

Wess-Zumino-Witten models and their correspondence to Chern-Simons theories

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DAMTP Part III essay - [GitHub](#)



Introduction

- Quantum Field Theories (QFTs) are mostly **perturbative** (Cons: confinement, mass gap in QCD, etc.)
- Non-perturbative QFTs: (mostly 2D) Conformal Field Theories (CFTs) & Topological QFTs.
- Rational 2D-CFTs [Belavin, Polyakov, Zamolodchikov '84]:
 - Use **infinite-dimensional symmetries** to reduce infinite problems to effective finite ones.
 - Hall-mark of chiral symmetry structures that realise sub-algebras of **Virasoro algebra**.
- An extended model of 2D-CFTs, realising **affine Kac-Moody algebra** (the simplest extension of semi-simple Lie algebras), namely **Wess - Zumino - (Novikov) - Witten (WZW) models**.

Contents

- Classical WZW actions = Non-linear sigma model + Wess - Zumino term
- Quantum theory - Current algebra (OPE)
- Affine Kac-Moody algebra
- Commutation relations for conserved currents
- Sugawara construction
- The algebra from language of “modes”
- Chern-Simons theories & correspondence to WZW models
- Further applications & Outlooks

Classical theory

Nonlinear sigma model

$$S_0 = \frac{1}{4\lambda^2} \int_{S^2} d^2x \operatorname{tr}(g^{-1} \partial_\mu g g^{-1} \partial^\mu g).$$

$g \equiv g(z, \bar{z}) : S^2 \rightarrow G$: element of semi-simple Lie group

Classical E.O.M.:

$$\partial_\mu (g^{-1} \partial^\mu g) = 0.$$

Conserved current $J^\mu = g^{-1} \partial^\mu g$ which equivalent to $J'^\mu = \partial^\mu g g^{-1}$,

Traceless tensor in CFT
 \Rightarrow These 2 terms vanish.

$$\partial_z \tilde{J}_{\bar{z}} + \partial_{\bar{z}} \tilde{J}_z = 0.$$

$$\begin{cases} \partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{z\bar{z}} = 0, \\ \partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0. \end{cases}$$

Problem: Not expected to vanish in general non-Abelian Lie algebra.

\rightarrow Not **separately conserved** and the theory does not have **(anti-)holomorphic currents**.

Classical theory

Wess - Zumino (WZ) term

$$\Gamma[\tilde{g}] = i \int_{B^3} \tilde{g}^* \omega = -\frac{i}{12\pi} \int_{B^3} \text{tr}(\tilde{g}^{-1} d\tilde{g} \wedge \tilde{g}^{-1} d\tilde{g} \wedge \tilde{g}^{-1} d\tilde{g}),$$

left-right invariant 3-form $\tilde{g} : B^3 \rightarrow G$.

With $\partial B^3 = S^2$, glueing with another WZ term along their boundaries $\mathcal{M} = (B^3 \sqcup B'^3)/S^2 = S^3$.

$$\Delta\Gamma = \Gamma[\tilde{g}]|_{B^3} - \Gamma[\tilde{g}']|_{B'^3} = \Gamma[\tilde{f}]|_{S^3} = 2\pi i.$$

S^3 deform into $SU(2)$ subgroup of G .
Homotopy group $\pi_3(G) = \mathbb{Z}$

Combined action

$$S[g] \equiv S_0[g] + k\Gamma[g], \quad k \in \mathbb{R}.$$

$\tilde{f} : S^3 \rightarrow G$, taken to be identity.

Path-integral

$$Z = \int \mathcal{D}g e^{-S[g]}.$$

$k \in \mathbb{Z}$: Level of model.

Next: WZW action & E.O.M.

Classical theory

WZW action

$$S[g] \equiv S_0[g] + k\Gamma[g], \quad k \in \mathbb{Z}.$$

Varying action

$$\delta\Gamma = \frac{-i}{4\pi} \int_{S^2} dzd\bar{z} \operatorname{tr} (g^{-1} \delta g (-\bar{\partial}(g^{-1} \partial g) + \partial(g^{-1} \bar{\partial} g))). \quad \delta S_0 = \frac{-i}{4\lambda^2} \int_{S^2} dzd\bar{z} \operatorname{tr} (g^{-1} \delta g (\partial(g^{-1} \bar{\partial} g) + \bar{\partial}(g^{-1} \partial g))).$$

Classical E.O.M.

