allows us to change the question of maps between manifolds to maps between Euclidean spaces, which is simply the subject of multivariable calculus.

Definition 2.12. Let $f : M \to N$ be a smooth map between manifolds of the same dimension. If df_x as a linear map described above is singular, that is, the determinant of the matrix is 0, then $x \in M$ is called a *critical point*. We say $y \in N$ is a *critical value* if some $x \in f^{-1}(y)$ is critical. Conversely, if df_x is not singular at $x \in M$, then x is called a *regular point*, and $y \in N$ is called a *regular value* if all $x \in f^{-1}(y)$ are regular.

It is of utmost important that the reader distinguish between regular *points* (which are in the domain) and regular *values* (which are in the co-domain), as it will avoid a lot of confusion over some of the following proofs.
We might wonder how many critical points there can be for such a function $f : M \to N$. In general, critical points are difficult to work with because they prevent us from inverting our functions, which is usually very important, and they also make it more difficult to define the degree, which we explain in the next section. Hence we do not want there to be too many critical points. Fortunately, the following theorem (which we state without proof) tells us that this is not really too much to ask for.

Theorem 2.13 (Sard)**.** *Let $f : U \to \mathbb{R}^m$ be a smooth map, defined on $U \subset \mathbb{R}^n$, and let $C \subset U$ be the set of points $x \in U$ such that the rank of df_x is less than m. Then $f(C)$ has Lebesgue measure zero, and hence $\mathbb{R}^m \backslash f(C)$ is everywhere dense.*

Remark 2.14. At this point it is appropriate to comment on the smoothness hypothesis made above. We have made smoothness hypotheses so that, in the future, we can apply Sard's theorem and know that we are not only dealing with critical values, and that regular values will be able to be found in the following section so that our definitions can make sense. However, though we do not give the proof of Sard's theorem, we note that it uses smoothness only insofar as it allows us to take many derivatives. To be precise, if n and m are as above, then it suffices for there to be more than $n/m - 1$ derivatives. With this in mind, sometimes it is possible to replace smoothness conditions with this condition if smoothness is too much to ask for.

We also note the following:

Lemma 2.15. *Suppose X is an oriented manifold with boundary of dimension $k+1$. Suppose the boundary of X is a manifold M oriented as the boundary of X. If N is of dimension k and $f : M \to N$ extends to a smooth map $F : X \to N$ and if y is a regular value of both f and F, then $F^{-1}(y)$ is a smooth 1-manifold, and $f^{-1}(y)$ is equal to the endpoints of the line segments of $F^{-1}(y)$ (which, by the classification of 1-manifolds, is indeed the union of line segments and circles).*

Theorem 2.16. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an orientation preserving diffeomorphism, then f is smoothly isotopic to the identity.*

Proof. Assuming without loss of generality that $f(0) = 0$, the isotopy $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ is given by

$$F(x,t) = \frac{f(xt)}{t}, 0 < t \leq 1$$
$$F(x,0) = df_0(x).$$

Indeed, this is a smooth isotopy between f and df_0. Also note df_0 is isotopic to the identity, being an orientation preserving linear transformation. Hence f is smoothly isotopic to the identity. □

Lemma 2.17 (Homogeneity)**.** *If y and z are arbitrary interior points of a smooth connected manifold N, then there exists a diffeomorphism $h : N \to N$ such that h is smoothly isotopic to the identity and $h(y) = z$.*

Proof. First we show that there exists a smooth isotopy from $\mathbb{R}^n$ to itself fixing all points outside of an open ball of fixed radius while sliding the origin to any predetermined point inside some ball. This part is constructive. Note that a rotation is a diffeomorphism, so that it suffices to show that we can slide the origin to a point $(a, 0)$ where $a \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^{n-1} \to \mathbb{R}$ be functions that are 1 at the origin and 0 outside a ball of radius 1. Define $h_t : \mathbb{R}^n \to \mathbb{R}^n$ by $h_t(x, y) = (x + tf(x)g(y)a, y)$ (where we take y to be in $\mathbb{R}^{n-1}$ and x to be in $\mathbb{R}$). The Jacobian at (x, y) is equal to

$$\begin{pmatrix} 1 + tg(y)f'(x)a & 1 + af(x)t\frac{\partial g}{\partial y_1}\mid_{(x,y)} & \cdots & 1 + af(x)t\frac{\partial g}{\partial y_{n-1}}\mid_{(x,y)} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The determinant of this is clearly $1 + tg(y)f'(x)a$. Hence, let M be greater than $g(y)$ inside of the unit ball and N be greater than $f'(x)$ inside of the unit ball. If a is such that $|MNa| < 1/2$, then for all $t \in [0, 1]$ the determinant is nonzero, meaning by the inverse function theorem that it is a diffeomorphism. Note further that h_0 is the identity and that $h_1(0, 0) = (a, 0)$, so that this map does indeed slide the origin to $(a, 0)$ as desired. Analyzing the proof more closely, we see that our range for z is predetermined (ie fixing f and g, we can pick any z inside of some radius, so that the fact that we are starting at the origin does not influence which values we can pick for z). The other requirements for h_t to be a smooth isotopy being easily verified, so that $h_1 = h$ is our desired function that carries the origin to $(y, 0)$.
Now, say $x \sim y$ if there is an isotopy carrying x to y. This defines an equivalence relation. Since x is an interior point, there is a ball that is a subset of N that contains it. Shrinking the maps appropriately, we see that $x \sim y$ for any y in this ball that is a subset of N. Let $A = \{y \mid x \sim y\}$. By the above, this is the union of balls, and is hence open. In fact, any equivalence class is open, but since our manifold is connected, it follows that there can only be one equivalence class, thus proving the lemma. □

We see from the linearity of the derivative that the tangent space, being the image of a linear space under a linear transformation, is clearly a linear space. Hence one might require that we have some basis for TM_u, and one might wonder what implications this has on the manifold in general. This leads us to the notion of orientation.

