COCYCLE CONSTRUCTIONS FOR TOPOLOGICAL FIELD THEORIES

BY

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DISSERTATION

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Abstract

In this thesis, we explore a chain-level construction in smooth Deligne cohomology that produces data for extended topological field theories. For closed oriented manifolds Σ, this construction takes the form of a chain map τ_Σ from the smooth Čech-Deligne complex on a manifold X to a degree-shifted version of the same complex on the mapping space X^Σ. More generally, if Σ has boundary $\partial\Sigma$, the construction produces a chain null-homotopy of the chain map $\tau_{\partial\Sigma}$ associated to the boundary.

To my parents.

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Chapter 1

Introduction

This work was born out of an attempt to generalize some simple examples of topological field theories. For example, given a manifold X and a principal S^1-bundle $P \to X$ with connection, one can construct a 1-dimensional field theory Z_P over X, i.e. a field theory whose domain is the category of 0-dimensional manifolds and 1-dimensional bordisms *over* X. The target of Z_P is the category of S^1-torsors; its value on $pt \xrightarrow{x} X$ is the fiber P_x, and its value on a path $\gamma\colon x \rightsquigarrow y$ is is the parallel transport operator $\Gamma(\gamma)_x^y\colon P_x \to P_y$.

There is a similar story one dimension higher. For details of the following constructions, see Brylinski [Bry08]. We will also return to this example, and provide definitions of the terms involved, in chapter 4.3. Let $\mathcal{G}$ be an S^1-gerbe over X, equipped with a connective structure and curving. Connective structures and curvings are the gerby analogues of connections on a principal bundles, and give rise to a notion of holonomy in $\mathcal{G}$. This in turn gives rise to a 2-dimensional topological field theory $Z_\mathcal{G}$ over X, again taking values in the category of S^1-torsors. The construction works as follows.

If Σ is a closed, oriented 2-manifold, then for any map $\phi\colon \Sigma \to X$, the holonomy in $\mathcal{G}$ around ϕ is an element $\mathrm{Hol}_\Sigma(\phi) \in S^1$. However, if Σ has boundary C, the holonomy takes values not in S^1, but in a principal S^1-bundle L_C over the mapping space X^C:

$$\begin{array}{ccc} & & L_C \\ & \overset{\mathrm{Hol}_\Sigma}{\nearrow} & \downarrow \\ X^\Sigma & \longrightarrow & X^C \end{array}$$

For any closed oriented 1-manifold C, and any map $\gamma\colon C \to X$, one defines $Z(C, \gamma)$ to be the fiber of L_C over $\gamma \in X^C$. For any compact oriented 2-manifold Σ, and any map $\phi\colon \Sigma \to X$, one defines

$$Z(\Sigma, \phi) = \mathrm{Hol}_{\mathcal{G}}(\phi) \in Z(\partial\Sigma, \phi|_{\partial\Sigma}).$$

The element $Z(\Sigma, \phi)$ may generally be interpreted as a morphism

$$Z(C_0, \gamma_0) \xrightarrow{Z(\Sigma,\phi)} Z(C_1, \gamma_1)$$

in the category of S^1-torsors, for any decomposition $\partial\Sigma = C_1 \amalg \overline{C_0}$, where $\gamma_i = \phi|_{C_i}$. The fact that this interpretation is possible, and gives rise to a topological field theory over X, boils down to some simple statements about the dependence of L_C on C, and of Hol_Σ on Σ, namely that these constructions are well behaved under disjoint unions, gluings, and orientation reversal.

The situation may be summarized as follows: if we agree to call principal S^1-bundles 0-gerbes, then for $n = 0$ and 1, every n-gerbe on X gives rise to an $(n+1)$-dimensional field theory on X taking values in S^1-torsors. This naturally leads to the question of whether a similar correspondance can be found for higher n. One might also wonder whether it is possible to get these constructions to produce extended field theories for $n > 0$.

This sounds plausible, but to provide a precise statement, let alone a precise answer, would seemingly require a lengthy excursion into the world of n-categories. So, rather than focusing directly on topological field theories, we will focus instead on generalizing the constructions that produce the bundle L_C and the section Hol_Σ mentioned above. The idea is to show that, given a fixed n-gerbe $\mathcal{G}$ (equipped with the appropriate notion of connection) on X, associated to any closed oriented d-manifold Σ is an $(n-d)$-gerbe $\mathcal{G}_\Sigma$ on X^Σ. If Σ has boundary C, then associated to C is an $(n-d+1)$-gerbe $\mathcal{G}_C$ on X^C, and associated to Σ is a section Hol_Σ of $\mathcal{G}_C$ over X^Σ.

Suppose we had such a construction. Then from any n-gerbe with connection on X, we could build, at least heuristically, an extended topological field over X. By extended, we mean that the theory would assign data to manifolds of dimension 0 through n. As above, fix an n-gerbe $\mathcal{G}$ with connection on X. For any finite, oriented 0-dimensional manifold Σ, our construction produces an n-gerbe $\mathcal{G}_\Sigma$ on X^Σ. For any particular map $\phi\colon \Sigma \to X$, we take the value of our field theory at ϕ to be the fiber of $\mathcal{G}_\Sigma$ over $\phi \in X^\Sigma$. We will denote this fiber by $\mathcal{G}_\Sigma(\phi)$.

Next, if $W\colon \Sigma_0 \to \Sigma_1$ is an oriented 1-dimensional bordism, our construction gives rise to a section Hol_W of $\mathcal{G}_{\Sigma_1 \amalg \overline{\Sigma_0}}$ over X^W. If we require that

$$\mathcal{G}_{\Sigma_1 \amalg \overline{\Sigma_0}} \cong \mathrm{Hom}(\mathcal{G}_{\Sigma_0}, \mathcal{G}_{\Sigma_1}),$$

then Hol_W determines a morphism $\mathcal{G}_{\Sigma_0} \to \mathcal{G}_{\Sigma_1}$. Thus for any particular $\Phi \colon W \to X$, thought of as a bordism over X from $\phi_0 = \Phi|_{\Sigma_0}$ to $\phi_1 = \Phi|_{\Sigma_1}$, our construction produces a map of fibers

$$\mathrm{Hol}_W(\Phi) \colon \mathcal{G}_{\Sigma_0}(\phi_0) \to \mathcal{G}_{\Sigma_1}(\phi_1).$$

If $V \colon \Sigma_1 \to \Sigma_2$ is another 1-dimensional bordism, then we should require the composition

$$\mathcal{G}_{\Sigma_0} \xrightarrow{\mathrm{Hol}_W} \mathcal{G}_{\Sigma_1} \xrightarrow{\mathrm{Hol}_V} \mathcal{G}_{\Sigma_2},$$

which is defined over $X^{W \amalg_{\Sigma_1} V}$, to equal (or at least be equivalent to) $\mathrm{Hol}_{W \amalg_{\Sigma_1} V}$.

Next, suppose W_0 and W_1 are both 1-dimensional bordisms $\Sigma_0 \to \Sigma_1$. Let $\Sigma = \Sigma_1 \amalg \overline{\Sigma_0}$, and let $W = W_1 \amalg_\Sigma \overline{W_0}$. Then W is closed, and thus there is an associated $(n-1)$-gerbe $\mathcal{G}_W$ over X^W. But an $(n-1)$-gerbe is the same thing as a section of the trivial n-gerbe. Moreover, the n-gerbe of automorphisms of $\mathcal{G}_{\Sigma_0}$ is trivialized, as it carries a canonical global section. Thus $\mathcal{G}_W$ determines an automorphism of $\mathcal{G}_{\Sigma_0}$, and we should require this automorphism to equal (or at least be equivalent to) the composite

$$\mathcal{G}_{\Sigma_0} \xrightarrow{\mathrm{Hol}_{W_1}} \mathcal{G}_{\Sigma_1} \xrightarrow{\mathrm{Hol}_{\overline{W_0}}} \mathcal{G}_{\Sigma_0}.$$

This is the same as saying that $\mathcal{G}_W$ is equivalent to the $(n-1)$-gerbe of 2-morphisms

$$\mathrm{Hol}_{W_0} \to \mathrm{Hol}_{W_1}.$$

Now, suppose that N is a 2-dimensional manifold with boundary W. Then associated to N is a section Hol_N of $\mathcal{G}_W$ over X^N, which, as mentioned above, is the same thing as a 2-morphism

$\mathrm{Hol}_{W_0} \to \mathrm{Hol}_{W_1}$. The diagram below illustrates the situation.

The same story continues all the way up to dimension n, and gives rise, roughly, to an extended topological field theory.

We will approach these constructions at the level of Čech cochains. Let $\mathbb{T}$ denote the sheaf of smooth S^1-valued functions, and for any $r \geq 0$ let $\mathbb{T}_D^\infty(r)$ denote the complex of sheaves

$$\mathbb{T} \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^r.$$

We will call $\mathbb{T}_D^\infty(r)$ the smooth Deligne complex; the ∞ and D stand for "smooth" and "Deligne" respectively.

For any open cover $\mathcal{U}$ of X, let $\check{C}^n(\mathcal{U}; \mathbb{T}_D^\infty(r))$ denote the n-th group of the totalization of the Čech double complex associated to $\mathbb{T}_D^\infty(r)$. Let D denote the total differential in this complex, and let $\check{Z}^n = \ker(D) \subseteq \check{C}^n$. The following proposition is a rephrasing of standard results.

Proposition 1.0.1. *Suppose that $\mathcal{U}$ is a good open cover of X, meaning that every non-empty intersection of finitely many sets in $\mathcal{U}$ is contractible. For $r \geq 0$, let $\mathcal{C}^1(r)$ denote the category associated to the (two-term) chain complex*

$$\check{C}^0(\mathcal{U}; \mathbb{T}_D^\infty(r)) \xrightarrow{D} \check{Z}^1(\mathcal{U}; \mathbb{T}_D^\infty(r)).$$

Then

- $\mathcal{C}^1(0)$ *is equivalent to the category of principal S^1-bundles on X,*
- $\mathcal{C}^1(1)$ *is equivalent to the category of principal S^1-bundles with connection,*
- $\mathcal{C}^1(r)$ *for $r \geq 2$ is equivalent to the category of principal S^1-bundles with flat connection.*

A similar result holds for S^1-gerbes on X, which are naturally organized into a 2-category.

Proposition 1.0.2. *For $r \geq 0$, let $\mathcal{C}^2(r)$ denote the 2-category associated to the chain complex*

$$\check{C}^0(\mathcal{U};\, \mathbb{T}^\infty_D(r)) \xrightarrow{D} \check{C}^1(\mathcal{U};\, \mathbb{T}^\infty_D(r)) \xrightarrow{D} \check{Z}^2(\mathcal{U};\, \mathbb{T}^\infty_D(r)).$$

Then,

- $\mathcal{C}^2(0)$ *is equivalent to the 2-category of gerbes on X,*
- $\mathcal{C}^2(1)$ *is equivalent to the 2-category of gerbes with connective structure,*
- $\mathcal{C}^2(2)$ *is equivalent to the 2-category of gerbes with connective structure and curving,*
- $\mathcal{C}^2(r)$*, for $r \geq 3$, is equivalent to the 2-category of gerbes with connective structure and flat curving.*

In light of these facts, we will adopt the point of view that an n-gerbe with connection on X is the same thing as a cocycle $\mathbf{A} \in \check{Z}^{n+1}(\mathcal{U};\, \mathbb{T}^\infty_D(n+1))$.

The main goal of this work is to define a homomorphism

$$\check{C}^{n+d}(\mathcal{U};\, \mathbb{T}^\infty_D(r+d)) \xrightarrow{\tau^n_\Sigma} \check{C}^n(\mathcal{U}^\Sigma;\, \mathbb{T}^\infty_D(r))$$

for any compact, oriented, d-dimensional manifold Σ, and for any non-negative integers n and r. Note that the super-script n in τ^n_Σ denotes the cohomological degree of the *output* of τ. The key result about τ is that it has the following chain homotopy property. This is restated as theorem 3.0.5 in chapter 3.

Theorem 1.0.3. *If Σ has boundary $\partial\Sigma$, then*

$$D \circ \tau^{n-1}_\Sigma + (-1)^n \tau^n_\Sigma \circ D = \rho^* \tau^n_{\partial\Sigma},$$

where ρ^* denotes the pull-back along the restriction map $X^\Sigma \to X^{\partial\Sigma}$.

In particular, if $\mathbf{A} \in \check{C}^{n+d}(X; \mathbb{T}^\infty_D(n+d))$ is closed, i.e. it satisfies $D\mathbf{A} = 0$, then for any d-dimensional Σ, $D\tau^n_\Sigma(\mathbf{A}) = \tau^{n+1}_{\partial\Sigma}(\mathbf{A})$. If we think of $\mathbf{A}$ as representing an $(n+d)$-gerbe on X, and if $\partial\Sigma = \emptyset$, then $\tau^n_\Sigma(\mathbf{A})$ represents an n-gerbe on X^Σ. If Σ has non-empty boundary C, then $\tau^n_\Sigma(\mathbf{A})$ represents a section over X^Σ of the $(n+1)$-gerbe on X^C represented by $\tau^{n+1}_C(\mathbf{A})$.

We will also show that τ_Σ respects gluings and orientation reversals in Σ, in the following sense.

Proposition 1.0.4. *Let $-\Sigma$ denote Σ equipped with the opposite orientation. Then for all n, $\tau^n_{-\Sigma} = -\tau^n_\Sigma$.*

This is proposition 3.0.9 in Chapter 3, and the following is proposition 3.0.10.

Proposition 1.0.5. *Suppose $\Sigma = \Sigma_1 \amalg_C \Sigma_2$ for some codimension 1 submanifold C. For $i = 1, 2$, let $r_i \colon X^\Sigma \to X^{\Sigma_i}$ denote the restriction map. Then*

$$\tau^n_\Sigma = r_1^* \tau^n_{\Sigma_1} + r_2^* \tau^n_{\Sigma_2}.$$

Proposition 1.0.5 encodes, at the level of Čech cochains, the consistency requirements we mentioned above in the discussion of how to produce an extended topological field theory from an n-gerbe. For example, if $W \colon \Sigma_0 \to \Sigma_1$ and $V \colon \Sigma_1 \to \Sigma_2$ are a pair of composable bordisms, we have

$$D\tau_W = \tau_{\Sigma_1} - \tau_{\Sigma_0}$$

so that τ_W determines a morphism $\tau_{\Sigma_0} \to \tau_{\Sigma_1}$, and likewise for τ_V. Moreover, we have

$$\tau_{W \amalg_{\Sigma_1} V} = \tau_W + \tau_V,$$

so that the composite of the morphisms determined by τ_W and τ_V is equal to the morphism determined by $\tau_{(W \amalg_{\Sigma_1} V)}$.

Finally, we will show that, in low dimensions, τ gives cocycle formulae for the constructions mentioned at the beginning of this introduction. The following is restated as proposition 4.2.1 in chapter 4.

Proposition 1.0.6. *Let Σ denote the manifold $[a, b]$. If a principal S^1-bundle with connection (L, ω) over X is given by the cocycle $\mathbf{A}$, then the bundle $\mathrm{Hom}(ev_a^* L, ev_b^* L)$ is given by the cocycle $\tau^1_{\partial\Sigma}(\mathbf{A})$, and the section of this bundle corresponding to parallel transport is given by $\tau^0_\Sigma(\mathbf{A})$.*

The follwing is proposition 4.3.9 of chapter 4.