$$\partial(g^{-1} \bar{\partial} g) \left(1 + \frac{\lambda^2 k}{\pi}\right) + \bar{\partial}(g^{-1} \partial g) \left(1 - \frac{\lambda^2 k}{\pi}\right) = 0.$$

Choosing $\lambda^2 = \pi/k$ ($k > 0$)

separately anti(-holomorphic) conserved current $\bar{J} = kg^{-1} \bar{\partial} g$, $J = -k \partial g g^{-1}$.

$$\partial(g^{-1} \bar{\partial} g) = 0 = g \partial(g^{-1} \bar{\partial} g) g^{-1} = \bar{\partial}(\partial g g^{-1}).$$

This exhibits the local $G(z) \times G(\bar{z})$ symmetry

$$g(z, \bar{z}) \mapsto g_L(z) g(z, \bar{z}) g_R^{-1}(z).$$

Quantum theory - Current algebra

Quantization: Consider **Operator Product Expansion (OPE)** structure instead of canonical quantization

$$J^a(z)J^b(w) \sim \sum_p \frac{X_p(w)}{(z-w)^p}.$$

Conformal dimension $2 - p$ is non-negative

constrain by current commutation and association

$$J^a(z)J^b(w) \sim \frac{\kappa^{ab}}{(z-w)^2} + \frac{if^ab_c J^c(w)}{z-w}.$$

OPE for (anti-)holomorphic current

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{if^ab_c J^c(w)}{z-w},$$

$$\bar{J}^a(\bar{z})\bar{J}^b(\bar{w}) \sim \frac{k\delta^{ab}}{(\bar{z}-\bar{w})^2} + \frac{if^ab_c \bar{J}^c(\bar{w})}{\bar{z}-\bar{w}}.$$

Affine Kac-Moody algebra

Under group multiplication, we derive **“Polyakov-Wiegmann identity”**:

$$S[gh] = S[g] + S[h] + W_k[g, h].$$

With convention of $W_k[g, h] \equiv kW_1[g, h] \equiv \frac{k}{8\pi} \int_{S^2} dzd\bar{z} \text{tr} (g^{-1} \bar{\partial} g \wedge \partial h h^{-1})$.

W_k transform as a **2-cocycle**:

$$\begin{aligned} S[(gh)l] &= S[g(hl)] \\ \iff W_k[gh, l] + W_l[g, h] &= W_k[g, hl] + W_k[h, l]. \end{aligned}$$

Virasoro algebra is central extension of Witt algebra

Exact sequence of the algebra & group

$$0 \rightarrow \mathbb{R} \rightarrow \hat{\mathfrak{g}} \rightarrow \mathcal{L}\mathfrak{g} \rightarrow 0,$$

$$1 \rightarrow U(1) \rightarrow \hat{G} \rightarrow LG \rightarrow 1.$$

Virasoro algebra structure

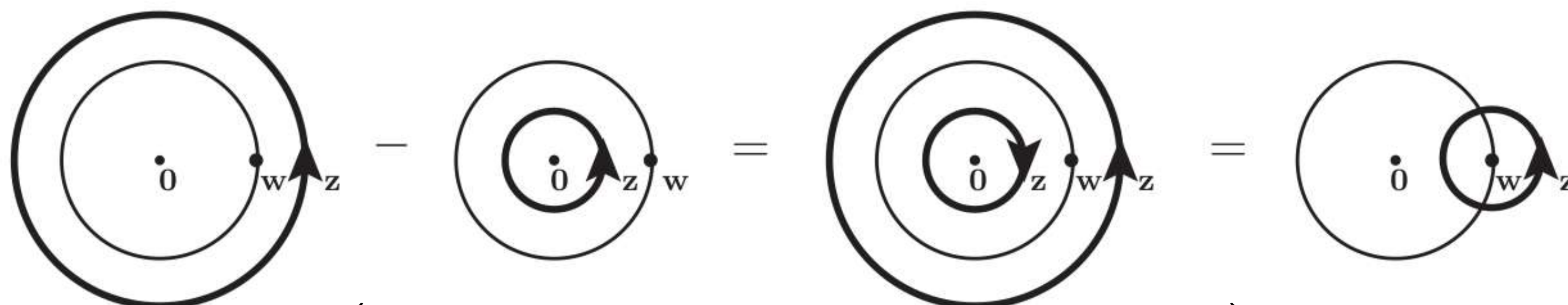
$$[X \otimes t^n, Y \otimes t^m] = [X, Y] \otimes t^{n+m} + \kappa(X, Y) n \delta_{n+m, 0} \mathcal{C}.$$