Stepping back a little bit, let V be any vector space of dimension k, and suppose $(v_1, v_2, \ldots, v_k)$ is some basis for V, and let $(w_1, w_2, \ldots, w_k)$ be some other basis. Then consider the unique linear transformation defined by the k equations

$$w_i = \sum_j a_{ij} v_j.$$

Let $A = (a_{ij})_{1\le i,j\le k}$. Note that this linear transformation is unique, because any vector in V can be written as a linear combination of $(v_1, v_2, \dots, v_k)$, since this collection of vectors is a basis. Also note that this matrix is nonsingular.

Definition 2.18. Consider a vector space V. An *orientation* is an ordering of basis vectors of V. We say that two bases determine the *same orientation* if the unique linear transformation that takes one to the other has positive determinant, and *opposite orientation* if it has negative determinant.

For example, we let the standard orientation for $\mathbb{R}^n$ be the orientation determined by the standard basis; that is, the basis $\{(1, 0, \dots, 0), (0, 1 \dots, 0), \dots, (0, 0, \dots, 1)\}$, also often referred to as $e_1, e_2, \dots, e_n$, respectively.
At this point, the reader should pause to understand why the notions of same and opposite orientations on a tangent space are the "right" ones. With this in mind, we can now take our vector space V to be the vector space TM^k_u, where $u \in M$, and begin to define the notion of orientation of a manifold. To define the notion of orientation on a manifold, note that the derivative of a chart of a manifold at some point x on the manifold defines an isomorphism between $\mathbb{R}^k$ and the tangent space of M^k at x. Precisely,

Definition 2.19. Let $M^k \subset \mathbb{R}^n$ be a k-dimensional differentiable manifold. By the above, if $\phi_i : U_i \cap M \to \mathbb{R}^k$ is a chart, then it determines a tangent space at any $x \in U_i \cap M^k$. An *orientation of a tangent space* is an ordering of basis vectors in a tangent space, with two orientations being the same or different according to the above definition. Since $d\phi_{i,u} : TM_u \to \mathbb{R}^k$ is non-singular (since a chart must be invertible), $d\phi_i$ is an isomorphism of tangent spaces at every $u \in U_i \cap M^k$ and $\mathbb{R}^k$. A chart therefore gives a *local orientation*, that is, an ordering of the basis vectors for any $x \in U_i$.
Now take an open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of M^k together with charts $\phi_i : U_i \cap M^k \to \mathbb{R}^k$. The collection of charts (U_i, ϕ_i) is *compatible* if $\det d(\phi_i \circ \phi_j^{-1}) > 0$ whenever $U_i \cap U_j \cap M^k \neq \emptyset$. A manifold is *orientable* if such a collection of charts exists.

In general, local orientations will always exist, but compatibility is not ensured.

One might also ask about how one should orient the boundary of a manifold. Suppose that, using the above guidelines, we have found an orientation for our manifold M but not its boundary. Recall that if M^k is k dimensional, then ∂M^k is $k-1$ dimensional. Hence, it makes sense to talk about the tangent space of ∂M^k, as an $k-1$ dimensional space. If $x \in \partial M^k$ and $f : U \cap M^k \to H^k$ is a chart, then f is also clearly a map to $\mathbb{R}^k$ since $H^k \subset \mathbb{R}^k$, and df_x determines an k dimensional linear space. Since $T(\partial M)_x$ is an $k-1$ dimensional subspace. So consider a vector in TM_x that is not in $T(\partial M)_x$. We say that this vector is *outward* if it corresponds under df_x to a vector in $\mathbb{R}^{n-1} \times \{x \in \mathbb{R} \mid x > 0\}$. Otherwise, we say the vector is *inward*. Now, for $x \in \partial M$, pick a consistent k-dimensional basis for TM_x, with the restriction that the vector linearly independent of $T(\partial M)_x$ is outward. This determines an orientation for $T(\partial M)_x$.

Examples 2.20. *(i)* The circle is orientable; indeed, with the atlas in the above examples, one can show that the orientations determined by these charts are consistent.

(ii) The cylinder is orientable. In particular, we can orient the boundary of the cylinder as follows. Consider a chart for the cylinder, and suppose everything except the boundaries are oriented. Then the orientation for one boundary component, ie a circle as a 1 manifold, is the orientation induced by the single vector tangent to the circle such that, when combined with an outward vector, provides an orientation consistent with the tangent spaces not on the boundary. If we keep moving up with this same orientation, we run into a slight problem at the other end; the previously outward pointing vector now points inward. So to orient this boundary component, reverse *both* vectors; the inward pointing vector points outward, the tangent vector points in the opposite direction, and orientation is preserved. But, the boundary components of the cylinder have opposite orientations.
(iii) The Mobius band is not orientable. To see this, consider any chart of the Mobius band. The fact that this is an open cover means that the orientation on one chart determines the orientation on some other intersecting chart. Actually, since all charts are connected in this way, the orientation on one chart determines the orientation on all charts. But this is a problem with the Mobius band; if we circle it once, we have that a single chart would need to determine two orientations, which is impossible.
(iv) Using a similar argument, we have that the Klein bottle is not orientable.

We conclude this section by introducing the notion of a vector field on a Manifold.

Definition 2.21. Let $v : U \to \mathbb{R}^n$ be a function such that $v(z) \in TM_z^k \subseteq \mathbb{R}^n$, where we note that $TM_z^k \cong \mathbb{R}^k \subseteq \mathbb{R}^m$. The image of this function under M^k defines a *vector field* on M^k. Hence we can view v as a map from M to $\mathbb{R}^k$ and say that a vector field is *smooth* if v is smooth. A *zero* of a vector field is a point $z \in M$ such that $v(z) = 0$, and it is *non-degenerate* if the derivative as described above is non-singular.