Proposition 1.0.7. *Let $C = S^1$. If a gerbe $\mathcal{G}$ with connective structure and curving on X is given by a cocycle $\mathbf{A} \in \check{C}^2(\mathcal{U}; \mathbb{T}^\infty_D(2))$, then the holonomy bundle L_C over the free loop space $\mathcal{L}X = X^C$ is given by the cocycle $\tau^1_C(\mathbf{A})$.*

More generally, if C is a union of copies of S^1, then $\tau^1_C(\mathbf{A})$ is a cocycle representing the tensor product of the holonomy bundles for the various components. If Σ is an oriented 2-dimensional manifold with boundary $\partial\Sigma = C$, then by virtue of the relation

$$D\tau^0_\Sigma(\mathbf{A}) = \tau^1_C(\mathbf{A}),$$

the cochain $\tau^0_\Sigma(\mathbf{A})$ determines a section of the pull-back of L_C to X^Σ.

In chapter 2 we recall some definitions and set up notation, which we use in chapter 3 to define τ and to prove the results mentioned above. Finally, in chapter 4, we compare τ in low dimensions with known formulas for parallel transport and holonomy in bundles and gerbes. Throughout, all manifolds are assumed to be smooth, as are all maps of manifolds.

Chapter 2

Definitions and Notation

2.1 Partial Integration

In this section, we define the notion of a complex of sheaves *with integrals*, and show that the Deligne complex and the de Rham complex are examples.

Definition 2.1.1. A complex of sheaves with integrals on a manifold X consists of

- a complex of sheaves

$$\mathcal{A} = \left(\mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \cdots\right)$$

  on X

- for any compact manifold Σ, a complex of sheaves

$$\mathcal{A}_\Sigma = \left(\mathcal{A}^0_\Sigma \xrightarrow{d} \mathcal{A}^1_\Sigma \xrightarrow{d} \cdots\right)$$

  on the smooth mapping space X^Σ

- for any map $f\colon \Gamma \to \Sigma$ of compact manifolds, a map of complexes

$$\tilde{f}^*\colon \mathcal{A}_\Gamma \to \tilde{f}_* \mathcal{A}_\Sigma,$$

  where $\tilde{f}$ denotes the induced map $X^\Sigma \to X^\Gamma$

- for any compact oriented l-dimensional manifold Σ, any open $U \subseteq X$, and any $n \geq 0$, an integration map

$$\int_\Sigma \colon \mathcal{A}^{n+l}(U) \to \mathcal{A}^n_\Sigma(U^\Sigma).$$

We require the following four properties of the integration map.

1. (Chain homotopy) For any $A \in \mathcal{A}^{n+l}(U)$,

$$d \int_{\Sigma} A = \int_{\Sigma} dA + (-1)^n \, \tilde{\imath}^* \int_{\partial\Sigma} A$$

   where $\imath$ denotes the inclusion $\partial\Sigma \hookrightarrow \Sigma$.

2. (Orientation) If $-\Sigma$ denotes Σ with the opposite orientation, then

$$\int_{-\Sigma} A = -\int_{\Sigma} A.$$

3. (Naturality) For any $V \subseteq U$, the following square is commutative.

$$\begin{array}{ccc} \mathcal{A}^{n+l}(U) & \xrightarrow{\int_\Sigma} & \mathcal{A}^n(U^\Sigma) \\ \downarrow res & & \downarrow res \\ \mathcal{A}^{n+l}(V) & \xrightarrow{\int_\Sigma} & \mathcal{A}^n(V^\Sigma) \end{array}$$

4. (Additivity) For any smooth triangulation S of Σ,

$$\int_{\Sigma} A = \sum_s \tilde{\imath}_s^* \int_s A$$

   where the sum runs over all top-dimensional simplices s of S, and $\imath_s$ denotes the inclusion $s \hookrightarrow \Sigma$.

Example 2.1.2. We now show that the de Rham complex on X

$$\Omega = (\Omega^0 \to \Omega^1 \to \cdots)$$

is a complex of sheaves with integrals. Here Ω^n denotes the sheaf of smooth n-forms on X. For any compact manifold Σ, we define

$$\Omega_\Sigma = (\Omega^0_\Sigma \to \Omega^1_\Sigma \to \cdots)$$

to be the de Rham complex on X^Σ. Here, we use the fact that X^Σ is a smooth (albeit infinite dimensional) manifold, modeled on a complete, Hausdorff, locally convex topological vector space (sections 2-4 of Milnor [Mil84]). Such manifolds admit a good theory of differential forms (see, for example, section 1.4 of Brylinski [Bry08]), which we review briefly below. At any rate, Ω^n_Σ is the sheaf of n-forms on X^Σ, and for $f\colon \Gamma \to \Sigma$, the map of complexes $\tilde{f}^*\colon \Omega_\Gamma \to \tilde{f}_*\Omega_\Sigma$ is given by pull-back of forms.

If E and F are topological vector spaces, $U \subseteq E$ is an open set, and $f\colon U \to F$ is continuous, then f is said to be continuously differentiable if for all $x \in U$ and $e \in E$, the limit

$$Df(x, e) = \lim_{t\to 0} \frac{f(x + te) - f(x)}{t}$$

exists and depends continuously on x and e. In this case, the map $Df\colon U \times E \to F$ is called the first derivative of f; higher derivatives are defined similarly, and f is said to be smooth if it is continuously differentiable to all orders. An n-form on $U \subseteq E$ is then a smooth function $f\colon U \times E^n \to \mathbb{R}$ that is multilinear and alternating on E^n. The exterior derivative of an n-form f is defined by

$$\begin{aligned} df(x; v_0, \ldots, v_n) &= \sum_{j=0}^{n} (-1)^j D\Big[f(-; v_0, \ldots, \widehat{v_j}, \ldots, v_n)\Big](x, v_j) \\ &= \sum_{j=0}^{n} (-1)^j \lim_{t\to 0} \frac{f(x + tv_j; \ldots, \widehat{v_j}, \ldots) - f(x; \ldots, \widehat{v_j}, \ldots)}{t} \end{aligned}$$

For any $\phi \in X^\Sigma$, let $T_\phi X^\Sigma$ denote the vector space consisting of all smooth sections of TX along ϕ. These spaces form the local models for X^Σ; in particular, the theory of n-forms on X^Σ is defined in terms of these models. For any open set $U \subseteq X^\Sigma$, let TU denote the restriction to U of the tangent bundle of X^Σ, and let $TU^n = TU \oplus \cdots \oplus TU$, with n summands. Then an n-form $f \in \Omega^n(U)$ is a smooth map $f\colon TU^n \to \mathbb{R}$ that is multilinear and alternating on each fiber.

We must still define the integration maps for the de Rham complex. For any oriented l-dimensional Σ and any open set $U \subseteq X$, we define the integration map

$$\int_\Sigma \colon \Omega^{n+l}(U) \to \Omega^n(U^\Sigma)$$

by setting $\int_\Sigma A = I_\Sigma(A)$, where $I_\Sigma(A)$ is the n-form on U^Σ given by

$$I_\Sigma(A)(\phi; V_1, \ldots, V_n) = \int_\Sigma \phi^* \iota(V_1, \ldots, V_n) A \tag{2.1}$$

for any $\phi \in U^\Sigma$ and any $V_1, \ldots, V_n \in T_\phi X^\Sigma$. Note that the integral sign on the right hand side of (2.1) denotes ordinary integration, and $\iota(V_1, \ldots, V_n)A$ denotes the contraction of A with $V_1, \ldots, V_n$, in that order.

Properties 2-4 of definition 2.1.1 are standard facts about ordinary integration. Property 1, however, requires some proof. By properties 3 and 4, it suffices to show that property 1 holds when Σ is an l-simplex and U is a coordinate patch in X. In other words, we may replace U with the Euclidean space E of the same dimension as X.

Lemma 2.1.3. *Let A be an $(n+l)$-form on a Euclidean space E, and let σ denote an oriented l-simplex. As above, let $I_\sigma(A)$ denote the n-form on E^σ given by*

$$I_\sigma(A)(\phi; V_1, \ldots, V_n) = \int_\sigma \phi^* \iota(V_1, \ldots, V_n) A.$$

Then

$$dI_\sigma(A) = I_\sigma(dA) + (-1)^n I_{\partial\sigma}(A).$$

Remark 2.1.4. Fix $\phi \in E^\sigma$. By means of the identification $T_e E \cong E$ for all $e \in E$, we may identify $T_\phi E^\sigma \cong E^\sigma$. In other words, a tangent vector $V \in T_\phi E^\sigma$ is identified with a map $V : \sigma \to E$. Given $\phi \in E^\sigma$ and $V_1, \ldots, V_n \in T_\phi E^\sigma$, consider the map $\psi \colon \mathbb{R}^n \times \sigma \to E$ given by

$$\psi(t_1, \ldots, t_n, s) = \phi(s) + \sum_j t_j V_j(s).$$

This map has the important property that

$$I_\sigma(A)(\phi; V_1, \ldots, V_n) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^n} \int_{[0,\epsilon]^n \times \sigma} \psi^* A.$$

Proof of Lemma 2.1.3. Fix $\phi \in E^\sigma$ and tangent vectors $V_0, \ldots, V_n \in T_\phi E^\sigma$. Let $\psi \colon \mathbb{R}^{n+1} \times \sigma \to E$

be given by

$$\psi(t_0, \ldots, t_n, s) = \phi(s) + \sum_j t_j V_j(s)$$

For any $x \in \mathbb{R}$ and any $0 \leq j \leq n$, let $i_x^j \colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ denote the map given by

$$i_x^j(t_1, \ldots, t_n) = (t_1, \ldots, t_j, x, t_{j+1}, \ldots, t_n),$$

and let $\psi_x^j = \psi \circ (i_x^j \times \mathrm{id}_\sigma) \colon \mathbb{R}^n \times \sigma \to E$. Then by Remark 2.1.4,

$$I_\sigma(A)(\phi; V_0, \ldots, \widehat{V}_j, \ldots, V_n) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^n} \int_{[0,\epsilon]^n \times \sigma} (\psi_0^j)^* A$$

and

$$I_\sigma(A)(\phi + \delta V_j; V_0, \ldots, \widehat{V}_j, \ldots, V_n) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^n} \int_{[0,\epsilon]^n \times \sigma} (\psi_\delta^j)^* A$$

to second order in δ. It follows that

$$V_j \left[I_\sigma(A)(\phi; V_0, \ldots, \widehat{V}_j, \ldots, V_n) \right] = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{n+1}} \int_{[0,\epsilon]^n \times \sigma} (\psi_\epsilon^j)^* A - (\psi_0^j)^* A,$$

where the term on the left denotes the directional derivative of $I_\sigma(A)(\phi; \ldots)$ in the direction of V_j.

Finally, we have

$$
\begin{aligned}
&I_\sigma(dA)(\phi; V_0, \dots, V_n) \\
&\quad = \int_\sigma \phi^* \iota(V_0, \dots, V_n) dA \\
&\quad = \lim_{\epsilon\to 0} \frac{1}{\epsilon^{n+1}} \int_{[0,\epsilon]^{n+1}\times\sigma} \psi^* dA \\
&\quad = \lim_{\epsilon\to 0} \frac{1}{\epsilon^{n+1}} \int_{\partial([0,\epsilon]^{n+1}\times\sigma)} \psi^* A \\
&\quad = \lim_{\epsilon\to 0} \frac{1}{\epsilon^{n+1}} \left[\sum_\alpha (-1)^\alpha \int_{[0,\epsilon]^n\times\sigma} (\psi_\epsilon^\alpha)^* A - (\psi_0^\alpha)^* A \right] \\
&\qquad + \lim_{\epsilon\to 0} \frac{1}{\epsilon^{n+1}} (-1)^{n+1} \int_{[0,\epsilon]^{n+1}\times\partial\sigma} \psi^* A \\
&\quad = \sum_\alpha (-1)^\alpha V_\alpha \left[I_\sigma(A)(\phi; V_0, \dots, \widehat{V_\alpha}, \dots, V_n) \right] \\
&\qquad + (-1)^{n+1} \int_{\partial\sigma} \phi^* \iota(V_0, \dots, V_n) A \\
&\quad = dI_\sigma(A)(\phi; V_0, \dots, V_n) \\
&\qquad + (-1)^{n+1} I_{\partial\sigma}(A)(\phi; V_0, \dots, V_n)
\end{aligned}
$$

□

This shows that the de Rham complex has integrals. However, we are more interested in the following.

Example 2.1.5. The truncated Deligne complex

$$\mathbb{T}_D^\infty(r) = (\mathbb{T} \to \Omega^1 \to \cdots \to \Omega^r \to 0 \to \cdots)$$

has integrals. Here, $\mathbb{T}$ denotes the sheaf of smooth S^1-valued functions; we will identify S^1 with $\mathbb{R}/\mathbb{Z}$, so that $\mathbb{T} \cong \Omega^0/\mathbb{Z}$. The map $\mathbb{T} \to \Omega^1$ is then induced by the usual exterior derivative $\Omega^0 \to \Omega^1$. For Σ an l-dimensional manifold, we define $\mathbb{T}_D^\infty(r)_\Sigma$ to be the complex

$$\mathbb{T}_\Sigma \to \Omega^1_\Sigma \to \cdots \to \Omega^{r-l}_\Sigma \to 0 \to \cdots$$

The integration maps are defined as they were for the de Rham complex; when $n = 0$, the de Rham integration map is followed by the projection $\Omega^0_\Sigma \to \mathbb{T}_\Sigma$.

As before, the only thing to check is property 1 of definition 2.1.1. We must show that for any $U \subseteq X$ and any $A \in \Omega^{n+l}(U)$, we have

$$d \int_\Sigma A = \int_\Sigma dA + (-1)^n \int_{\partial\Sigma} A. \tag{2.2}$$

When $0 < n < r - l$, this is identical to the de Rham version. When $n = 0$, equation (2.2) is obtained from the de Rham version by composition with $\Omega^0 \to \mathbb{T}$. Finally, for $n \geq r - l$, (2.2) is an equation in the zero sheaf on X^Σ.

2.2 Čech Cohomology

Fix an open cover $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ of X. Recall that for any sheaf of abelian groups $\mathcal{F}$ on X, the group of Čech n-cochains on $\mathcal{U}$ with coefficients in $\mathcal{F}$ is

$$\check{C}^n(\mathcal{U};\, \mathcal{F}) = \prod_{i_0 \cdots i_n} \mathcal{F}(U_{i_0 \cdots i_n}),$$

where the product ranges over all tuples of $n + 1$ indices in $\mathcal{I}$, and

$$U_{i_0 \cdots i_n} = U_{i_0} \cap \cdots \cap U_{i_n}.$$

For any $f \in \check{C}^n(\mathcal{U};\, \mathcal{F})$, we denote by $f_{i_0 \cdots i_n}$ the value of f on $U_{i_0 \cdots i_n}$. The Čech cochain groups form a cochain complex, with coboundary map $\delta\colon \check{C}^n \to \check{C}^{n+1}$ given by

$$(\delta f)_{i_0 \cdots i_{n+1}} = \sum_{j=0}^{n+1} (-1)^j f_{i_0 \cdots \widehat{i_j} \cdots i_{n+1}}.$$

Let $N\check{C}^*(\mathcal{U};\, \mathcal{F})$ denote the subcomplex given by

$$N\check{C}^n(\mathcal{U};\, \mathcal{F}) = \{f \mid f_{i_0 \cdots i_n} = 0 \text{ whenever } i_j = i_{j+1} \text{ for some } j\}.$$

We will call this the *normalized* Čech cochain complex.

