Manes

Relative functorial semantics, III. Triples vs. theories.

by F. E. J. Linton

1. The construction of Kleisli associates with each triple $\mathbb{T} = (T, \eta, \mu)$ on a category $\mathcal{A}$ a category $\mathrm{Kl}(\mathbb{T})$ (cf. [1] and [3]), having the same objects as $\mathcal{A}$, and a functor $f^{\mathbb{T}}: \mathcal{A} \longrightarrow \mathrm{Kl}(\mathbb{T})$, working as the identity on the objects and having a right adjoint $u^{\mathbb{T}}$ that, on objects, works like T, the adjunction isomorphisms being the <u>identity</u> maps

$$\mathrm{Kl}(\mathbb{T})(f^{\mathbb{T}}A, B) = \mathrm{Kl}(\mathbb{T})(A, B) \underset{\mathrm{def}}{=} \mathcal{A}(A, TB) = \mathcal{A}(A, u^{\mathbb{T}}B) .$$

2. This note records a simple and informative conceptual argument for the complete identification (announced in [2] and arduously established in [3]) of the Eilenberg-Moore category $\mathcal{A}^{\mathbb{T}}$ of algebras over $\mathbb{T}$ (cf. [0]) with the category of Lawvere-style algebras over the Kleisli category $\mathrm{Kl}(\mathbb{T})$. It will be recalled that the former is equipped with a canonical "underlying $\mathcal{A}$-object" functor $U^{\mathbb{T}}: \mathcal{A}^{\mathbb{T}} \longrightarrow \mathcal{A}$, and that the latter is, by definition, any $\mathcal{A}$-valued functor serving as a pullback of the diagram

(1)
$$\begin{tikzcd} & \mathcal{S}^{\mathrm{Kl}(\mathbb{T})} \arrow[d, "\mathcal{S}^{f^{\mathbb{T}}}"] \\ \mathcal{A} \arrow[r, "Y"'] & \mathcal{S}^{\mathcal{A}} \end{tikzcd}$$

in which Y is the Yoneda embedding and <u>the functor category notation is used to indicate categories of contravariant functors</u>.

3. To see that $U^{\mathbb{T}}$ serves as pullback of (1), use is first made of the Yoneda Lemma and the category $(\mathcal{S}^{\mathcal{A}})_{\check{\mathbb{T}}}$ of Eilenberg-Moore coalgebras over the "composition with the ingredients of $\mathbb{T}$" cotriple $\check{\mathbb{T}} = (\check{T}, \check{\eta}, \check{\mu})$ on the (contravariant functor) category $\mathcal{S}^{\mathcal{A}}$. Here

$$\check{T}(X) = X \circ T , \quad \check{\eta}_X = X \circ \eta , \quad \check{\mu}_X = X \circ \mu .$$

Each Eilenberg-Moore $\mathbb{T}$-algebra $\mathbb{B} = (B, \beta)$ becomes [3] a coalgebra

$$\check{Y}(\mathbb{B}) = (Y_B, \check{\beta}) = (\mathcal{A}(-, B), \ \mathcal{A}(-, B) \xrightarrow{T} \mathcal{A}(T-, TB) \xrightarrow{\beta} \mathcal{A}(T-, B))$$

over the cotriple $\check{\mathbb{T}}$. In this way, there arises a functor

$$\check{Y}: \mathfrak{A}^{\mathbb{T}} \longrightarrow (\mathfrak{S}^{\mathfrak{A}})_{\check{\mathbb{T}}}$$

lifting Y over $U^{\mathbb{T}}: \mathfrak{A}^{\mathbb{T}} \to \mathfrak{A}$ and $U_{\check{\mathbb{T}}}: (\mathfrak{S}^{\mathfrak{A}})_{\check{\mathbb{T}}} \to \mathfrak{S}^{\mathfrak{A}}$. The Yoneda Lemma now indicates: first, that a $\check{\mathbb{T}}$-coalgebra structure on a represented functor YB "is" nothing more than a $\mathbb{T}$-algebra structure on B; next, that $\check{Y}$ is fully faithful; and last, that $\check{Y}$ makes $U^{\mathbb{T}}: \mathfrak{A}^{\mathbb{T}} \to \mathfrak{A}$ the pullback of the diagram

(2)
$$\begin{tikzcd} & (\mathfrak{S}^{\mathfrak{A}})_{\check{\mathbb{T}}} \arrow[d, "U_{\check{\mathbb{T}}}"] \\ \mathfrak{A} \arrow[r, "Y"'] & \mathfrak{S}^{\mathfrak{A}} . \end{tikzcd}$$

4. For $U^{\mathbb{T}}$ to be the pullback of diagram (1), therefore, it would be nice if $\mathfrak{S}^{\mathrm{Kl}(\mathbb{T})}$ and $(\mathfrak{S}^{\mathfrak{A}})_{\check{\mathbb{T}}}$ were isomorphic as categories over $\mathfrak{S}^{\mathfrak{A}}$. It is nice: they are. The adjointness between $f^{\mathbb{T}}$ and $u^{\mathbb{T}}$, with adjunction triple $\mathbb{T}$ on $\mathfrak{A}$, provides an adjunction making $\mathfrak{S}^{u^{\mathbb{T}}}$ right adjoint to $\mathfrak{S}^{f^{\mathbb{T}}}$, with adjunction cotriple $\check{\mathbb{T}}$ on $\mathfrak{S}^{\mathfrak{A}}$. Moreover, $\mathfrak{S}^{f^{\mathbb{T}}}$ is easily seen to create equalizers of $\mathfrak{S}^{f^{\mathbb{T}}}$-split pairs, $f^{\mathbb{T}}$ being a bijection on the object classes. Thus, Beck's Theorem (cf. [3]), in its cotriple version, completes this proof and ends the argument. Of course, Beck's Theorem could have been applied directly to the pullback of (1), but checking its hypotheses would have been more tedious, and the isomorphism of this paragraph would have escaped notice.

5. P.S.: The reader whom our notation (and references) successfully misled into assuming, comfortably, that $\mathfrak{S}$ referred to his favorite category of sets and functions is hereby invited to choose an arbitrary multilinear category $\mathfrak{S}$ and to place himself in the cosmos of $\mathfrak{S}$-categories, where, bearing in mind that, even though $\mathrm{Kl}(\mathbb{T})$ remains (cf. [5]) an $\mathfrak{S}$-category when $\mathbb{T}$ is an $\mathfrak{S}$-triple on the $\mathfrak{S}$-category $\mathfrak{A}$, the constructions of $\mathfrak{A}^{\mathbb{T}}$, the functor categories, the pullback of (1), and $(\mathfrak{S}^{\mathfrak{A}})_{\check{\mathbb{T}}}$ may force him to enter the larger cosmos (cf. [7]) of pro-$\mathfrak{S}$-categories (so that Street's suggested procedure [8] isn't readily applied), he may nevertheless assure

himself, using [4] and [6] for the Yoneda Lamma and Beck's Theorem, that the argument here presented remains entirely valid.

6. References.

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