Let us be slightly more precise concerning what it means for v to be smooth. A vector field assigns, to every point $x \in M^k$, a vector in TM_x^k, a vector space isomorphic to $\mathbb{R}^k$. In particular, suppose we have a vector field defined on a k dimensional manifold M^k. Suppose we have a chart $\phi : U \to M^k$ where $U \subset \mathbb{R}^k$ is open. Then by the above $d\phi_{\phi^{-1}(z)} : \mathbb{R}^k \to TM_z^k$ is an isomorphism of vector spaces. So suppose there is some vector field v defined on an open subset of M^k for which we have a chart ϕ. Then this vector field v can be written using our chart ϕ and an appropriate vector field on $\mathbb{R}^k$, say $v' : U \to \mathbb{R}^k$, by setting $v = d\phi_{\phi^{-1}(z)} \circ v' \circ \phi^{-1}$. Hence not only can we talk about v as smooth, but we can also talk about the derivative as a linear map by considering $d(d\phi_{\phi^{-1}(z)} \circ v' \circ \phi^{-1})$.

3. Degrees and their Invariance

Definition 3.1. Suppose that $f : M \to N$ is a smooth map between manifolds M and N of the same dimension. Consider some y in N such that, for every x such that $f(x) = y$, the linear map $df_x : TM_x \to TN_y$ is non-singular, i.e., an isomorphism. If $\det(df_x) > 0$ then let sign $df_x = 1$ and if $\det(df_x) < 0$ then let sign $df_x = -1$. We define the *degree of f at y* to be

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \text{sign } df_x.$$

There are two fundamental properties of the degree which we will demonstrate. The first is that the degree of a function does not depend on the choice of regular value; that is, $\deg(f, y) = \deg(f, z)$ if y and z are both regular values. The second is two functions which are homotopic have the same degree. We prove this second property first, but after some introductory lemmas.

Lemma 3.2. *The degree of f at a regular value is locally constant.*

Proof. Let $\{x_1, \ldots, x_n\} = f^{-1}(y)$. Take disjoint small open neighborhoods U_i each containing exactly one x_i, which we assume without loss of generality are all diffeomorphic to the same neighborhood V of N, and in particular, where the diffeomorphism is non-singular. Let $U = U_1 \cup \ldots \cup U_n$ and let $W = f(M \backslash U)$. Then the degree is locally constant in the neighborhood $V \backslash W$. Consider any point y' in this neighborhood. Since our diffeomorphisms are non-singular, each U_i contains exactly one point mapped to y'. By selection, there are no other such open sets that do. Hence the number of points in the inverse image of a point is locally constant. That the orientations at all of these points are all the same is clear. This proves the lemma. □

Theorem 3.3. *As in lemma 2.16, let X be an oriented manifold with boundary of dimension k, with the boundary of X being a manifold M that is oriented as the boundary of X. If N is of dimension $k-1$ and $f : M \to N$ extends to a smooth map $F : X \to N$, then $\deg(f; y) = 0$ for every regular value y.*

Proof. Take $y \in N$. As indicated above, for such functions F and f, we have that $f^{-1}(y)$ is equal to the endpoints of the line segment of $F^{-1}(y)$. Given this, we can refer to these endpoints as a and b. We wish to show that, for every

$$\text{sign } df_a + \text{sign } df_b = 0.$$

Given this, we would have that, summing over all points in the inverse image of y, that $\deg(f, y) = 0$.

Let $x \in F^{-1}(y)$. Since $F^{-1}(y)$ is 1-dimensional, it makes sense to talk about a vector tangent to $F^{-1}(y)$ at x. Let v_1 be this vector, and let $(v_1, v_2, \ldots, v_n)$ form a positively oriented basis of TM_x such that dF_x takes $(v_2, \ldots, v_n)$ into a positively oriented basis for TN_y. That this can be done for all $x \in F^{-1}(y)$ follows from the regularity of y.

Now, v_1, being tangent to $F^{-1}(y)$, is a smooth function of $x \in F^{-1}(y)$, say $v_1(x)$, which is defined above for all points outside of the boundary. On the boundary, however, we can extend $v_1(x)$ to be defined by continuity; that is, we define $v_1(a)$ and $v_1(b)$ via a limit. At one boundary point, v_1 points outward, and at the other it points inward. But since the orientation is preserved for all values of x, we have that opposite orientations are determined on the boundaries. Hence, $\text{sign } df_a = 1, \text{sign } df_b = -1$, so that the sum of the two is equal to 0.

On the other hand, suppose that y is not a regular value for F. By Sard's theorem, we can ensure the existence of some neighborhood $U \subset N$ for which the degree is locally constant within U and such that there is a $y' \in U$ that is regular for both f and F. Applying the above results gives us that $\deg(y'; f) = 0$ so that $\deg(y; f) = 0$ as well. □

Theorem 3.4. *The value $\deg(f; y)$ is equal to the value of $\deg(g; y)$ if f is homotopic to g.*

Proof. Consider $M \times [0,1]$ as a manifold and the homotopy $F : M \times [0,1] \to N$ as a smooth map. Then the degrees of F restricted to the boundary component is equal to 0 by the above, and as the proof demonstrates, one end of the boundary gets the reversed orientation; but we can say that $F \mid_{M\times\{0\}} = f$ and $F \mid_{M\times\{1\}} = g$, which gives us that $\deg(F \mid_{\partial(M\times[0,1])}; y) = \deg(g; y) - \deg(f; y) = 0$. Hence $\deg(f; y) = \deg(g; y)$. □

Corollary 3.5. *The value of* $\deg(f; y)$ *does not depend on choice of regular value.*

Proof. Consider two regular values of f, y and z. Let $h : N \to N$ be a diffeomorphism which is isotopic to the identity and for which $h(y) = z$, the existence of which is given by the homogeneity lemma. The homotopy between h and the identity is also a homotopy between $h \circ f$ and f. Hence $\deg(h \circ f, z) = \deg(f, z)$. But h is bijective and orientation preserving, which means that $\deg(h \circ f, h(y)) = \deg(h \circ f, y) = \deg(f, y)$, the last step following again from homotopy. Since $h(y) = z$, we have $\deg(f, z) = \deg(f, y)$. □

Henceforth we will refer to this common value as $\deg f$. The theorem above therefore indicates that f being homotopic to g implies that $\deg f = \deg g$.