Proposition 2.2.1. *The inclusion* $N\check{C}^*(\mathcal{U}; \mathcal{F}) \hookrightarrow \check{C}^*(\mathcal{U}; \mathcal{F})$ *is an equivalence.*

Proof. The Čech cochain groups together form a cosimplicial abelian group; for any map of finite ordinals $\phi\colon [n] \to [m]$, the associated map ϕ_* on Čech cochains is given by

$$(\phi_* f)_{i_0 \cdots i_m} = f_{i_{\phi(0)} \cdots i_{\phi(n)}}.$$

In particular, if $s^j : [n] \to [n-1]$ denotes the usual degeneracy map, then we have

$$(s^j_* f)_{i_0 \cdots i_{n-1}} = f_{i_0 \cdots i_j i_j \cdots i_{n-1}}.$$

Thus

$$N\check{C}^n(\mathcal{U}; \mathcal{F}) = \check{C}^n(\mathcal{U}; \mathcal{F}) \cap \bigcap_j \ker(s^j_*). \tag{2.3}$$

It is a general fact that for any cosimplicial abelian group $\mathcal{C}$, the subcomplex $N\mathcal{C}$ defined by equation (2.3) is equivalent to $\mathcal{C}$. The subcomplex $N\mathcal{C}$ is usually called the Moore complex, or the normalized Moore complex, of $\mathcal{C}$. □

Remark 2.2.2. It turns out that the formula we will eventually give for τ is not well-defined on general Čech cochains. It is, however, well defined on normalized Čech cochains, and, fortuitously, produces normalized Čech cochains as its output. As such, from here on we will replace the Čech complex with its normalized subcomplex; everywhere we write $\check{C}^n(\mathcal{U}; \mathcal{F})$, the group $N\check{C}^n(\mathcal{U}; \mathcal{F})$ is to be understood.

Now, let

$$\mathcal{A} = \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \xrightarrow{d} \cdots$$

be a complex of sheaves of abelian groups on X. Associated to this complex is the Čech double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \\
\check{C}^2(\mathcal{U};\,\mathcal{A}^0) & \xrightarrow{d} & \check{C}^2(\mathcal{U};\,\mathcal{A}^1) & \xrightarrow{d} & \check{C}^2(\mathcal{U};\,\mathcal{A}^2) & \xrightarrow{d} & \cdots \\
\delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \\
\check{C}^1(\mathcal{U};\,\mathcal{A}^0) & \xrightarrow{d} & \check{C}^1(\mathcal{U};\,\mathcal{A}^1) & \xrightarrow{d} & \check{C}^1(\mathcal{U};\,\mathcal{A}^2) & \xrightarrow{d} & \cdots \\
\delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \\
\check{C}^0(\mathcal{U};\,\mathcal{A}^0) & \xrightarrow{d} & \check{C}^0(\mathcal{U};\,\mathcal{A}^1) & \xrightarrow{d} & \check{C}^0(\mathcal{U};\,\mathcal{A}^2) & \xrightarrow{d} & \cdots
\end{array}$$

We will use the notation $\check{C}^*(\mathcal{U};\,\mathcal{A})$ for the *totalization* of this double complex; the totalization is the chain complex whose n-th group is

$$\check{C}^n(\mathcal{U};\,\mathcal{A}) = \bigoplus_{p+q=n} \check{C}^p(\mathcal{U};\,\mathcal{A}^q).$$

We will use D to denote the total differential in $\check{C}^*(\mathcal{U};\,\mathcal{A})$. It is given by $D = d + \hat{\delta}$, where $\hat{\delta} = (-1)^q\delta$ on $\check{C}^p(\mathcal{U};\,\mathcal{A}^q)$.

2.3 Triangulations

The definition of τ_Σ will depend on having a good theory of triangulations of Σ. In particular, we will require that any two triangulations have a common refinement. If Σ is equipped with a PL structure, then this comes for free. Fortunately, by Whitehead's theorem on triangulations, every smooth manifold carries an essentially unique PL structure. This PL structure is compatible with the smooth structure on Σ in the following sense. For any triangulation S arising from the PL structure, and for any simplex $s \in S$, the inclusion $s \hookrightarrow \Sigma$ is smooth. We will call a smooth manifold equipped with such a PL structure a smooth PL manifold. In the definition of τ_Σ, we will require Σ to be a smooth PL manifold.

Now, suppose Σ is a smooth PL manifold. Let X be a smooth manifold, and let $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ be an open cover of X. Then there is an associated open cover $\mathcal{U}^\Sigma$ of X^Σ, defined as follows.

Definition 2.3.1. Let S be a triangulation of Σ. An $\mathcal{I}$-*labeling* of S is a function $i\colon S \to \mathcal{I}$. We call the pair $I = (S, i)$ an $\mathcal{I}$-*labeled triangulation* of Σ, and for any such I, we define an open set

$U_I^\Sigma \subseteq X^\Sigma$ by

$$U_I^\Sigma = \{\phi\colon \Sigma \to X \mid \phi(s) \subseteq U_{i(s)} \text{ for all } s \in S\}.$$

We then define $\mathcal{U}^\Sigma = \{U_I^\Sigma\}$, where I ranges over all $\mathcal{I}$-labeled triangulations of Σ. That $\mathcal{U}^\Sigma$ is an open cover of X^Σ is an immediate consequence of the fact that for any open cover $\mathcal{O}$ of Σ, there exists a triangulation S of Σ subordinate to $\mathcal{O}$.

Suppose T is a triangulation of Σ, and that S is a refinement of T. Then for every simplex $s \in S$ there exists a simplex $t \in T$ such that $s \subseteq t$; consequently there exists a unique such t of minimal dimension. We call this minimal simplex t the parent of s. The mapping $s \mapsto t$ defines a function $S \to T$. Via this function, we may extend any $\mathcal{I}$-labeling i of T to an $\mathcal{I}$-labeling of S; by abuse of notation, we will denote this induced $\mathcal{I}$-labeling by i as well.

The whole reason for talking about smooth PL manifolds is that any finite set of triangulations of a PL manifold has a common refinement. It follows that any intersection in $\mathcal{U}^\Sigma$ can be expressed in terms of a single triangulation.

Proposition 2.3.2. *Suppose $\mathbf{I} = (I^0, \ldots, I^n)$ is a tuple of $\mathcal{I}$-labeled triangulations. Let S be any triangulation that simultaneously refines each of $I^0, \ldots, I^n$. Then the intersection*

$$U_{\mathbf{I}}^\Sigma = U_{I^0}^\Sigma \cap \cdots \cap U_{I^n}^\Sigma$$

is given by

$$U_{\mathbf{I}}^\Sigma = \{\phi\colon \Sigma \to X \mid \phi(s) \subseteq U_{i^0(s)\cdots i^n(s)} \text{ for all } s \in S\}.$$

Proof. Let T^j denote the underlying triangulation of I^j. Since S is a refinement of T^j, each simplex $t \in T^j$ is equal to the union of those simplices $s \in S$ whose parent is t. Thus the condition that $\phi(t) \subseteq U_{i^j(t)}$ is equivalent to the condition that $\phi(s) \subseteq U_{i^j(s)}$ for all $s \in S$ whose parent is t. □

Now, suppose Σ has non-empty boundary. Any triangulation S of Σ restricts to give a triangulation ∂S of $\partial\Sigma$; by the same token, any $\mathcal{I}$-labeled triangulation I of Σ restricts to an $\mathcal{I}$-labeled triangulation ∂I of $\partial\Sigma$.

Proposition 2.3.3. *Let $\imath\colon \partial\Sigma \to \Sigma$ denote the inclusion, and $\tilde{\imath}\colon X^\Sigma \to X^{\partial\Sigma}$ the induced restriction* *map. Then for any $\mathcal{I}$-labeled triangulation I of Σ,*

$$U_I^\Sigma \subseteq \tilde{\imath}^{-1}(U_{\partial I}^{\partial\Sigma}).$$

Proof. Immediate. □

Let $\mathcal{A}$ be a complex of sheaves with integrals on X. Then for any tuple $\mathbf{I} = (I^0, \ldots, I^{n-q})$ of $\mathcal{I}$-labeled triangulations of Σ, we have the composite map

$$\mathcal{A}^q_{\partial\Sigma}(U^{\partial\Sigma}_{\partial\mathbf{I}}) \xrightarrow{\tilde{\imath}^*} \mathcal{A}^q_\Sigma(\tilde{\imath}^{-1}(U^{\partial\Sigma}_{\partial\mathbf{I}})) \xrightarrow{res} \mathcal{A}^q_\Sigma(U^\Sigma_{\mathbf{I}}).$$

Collectively, as $\mathbf{I}$ ranges over all such tuples of $\mathcal{I}$-labeled triangulations of Σ, these maps define a map in Čech cohomology

$$\tilde{\imath}^* \colon \check{C}^n(\mathcal{U}^{\partial\Sigma}; \mathcal{A}_{\partial\Sigma}) \to \check{C}^n(\mathcal{U}^\Sigma; \mathcal{A}_\Sigma).$$

This is the pullback map referenced in the statement of theorem 3.0.5.

Another concept we will use in the definition of τ is that of a coflag in a triangulation.

Definition 2.3.4. Suppose Σ is an oriented d-dimensional smooth PL manifold, and that S is a triangulation of Σ. An l-coflag in S is a chain

$$\mathbf{c} = (c_l \subset \cdots \subset c_d)$$

of simplices of S, where c_j has dimension j for all $l \le j \le d$. Note that each simplex c_j is necessarily a face of the subsequent simplex c_{j+1}. We orient each simplex c_j as follows. c_d is a top-dimensional simplex; it inherits the orientation of Σ. Then, for any $l \le j < d$, assuming we have assigned an orientation to c_{j+1}, we assign to c_j the orientation it inherits as part of the boundary of c_{j+1}. We denote the set of all l-coflags by $C_l = C_l(S)$.

Notation 2.3.5. If $\mathcal{A}$ is a complex of sheaves with integrals, $\mathbf{c} = (c_l \subset \cdots \subset c_d)$ is an l-coflag in Σ, and A is an element of $\mathcal{A}^{n+l}(U)$ for some $U \subseteq X$, we will use the notation

$$\int_{\mathbf{c}} A$$

for the integral over c_l with respect to the orientation it inherits from $\mathbf{c}$.

Definition 2.3.6. Suppose T is a triangulation of Σ, and S is a refinement of T. Let $\mathbf{c} = (c_l \subset \cdots \subset c_d)$ be an l-coflag of S. For all j, let $t_j \in T$ denote the parent of c_j, and let $\mathbf{t} = (t_l \subset \cdots \subset t_d)$. If $\mathbf{t}$ is itself a coflag, i.e. if for all j the dimension of t_j is j, then we say $\mathbf{c}$ is an *essential child of* $\mathbf{t}$. Otherwise, we say $\mathbf{c}$ is inessential.

Proposition 2.3.7. *Let S, T, $\mathbf{c}$ and $\mathbf{t}$ be as in definition 2.3.6. If $\mathbf{c}$ is inessential, then $t_j = t_{j+1}$ for some $l \leq j < d$.*

Proof. Since $t_j \subset t_{j+1}$, it suffices to show $\dim(t_j) = \dim(t_{j+1})$ for some j. First, note that $\dim(t_j) \leq \dim(t_{j+1})$, so the sequence $\dim(t_l), \ldots, \dim(t_d)$ is monotonically increasing. Also, note that $\dim(t_d) = d$. In general, since $c_j \subset t_j$, we have $\dim(t_j) \geq \dim(c_j) = j$. But because $\mathbf{t}$ is *not* a coflag, this inequality must be strict for at least one j. It follows that $\dim(t_j) = \dim(t_{j+1})$ for some j. □

Proposition 2.3.8. *Let T be a triangulation of Σ, let S be a refinement of T, let $\mathbf{t} = (t_l \subset \cdots \subset t_d)$ be an l-coflag of T, and let c be an l-simplex of S with $c \subseteq t_l$. Then there is a unique l-coflag $\mathbf{s} = (s_l \subset \cdots s_d)$ of S with $s_l = c$, such that $\mathbf{s}$ is an essential child of $\mathbf{t}$.*

Proof. By induction, it will suffice to show that there is a unique $(l+1)$-simplex $s_{l+1} \in S$ such that $c \subset s_{l+1} \subseteq t_{l+1}$. Let $\operatorname{int}(c) = c \setminus \partial c$, and fix any point $x \in \operatorname{int}(c)$. Then x is not contained in any l-simplex of S other than c. Moreover, for any $(l+1)$-simplex $s_{l+1} \in S$ with $x \in s_{l+1}$, we must have $c \subseteq s_{l+1}$.

Note that t_{l+1} is equal to a union of $(l+1)$-simplices of S. So there must exist some $(l+1)$-simplex $s_{l+1} \in S$ such that $x \in s_{l+1} \subseteq t_{l+1}$. As we just mentioned, this implies $c \subset s_{l+1}$. To see that s_{l+1} is unique, consider the tangent space

$$V = T_x\, t_{l+1} = T_x\, s_{l+1} \subseteq T_x\Sigma.$$

Note that $x \in \operatorname{int}(t_l) \subseteq \partial\, t_{l+1}$, i.e. x belongs to the interior of a top-dimensional face of t_{l+1}. It follows that the set of vectors in V that point to the interior of t_{l+1} form an open half-space $O \subseteq V$. But we also have $x \in \operatorname{int}(c) \subseteq \partial s_{l+1}$, so the set of vectors that point to the interior of s_{l+1} form an open half-space $P \subseteq V$. Moreover, since $s_{l+1} \subseteq t_{l+1}$, we must have $P = O$. If s'_{l+1} were another $(l+1)$-simplex of S with $x \in s'_{l+1} \subseteq t_{l+1}$, then we would still have $\mathrm{int}(c) \subseteq \partial s'_{l+1}$, and the open half-space of vectors pointing to the interior of s'_{l+1} would be contained in O, and so would equal P. But this is impossible, as the $(l+1)$-simplices of S have disjoint interiors. □

The following corollary is an immediate consequence of proposition 2.3.8.

Corollary 2.3.9. *Let $\mathcal{A}$ be a complex of sheaves with integrals on X, and let $A \in \mathcal{A}^{n+l}(U)$ for some $U \subseteq X$. Let T be a triangulation of Σ, and let S be a refinement of T. Then for any l-coflag $\mathbf{t}$ in T, we have*

$$\int_{\mathbf{t}} A = \sum_{\mathbf{s} \subseteq \mathbf{t}} \int_{\mathbf{s}} A,$$

where the sum on the right ranges over the essential children $\mathbf{s}$ of $\mathbf{t}$.

Proposition 2.3.10. *Let S be a triangulation of Σ. For any l-coflag $\mathbf{c}$ of S, and any $l < n < d$, there is exactly one l-coflag $\bar{\mathbf{c}}$ of S such that $c_r = \bar{c}_r$ for all $r \neq n$, but $c_n \neq \bar{c}_n$. Moreover, the orientation c_l inherits from $\mathbf{c}$ is opposite the orientation it inherits from $\bar{\mathbf{c}}$. When $n = d$, the same conclusions hold if $c_{d-1} \not\subseteq \partial\Sigma$. If $c_{d-1} \subseteq \partial\Sigma$, then c_d is the only d-simplex of S containing c_{d-1}.*

Proof. For the $n < d$ case, the existence and uniqueness of $\bar{\mathbf{c}}$ is a consequence of the simplicial identites, but we will give a slightly more direct argument. Let $v_0, \ldots, v_{n+1}$ denote the vertices of c_{n+1}. Simplices of S that are contained in c_{n+1} are uniquely determined by which of these vertices they contain. Let $[v_{i_0}, \ldots, v_{i_r}]$ denote the r-simplex whose vertices are $v_{i_0}, \ldots, v_{i_r}$. Then for some $i < j$, we must have $c_{n-1} = [v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_{n+1}]$. There are exactly two n-simplices contained in c_{n+1} that contain c_{n-1}, namely $[v_0, \ldots, \widehat{v_i}, \ldots, v_{n+1}]$ and $[v_0, \ldots, \widehat{v_j}, \ldots, v_{n+1}]$. One of these must be c_n, the other is the desired $\bar{c}_n$.