Commutation relations for conserved currents

Laurent expansion around origin

$$J^a(z) \equiv \sum_{n \in \mathbb{Z}} J_n^a z^{-n-1}.$$

generalised modes of expansion $J_n^a = \oint dz J^a(z) z^n.$



$$\begin{aligned} [J_n^a, J_m^b] &= \frac{1}{(2\pi i)^2} \left(\oint dz \oint_{|z| > |w|} dw - \oint dz \oint_{|w| > |z|} dw \right) z^n w^m J^a(z) J^b(w) \\ &= \frac{1}{(2\pi i)^2} \oint_0 dw \oint_w dz z^n w^m \left[\frac{i f^{ab}_c J^c(z)}{z-w} + \frac{k \delta^{ab}}{(z-w)^2} \right] \\ &= i f^{ab}_c J_{n+m}^c + k n \delta^{ab} \delta_{n+m,0}. \end{aligned}$$

Virasoro algebra structure

$$[J^a \otimes s_n, J^b \otimes s_m] = [J^a, J^b] \otimes s_{n+m} + k n \delta^{ab} \delta_{n+m,0}.$$

when $n = m = 0$, back to
finite-dim Lie algebra

$$[J_0^a, J_0^b] = i f^{ab}_c J_0^c.$$

Sugawara construction

Energy-momentum tensor

Classical version

$$T(z) = \gamma' J^a(z) J^a(z).$$

Quantum version

$$T(z) = \gamma(J^a J^a)(z) \equiv \frac{\gamma}{2\pi i} \oint_z \frac{dx}{x-z} J^a(x) J^a(z).$$

TJ OPE:

$$\overline{T(z) J^a(w)} = \overline{J^a(w) T(z)} \sim 2\gamma(k + h^\vee) \left(\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} \right).$$

Coexter number (quad. Casimir of ad. rep.)

$$f^{ab}_c f^{bc}_d = 2h^\vee \delta^{ad}.$$

TT OPE:

$$\overline{T(z) T(w)} \sim \frac{k \dim \mathfrak{g}}{2(k + h^\vee)(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}.$$

$$\gamma \equiv \frac{1}{2(k + h^\vee)}.$$

Central charge

$$c \equiv \frac{k \dim \mathfrak{g}}{k + h^\vee}.$$

The algebra from language of “modes”

Recall: Commutation relation for conserved currents

$$[J_n^a, J_m^b] = if^{ab}_c J_{n+m}^c + kn\delta^{ab}\delta_{n+m,0}.$$

Normal ordering term

$$L_n \equiv \gamma \left(\sum_{m \leq -1} J_m^a J_{n-m}^a + \sum_{m \geq 0} J_{n-m}^a J_m^a \right) = \gamma \sum_{m \in \mathbb{Z}} : J_m^a J_{n-m}^a :.$$

A good exercise: The commutation relation for conserved currents

$$[L_n, J_m^a] = -2\gamma(k + h^\vee)mJ_{n+m}^a = -mJ_{n+m}^a.$$

Virasoro algebra (Vir)

$$[L_n, L_m] = \gamma \left[L_n, \sum_{l \leq -1} J_l^a J_{n-l}^a + \sum_{l \geq 0} J_{n-l}^a J_l^a \right]$$

$$= (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}.$$

$$c = \frac{k \dim \mathfrak{g}}{k + h^\vee} = 2k\gamma \dim(\mathfrak{g}).$$

→ Affine Kac-Moody algebra is the Lie ideal of the universal enveloping algebra: $\text{Vir} \ltimes \hat{\mathfrak{g}}_k$.

Free field construction

At quantum level, **n-free fermions** is just the same theory to $\mathfrak{so}(n)_1$ of the WZW model

Action of n massless free fermion with $SO(N)$ symmetry:

$$S = \int_{S^2} d^2z \bar{\psi}^i \gamma^\mu \partial_\mu \psi^i, \quad (i = 1, \dots, n).$$

Generator of fun. rep. of $\mathfrak{so}(n)$

(Anti-)holomorphic conserved current

$$J^a(z) = \frac{1}{2} T_{ij}^a(\psi^i \psi^j)(z), \quad \bar{J}^a(\bar{z}) = \frac{1}{2} T_{ij}^a(\bar{\psi}^i \bar{\psi}^j)(\bar{z}).$$

OPE

$$\psi^i(z) \psi^j(w) \sim \frac{\delta^{ij}}{z-w}, \quad J^a(z) \psi^j(w) = -\frac{T_{lj}^a \psi^l(w)}{z-w}.$$

In $\mathfrak{so}(n)_1$, it is $\frac{(n-1)n}{n(n-1)} = 1$

$$J^a(z) J^b(w) \sim \frac{\mathcal{C}(\lambda) \dim(\lambda)}{2 \dim(\mathfrak{g})} \frac{\delta^{ab}}{(z-w)^2} + \frac{i f^{ab}_c J^c(w)}{z-w},$$

→ “**Non-abelian bosonization**” transform local fermionic to bosonic fields & retained all symmetries.