Definition 3.6. Let v be a smooth vector field on an open set $U \subset \mathbb{R}^n$. Let $z \in U$ be an isolated zero of the vector field v. Let N_ε be a small disk containing a zero, where we take small to mean that there are no other zeros inside of N_ε. Define $\hat{v} : \partial N_\varepsilon \to S^{n-1}$, and let the boundary of each N_ε be oriented as the boundary of the disk containing the zero. The *index of* v *at* z is the degree of $\hat{v}$.

One might wonder whether the above definition is well defined for any $\varepsilon > 0$ sufficiently small. Consider let $\hat{v_1} : \partial N_{\varepsilon_1} \to S^{n-1}$ and $\hat{v_2} : \partial N_{\varepsilon_2} \to S^{n-1}$ be two maps defined using the above construction. Then since the vectors in v vary smoothly, v being a smooth vector field, we have that $\hat{v_1}$ is homotopic to $\hat{v_2}$. Thus by the above, the index is well-defined. We will generally refer to the index at a zero z by $\mathfrak{i}_z$.
Now, suppose that a non-degenerate vector field v is defined on an open set $U \subset \mathbb{R}^n$. Consider a zero of the vector field. We note

Theorem 3.7. *If v is a vector field with a zero z and $f : U \to V$ is a diffeomoprhism, then the index of v at an isolated zero z is the same as the index of $v' = df \circ v \circ f^{-1}$ at $f(z)$.*

With this in mind, let $M^k \subset \mathbb{R}^n$ be a manifold, v a vector field on that manifold with a zero z and a chart $f : U \to \mathbb{R}^n$ for U containing z. Each vector field v on M corresponds to a vector field w on U via the chart. Defined as such, the following seems appropriate.

Definition 3.8. We define the *index of* v *at* z to be the index of $w = df^{-1} \circ v \circ f$ at $f^{-1}(z)$, noting that w is defined on an open set in $\mathbb{R}^k$.

So, suppose our vector field $v : U \to \mathbb{R}^n$ is defined on an open subset of $\mathbb{R}^n$. Consider the index at a zero. Recall from above that any orientation preserving diffeomorphism is smoothly isotopic to the identity. So think of v as a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. If, in a neighborhood around the zero, v preserves orientation, that is, the determinant of the linear map is positive for all points inside the neighborhood, then v is smoothly isotopic to the identity inside the neighborhood. If v

reverses orientation, then v is smoothly isotopic to a reflection inside of the neighborhood. But the key point is that if v is non-degenerate, then in a small enough neighborhood, by the smoothness of the vector field, exactly one of the two must hold. Therefore, in some small ball of radius ε around the zero, we have that v can either be deformed isotopically into the identity or a reflection. But since the degree is invariant under isotopy, we have that in the former case, the index must be +1, and the latter case the index must be -1. This suggests that perhaps, for any vector field at an isolated non-degenerate zero, the index is ± 1. This is indeed true, though we state it without proof.

Lemma 3.9. *At a non-degenerate zero of a vector field v, the index is $+1$ if the determinant dv_z is positive and -1 if the determinant is negative.*

Definition 3.10. For a manifold with boundary $M^n \subset \mathbb{R}^n$, we define the *Gauss Mapping* $g : \partial M^n \to S^{n-1}$ as that which maps a point x in the boundary to the normal unit vector; that is, the outward vector that is perpendicular to $T(\partial M_x)$. We emphasize that M is an n dimensional manifold embedded in n dimensional space, that is, the dimension of the manifold is the same as the dimension of the space, so that the definition of the Gauss mapping makes sense and is well defined.

Theorem 3.11. *Let v be a smooth vector field on a manifold with boundary M with only nondegenerate zeros such that v points outward along the boundary. Then the index sum is equal to the degree of the Gauss mapping g, and hence is the same for any smooth vector field v.*

Proof. Around every zero of our vector field, pick some $\varepsilon > 0$ such that the ball of radius ε around the zero contains no other zeros. Removing all of these ε balls, we end up having a new manifold with boundary. Let N be the manifold M with all of the ε balls removed. We have $\hat{v} : N \to S^{n-1}$ defined by $\frac{v(x)}{||v(x)||}$ is a smooth function that maps our smooth vector field to S^{m-1}. But this is true on both the boundary components and on the whole manifold; hence by theorem 3.3, since the vector field is a smooth function that extends off of the boundary to another smooth function, when restricted to the various boundary components, the degree of $\frac{v(x)}{||v(x)||}$ must be 0. Consider the boundary component of M. Since the vector field points outward, it is homotopic to the Gauss mapping and hence has the same degree as the Gauss mapping. Note that the index sum is equal to the degree of $\frac{v(x)}{||v(x)||}$ on all of the other boundary components. However, when computing the index sum, the spheres are oriented as the boundary, which means the standard orientation has a vector pointing outward, that is, away from the zero. On the other hand, doing the same for the degree of $\frac{v(x)}{||v(x)||}$ when restricted to the boundary components of the new manifold, orienting the new boundary components gives us a vector pointing inward. So, when computing the degree, since each boundary components have opposite orientations as they did for computing the index, we see that the index sum is equal to the opposite of the degree of $\frac{v(x)}{||v(x)||}$. Hence since the degree of $\frac{v(x)}{||v(x)||}$ restricted to the boundary of the new manifold is zero, and since it is also equal to the degree along all of the various boundary components we have

$$\deg g - \sum_z \mathfrak{i}_z = 0$$

which gives the desired result. ∎

We now use this approach to yield an important theorem

Theorem 3.12. *Let v be a smooth vector field on a manifold without boundary M^k of dimension k with only nondegenerate zeros. Let $N_\varepsilon = \{x \in \mathbb{R}^n \mid ||x-y|| \leq \varepsilon, y \in M^k\}$; in other words, N_ε thickens M^k to be in $\mathbb{R}^n$ (We note, without proof, that for ε sufficiently small, N_ε is a closed manifold with boundary). Then the index sum $\sum_z \mathfrak{i}_z$ is equal to the degree of the Gauss mapping $g : \partial N_\varepsilon \to S_n$.*