The statement about orientations follows from the standard maxim that the boundary of the boundary is empty; in this case, that every simplex in the boundary of the boundary of c_{n+1} appears twice with opposite orientation.

The $n = d$ case is essentially just the statement that any $(d-1)$-simplex c_{d-1} of S that is contained in $\partial\Sigma$ has just one "side" in Σ, and hence is contained in just one d-simplex. If c_{d-1} is not contained in the boundary, then it has two sides in Σ, and so is contained in two d-simplexes, one on each side. In this case, c_{d-1} will inherit opposing orientations from the simplices on either side. □

2.4 Staircases

Definition 2.4.1. Let $\mathbf{i} = (i^0, \ldots, i^p)$ be an arbitrary tuple, and fix integers $0 \leq l \leq d$. A *(descending) staircase* is a sequence $\mathbf{k} = (k_0, \ldots, k_m)$ of points in $\{i^0, \ldots, i^p\} \times [l, d]$, such that $k_0 = (i^0, d)$, $k_m = (i^p, l)$, and for any $0 \leq j < m$, if $k_j = (i^r, s)$, then $k_{j+1} = (i^{r+1}, s)$ or $k_{j+1} = (i^r, s-1)$. Figure 2.1 gives an example. We denote the set of all such staircases by $K^{\mathbf{i}}_{d,l}$.

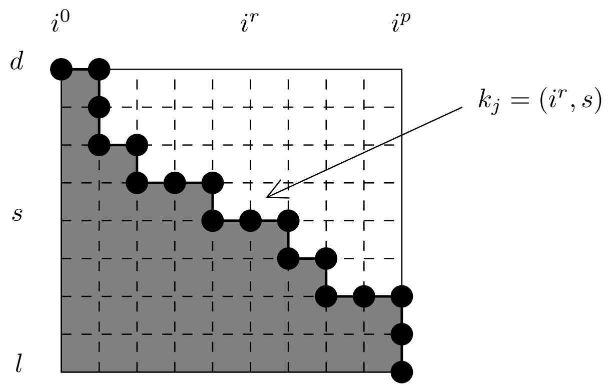

Figure 2.1: A typical staircase in $K^{\mathbf{i}}_{d,l}$.

Note that if $k_j = (i^r, s)$, then $r = j - (d - s)$, so that r is determined by $\mathbf{k}$ and j. We call r and s the first and second coordinates of k_j. Note also that the index m of the final point of $\mathbf{k}$ is given by $m = p + d - l$. We denote this value by $m_{\mathbf{k}}$. Lastly, we define $(-1)^{\mathbf{k}} = (-1)^A$, where A is the area of the region inside the rectangle $\{i^0, \ldots, i^p\} \times [l, d]$ that lies below the staircase. This is the shaded region in Figure 2.1.

Fix $\mathbf{k} = (k_0, \ldots, k_m) \in K^{\mathbf{i}}_{d,l}$. For any $0 < j < m$, we call k_j a *vertical* (respectively *horizontal*) point of $\mathbf{k}$ if the first (respectively second) coordinates of k_{j-1}, k_j, and k_{j+1} are all equal. We call k_j a *corner* point of $\mathbf{k}$ if it is neither horizontal nor vertical. Similarly, the endpoints k_0 and k_m are called horizontal or vertical, depending on the position of the adjacent point. The end points are never corner points. For example, the point marked k_j in Figure 2.1 is a horizontal point, whereas the two neighboring points are corners. The initial point k_0 is horizontal, and the final point k_m is vertical.

Now, let X be a manifold with open cover $\mathcal{U}$, let $\mathcal{F}$ be a sheaf on X, let S be a triangulation of a compact manifold Σ, let $\mathbf{c} = (c_l, \ldots, c_d)$ be an l-coflag in S, and let $\mathbf{i} = (i^0, \ldots, i^p)$ be a tuple of $\mathcal{I}$-labelings of S. For any point $z = (i^r, s) \in \{i^0, \ldots, i^p\} \times [l, d]$, let $i_z = i^r(c_s)$. Then for any $f \in \check{C}^{p+d-l}(\mathcal{U}; \mathcal{F})$, and any $\mathbf{k} = (k_0, \ldots, k_m) \in K^{\mathbf{i}}_{d,l}$, we define

$$T(f, \mathbf{c}, \mathbf{k}) = f_{i_{k_0} \cdots i_{k_m}}.$$

If $f \in \check{C}^{p+d-l+1}(\mathcal{U}; \mathcal{F})$ and $0 \leq j \leq m$, we define

$$\widehat{T}(f, \mathbf{c}, \mathbf{k}, j) = f_{i_{k_0} \cdots \widehat{i_{k_j}} \cdots i_{k_m}}.$$

Lastly, we define

$$\widehat{T}_h(f, \mathbf{c}, \mathbf{k}, j) = \begin{cases} \widehat{T}(f, \mathbf{c}, \mathbf{k}, j) & k_j \text{ is a horizontal point of } \mathbf{k} \\ 0 & \text{otherwise,} \end{cases}$$

and we define $\widehat{T}_v$ and $\widehat{T}_c$ similarly, for vertical and corner points respectively.

Chapter 3

Main Results

We now give the general formula for τ. For the rest of this section, consider the following data to be fixed.

- A smooth manifold X.
- An open cover $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ of X.
- A complex $\mathcal{A}^* = \mathcal{A}^0 \to \mathcal{A}^1 \to \cdots$ of sheaves with integrals on X.
- A compact oriented smooth PL manifold Σ of dimension d.

For any Čech data $\mathbf{A} = (A^0, \ldots, A^{n+d}) \in \check{C}^{n+d}(\mathcal{U}; \mathcal{A}^*)$ on X, we will construct corresponding Čech data

$$\tau^n_\Sigma(\mathbf{A}) = (\tau^{n,0}_\Sigma(\mathbf{A}), \ldots, \tau^{n,n}_\Sigma(\mathbf{A})) \in \check{C}^n(\mathcal{U}^\Sigma; \mathcal{A}^*)$$

on the mapping space X^Σ. In particular, we must define

$$\tau^{n,q}_\Sigma(\mathbf{A}) \in \check{C}^{n-q}(\mathcal{U}^\Sigma; \mathcal{A}^q)$$

for every $0 \leq q \leq n$. So, given any tuple $\mathbf{I} = (I^q, \ldots, I^n)$ of $\mathcal{I}$-labeled triangulations, let S be a triangulation that simultaneously refines each of $I^q, \ldots, I^n$ and let $\mathbf{i} = (i^q, \ldots, i^n)$ denote the corresponding tuple of $\mathcal{I}$-labelings of S (see proposition 2.3.2). Then we define $\tau^{n,q}_\Sigma(\mathbf{A})_\mathbf{I} \in \mathcal{A}^q(U^\Sigma_\mathbf{I})$ by

$$\tau^{n,q}_\Sigma(\mathbf{A})_\mathbf{I} = \sum_{l=0}^{d} \sum_{\mathbf{c}\in C_l} \sum_{\mathbf{k}\in K^{\mathbf{i}}_{d,l}} (-1)^{\mathbf{k}} \gamma^{n,q}_{d,l} \int_\mathbf{c} T(A^{q+l}, \mathbf{c}, \mathbf{k}), \tag{3.1}$$

where C_l is the set of l-coflags in S (see definition 2.3.4), and

$$\gamma^{n,q}_{d,l} = (-1)^{l+nd+ld+qd+\frac{n(n-1)}{2}}.$$

For a description of the set $K^{\mathbf{i}}_{d,l}$ and of the notation $T(A^{q+l}, \mathbf{c}, \mathbf{k})$, see section 2.4. The integral in equation (3.1) is part of the structure of $\mathcal{A}^*$ as a complex of sheaves with integrals; this is the subject of section 2.1. The subscript $\mathbf{c}$ on the integral is explained in notation 2.3.5.

Remark 3.0.2. The number $\gamma^{n,q}_{d,l}$ defined above is a solution to the following recurrences, which will appear in the proof of theorem 3.0.5.

$$\gamma^{n-1,q-1}_{d,l} = (-1)^{n+1}\gamma^{n,q}_{d,l}$$

$$\gamma^{n-1,q-1}_{d,l+1} = (-1)^{n+d}\gamma^{n,q}_{d,l}$$

$$\gamma^{n-1,q}_{d,l} = (-1)^{d+n+1}\gamma^{n,q}_{d,l}$$

$$\gamma^{n,q}_{d-1,l} = (-1)^{q+l+n}\gamma^{n,q}_{d,l}$$

The following proposition shows that $\tau^{n,q}_{\Sigma}(\mathbf{A})_{\mathbf{I}}$ is independent of the choice of refinement S.

Proposition 3.0.3. *Fix a triangulation S of Σ, a refinement S' of S, a tuple $\mathbf{i} = (i^q, \ldots, i^n)$ of $\mathcal{I}$-labelings of S, and a staircase $\mathbf{k} \in K^{\mathbf{i}}_{d,l}$. Then*

$$\sum_{\mathbf{c} \in C_l(S)} \int_{\mathbf{c}} T(A^{q+l}, \mathbf{c}, \mathbf{k}) = \sum_{\mathbf{c}' \in C_l(S')} \int_{\mathbf{c}'} T(A^{q+l}, \mathbf{c}', \mathbf{k}). \tag{3.2}$$

Proof. First, let $\mathbf{c}' = (c'_l, \ldots, c'_d)$ be any inessential coflag in S', and let $c_j \in S$ denote the parent of c'_j for each $l \leq j \leq d$. By proposition 2.3.7, for at least one j, we must have $c_j = c_{j+1}$. Let i^r be the label on the column at which the staircase $\mathbf{k}$ descends from row $j+1$ to row j, as in Figure 3.1. Then $i^r(c'_{j+1}) = i^r(c'_j)$, so that

$$T(A^{q+l}, \mathbf{c}', \mathbf{k}) = A^{q+l}_{i^q(c'_d)\cdots i^r(c'_{j+1}) i^r(c'_j) \cdots i^n(c'_l)}$$

is zero by virtue of having repeated adjacent indices (recall, as per remark 2.2.2, that we are working exclusively with normalized Čech cochains). Thus in the right hand side of equation 3.2, the contribution from the inessential coflags vanishes.

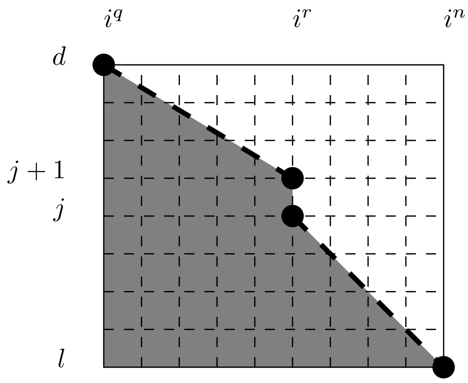

Figure 3.1: This staircase descends from row $j+1$ to row j in column i^r, as in proposition 3.0.3.

Next, let $\mathbf{c} = (c_l, \ldots, c_d)$ be any coflag in $C_l(S)$. For any essential child $\mathbf{c}'$ of $\mathbf{c}$, we have

$$T(A^{q+l}, \mathbf{c}, \mathbf{k}) = T(A^{q+l}, \mathbf{c}', \mathbf{k}).$$

This is because the indices match: $i^r(c_j') = i^r(c_j)$ for all r and j.

By corollary 2.3.9, we then have

$$\int_{\mathbf{c}} T(A^{q+l}, \mathbf{c}, \mathbf{k}) = \sum_{\mathbf{c}' \subseteq \mathbf{c}} \int_{\mathbf{c}'} T(A^{q+l}, \mathbf{c}', \mathbf{k}),$$

where the sum on the right is over the essential children $\mathbf{c}'$ of $\mathbf{c}$. It remains only to note that each essential coflag in S' is the child of a unique coflag in S, so that

$$\sum_{\mathbf{c}' \in C_l(S')} T(A^{q+l}, \mathbf{c}', \mathbf{k}) = \sum_{\mathbf{c} \in C_l(S)} \sum_{\mathbf{c}' \subseteq \mathbf{c}} T(A^{q+l}, \mathbf{c}', \mathbf{k}).$$

□

Proposition 3.0.4. *If* $\mathbf{I} = (I^q, \ldots, I^n)$ *has the property that* $I^r = I^{r+1}$ *for some* r*, then* $\tau_{\Sigma}^{n,q}(\mathbf{A})_{\mathbf{I}} = 0$*. In other words,* τ *takes values in the group of normalized Čech cochains.*

Proof. Fix a staircase $\mathbf{k} \in K_{d,l}^{\mathbf{i}}$, and let j denote the row in which $\mathbf{k}$ advances from the column labeled i^r to the column labeled i^{r+1} (see figure 3.2). For any coflag $\mathbf{c} = (c_l, \ldots, c_d)$, the expression

$$T(A^{q+l}, \mathbf{c}, \mathbf{k}) = A^{q+l}_{\cdots i^r(c_j) i^{r+1}(c_j) \cdots}$$

contains adjacent copies of the repeated index $i^r(c_j) = i^{r+1}(c_j)$, and hence is zero. Thus every summand in expression (3.1) vanishes. $\square$

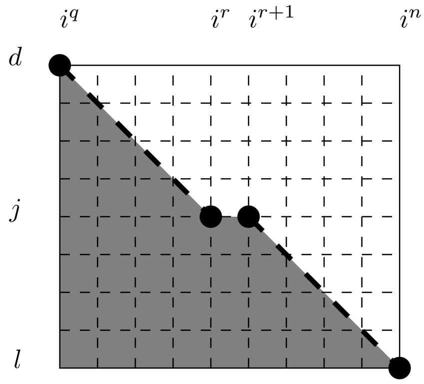

Figure 3.2: This staircase advances from column i^r to column i^{r+1} in row j, as in proposition 3.0.4.