Chern-Simons theory

2+1 dim Chern-Simons (CS) theory:

since, $\pi(G) = \mathbb{Z}$,
gauge invariant $\Rightarrow k \in \mathbb{Z}$

$$S_{CS} = \frac{k}{4\pi} \int_{\mathcal{M}} \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

A : Connection of the principal
G-bundle on \mathcal{M}

Canonical Quantization:

$$S_{CS} = \frac{k}{4\pi} \int_{\mathbb{R}} dt \int_{\Sigma} \text{tr}(\epsilon^{ij} A_i \partial_t A_j).$$

$$[A_i^a(x), A_j^b(y)] = \frac{\pi i}{k} \delta^{ab} \epsilon_{ij} \delta^2(x - y).$$

$\mathcal{M} = \mathbb{R} \times \Sigma$ (time x 2D surface) form of $S = \int p_i \partial_t q^i$

$$[A_z^a(z_1, \bar{z}_1), A_{\bar{z}}^b(z_2, \bar{z}_2)] = \frac{\pi}{k} \delta^{ab} \delta(z_1 - z_2) \delta(\bar{z}_1 - \bar{z}_2).$$

Hilbert space \mathcal{H} of all holomorphic functional $\Psi(A_{\bar{z}})$ & its functional derivatives of A_z

$$A_z^a \Psi(A_{\bar{z}}) = \frac{\pi}{k} \frac{\delta}{\delta A_{\bar{z}}^a} \Psi(A_{\bar{z}}).$$

$$S_{CS} = \frac{k}{4\pi} \int dt \int \text{tr}(A_z \partial_t A_{\bar{z}} + A_t F_{z\bar{z}}).$$

$$\left(\partial_{\bar{z}} \frac{\delta}{\delta A_{\bar{z}}} + \left[A_{\bar{z}}, \frac{\delta}{\delta A_{\bar{z}}} \right] \right) \Psi(A_z) = \frac{k}{\pi} \partial_z A_{\bar{z}} \Psi(A_z).$$

Identity

Correspondence to WZW model

WZW partition function

$$Z[A] = \left\langle \exp \left(\frac{1}{\pi} \int A_{\bar{w}}^b J_w^b \right) \right\rangle_{\text{WZW}}.$$

From the known $J^a(z)J^b(w)$, we derive $\partial_{\bar{z}}J^a(z)J^b(w)$ and use relation

$$\frac{\delta}{\delta A_{\bar{z}}^a} Z[A] = \frac{1}{\pi} J^a(z) Z[A].$$

We derive

$$\left(\partial_{\bar{z}} \frac{\delta}{\delta A_{\bar{z}}} + \left[A_{\bar{z}}, \frac{\delta}{\delta A_{\bar{z}}} \right] \right) \Psi(A_z) = \frac{k}{\pi} \partial_z A_{\bar{z}} \Psi(A_z).$$

Identity

- Hilbert states of CS theory connect to generating function (for a given source) of WZW models.
- Exploit it to calculate CS observables (e.g. n-Wilson loops on S^3) using WZW models.

Further applications & Outlooks

- Other aspects (refer to my essay - [GitHub](#)):
 - Representation (Primary fields, Verma modules, why called “Rational” CFT?)
 - Knizhnik–Zamolodchikov (KZ) equations (4-point correlation function)
 - Mathematical construction of Affine Kac-Moody algebra - in the Appendix
- Further developments:
 - $SL(2, \mathbb{R})/U(1)$ gauged WZW model as Witten's 2-dim Euclidean black holes [[Witten '92](#)]
 - WZW models of universal cover group of $SL(2, \mathbb{R})$ describes bosonic strings on AdS_3 [[Maldacena & Oorugi '01](#)], [[Maldacena et. al '01](#)], [[Maldacena & Oorugi '02](#)]
 - Plateau transition in the integer quantum Hall effect [[Zirnbauer '19](#)]