Proof. To show this, we thicken our manifold M appropriately so that it becomes a manifold with boundary of the same dimension as our ambient space. We can define $r : N_\varepsilon \to M$ by letting $r(x)$ be the closest point to x in M. Note that, due to our smoothness conditions of the manifold, this map is well defined for a small enough ε. Since it is also 0 for all $x \in M$, we can therefore use it to create a new vector field on N_ε, the thickened manifold, that is outward on the boundary and which has the same zeros and indices as v.
Explicitly, consider $f(x) = ||x - r(x)||^2$. By a computation of the gradient, we see $\nabla f(x) = 2(x - r(x))$. Note that $f^{-1}(\varepsilon^2)$ puts us on ∂N_ε, since these are precisely the points for which $||x - r(x)|| = \varepsilon$. Since the gradient gives us the normal vector, we have that the Gauss map on $f^{-1}(\varepsilon^2)$ described above is given by

$$g(x) = \frac{\nabla f(x)}{||\nabla f(x)||} = \frac{2(x-r(x))}{||2(x-r(x))||} = \frac{x-r(x)}{||x-r(x)||} = \frac{x-r(x)}{\varepsilon}.$$

Note that on M, $r(x) = x$. Hence we can extend v to a vector field on N_ε using a vector field $w(x) = (x - r(x)) + v(r(x))$. Note that this is actually an extension of the vector field; vectors were n dimensional when first defined on M^k via v, and they are n dimensional when defined on N via w. Note

$$w(x) \cdot g(x) = [(x-r(x)) + v(r(x))] \cdot [\frac{x-r(x)}{\varepsilon}] = \frac{||x-r(x)||^2}{\varepsilon} + v(r(x)) \cdot (x - r(x)).$$

Clearly $v(r(x))$ and $x - r(x)$ are orthogonal; the closest point in Euclidean space is always orthogonal to any tangent vectors, and every vector in a vector field is a tangent vector. So on the boundary of N_ε, we see that $||x - r(x)|| = \varepsilon$ and $v(r(x)) \cdot (x - r(x)) = 0$. Hence the above equation simplifies to $w(x) \cdot g(x) = \varepsilon > 0$, so $w(x)$ points outward at the boundary.
Now, our vector field w can only have a zero if v has a zero, because the two vectors $(x - r(x))$ as $v(r(x))$ are orthogonal, and hence their sum cannot be zero. So, consider the derivative of w. Note that $TN_z = \mathbb{R}^n$. If $h \in TM_z$, then $dw_z(h) = dv_z(h)$, and if $h \in TM_z^\perp$, $dw_z = h$. Hence the derivatives at a zero have the same determinant, which means their index sums are the same since the vector field is non-degenerate. Applying the previous theorem to N_ε, a n manifold living in n dimensional space with boundary, we get the desired result. □

Putting this together, we have proved:

Theorem 3.13. *The sum of the indices is invariant; that is, we have that the sum of the indices of a smooth vector field v is the same for any v on a manifold M¿*

4. Fun with the Euler Characteristic

In this section we analyze the Euler characteristic more closely. Recall that the definition from the introduction. Consider some triangulated space X, and let $C_n(X)$ be a real vector space with n-simplices $[x_0, \dots, x_n]$ (where an $n-$ simplex is just a collection of n vertices). The x_is refer to the vertices chosen, and each different combination forms a different basis vector of our vector space, which, emphasizing further, we are taking to be over $\mathbb{R}$. Hence we can think of the dimension of the vector space as being the different number of such combinations. To keep ordering straight, we adopt the following convention: let σ be a permutation of $0, 1, \dots, n$, representable by a matrix consisting of exactly one 1 in each column and each row, and zeros elsewhere, and note that σ is representable by a matrix. Then given a k simplex,

$$[n_0, \dots, n_k] = \det(\sigma)[n_{\sigma(0)}, \dots, n_{\sigma(k)}].$$

Definition 4.1. Suppose we have a sequence of vector spaces $(C_n)_{n\in\mathbb{Z}}$ and linear maps $\partial_n : C_n \to C_{n-1}$ such that $\partial_n \circ \partial_{n+1} = 0$ for all n. Such a sequence is called a *chain complex*. The homomorphisms are called *boundary operators*.

Remark 4.2. One can substitute groups for vector spaces and homomorphisms for linear transformations, but for this paper the above definition is convenient.

In theory, one could take these sequences to have no starting or stopping elements. However, if we only have k groups, then we can take $C_n = 0$ for $n > k$ and obtain a chain complex.

Now consider the following linear homomorphisms:

$$\partial_n([m_0, \dots, m_n]) = \sum_{k=0}^{n}(-1)^k([m_0, \dots, \widehat{m_k}, \dots, m_n])$$

where $\hat{m_k}$ means that we take the $n-1$ simplex formed by deleting the kth vertex. That this defines a homomorphism is straightforward, but one can also show that $\partial_n \circ \partial_{n+1} = 0$. Indeed, note that

$$\begin{aligned}\partial_n \circ \partial_{n+1}([m_0, \dots, m_{n+1}]) &= \partial_n\left(\sum_{k=0}^{n}(-1)^k([m_0, \dots, \widehat{m_k}, \dots, m_n])\right) \\ &= \sum_{k=0}^{n}(-1)^k(\partial_n([m_0, \dots, \widehat{m_k}, \dots, m_n])).\end{aligned}$$

Pick an arbitrary k, and consider $\partial_n([m_0, \dots, \widehat{m_k}, \dots, m_n])$. For each of our $k-2$ simplices $[m_0, \dots, \widehat{m_j}, \widehat{m_k}, \dots, m_n]$, note that there is exactly on other term in the larger sum with the same elements removed, that is, $[m_0, \dots, \widehat{m_k}, \widehat{m_j}, \dots, m_n]$, and since each of these obtain opposite signs, we have $\partial_n \circ \partial_{n+1} = 0$ as desired.