Our main result is

Theorem 3.0.5. *If Σ has boundary $\partial\Sigma$, then for any $\mathbf{A} \in \check{C}^{n+d-1}(\mathcal{U}^{\Sigma}; \mathcal{A}^*)$,*

$$D\tau_{\Sigma}^{n-1}(\mathbf{A}) + (-1)^n \tau_{\Sigma}^{n}(D\mathbf{A}) = \rho^* \tau_{\partial\Sigma}^{n}(\mathbf{A}), \tag{3.3}$$

where ρ^* is the pullback

$$\check{C}^n(\mathcal{U}^{\partial\Sigma}; \mathcal{A}^*_{\partial\Sigma}) \to \check{C}^n(\mathcal{U}^{\Sigma}; \mathcal{A}^*_{\Sigma}).$$

Proof. We begin by separately computing the various terms in equation 3.3 above. Recall from section 2.2 the notational convention $D = d + \hat{\delta}$, where $\hat{\delta} = (-1)^q \delta$ in degree (p, q). Then for any $0 \le q \le n$ and any $\mathbf{I} = (I^q, \ldots, I^n)$, we have

$$\tau_{\Sigma}^{n,q}(d\mathbf{A})_{\mathbf{I}} = \sum_{l=0}^{d} \sum_{C_l} \sum_{K^{\mathbf{i}}_{d,l}} (-1)^{\mathbf{k}} \, \gamma_{d,l}^{n,q} \int_{\mathbf{c}} T(dA^{q+l-1}, \mathbf{c}, \mathbf{k}) \tag{3.4}$$

and

$$\begin{aligned}\tau_\Sigma^{n,q}(\hat{\delta}\mathbf{A})_\mathbf{I} &= \sum_{l=0}^{d}\sum_{C_l}\sum_{K^{\mathbf{i}}_{d,l}}(-1)^{\mathbf{k}+q+l}\,\gamma_{d,l}^{n,q}\int_\mathbf{c} T(\delta A^{q+l},\mathbf{c},\mathbf{k})\\ &= \sum_{l=0}^{d}\sum_{C_l}\sum_{K^{\mathbf{i}}_{d,l}}\sum_{j=0}^{m_\mathbf{k}}(-1)^{\mathbf{k}+j+q+l}\,\gamma_{d,l}^{n,q}\int_\mathbf{c}\widehat{T}(A^{q+l},\mathbf{c},\mathbf{k},j).\end{aligned} \tag{3.5}$$

The sum of these two terms is $\tau_\Sigma^{n,q}(D\mathbf{A})_\mathbf{I}$. On the other hand, the degree q part of $D\tau_\Sigma^n(\mathbf{A})$ is $d\tau^{n-1,q-1}(\mathbf{A})+\hat{\delta}\tau^{n-1,q}(\mathbf{A})$. The first of these two terms is given by

$$\begin{aligned}&d\tau^{n-1,q-1}(\mathbf{A})_\mathbf{I}\\ &\quad=\sum_{l=0}^{d}\sum_{C_l}\sum_{K^{\mathbf{i}}_{d,l}}(-1)^{\mathbf{k}}\,\gamma_{d,l}^{n-1,q-1}d\int_\mathbf{c} T(A^{q+l-1},\mathbf{c},\mathbf{k})\\ &\quad=\sum_{l=0}^{d}\sum_{C_l}\sum_{K^{\mathbf{i}}_{d,l}}(-1)^{\mathbf{k}}\,\gamma_{d,l}^{n-1,q-1}\int_\mathbf{c} T(dA^{q+l-1},\mathbf{c},\mathbf{k})\\ &\qquad+\sum_{l=1}^{d}\sum_{C_l}\sum_{K^{\mathbf{i}}_{d,l}}(-1)^{\mathbf{k}+q-1}\,\gamma_{d,l}^{n-1,q-1}\int_{\partial\mathbf{c}} T(A^{q+l-1},\mathbf{c},\mathbf{k}).\end{aligned} \tag{3.6}$$

Here, the integral over $\partial\mathbf{c}$ should be interpreted as the integral over the boundary of the bottom (l-dimensional) simplex of $\mathbf{c}$. However, a slightly different interpretation, which is nevertheless equivalent, will be useful in the next paragraph. If $\mathbf{c}=(c_l,\ldots,c_d)$, we can define $\partial\mathbf{c}$ to be the set of $(l-1)$-coflags of the form $\mathbf{c}'=(c'_{l-1},c_l,\ldots,c_d)$, where of course c'_{l-1} is a face of c_l. Then the integral over $\partial\mathbf{c}$ is equal to the sum of the integrals over $\mathbf{c}'$ for all $\mathbf{c}'$ in $\partial\mathbf{c}$.

For any $\mathbf{k}\in K^{\mathbf{i}}_{d,l}$, let $\overline{\mathbf{k}}\in K^{\mathbf{i}}_{d,l-1}$ be obtained by appending $(i^n,l-1)$ to the end of $\mathbf{k}$, as in Figure 3.3. Each $\overline{\mathbf{k}}\in K^{\mathbf{i}}_{d,l-1}$ whose last point is vertical arises in this way from a unique $\mathbf{k}\in K^{\mathbf{i}}_{d,l}$. Note that $(-1)^{\mathbf{k}}=(-1)^{\overline{\mathbf{k}}+n-q}$. Moreover, each $\overline{\mathbf{c}}\in C_{l-1}$ belongs to $\partial\mathbf{c}$ (in the sense mentioned above) for exactly one $\mathbf{c}\in C_l$, and

$$T(\cdot,\mathbf{c},\mathbf{k})=\widehat{T}(\cdot,\overline{\mathbf{c}},\overline{\mathbf{k}},m_{\overline{\mathbf{k}}}).$$

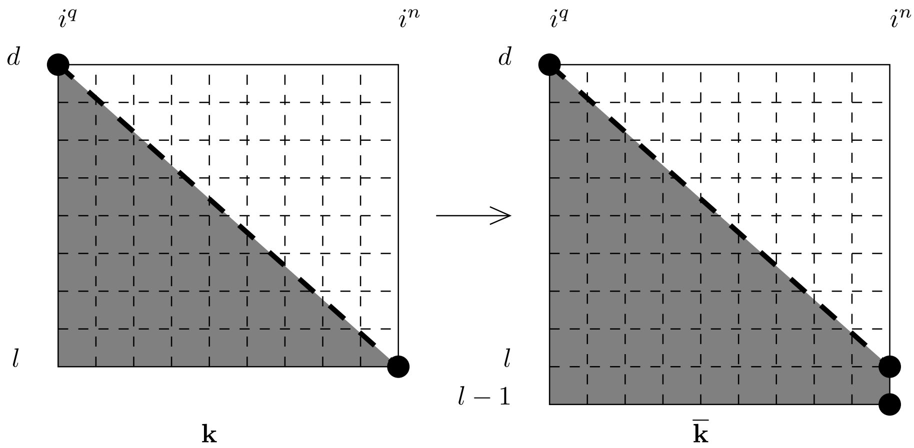

Figure 3.3: $\overline{\mathbf{k}}$ is obtained from $\mathbf{k}$ by appending $(i^n, l-1)$.

This allows us to rewrite the last line of (3.6) as

$$\sum_{l=1}^{d} \sum_{\overline{\mathbf{c}} \in C_{l-1}} \sum_{\overline{\mathbf{k}} \in K^{\mathbf{i}}_{d,l-1}} (-1)^{\overline{\mathbf{k}}+n-1} \gamma^{n-1,q-1}_{d,l} \int_{\overline{\mathbf{c}}} \widehat{T}_v(A^{q+l-1}, \overline{\mathbf{c}}, \overline{\mathbf{k}}, m_{\overline{\mathbf{k}}}). \tag{3.7}$$

We shift the index l down by one (and drop all the bars from the notation), and then substitute back into (3.6) to obtain

$$\begin{aligned} d\tau^{n-1,q-1}(\mathbf{A})_{\mathbf{I}} & \\ = \sum_{l=0}^{d} \sum_{C_l} \sum_{K^{\mathbf{i}}_{d,l}} & (-1)^{\mathbf{k}} \gamma^{n-1,q-1}_{d,l} \int_{\mathbf{c}} T(dA^{q+l-1}, \mathbf{c}, \mathbf{k}) \\ & + \sum_{l=0}^{d-1} \sum_{C_l} \sum_{K^{\mathbf{i}}_{d,l}} (-1)^{\mathbf{k}+n-1} \gamma^{n-1,q-1}_{d,l+1} \int_{\mathbf{c}} \widehat{T}_v(A^{q+l}, \mathbf{c}, \mathbf{k}, m_{\mathbf{k}}) \\ = \sum_{l=0}^{d} \sum_{C_l} \sum_{K^{\mathbf{i}}_{d,l}} & (-1)^{\mathbf{k}+n+1} \gamma^{n,q}_{d,l} \int_{\mathbf{c}} T(dA^{q+l-1}, \mathbf{c}, \mathbf{k}) \\ & + \sum_{l=0}^{d-1} \sum_{C_l} \sum_{K^{\mathbf{i}}_{d,l}} (-1)^{\mathbf{k}+d-1} \gamma^{n,q}_{d,l} \int_{\mathbf{c}} \widehat{T}_v(A^{q+l}, \mathbf{c}, \mathbf{k}, m_{\mathbf{k}}). \end{aligned} \tag{3.8}$$

Here, in the second equality, we have used the relations

$$\gamma^{n-1,q-1}_{d,l} = (-1)^{n+1} \gamma^{n,q}_{d,l} \qquad \text{and} \qquad \gamma^{n-1,q-1}_{d,l+1} = (-1)^{n+d} \gamma^{n,q}_{d,l},$$

which were mentioned in remark 3.0.2.

Next, we consider $\hat{\delta}\tau^{n-1,q}(\mathbf{A})$. Using the notation $\delta_p\mathbf{i} = (i^q, \ldots, \widehat{i^p}, \ldots, i^n)$, we have

$$\hat{\delta}\tau^{n-1,q}(\mathbf{A})_{\mathbf{I}} = \sum_{p=q}^{n} \sum_{l=0}^{d} \sum_{C_l} \sum_{K_{d,l}^{\delta_p \mathbf{i}}} (-1)^{\mathbf{k}+p} \, \gamma_{d,l}^{n-1,q} \int_{\mathbf{c}} T(A^{q+l}, \mathbf{c}, \mathbf{k}). \tag{3.9}$$

We change variables as follows. Fix $q \leq p \leq n$ and $\mathbf{k} \in K_{d,l}^{\delta_p \mathbf{i}}$, and let s be the row in which $\mathbf{k}$

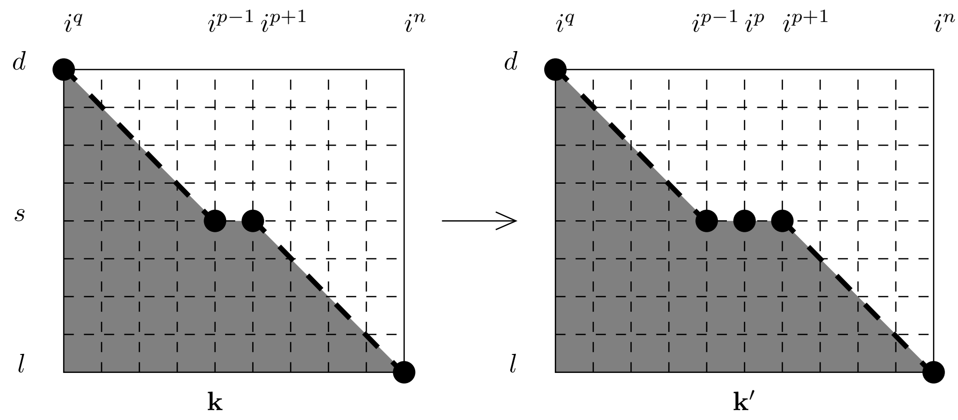

Figure 3.4: $\mathbf{k}'$ is obtained from $\mathbf{k}$ by inserting (i^p, s).

advances from i^{p-1} to i^{p+1}. Let $\mathbf{k}' \in K_{d,l}^{\mathbf{i}}$ be obtained from $\mathbf{k}$ by inserting (i^p, s), as in Figure 3.4. Note that the position of the newly inserted point is $j = p - q + d - s$. Then we have

$$T(\cdot, \mathbf{c}, \mathbf{k}) = \widehat{T}(\cdot, \mathbf{c}, \mathbf{k}', j)$$

and $(-1)^{\mathbf{k}'} = (-1)^{\mathbf{k}+s-l}$. Moreover, the assignment $(\mathbf{k}, p) \mapsto (\mathbf{k}', j)$ gives a bijection between the set of all pairs $(\mathbf{k}, p)$ and the set of pairs $(\mathbf{k}', j)$ for which the j^{th} point of $\mathbf{k}'$ is a horizontal point. It follows that

$$\begin{aligned} &\hat{\delta}\tau^{n-1,q}(\mathbf{A})_{\mathbf{I}} \\ &\quad = \sum_{l=0}^{d} \sum_{C_l} \sum_{K_{d,l}^{\mathbf{i}}} \sum_{j=0}^{m_{\mathbf{k}}} (-1)^{\mathbf{k}+j+q+l-d} \, \gamma_{d,l}^{n-1,q} \int_{\mathbf{c}} \widehat{T}_h(A^{q+l}, \mathbf{c}, \mathbf{k}, j) \\ &\quad = \sum_{l=0}^{d} \sum_{C_l} \sum_{K_{d,l}^{\mathbf{i}}} \sum_{j=0}^{m_{\mathbf{k}}} (-1)^{\mathbf{k}+j+q+l+n+1} \, \gamma_{d,l}^{n,q} \int_{\mathbf{c}} \widehat{T}_h(A^{q+l}, \mathbf{c}, \mathbf{k}, j), \end{aligned} \tag{3.10}$$

where in the second equality we have applied the relation

$$\gamma_{d,l}^{n-1,q} = (-1)^{d+n+1}\gamma_{d,l}^{n,q}$$

from remark 3.0.2.

Combining equations 3.4, 3.5, 3.8, and 3.10 then gives

$$\begin{aligned}
&d\tau_{\Sigma}^{n-1,q-1}(\mathbf{A})_{\mathbf{I}} + \hat{\delta}\tau_{\Sigma}^{n-1,q}(\mathbf{A})_{\mathbf{I}} + (-1)^d\tau_{\Sigma}^{n,q}(D\mathbf{A})_{\mathbf{I}} \\
&\quad= \sum_{l=0}^{d}\sum_{C_l}\sum_{K_{d,l}^{\mathbf{i}}} (-1)^{\mathbf{k}+n+1}\,\gamma_{d,l}^{n,q}\int_{\mathbf{c}} T(dA^{q+l-1},\mathbf{c},\mathbf{k}) \\
&\qquad+ \sum_{l=0}^{d-1}\sum_{C_l}\sum_{K_{d,l}^{\mathbf{i}}} (-1)^{\mathbf{k}+d-1}\,\gamma_{d,l}^{n,q}\int_{\mathbf{c}} \widehat{T}_v(A^{q+l},\mathbf{c},\mathbf{k},m_{\mathbf{k}}) \\
&\qquad+ \sum_{l=0}^{d}\sum_{C_l}\sum_{K_{d,l}^{\mathbf{i}}}\sum_{j=0}^{m_{\mathbf{k}}} (-1)^{\mathbf{k}+j+q+l+n+1}\,\gamma_{d,l}^{n,q}\int_{\mathbf{c}} \widehat{T}_h(A^{q+l},\mathbf{c},\mathbf{k},j) \\
&\qquad+ \sum_{l=0}^{d}\sum_{C_l}\sum_{K_{d,l}^{\mathbf{i}}} (-1)^{\mathbf{k}+n}\,\gamma_{d,l}^{n,q}\int_{\mathbf{c}} T(dA^{q+l-1},\mathbf{c},\mathbf{k}) \\
&\qquad+ \sum_{l=0}^{d}\sum_{C_l}\sum_{K_{d,l}^{\mathbf{i}}}\sum_{j=0}^{m_{\mathbf{k}}} (-1)^{\mathbf{k}+j+q+l+n}\,\gamma_{d,l}^{n,q}\int_{\mathbf{c}} \widehat{T}(A^{q+l},\mathbf{c},\mathbf{k},j).
\end{aligned} \tag{3.11}$$

Most of these terms immediately cancel. In particular, the first and fourth lines cancel exactly. The third line cancels the $\widehat{T}_h$ terms on the fifth line. The second line cancels the $\widehat{T}_v$ terms with $j = m_{\mathbf{k}} = n - q + d - l$ on the fifth line. The remaining terms are then

$$\begin{aligned}
&d\tau_{\Sigma}^{n-1,q-1}(\mathbf{A})_{\mathbf{I}} + \hat{\delta}\tau_{\Sigma}^{n-1,q}(\mathbf{A})_{\mathbf{I}} + (-1)^d\tau_{\Sigma}^{n,q}(D\mathbf{A})_{\mathbf{I}} \\
&\quad= \sum_{l=0}^{d-1}\sum_{C_l}\sum_{K_{d,l}^{\mathbf{i}}}\sum_{j=0}^{m_{\mathbf{k}}-1} (-1)^{\mathbf{k}+j+q+l+n}\,\gamma_{d,l}^{n,q}\int_{\mathbf{c}} \widehat{T}_v(A^{q+l},\mathbf{c},\mathbf{k},j) \\
&\qquad+ \sum_{l=0}^{d-1}\sum_{C_l}\sum_{K_{d,l}^{\mathbf{i}}}\sum_{j=1}^{m_{\mathbf{k}}-1} (-1)^{\mathbf{k}+j+q+l+n}\,\gamma_{d,l}^{n,q}\int_{\mathbf{c}} \widehat{T}_c(A^{q+l},\mathbf{c},\mathbf{k},j).
\end{aligned} \tag{3.12}$$

The upper bound on l is $d-1$ rather than d simply because there are no vertical or corner points in staircases with $d = l$. The bounds on j in the last line have been modified for a similar reason

— there are no corner points in position 0 or $m_{\mathbf{k}}$. At any rate, most of the terms in (3.12) are zero. First, we have

Lemma 3.0.6. *For any* $\mathbf{c} \in C_l$ *and any* $0 < j < m_{\mathbf{k}}$*,*

$$\sum_{\mathbf{k} \in K^{\mathbf{i}}_{d,l}} (-1)^{\mathbf{k}} \, \widehat{T}_c(A^{q+l}, \mathbf{c}, \mathbf{k}, j) = 0.$$

Proof. Suppose $\mathbf{k}$ has an outward corner point in position j, and let $\mathbf{k}'$ be the corresponding staircase with an inward corner, as in Figure 3.5. Then

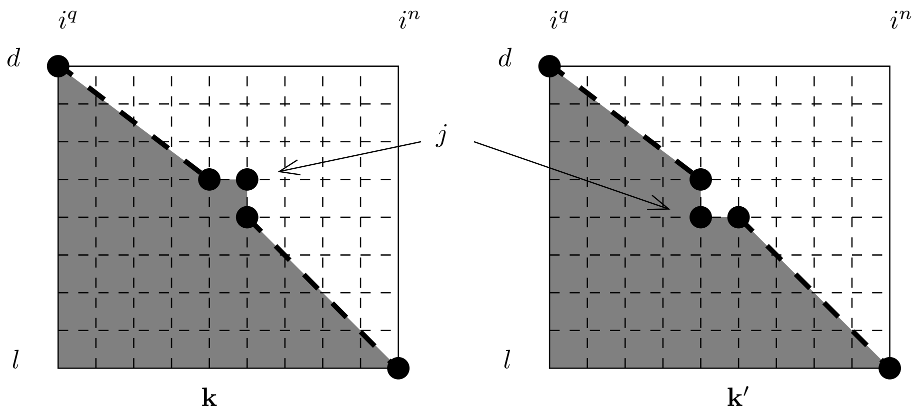

Figure 3.5: $\mathbf{k}$ has an outward corner at j, and $\mathbf{k}'$ has an inward corner.

$$\hat{T}_c(\cdot, \mathbf{c}, \mathbf{k}, j) = \hat{T}_c(\cdot, \mathbf{c}, \mathbf{k}', j),$$

and $(-1)^{\mathbf{k}} + (-1)^{\mathbf{k}'} = 0$, so that the contribution to the sum from $\mathbf{k}$ cancels the contribution from $\mathbf{k}'$. □

This shows that the last line of (3.12) vanishes. Next, we have

Lemma 3.0.7. *If* $\mathbf{k}$ *has a vertical point at position* j*, with* $0 < j < m_{\mathbf{k}}$*, then*

$$\sum_{\mathbf{c} \in C_l} \int_{\mathbf{c}} \widehat{T}_v(A^{q+l}, \mathbf{c}, \mathbf{k}, j) = 0. \tag{3.13}$$

Proof. Let s denote the row number of the j^{th} point of $\mathbf{k}$, as in Figure 3.6. Then the j^{th} point is the only point in row s. It follows that for any $\mathbf{c} \in C_l$, the value of $\hat{T}(A^{q+l}, \mathbf{c}, \mathbf{k}, j)$ depends only on the simplices c_r for $r \neq s$.

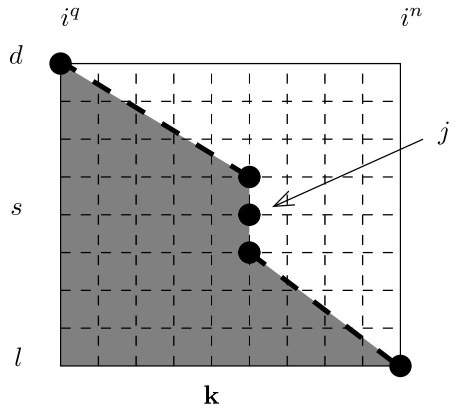

Figure 3.6: $\mathbf{k}$ has a vertical point at position j, in row s.

Recall proposition 2.3.10: for any coflag $\mathbf{c} \in C_l$, there is exactly one coflag $\overline{\mathbf{c}} \in C_l$ such that $c_r = \overline{c}_r$ for all $r \neq s$, but $c_s \neq \overline{c}_s$. In particular, this means that

$$\hat{T}(A^{q+l}, \mathbf{c}, \mathbf{k}, j) = \hat{T}(A^{q+l}, \overline{\mathbf{c}}, \mathbf{k}, j).$$

On the other hand, the bottom simplex $c_l = \overline{c}_l$ is given opposing orientations by $\mathbf{c}$ and $\overline{\mathbf{c}}$, so that in (3.13) the integrals over these two coflags cancel exactly. □

This shows that, in the middle line of (3.12), every term with $j > 0$ vanishes, leaving

$$\begin{aligned} &d\tau_{\Sigma}^{n-1,q-1}(\mathbf{A})_{\mathbf{I}} + \hat{\delta}\tau_{\Sigma}^{n-1,q}(\mathbf{A})_{\mathbf{I}} + (-1)^d \tau_{\Sigma}^{n,q}(D\mathbf{A})_{\mathbf{I}} \\ &\quad = \sum_{l=0}^{d-1} \sum_{C_l} \sum_{K_{d,l}^{\mathbf{i}}} (-1)^{\mathbf{k}+q+l+n}\, \gamma_{d,l}^{n,q} \int_{\mathbf{c}} \widehat{T}_v(A^{q+l}, \mathbf{c}, \mathbf{k}, 0). \end{aligned} \tag{3.14}$$

Next, we have

Lemma 3.0.8. *Let $\mathbf{k} \in K_{d,l}^{\mathbf{i}}$ have a vertical point at position 0, let $\mathbf{k}' \in K_{d-1,l}^{\mathbf{i}}$ be obtained from $\mathbf{k}$ by omitting the first point, as in Figure 3.7. Then*

$$\sum_{\mathbf{c} \in C_l(S)} \int_{\mathbf{c}} \hat{T}_v(A^{q+l}, \mathbf{c}, \mathbf{k}, 0) = \sum_{\mathbf{c}' \in C_l(\partial S)} \int_{\mathbf{c}'} T(A^{q+l}, \mathbf{c}', \mathbf{k}'). \tag{3.15}$$

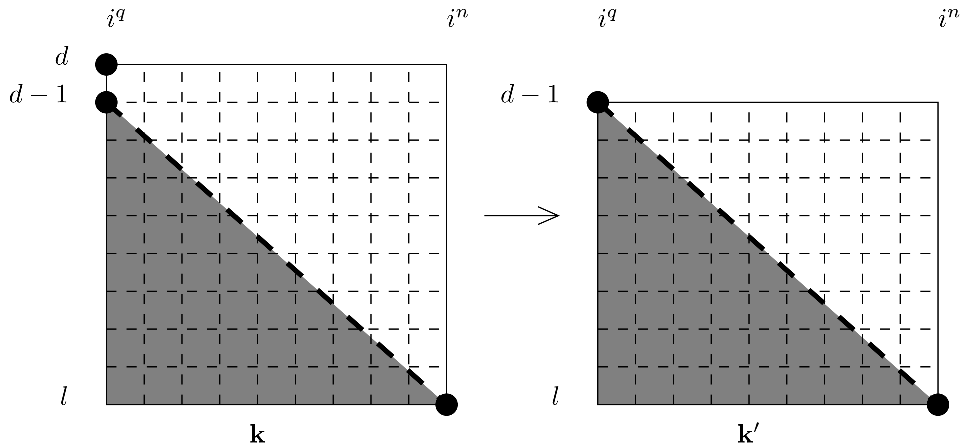

Figure 3.7: $\mathbf{k}'$ is obtained by omitting the first point of $\mathbf{k}$.

Proof. For any $\mathbf{c} \in C_l(S)$, let $\mathbf{c}' = (c_l, \ldots, c_{d-1})$. Then the assigment $\mathbf{c} \mapsto \mathbf{c}'$ gives rise to a bijection between the set of $\mathbf{c}$ with $c_{d-1} \subseteq \partial\Sigma$ and the set $C_l(\partial S)$ (see proposition 2.3.10). Moreover, since the first point of $\mathbf{k}$ is the only point in row d, we have

$$\hat{T}_v(A^{q+l}, \mathbf{c}, \mathbf{k}, 0) = T(A^{q+l}, \mathbf{c}', \mathbf{k}')$$

for any $\mathbf{c} \in C_l(S)$. Now, for any $\mathbf{c}$ with $c_{d-1} \nsubseteq \partial\Sigma$, there exists exactly one coflag $\overline{\mathbf{c}} \in C_l(S)$ such that $\mathbf{c}' = \overline{\mathbf{c}}'$, but $c_d \neq \overline{c}_d$ (again, see proposition 2.3.10). Moreover, the bottom simplex $c_l = \overline{c}_l$ is given opposing orientations by $\mathbf{c}$ and $\overline{\mathbf{c}}$, so that

$$\int_{\mathbf{c}} \hat{T}_v(A^{q+l}, \mathbf{c}, \mathbf{k}, 0) + \int_{\overline{\mathbf{c}}} \hat{T}_v(A^{q+l}, \overline{\mathbf{c}}, \mathbf{k}, 0) = 0.$$

Thus in the left hand side of (3.15), the terms involving $\mathbf{c}$ with $c_{d-1} \nsubseteq \partial\Sigma$ cancel in pairs, and the remaining terms in the left hand side exactly match the terms on the right hand side, via the correspondence $\mathbf{c} \mapsto \mathbf{c}'$. □

The correspondence $\mathbf{k} \mapsto \mathbf{k}'$ from the statement of Lemma 3.0.8 gives a bijection between the set of $\mathbf{k}$ with a vertical point at position 0, and the set of all $\mathbf{k}' \in K^{\mathbf{i}}_{d-1,l}$. This fact, coupled with equation (3.14) and the relation

$$\gamma^{n,q}_{d-1,l} = (-1)^{q+l+n} \gamma^{n,q}_{d,l}$$

from remark 3.0.2, gives

$$\begin{aligned} &d\tau_\Sigma^{n-1,q-1}(\mathbf{A})_\mathbf{I} + \hat{\delta}\tau_\Sigma^{n-1,q}(\mathbf{A})_\mathbf{I} + (-1)^d \tau_\Sigma^{n,q}(D\mathbf{A})_\mathbf{I} \\ &\quad = \sum_{l=0}^{d-1} \sum_{C_l(\partial S)} \sum_{K^\mathbf{i}_{d-1,l}} (-1)^\mathbf{k}\, \gamma^{n,q}_{d-1,l} \int_\mathbf{c} T(A^{q+l}, \mathbf{c}, \mathbf{k}) \\ &\quad = \rho^* \tau^{n,q}_{\partial\Sigma}(\mathbf{A})_\mathbf{I}. \end{aligned} \tag{3.16}$$

This completes the proof of Theorem 3.0.5. □

We finish the section with a few easy propositions; these will be useful in drawing analogies between τ and topological field theories.

Proposition 3.0.9. *Let $\overline{\Sigma}$ denote Σ with the opposite orientation. Then*

$$\tau^n_{\overline{\Sigma}} = -\tau^n_\Sigma.$$

Proof. Note that for any l-coflag $\mathbf{c} = (c_l, \ldots, c_d)$ in Σ, the orientation c_l inherits from $\mathbf{c}$ depends on the orientation of Σ: switching the orientation of Σ switches the orientation of c_l. Then, because τ_Σ is a sum over the l-coflags $\mathbf{c}$ in Σ, where the contribution from each $\mathbf{c}$ is an integral over c_l in its inherited orientation, the proposition follows as an immediate consequence of property 2 (orientation) for complexes of sheaves with integrals (see definition 2.1.1). □

Proposition 3.0.10. *Suppose $\Sigma = \Sigma_1 \amalg_C \Sigma_2$ for some $(d-1)$-dimensional manifold C. For $i = 1, 2$, let $r_i \colon X^\Sigma \to X^{\Sigma_i}$ denote the natural restriction map. Then*

$$\tau^n_\Sigma = r_1^* \tau^n_{\Sigma_1} + r_2^* \tau^n_{\Sigma_2}.$$

Chapter 4

Low dimensional examples

In this chapter, we will show that when $\mathbf{A} \in \check{C}^1(\mathcal{U}; \mathbb{T}^\infty_D(1))$ satisfies $D\mathbf{A} = 0$, then $\mathbf{A}$ gives cocycle data for a principal S^1-bundle L with connection on X, and that for compact 1-dimensional manifolds Σ, $\tau^0_\Sigma(\mathbf{A})$ computes parallel transport along Σ in L. More precisely, $\tau^0_\Sigma(\mathbf{A})$ defines a morphism of principal S^1-bundles on X^Σ, as follows. Fix an orientation of $\partial\Sigma$, and define bundles L_+ and L_- on X^Σ by

$$L_\pm = \bigotimes_{p \in \partial_\pm} ev_p^* L$$

where $\partial_\pm$ denotes the positively or negatively oriented points in $\partial\Sigma$. Then $\tau^1_{\partial\Sigma}(\mathbf{A})$ gives cocycle data for the bundle $\mathrm{Hom}(L_-, L_+)$. The statement that $D\tau^0_\Sigma(\mathbf{A}) = \tau^1_{\partial\Sigma}(\mathbf{A})$ is then equivalent to the statement that $\tau^0_\Sigma(\mathbf{A})$ defines a global section of the bundle $\mathrm{Hom}(L_-, L_+)$. If $\partial\Sigma$ is empty, then $\mathrm{Hom}(L_-, L_+)$ is canonically trivialized, so that $\tau^0_\Sigma(\mathbf{A})$ determines a global S^1-valued function on X^Σ. This function computes the holonomy in L of maps $\Sigma \to X$.

Similarly, when $\mathbf{A} \in \check{C}^2(\mathcal{U}; \mathbb{T}^\infty_D(2))$ satisfies $D\mathbf{A} = 0$, then $\mathbf{A}$ gives cocycle data for an S^1-gerbe $\mathscr{G}$ with differential structure (a connection and a curving). If P is a finite oriented 0-manifold, we define gerbes $\mathscr{G}_\pm$ on X^P by

$$\mathscr{G}_\pm = \bigotimes_{p \in P_\pm} ev_p^* \mathscr{G}.$$

Then $\tau^2_P(\mathbf{A})$ gives cocycle data for the gerbe $\mathrm{Hom}(\mathscr{G}_-, \mathscr{G}_+)$. If C is a compact 1-manifold with boundary $\partial C = P$, then we have $D\tau^1_C(\mathbf{A}) = \tau^2_P(\mathbf{A})$, which means that $\tau^1_C(\mathbf{A})$ determines a global section of the gerbe $\mathrm{Hom}(\mathscr{G}_-, \mathscr{G}_+)$ on X^C. If P is empty, then $\mathrm{Hom}(\mathscr{G}_-, \mathscr{G}_+)$ is canonically equivalent to the trivial gerbe, so a global section is just a principal S^1-bundle on $X^{\mathbb{C}}$. In fact, this is equivalent to the usual construction of a line bundle on the loopspace given an S^1-gerbe over the base.