Definition 4.3. Suppose we have some chain complex consisting of groups C_n and homomorphisms ∂_n. We define the *ith homology group* to be

$$H_n(X; \mathbb{R}) = \ker \partial_n / \mathrm{im}\ (\partial_{n+1})$$

Definition 4.4. A *n-cell* is an (assumed to be closed) n-dimensional object homeomorphic to a closed n-ball; if $n = 0$ then the n-cell is taken to be a single point. A *0-skeleton* therefore consists of a collection of points. Now, supposing we have a $k-1$-skeleton, we inductively define a *k-skeleton* to be a collection of k cells whose boundary components are attached at elements of the $k-1$-skeleton. Note immediately that this makes sense, since the boundary of segments are points, the boundary of disks are segments, and so on. A space formed in this way is called a

CW-complex. If n is finite then the topology is on a CW-complex X is usual, but if n is infinite, then we require a set $A \subset X$ is open if and only if $A \cap X^n$ is open for all n.

Remark 4.5. The notion of "attach" was not formulated entirely precisely in the previous definition, so some further clarification is necessary. Suppose we have X^{n-1} and $n-1$-skeleton, and if S^{n-1} is the boundary of an $n-$cell, we have $g : S^{n-1} \to X^{n-1}$ a continuous map. In this case, attaching an $n-$cell is the consideration of $S^{n-1} \cup X^{n-1}$ with the identification of $x \in S^{n-1}$ with $g(x) \in X^{n-1}$.

Proposition 4.6. *Let C_n be a chain complex of finite dimensional real vector spaces, and take $C_n = 0$ if $|n| > N$. Then*

$$\sum_{i=-N}^{N} (-1)^i \dim C_i = \sum_{i=-N}^{N} (-1)^i \dim H_i.$$

Proof. Since $H_i = \ker \partial_i / \mathrm{im}\ \partial_{i+1}$, we have

$$\begin{aligned}
\sum_{-N}^{N} (-1)^i \dim H_i &= \sum_{i=-N}^{N} (-1)^i [\dim \ker \partial_i - \dim \mathrm{im}\ \partial_{i+1}] \\
&= \sum_{i=-N}^{N} (-1)^i \dim \ker \partial_i + \sum_{i=-N}^{N} (-1)^{i+1} \dim \mathrm{im}\ \partial_{i+1} \\
&= \left(\sum_{i=-N+1}^{N} (-1)^i [\dim \ker \partial_i + \dim \mathrm{im}\ \partial_i] \right) + (-1)^{-N} \ker \partial_{-N} + (-1)^{N+1} \mathrm{im}\ \partial_{N+1}.
\end{aligned}$$

By rank nullity, we have $\dim \ker \partial_i + \dim \mathrm{im}\ \partial_i = \dim C_i$. Also, by assumption, we have C_{N+1} and C_{-N-1} are both 0. Hence $\mathrm{im}\ \partial_{-N} = 0$, since this boundary operator maps to $-N-1$ which is a 0-dimensional space, and $\mathrm{im}\ \partial_{N+1} = 0$, since this boundary operator maps from $N+1$ which is a 0 dimensional space. Thus, as desired,

$$\begin{aligned}
\sum_{i=-N}^{N} (-1)^i \dim H_i &= \left(\sum_{i=-N+1}^{N} (-1)^i \dim C_i \right) + (-1)^{-N} \dim \ker \partial_{-N} + (-1)^{-N} \dim \mathrm{im}\ \partial_{-N} + 0 \\
&= \sum_{i=-N}^{N} (-1)^i \dim C_i
\end{aligned}$$

□

Definition 4.7. Let $H_i(M)$ be the ith homology group of M. The *Euler characteristic* is $\sum_{i=0}^{m} (-1)^i \dim H_i(M)$.

Corollary 4.8. *The two definitions of the Euler characteristic are equivalent.*

Proof. The dimensions of C_i correspond to the number of vertices, edges, faces, etc. used in a triangulation. Hence the result follows directly from proposition 3.5. □

Observe that the construction of the CW-complex is directly related to the triangulation of a manifold. Indeed, take some triangulation of an arbitrary manifold. The vertices correspond to the 0-cells, the edges correspond to the 1-cells, the faces correspond to the two cells, and so on.

Now, we associate a vector space to each CW-complex, that is, we consider some CW space X, and let $C_n^{CW}(X)$ be a real vector space over a basis consisting of n-cells, analogously to simplicial complexes. Not only is this the case, but there are boundary operators for this chain complex that allow us to define homology over $\mathbb{R}$ in the same way. This is encapsulated in the following theorem.

Theorem 4.9. *For any X that is both a triangulated space and a CW-space, CW-homology is isomorphic to simplicial homology.*

See [3] for more details. Thus the above proposition carries over unchanged. In the next section we explain how CW-homology can be used find the Euler characteristic based on a special class of functions.

5. Morse Theory

In this section we explain the relevance of Morse theory to the discussion of the Euler Characteristic. Let $f : M \to \mathbb{R}$ be a twice differentiable function. Recall from multivariable calculus the definition of the *Hessian* as the matrix of cross partials. We say p is critical if $df_p : TM_p \to \mathbb{R}$ is zero, and non-degenerate if the Hessian is non-singular. Since the Hessian is symmetric, by the spectral theorem, it is diagonalizable, and at non-degenerate critical point, all eigenvalues are nonzero.

Definition 5.1. For a function $f : M \to \mathbb{R}$ as above with only non-degenerate critical points, the *index of f at a critical point p* is the number of negative eigenvalues of the Hessian. Such a function is called a *Morse function.*

Note that Morse functions always exist; see [2]. Note that the definition of the index of a critical point is not to be confused with the index of a zero of a vector field, though we will often refer to the index of a critical point of f at j using notation $\mathfrak{i}_{f,j}$. Roughly speaking, Morse theory studies the indices of these critical points. This might seem somewhat complicated, but actually given appropriate charts, life isn't so bad after all.