Now suppose that Σ is a compact 2-manifold. For any choice of orientation on $\partial\Sigma$, we can form line bundles L_+ and L_- on X^Σ, given by

$$L_\pm = \bigotimes_{\Gamma \subseteq \partial_\pm} \operatorname{res}_\Gamma^* L_\Gamma,$$

where Γ ranges over the connected components of $\partial_\pm$, L_Γ is the bundle on X^Γ given by $\tau_\Gamma^1(\mathbf{A})$, and $\operatorname{res}_\Gamma \colon X^\Sigma \to X^\Gamma$ is the obvious map. Then $\tau_\Sigma^0(\mathbf{A})$ determines a global section of the bundle $\operatorname{Hom}(L_-, L_+)$. If $\partial\Sigma$ is empty, then $\operatorname{Hom}(L_-, L_+)$ is canonically trivialized, so $\tau_\Sigma^0(\mathbf{A})$ determines a global S^1-valued function on X^Σ. This function computes the gerby version of holonomy for maps $\phi \colon \Sigma \to X$.

There is a slightly different perspective we can take for 2-dimensional manifolds Σ. Namely, let P be a finite oriented 0-manifold, let C_0 and C_1 be a pair of compact oriented 1-manifolds with $\partial C_0 = \partial C_1 = P$, and let Σ be a compact oriented 2-manifold with $\partial\Sigma = C_1 \amalg_P \overline{C_0}$. Then $\tau_{C_0}^1(\mathbf{A})$ and $\tau_{C_1}^1(\mathbf{A})$ determine sections s_0 and s_1, respectively, of the gerbe $\operatorname{Hom}(\mathscr{G}_-, \mathscr{G}_+)$, and $\tau_\Sigma^0(\mathbf{A})$ determines a morphism $s_0 \to s_1$.

In effect, any cocycle in $\check{C}^1(\mathfrak{U}; \mathbb{T}_D^\infty(1))$ determines a 1-dimensional topological field theory over X, and any cocycle in $\check{C}^2(\mathfrak{U}; \mathbb{T}_D^\infty(2))$ determines an *extended* (0,2)-dimensional field theory over X. While I am personally unaware of a language for higher gerbes that would permit a direct extension of the story above to higher dimensions, the map τ determines something that is at least plausibly close to being an extended $(0,n)$-field theory over X for any cocycle in $\check{C}^n(\mathfrak{U}; \mathbb{T}_D^\infty(n))$.

Throughout this section, we will use additive notation for S^1 and $\mathbb{T}$, to make better contact with the formulas of chapter 3.

4.1 Principal S^1-bundles

It is a standard fact that a cocycle $A^0 \in \check{C}^1(\mathfrak{U}; \mathbb{T})$ defines a principal S^1-bundle $L \to X$. Only slightly less well known is the fact that a cochain $A^1 \in \check{C}^0(\mathfrak{U}; i\Omega^1)$ with $\delta A^1 = \operatorname{dlog} A^0$ defines a connection on L. We review these constructions briefly.

Fix $\mathbf{A} = (A^0, A^1) \in \check{C}^1(\mathfrak{U}; \mathbb{T}_D^\infty(1))$ such that $D\mathbf{A} = 0$. In other words, we require $A^0 \in \check{C}^1(\mathfrak{U}; \mathbb{T})$

and $A^1 \in \check{C}^0(\mathcal{U}; i\Omega^1)$ to satisfy

$$A^0_{ij} + A^0_{jk} + A^0_{ki} = 0 \qquad \text{and} \qquad \operatorname{dlog} A^0_{ij} = A^1_j - A^1_i$$

for all i,j, and k. By definition we also have $A^0_{ii} = 0$ for all i. We then define the principal S^1-bundle $l = L_{\mathbf{A}}$ to be the following coequalizer.

$$\begin{array}{ccccc}
\coprod_{i,j} U_{ij} \times S^1 & \rightrightarrows & \coprod_i U_i \times S^1 & \longrightarrow & L \\
\downarrow & & \downarrow & & \downarrow \\
\coprod_{i,j} U_{ij} & \rightrightarrows & \coprod_i U_i & \longrightarrow & X
\end{array}$$

The two horizontal maps are given, component-wise, by

$$U_{ij} \times S^1 \xrightarrow{id} U_j \times S^1$$

and

$$U_{ij} \times S^1 \xrightarrow{A^0_{ij}} U_i \times S^1.$$

The cocycle condition on A^0 ensures that the resulting space L is, in fact, a principal S^1-bundle. This bundle is determined by the fact that it has sections $s_i \colon U_i \to L$ that satisfy $s_j = s_i + A^0_{ij}$ over U_{ij}.

We next show that A^1 determines a connection on L. For this, we will take a slightly non-standard definition of connection, namely a map of sheaves on X

$$\omega \colon \Gamma_L \to i\Omega^1$$

such that

$$\omega(s + f) = \omega(s) + \operatorname{dlog} f$$

for all $s \in \Gamma_L(U)$ and all $f \colon U \to S^1$. This last condition is the same as saying that ω is $\mathbb{T}$-equivariant. Now, the idea is that the connection ω associated to A^1 should be uniquely determined by requiring $\omega(s_i) = A^1_i$ for all i. Since $\mathbb{T}$ acts freely and transitively on Γ_L, the value of ω on any other section of L will then be determined. Moreover, the only constraint on the values of $\omega(s_i)$ is that

$$\omega(s_j) = \omega(s_i + A^0_{ij}) = \omega(s_i) + \operatorname{dlog} A^0_{ij}$$

for all i and j. If $\omega(s_i) = A^1_i$, then this condition is implied by $D\mathbf{A} = 0$, as noted above.

So, we have given a construction of a principal bundle with connection for any cocycle $\mathbf{A}$ as above. We now show that this construction may be lifted to give an equivalence of categories. On the one hand, we have the category $\mathcal{P}(\mathcal{U})$ of principal S^1-bundles with connection on X, for which $\mathcal{U}$ is a trivializing open cover, and in which a morphism $(L, \omega) \to (L', \omega')$ is given by a bundle map $f\colon L \to L'$ such that $f^*\omega' = \omega$.

On the other hand, consider the homomorphism

$$\check{C}^0(\mathcal{U}; \mathbb{T}^\infty_D(1)) \xrightarrow{D} \check{Z}^1(\mathcal{U}; \mathbb{T}^\infty_D(1)),$$

where $\check{Z}^1 = \ker(D) \subseteq \check{C}^1$. Associated to this homomorphism is a category, whose objects are the elements of $\check{Z}^1$, in which a morphism $\mathbf{A} \to \mathbf{A}'$ is given by a cochain $f \in \check{C}^0$ such that $Df = \mathbf{A}' - \mathbf{A}$. Denote this category by $\mathcal{C}(\mathcal{U})$. Then the construction above lifts to an equivalence of categories $L\colon \mathcal{C}(\mathcal{U}) \to \mathcal{P}(\mathcal{U})$.

The value of L on a morphism $f\colon \mathbf{A} \to \mathbf{A}'$ is as follows. Let s_i and s'_i denote the local sections of $L_{\mathbf{A}}$ and $L_{\mathbf{A}'}$, respectively, that appear in the construction above. We define $Lf\colon L_{\mathbf{A}} \to L_{\mathbf{A}'}$ by setting

$$Lf(s_i) = s'_i - f_i.$$

For Lf to be well-defined, we must have

$$\begin{aligned} 0 &= Lf(s_j) - Lf(s_i + A^0_{ij}) \\ &= Lf(s_j) - Lf(s_i) - A^0_{ij} \\ &= s'_j - s'_i - f_j + f_i - A^0_{ij} \\ &= A'^0_{ij} - A^0_{ij} - f_j + f_i. \end{aligned}$$

This is exactly the condition that $\delta f = A'^0 - A^0$. We must also show that $Lf^*\omega' = \omega$. Note that

$$\begin{aligned}\omega'(Lf(s_i)) - \omega(s_i) &= \omega'(s_i' - f_i) - A_i^1 \\ &= A_i'^1 - A_i^1 - \operatorname{dlog} f_i.\end{aligned}$$

This last expression is zero for all i if and only if $\operatorname{dlog} f_i = A'^1 - A^1$. Collectively, the condition that $Df = \mathbf{A}' - \mathbf{A}$ is exactly the condition that Lf define a morphism $L_{\mathbf{A}} \to L_{\mathbf{A}'}$.

We now collect, for the sake of completeness, a few more facts about L. For any $\mathbf{A}, \mathbf{A}'$, we have

$$L_{\mathbf{A}+\mathbf{A}'} \cong L_{\mathbf{A}} \otimes L_{\mathbf{A}'}$$

and

$$L_{\mathbf{A}'-\mathbf{A}} \cong \operatorname{Hom}(L_{\mathbf{A}}, L_{\mathbf{A}'}).$$

In particular, if $\mathbf{A} = 0$, then $L_{\mathbf{A}}$ is the trivial bundle.

A morphism $f\colon 0 \to \mathbf{A}$ in $\mathcal{C}(\mathcal{U})$ is given by a cochain f with $Df = \mathbf{A}$. This is carried by L to a map of bundles $X \times S^1 \to L_{\mathbf{A}}$, which is essentially just a global section of $L_{\mathbf{A}}$. More generally, if $Df = \mathbf{A}' - \mathbf{A}$, then Lf determines a global section of the bundle $\operatorname{Hom}(L_{\mathbf{A}}, L_{\mathbf{A}'})$. We will frequently take this point of view in the sequel.

4.2 Parallel transport in S^1-bundles

In this section, we explain the connection between τ and parallel transport in S^1-bundles.

Let $L \to X$ be a principal S^1-bundle with connection ω, let $\Sigma = [a, b]$, and let $\gamma\colon \Sigma \to X$ be any smooth path. Then there is a natural S^1-equvariant map

$$\Gamma(\gamma)_a^b\colon L_{\gamma(a)} \to L_{\gamma(b)}$$

called parallel transport; it is uniquely determined by the requirement that $\Gamma(\gamma)_a^b s(a) = s(b)$ for any section s along γ such that $\omega(s) = 0$. This operator depends smoothly on the path γ, and so determines a global section of the bundle $\operatorname{Hom}(ev_a^* L, ev_b^* L)$ on X^Σ. When translated into the language of cocycles, these constructions take the following form.

Proposition 4.2.1. *Let Σ denote the manifold $[a,b]$. If a bundle with connection (L,ω) is given by a cocycle $\mathbf{A}$, then the bundle $\mathrm{Hom}(ev_a^*L, ev_b^*L)$ is given by the cocycle $\tau^{1,0}_{\partial\Sigma}(\mathbf{A})$, and the section of this bundle corresponding to parallel transport is given by $\tau^0_\Sigma(\mathbf{A})$.*

Proof. First, let P be the 0-manifold consisting of a single, positively oriented point p. Then $ev_p\colon X^P \to X$ is a homeomorphism, and the cover $\mathcal{U}^P$ is just the pullback of $\mathcal{U}$ along ev_p. The formula for $\tau^{1,0}_P(\mathbf{A})$ is given by

$$\begin{aligned}\tau^{1,0}_P(\mathbf{A})_{ij} &= \sum_{\mathbf{c}\in C_0} \sum_{\mathbf{k}\in K^{ij}_{0,0}} (-1)^{\mathbf{k}} \gamma^{1,0}_{0,0} \int_{\mathbf{c}} T(A^0, \mathbf{c}, \mathbf{k}) \\ &= A^0_{ij}(p)\end{aligned}$$

for any open sets $U_i, U_j \in \mathcal{U}$. In other words, $\tau^{1,0}_P(\mathbf{A}) = ev_p^* A^0$, which is a cocycle for ev_p^*L. Then, by Propositions 3.0.9 and 3.0.10, we have $\tau^{1,0}_{\partial\Sigma} = ev_b^*A^0 - ev_a^*A^0$, which is visibly a cocycle for $\mathrm{Hom}(ev_a^*L, ev_b^*L)$.

Now, fix an open set U^Σ_I in the cover $\mathcal{U}^\Sigma$; such a set is determined by a pair $I = (S,i)$, with S a triangulation of Σ and i an $\mathcal{I}$-labeling of S, and consists of all the paths $\gamma\colon \Sigma \to X$ such that $\gamma(c) \subseteq U_{i_c}$ for all $c \in S$. In particular, the endpoints a and b are 0-simplices of S, and so carry labels i_a and i_b. Let $s_a = s_{i_a} \circ ev_a$, which is a section of ev_a^*L over U^Σ_I, and define s_b similarly. Then $s_a \mapsto s_b$ is a section of $\mathrm{Hom}(ev_a^*L, ev_b^*L)$ over U^Σ_I; the cocycle associated to this family of sections is none other than $\tau^{1,0}_\Sigma(\mathbf{A})$. Since $D\tau^0_\Sigma(\mathbf{A}) = \tau^1_{\partial\Sigma}(\mathbf{A})$, $\tau^0_\Sigma(\mathbf{A})$ defines a global section of $\mathrm{Hom}(ev_a^*L, ev_b^*L)$ whose value on U^Σ_I is given by $s_a \mapsto s_b - \tau^0_\Sigma(\mathbf{A})_I$. We now show that this is exactly the formula for parallel transport in L.

Note that parallel transport satisfies

$$\Gamma(\gamma)^z_y \circ \Gamma(\gamma)^y_x = \Gamma(\gamma)^z_x$$

for any three points x, y, z along a path γ. In particular, for any 1-simplex $e \in S$, let x and y

denote the incoming and outgoing vertices of e, respectively, and define

$$\lambda_e = s_{i_y} - \Gamma(\gamma)_x^y\, s_{i_x}.$$

Then

$$\Gamma(\gamma)_a^b\, s_a = s_b - \sum_e \lambda_e,$$

so we must show that $\tau_\Sigma^0(\mathbf{A})_I = \sum_e \lambda_e$.

First, we compute λ_e. Let s_e denote the section along e given by $s_e = s_{i_e} \circ \gamma|_e$. Define

$$f(t) = \int_x^t \omega(s_e).$$

Then $\omega(s_e - f) = \omega(s_e) - df = 0$, which means that parallel transport along γ is given by

$$\Gamma(\gamma)_x^y\, s_e(x) = s_e(y) - f(y).$$

Note that $s_{i_x} = s_e(x) + A^0_{i_e i_x}$, and likewise $s_{i_y} = s_e(y) + A^0_{i_e i_y}$. Also, note that $f(y) = \int_e \gamma^* A^1_{i_e}$. Thus we have

$$\lambda_e = s_{i_x} - \Gamma(\gamma)_x^y\, s_{i_y} = A^0_{i_e i_y} - A^0_{i_e i_x} + \int_e \gamma^* A^1_{i_e}. \tag{4.1}$$

Now, recall that $\tau_\Sigma^0(\mathbf{A})_I$ is an S^1-valued function on U_I^Σ, given by

$$\tau_\Sigma^0(\mathbf{A})_I = \sum_{l=0}^{1} \sum_{\mathbf{c}\in C_l} \sum_{\mathbf{k}\in K^i_{1,l}} (-1)^{\mathbf{k}} \gamma_{1,l}^{0,0} \int_{\mathbf{c}} T(A^l, \mathbf{c}, \mathbf{k})$$

Consider the $l = 1$ summand first. A 1-coflag $\mathbf{c} \in C_1$ is nothing more than a 1-simplex e of S. The only element of $K^i_{1,1}$ is $\mathbf{k} = \{(i, 1)\}$, and

$$T(A^1, \mathbf{c}, \mathbf{k}) = A^1_{i_e}.$$

Also, note that $(-1)^{\mathbf{k}} \gamma_{1,1}^{0,0} = 1$. Thus the $l = 1$ contribution to $\tau_\Sigma^0(\mathbf{A})_I$ is

$$\sum_e \int_e \gamma^* A^1_{i_e}.$$

Next, consider the $l = 0$ summand. A 0-coflag $\mathbf{c} \in C_0$ consists of a 1-simplex e and a 0-simplex $v \subset e$. Again there is only one element of $K^i_{1,0}$, namely $\mathbf{k} = \{(i, 1), (i, 0)\}$. Integrating over $\mathbf{c}$ means evaluating at v, and multiplying by -1 if v is the incoming boundary point of e. Since $(-1)^{\mathbf{k}}\gamma^{0,0}_{1,0} = 1$, we see that the $l = 0$ contribution is

$$\sum_{v\subset e} \pm A^0_{i_e i_v}(\gamma(v)),$$

where the sign is positive if v is the outgoing vertex of e, and negative otherwise. So the total expression for $\tau^0_\Sigma(\mathbf{A})_I$ simplifies to

$$\tau^0_\Sigma(\mathbf{A})_I(\gamma) = \sum_e \int_e \gamma^* A^1_{i_e} + \sum_{v\subset e} \pm A^0_{i_e i_v}(\gamma(v)).$$

But this is exactly equal to $\sum_e \lambda_e$, as desired. □

4.3 S^1-Gerbes

We begin by reviewing the definition of an S^1-gerbe on a manifold X. For a detailed discussion, see Brylinski [Bry08] or Brylinski-McLaughlin [BM94].

Definition 4.3.1. A gerbe $\mathcal{G}$ on a space X is a stack of groupoids on X that is locally non-empty, and locally connected. A stack of groupoids is, roughly, a presheaf of groupoids that satisfies descent; see Moerdijk [Moe02] for details. The condition that $\mathcal{G}$ is locally non-empty says that for each point $x \in X$, there is a neighborhood U of X such that the groupoid $\mathcal{G}(U)$ is non-empty. The condition that $\mathcal{G}$ is locally connected says that for any open set $U \subseteq X$, any objects $a, b \in \mathcal{G}(U)$, and any point $x \in U$, there exists a neighborhood $V \subseteq U$ of x such that $a|_V$ and $b|_V$ are isomorphic as objects of $\mathcal{G}(V)$.

Definition 4.3.2. An S^1-gerbe on a manifold X is a gerbe on X equipped with isomorphisms $\phi_a \colon \mathrm{Aut}_{\mathcal{G}(U)}(a) \xrightarrow{\simeq} \mathbb{T}(U)$ for every open $U \subseteq X$ and every object $a \in \mathcal{G}(U)$. This isomorphism must commute with morphisms in $\mathcal{G}(U)$, and must be compatible with restriction to smaller open sets. In particular, each hom-set $\mathrm{Hom}_{\mathcal{G}(U)}(a, b)$ is a torsor for the group $\mathbb{T}(U)$. We will use the notation $f + \lambda$ for the action of $\lambda \in \mathcal{T}(U)$ on $f \in \mathrm{Hom}_{\mathcal{G}(U)}(a, b)$.

Remark 4.3.3. What we are calling an S^1-gerbe is more commonly called a *gerbe with band* $\mathbb{T}$, or a *gerbe bound by* $\mathbb{T}$. We use the term S^1-gerbe to emphasize their similarities to principal S^1-bundles.

Example 4.3.4. The standard example of an S^1-gerbe on X, which we call the trivial gerbe, assigns to each open set $U \subseteq X$ the category of principal S^1-bundles on U. We will denote the trivial S^1-gerbe by $\mathcal{T}$.

In close analogy with the situation for principal S^1-bundles, S^1-gerbes admit a classification in terms of Čech cocycles. Let $\mathcal{G}$ be an S^1-gerbe on X. The requirement that $\mathcal{G}$ be locally non-empty means that there exists an open cover $\mathcal{U} = \{U_i\}$ of X and an object $a_i \in \mathcal{G}(U_i)$ for all i. The requirement that $\mathcal{G}$ be locally connected means that, possibly after passing to a finer open cover, we can choose isomorphisms $f_{ij} \colon a_i|_{U_{ij}} \to b_i|_{U_{ij}}$ for all i, j. We will asssume that $f_{ii} = id$, but not that $f_{ij} = f_{ji}^{-1}$.

Suppose for a moment that these morphisms satisfy the cocycle condition $f_{jk} \circ f_{ij} = f_{ik}$ over U_{ijk}, i.e. that the following diagram commutes.

$$\begin{array}{ccc} a_i|_{U_{ijk}} & \xrightarrow{f_{ik}|_{U_{ijk}}} & a_k|_{U_{ijk}} \\ & {\scriptstyle f_{ij}|_{U_{ijk}}} \searrow \quad \nearrow {\scriptstyle f_{jk}|_{U_{ijk}}} & \\ & a_j|_{U_{ijk}} & \end{array} \tag{4.2}$$

Then the descent condition guarantees the existance of a "glued together" object $a \in \mathcal{G}(X)$, together with isomorphisms $g_i \colon a|_{U_i} \to a_i$, such that

$$f_{ij} \circ g_i|_{U_{ij}} = g_j|_{U_{ij}}.$$

We think of the object a as a global section of $\mathcal{G}$, and its existence implies that $\mathcal{G}$ is equivalent to the trivial gerbe $\mathcal{T}$ (see proposition 4.3.5).

Of course, it is not always possible to choose the morphisms f_{ij} so that they satisfy the cocycle condition (4.2). But because the hom-set

$$\mathrm{Hom}_{\mathcal{G}(U_{ijk})}(a_i|_{U_{ijk}}, a_k|_{U_{ijk}})$$

is a torsor for the group $\mathbb{T}(U_{ijk})$, there exists a unique element $\lambda_{ijk} \in \mathbb{T}(U_{ijk})$ such that over U_{ijk},

$$f_{jk} \circ f_{ij} = f_{ik} + \lambda_{ijk}.$$

Since we required $f_{ii} = id$, we have $\lambda_{iij} = 0$ and $\lambda_{ijj} = 0$ for all i and j. However, λ_{iji} is not necessarily zero, as we did not require $f_{ij} = f_{ji}^{-1}$. Thus as i, j and k vary, we obtain a normalized Čech cochain $\lambda \in \check{C}^2(\mathcal{U}; \mathbb{T})$. In fact, λ is a cocycle: over U_{ijkl}, we have

$$\begin{aligned} f_{kl} \circ f_{kj} \circ f_{ij} &= f_{il} + \lambda_{ijk} + \lambda_{ikl} \\ &= f_{il} + \lambda_{ijl} + \lambda_{jkl}, \end{aligned}$$

from which it follows that

$$(\delta\lambda)_{ijkl} = \lambda_{jkl} - \lambda_{ikl} + \lambda_{ijl} - \lambda_{ijk} = 0.$$

This cocycle classifies $\mathcal{G}$ up to equivalence. More precisely, we have the following, which is a restatement of theorem 5.2.8 in Brylinski [Bry08].

Proposition 4.3.5. *Suppose that $\mathcal{U}$ is a good cover of X, in the sense that every non-empty intersection of finitely many open sets in $\mathcal{U}$ is contractible. Then the construction above gives rise to a bijection between the set of S^1-gerbes up to equivalence, and the Čech cohomology group $\check{H}^2(\mathcal{U}; \mathbb{T})$. Under this correspondence, the zero cohomology class corresponds to the trivial S^1-gerbe $\mathcal{T}$.*

For S^1-gerbes on X, there are two auxiliary structures, known as connective structures and curvings, that play an analogous role to connections for principal S^1-bundles.

There is a nice theory of connections on S^1-gerbes, though they are usually called connective structures and curvings [Bry08]. These are analogous to ordinary connections on principal

S^1-bundles. Compare the following with the definition of connection given on page 37.

Definition 4.3.6. Let $\mathcal{G}$ be an S^1-gerbe on X. A *connective structure* Co on $\mathcal{G}$ is a map of stacks

$$\mathrm{Co}\colon \mathcal{G} \to \mathrm{Tors}(\Omega^1),$$

where $\mathrm{Tors}(\Omega^1)$ is the stack that associates to each open set $U \subseteq X$ the category of torsors for the group $\Omega^1(U)$. Thus Co assigns to each object $a \in \mathcal{G}(U)$ an $\Omega^1(U)$-torsor $\mathrm{Co}(a)$, and to each morphism $f\colon a \to b$ in $\mathcal{G}(U)$ a map of $\Omega^1(U)$-torsors $f_*\colon \mathrm{Co}(a) \to \mathrm{Co}(b)$, in a way that is compatible with restriction to smaller open sets. We require that Co be $\mathbb{T}$-equivariant, in the sense that for any $\lambda \in \mathbb{T}(U)$ and any morphism $f\colon a \to b$ in $\mathcal{G}(U)$, we have

$$(f + \lambda)_* = f_* + d\lambda.$$

Definition 4.3.7. Let $\mathcal{G}$ be an S^1-gerbe on X, and let Co be a connective structure on $\mathcal{G}$. A *curving* for Co is a rule K that assigns to each object $a \in \mathcal{G}(U)$, and to each element $\nabla \in \mathrm{Co}(a)$, a 2-form $K(a, \nabla) \in \Omega^2(U)$ such that $K(a, \nabla + \alpha) = K(a, \nabla) + d\alpha$ for any 2-form $\alpha \in \Omega^2(U)$. We also require K to be compatible with restrictions, and to satisfy $K(a, \nabla) = K(b, f_*\nabla)$ for any morphism $f\colon a \to b$ in $\mathcal{G}(U)$.

Proposition 4.3.8 (Theorem 3.5 in [BM94])**.** *Let $\mathcal{U}$ be a good open cover of X. Then S^1-gerbes with connective structure on X are classified up to equivalence by the Čech cohomology group $\check{H}^2(\mathcal{U}; \mathbb{T}^\infty_D(1))$, and S^1-gerbes with connective structure and curving are classified up to equivalence by $\check{H}^2(\mathcal{U}; \mathbb{T}^\infty_D(2))$.*

For S^1-gerbes on X with connective structure and curving, there is a theory of holonomy over 1-dimensional and 2-dimensional manifolds. This is covered, for example, in section 6.2 of [Bry08]. Let $(\mathcal{G}, \mathrm{Co}, K)$ be an S^1-gerbe with connective structure and curving. The notion of holonomy in $\mathcal{G}$ for 1-dimensional manifold C takes the form of a line bundle L_C on the mapping space X^C. We won't provide a definition for L_C here. However, the following formula, given as equation 11.4 in

[GR02], defines a Čech cocycle for L_C in terms of a cocycle for $(\mathcal{G}, \mathrm{Co}, K)$.

$$g_{IJ}(\phi) = \exp\left[i \sum_{\bar{b}} \int_{\bar{b}} \phi^* A_{j_{\bar{b}} i_{\bar{b}}} \right] \prod_{\bar{v} \in \bar{b}} \frac{g_{i_{\bar{v}} j_{\bar{v}} j_{\bar{b}}}(\phi(\bar{v}))}{g_{i_{\bar{v}} i_{\bar{b}} j_{\bar{b}}}(\phi(\bar{v}))} \tag{4.3}$$

Here I and J are $\mathcal{I}$-labeled triangulations of C. To interpret the right hand side, one must choose a triangulation S that refines both I and J, and let i and j denote the induced $\mathcal{I}$-labelings of S. $\bar{b}$ ranges over the 1-simplices of S, and the sum over $\bar{v} \in \bar{b}$ is, in our terminology, a sum over the 0-coflags of S. Implicit in the notation is the convention that if $\bar{v}$ is the incoming vertex of $\bar{b}$, then the multiplicand on the far right is to be inverted. The pair (g, A) is a cocycle in $\check{C}^2(\mathcal{U}; \mathbb{T}^\infty_D(1))$ that represents $(\mathcal{G}, \mathrm{Co})$. The curving K does not enter into the definition of L_C, so the corresponding cocycle data does not appear in formula (4.3). Finally, ϕ is any map $C \to X$ contained in the open set U^C_{IJ}. In other words, we require $\gamma(\bar{b}) \subseteq U_{i_{\bar{b}} j_{\bar{b}}}$ for all edges $\bar{b}$, and likewise for all vertices. Thus G is an element of $\check{C}^1(\mathcal{U}^{\mathbb{C}}; \mathbb{T})$.

Other than the use of multiplicative notation for S^1, and perhaps a few differing signs, equation 4.3 is essentially the same formula we gave for $\tau^{1,0}_C(g, A)_{IJ}$. The term in the exponential corresponds to the $l = 1$ term in the formula for τ_C, and the product over $\bar{v} \in \bar{b}$ corresponds to the $l = 0$ term in τ. There are a few minor differences, however. The staircases that correspond to the sequences of indices in the numerator and denominator on the right are *ascending* staircases; they are shown in figure 4.1. However, in the presense of the cocycle condition $\delta g = 0$, the expression in equation 4.3 is equal to the corresponding expression involving descending staircases. Similarly, the term $A_{j_{\bar{b}} i_{\bar{b}}}$, according to our formula for τ, should be $A_{i_{\bar{b}} j_{\bar{b}}}$. Again, the two terms agree (up to a sign) in the presence of the cocycle condition.

Figure 4.1: The two ascending staircases that appear in formula (4.3).

Thus we have shown the following.

Proposition 4.3.9. *Let $C = S^1$. If a gerbe $\mathcal{G}$ with connective structure and curving on X is given by the cocycle $\mathbf{A} \in \check{C}^2(\mathcal{U}; \mathbb{T}^\infty_D(2))$, then the holonomy bundle L_C over the free loop space $\mathcal{L}X = X^C$*

is given by the cocycle $\tau^1_C(\mathbf{A})$*. If* $C = \partial\Sigma$ *for some* Σ*, then* $\tau^{0,0}_\Sigma(\mathbf{A})$ *is a cocycle for the section* Hol_Σ *of* L_C *over* X^Σ*.*

If Σ is a compact oriented surface with boundary C, the holonomy in $(\mathcal{G}, \mathrm{Co}, K)$ for maps $\Sigma \to X$ takes the form of a section over X^Σ of the bundle L_C. The formula given for this section in [GR02] is as follows.

$$\mathcal{A}(\phi)_I = \exp\left[i \sum_c \int_c \phi^* B_{i_c} + i \sum_{b \subset c} \int_b \phi^* A_{i_c i_b} \right] \prod_{v \in b \subset c} g_{i_c i_b i_v}(\phi(v)) \tag{4.4}$$

Most of the same notational conventions are in effect. I is an $\mathcal{I}$-labeled triangulation of Σ, and c, b and v range over the 2, 1, and 0-simplices in I. B is a Čech cochain for the curving K in the gerbe. In other words, $(\mathcal{G}, \mathrm{Co}, K)$ is given by the Čech cocycle $(g, A, B) \in \check{C}^2(\mathcal{U}; \mathbb{T}^\infty_D(2))$. Lastly, ϕ is an element of U^Σ_I. Thus $\mathcal{A}$ is an element of $\check{C}^0(\mathcal{U}^\Sigma; \mathbb{T})$. Again, other than a few signs and the use of multiplicative notation, it is easy to see that equation (4.4) agrees with our formula for $\tau^{0,0}_\Sigma(g, a, b)_I$.

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