Lemma 5.2 (Morse)**.** *Given a k-dimensional manifold M, a function $f : M \to \mathbb{R}$ and a non-degenerate critical point p for f, there is some chart ϕ in a neighborhood of p such that $\phi(p) = 0$ and $f(x_1, \ldots, x_k) = f(p) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_k^2$, where i is the index of f at p.*

Definition 5.3. Let X and Y be two topological spaces. We say that X and Y are of the same *homotopy type* if there exist functions $f : X \to Y$and $g : Y \to X$ such that $f \circ g \sim \text{identity}_X$ and $g \circ f \sim \text{identity}_Y$.

Example 5.4. Of course, two isomorphic topological spaces are of the same homotopy type. Additionally, $\mathbb{R}^2$ and a point are of the same homotopy type. Let $f : p \to \mathbb{R}^2$ be the map that sends p point to the origin and $g : \mathbb{R}^2 \to p$ be the map that send every point to p. Then fg is the identity and gf is homotopic to the identity via the homotopy $h : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2$ defined by $h((x, y), t) = t(x, y)$.

We see that there are plenty of other examples from the following theorems.

Theorem 5.5. *If f is a smooth real valued function on a manifold M, and if a and b are points such that $a < b$ and $f^{-1}[a, b]$ has no critical points, then $f^{-1}(-\infty, a]$ is diffeomorphic to $f^{-1}(\infty, b]$; in particular, these two sets are of the same homotopy type.*

Theorem 5.6. *If f is a smooth real valued function on a manifold M, and p a non-degenerate critical point with index i, if $f^{-1}([f(p)-\varepsilon, f(p)+\varepsilon])$ is compact with exactly one critical point, then for all ε sufficiently small, $f^{-1}(-\infty, f(p)-\varepsilon]$ with an i-cell attached appropriately is of the same homotopy type as $f^{-1}(-\infty, f(p)+\varepsilon]$.*

Indeed, attaching an $n-$cell is meant in the same rigorous formulation as indicated in the remark describing the construction of CW-complexes. Now, one can show that homology groups are invariant under homotopy, meaning that two manifolds of the same homotopy type have the same Euler characteristic. Hence, suppose we have some Morse function $f : M \to \mathbb{R}$. Then M has some associated CW-structure, and it consists of an attached i cell for every critical point of index i, by the above lemma.
So, suppose we have a Morse function f as above. By the above theorems, a Morse function gives us $C_*^{CW}(M)$, which in turn allows us to compute the Euler characteristic. Thus let $\mathfrak{i}_{f,j}$ be the number of critical points of f with index j. We have:

Corollary 5.7. *For any closed manifold M and a Morse function f,*

$$\sum_{j=0}^{n}(-1)^j \mathfrak{i}_j = \chi(M).$$

Proof. The number of critical points with index j is the number of cells of dimension j which must be attached. By corollary 4.8, the alternating sum is equal to the Euler characteristic. □

Examples 5.8. We could calculate the Euler characteristics of many shapes, but in particular we use these theorems to calculate the Euler characteristics of a few broad manifolds.
1) Any odd dimensional closed manifold has Euler characteristic 0. To see this, consider some Morse function $f : M \to \mathbb{R}$. Then by the corollary, $\sum_{j=0}^{n}(-1)^j \mathfrak{i}_{f,j} = \chi(M)$. But $-f$ is also a Morse function, and hence $\sum_{j=0}^{n}(-1)^j \mathfrak{i}_{-f,j} = \chi(M)$. Any point x that is a critical point of f with index k is also a critical point of $-f$ with index $n-k$, and visa versa. Thus

$$\sum_{j=0}^{n}(-1)^j \mathfrak{i}_{f,j}+\sum_{j=0}^{n}(-1)^j \mathfrak{i}_{-f,j} = \sum_{j=0}^{n}(-1)^j \mathfrak{i}_{f,j}+\sum_{j=0}^{n}(-1)^j \mathfrak{i}_{f,n-j} = \sum_{j=0}^{n}(-1)^j \mathfrak{i}_{f,j}+(-1)^j \mathfrak{i}_{f,n-j} = 0,$$

where the last equality follows from n being odd. So $0 = 2\chi(M)$ and hence $\chi(M) = 0$, as claimed.
2) The Euler Characteristic of an even dimensional sphere is 2. To see this, consider the height function. This is a Morse function with two critical point. The first critical point has index 0 and the second critical point has index n. Since n is even, we have the Euler characteristic is $(-1)^0 \cdot 1 + (-1)^n \cdot 1 = 2$.
3) The Euler characteristic of $\mathbb{T}^2$ is -2; if we embed this manifold into $\mathbb{R}^3$, then we can define a projection function onto one of the axes, and this is a Morse function (though one should be careful, because the wrong projection is not a Morse function). At the bottom, we see that the function is increasing in two directions so its index is 0. There are two other points in the center which are saddle points, and at the top, looking in both directions on the Manifold we see the function is decreasing. Hence the Euler characteristic is $1^0 + (-1)^1 + (-1)^1 + (-1)^2 = -2$.

Similarly, for the two holed torus, everything is the same except there are now 4 saddle points, meaning that the Euler characteristic is now -4. In fact, the genus of any g holed torus is $2-2g$, since adding a hole to a torus adds two saddle points and thus decreases the genus by 2.

As a final example, we give the following proposition which is fundamental in the proof of Poincare-Hopf.

Theorem 5.9. *On any closed manifold M there is a vector field v such that the sum of the indices of the critical points of v is equal to the Euler characteristic of M.*

Proof. By the Morse Lemma, we know that there is a function $f : M \to \mathbb{R}$ such that, for any critical point p on M, we have

$$f(x_1, \dots, x_k) = f(p) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_k^2.$$

A point p is a critical point on f is if the gradient of f at p is zero, so it also happens to be a zero of the vector field defined by the gradient of the above function. If the index is k, there are k negative eigenvalues, so that the determinant is positive if k is even and negative if k is odd, which means that the index of the zero of the vector field is $(-1)^k$ by the above. Hence the sum of the indices of the vector field is equal to the alternating sum of indices of the critical points described above. Thus the sum of the indices of this vector field is equal to the Euler characteristic as desired. $\square$

6. The Poincare-Hopf Theorem

We now return to the Poincare-Hopf theorem.

Theorem 6.1 (Poincare-Hopf). *Let v be a vector field on M, where M is without boundary, which also only has isolated zeros. Then the sum of the indices of the zeros of the vector field is equal to the Euler characteristic of the manifold.*

Remark 6.2. The theorem is true from M with boundary, but requires some effort because the differentiability conditions in the below steps are not as immediate. The version of the theorem for manifolds with boundary requires that the vector field be pointing outward at the boundary.

Proof. Above, we show that the sum of the indices of a vector field is invariant, since it is equal to the degree of the Gauss map, which means it suffices to find a single example of a vector field and compute its index. Using Morse theory, we found that there is such a vector field, and that for this vector field, the sum of the indices at the zero is equal to the Euler characteristic. Hence the statement holds for manifolds with boundary with only non-degenerate zeros.

Now, suppose we have a vector field v defined on an open set $U \subset \mathbb{R}^k$ with a degenerate zero z. Let $\varepsilon > 0$ be so small that z is the only zero inside a ball of radius 2ε. Let $f : U \to [0, 1]$ be a smooth function such that $f(x) = 1$ in a ball of radius ε around z, and 0 outside of ball of radius 2ε. Consider the vector field $\hat{v}(x) = v(x) - f(x)y$, where y is some regular value of the vector field v (which exists by Sard's theorem). In fact, note that $||v(x)||$ is greater than some $\delta > 0$ for all x outside of the ball of radius ε and inside the ball of radius 2ε. Hence we can pick y so small such that $||y|| < \delta$, so that all of the zeros of the new vector field are inside of the ball of radius ε. Now, however, if we consider any zero of the new vector field, we see that since $f(x)$ is constant inside the ball of radius ε, that $d\hat{v}(z') = v(z') \neq 0$. Hence this is a nondegenerate vector field. Doing this for every zero, we thereby obtain a non-degenerate vector field. To finish this part of the theorem, we must show that the index of the vector field is unchanged. Specifically, let $\mathfrak{i}_{v,z}$ be the index of a zero of the original vector field. Note that, by the definition of the index, the index is equal to the degree of the map $\frac{v(x)}{||v(x)||}$ around $\partial B_{2\varepsilon}(z)$. We show that, if $\sum_{z'} \mathfrak{i}_{\hat{v},z'}$ is the sum of the indices of $\hat{v}$ at the zeros z' inside the ball of radius 2ε, then $\mathfrak{i}_{v,z} = \sum_{z'} \mathfrak{i}_{\hat{v},z'}$. This argument, however, is almost identical to the one in theorem 3.12. Let $\varepsilon' > 0$ be so small such that there are no zeros outside of a ball of radius ε' around each zero z'. Removing these balls, we consider the degrees of the maps at each of the boundary components. This must be zero, since oriented as the boundary, it extends to a smooth vector field. The degree on the outer component is simply $\mathfrak{i}_{\mathfrak{z}}$, since $v = \hat{v}$ on this component. Using an identical argument as above, we have on the other components the degree is $-\sum_{z'} \mathfrak{i}_{\hat{v},z'}$. Hence

$$\mathfrak{i}_{\mathfrak{v},\mathfrak{z}} - \sum_{z'} \mathfrak{i}_{\hat{v},z'} = 0$$

and the desired result follows.
After using an appropriate chart to transfer the above argument from an open subset $U \subset \mathbb{R}^k$ to a k dimensional manifold M^k, we have that the theorem holds even for degenerate zeros. $\square$

7. Consequences and Applications of the Theorem

To reference the question which began the section, we now consider shapes we are typically familiar with and see

Theorem 7.1 (Hairy Ball). *A smooth vector field on an even dimensional sphere must have at least 1 zero.*

Proof. By Poincare Hopf, the index sum is the Euler characteristic, which is 2. If there were no zeros, the index sum would be zero. Hence, there is a zero. $\square$

Additionally, for $g \geq 2$, we see that a smooth vector field on g-holed torus must have a zero, for the exact same reason. However, this is not true for the 1- holed torus, for which there is, indeed, smooth a vector field with no zero; letting the torus lie flat in $\mathbb{R}^3$, we can let all vectors have length 1 and lie parallel to the x-axis. This alludes to the possibility of rotating the 1 holed torus so that all points are rotated approximately 1 unit. When we apply this to the sphere, we see that this can not occur. Thus we have the following theorem as well.

Theorem 7.2. *Any rotation of the 2 dimensional sphere has exactly 2 fixed points.*

Proof. Assign a vector at each point x that represents the magnitude of the rotation of x, that is, the distance between the starting and ending position of x after it is rotated. This defines a smooth vector field on the sphere. If the Euler characteristic is not equal to zero, then there must be a zero for this vector field, corresponding to a point which does not move. Indeed, this is a nondegenerate vector field, so each zero has index +1 or -1. By symmetry, it has the same index at every zero, since the rotation looks the same at every zero. Since the Euler characteristic is 2, then there must be 2 zeros with index +1. $\square$

One could think about how to extend the above theorem to other manifolds. For example, as should be expected since the Euler characteristic of the circle is 0, one can rotate a circle so that no points are fixed.
Finally, note that zeros with a different index look different. Thus we see that, given the existence of a zero on our manifold M, being told the index gives us information as to what the index is, thereby limiting what the vector field can look like at that zero. If a zero of index 1 is added to a torus, for example, then a zero of index -1 has to be added, or multiple other zeros with index sum equal to -1. Hence we see that altering the vector field locally to include a zero must have global ramifications as to what the vector field looks like elsewhere, in particular at the other zero.

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