Remark 5.2.5.19, so that each element $g \in G$ determines an automorphism ρ_g of $\mathbb{E}_{k+1}^{\otimes}$. We now observe that there is a unique element $g \in G$ whose image in $\mathrm{O}(k+1)$ belongs to the identity component $\mathrm{SO}(k+1)$ and which satisfies $\rho_g \circ \iota = \iota'$. It will therefore suffice to show that ρ_g is homotopic to the identity, which follows from the fact that the action of G on $\mathbb{E}_{k+1}^{\otimes}$ factors through an action of $\mathrm{O}(k+1)$. Note that this argument yields homotopy equivalences

$$\mathrm{Pair}^{(k)}(A, B^{\mathrm{rev}}) \simeq \mathrm{Pair'}^{(k)}(A, B)$$

which depend functorially on $A, B \in \mathrm{Alg}^{\mathrm{aug}}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})$ but are not truly canonical (they depend on a choice of path from g to the identity in the group $\mathrm{SO}(k+1)$).

Lemma 5.2.5.36 motivates the following:

Definition 5.2.5.38. Let $\mathcal{C}$ be an $\mathbb{E}_k$-monoidal ∞-category for $0 < k < \infty$. Suppose we are given augmented algebra objects $A \in \mathrm{Alg}^{\mathrm{aug}}_{/\mathbb{E}_k}(\mathcal{C})$ and $B \in \mathrm{Alg}^{\mathrm{aug}}_{/\mathbb{E}_k}(\mathcal{C}^{\mathrm{rev}})$ and an element $\eta \in \mathrm{Pair'}^{(k)}(A, B^{\mathrm{rev}})$. We will say that η *exhibits B as a left Koszul dual of A* if, for every augmented algebra $B' \in \mathrm{Alg}^{\mathrm{aug}}_{/\mathbb{E}_k}(\mathcal{C}^{\mathrm{rev}})$, evaluation on η induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Alg}^{\mathrm{aug}}_{/\mathbb{E}_k}(\mathcal{C}^{\mathrm{rev}})}(B', B) \to \mathrm{Pair'}^{(k)}(A, B').$$

We will say that η *exhibits A as a right Koszul dual of B* if, for every object $A' \in \mathrm{Alg}^{\mathrm{aug}}_{/\mathbb{E}_k}(\mathcal{C})$, evaluation on η induces a homotopy equialence

$$\mathrm{Map}_{\mathrm{Alg}^{\mathrm{aug}}_{/\mathbb{E}_k}(\mathcal{C})}(A', A) \to \mathrm{Pair'}^{(k)}(A', B).$$

In the special case where the monoidal structure on $\mathcal{C}$ is symmetric, Definition 5.2.5.38 reduces to the the definition given at the beginning of this section (this follows immediately from Lemma 5.2.5.36).

Proof of Proposition 5.2.5.1. In the case $k = 0$, the desired result follows from Example 5.2.5.31. For $k > 0$, combine Lemma 5.2.5.36 with Proposition 5.2.5.33. □

5.2.6 Iterated Loop Spaces

Let X be a topological space equipped with a base point $*$ and let $k \geq 0$ be an integer. We let $\Omega^k X$ denote the k-fold loop space of X, which we will identify with the space of maps $f : [-1, 1]^k \to X$ which carry the boundary $\partial[-1, 1]^k$ to the base point of X. Then $\Omega^k X$ is equipped with an action of the topological operad ${}^t\mathbb{E}_k$: given a collection of rectilinear embeddings $\vec{\gamma} = \{\gamma_i : \square^k \to \square^k\}_{1 \leq i \leq n}$ with disjoint images, there is an induced map

$$\prod_{1 \leq i \leq n} \Omega^k X \to \Omega^k X$$

$$(f_1, \ldots, f_n) \mapsto f$$

$$f(y) = \begin{cases} f_i(z) & \text{if } y = \gamma_i(z) \text{ for some } i \\ * & \text{otherwise.} \end{cases}$$

It follows that the Kan complex $\mathrm{Sing}_\bullet \Omega^k(X)$ is equipped with an action of the simplicial operad $\mathrm{Sing}^t \mathbb{E}_k$. This action is envoded by a map

$$\theta_X : \mathrm{Sing}^t \mathbb{E}_k^{\otimes} \to \mathcal{K}\mathrm{an},$$

where $\mathcal{K}\mathrm{an}$ denotes the (simplicial) category of Kan complexes. Restricting our attention to the case where $X = |K|$, where K is a (pointed) Kan complex, we obtain a simplicial functor

$$\mathcal{K}\mathrm{an}_{*/} \times \mathrm{Sing}^t \mathbb{E}_k^{\otimes} \to \mathcal{K}\mathrm{an}\,.$$

$$(K, \langle n \rangle) \mapsto (\mathrm{Sing}_\bullet \Omega^k |K|)^n.$$

Passing to nerves and using the equivalence $\mathrm{N}(\mathcal{K}\mathrm{an}_{*/}) \to \mathcal{S}_*$, we obtain a functor

$$\mathcal{S}_* \times \mathbb{E}_k^{\otimes} \to \mathcal{S}\,.$$

For every pointed Kan complex K, the resulting map $\mathbb{E}_k^{\otimes} \to \mathcal{S}$ is evidently an $\mathbb{E}_k$-monoid object of $\mathcal{S}$ (in the sense of Definition 2.4.2.1). Consequently, $\mathrm{N}(\theta)$ is adjoint to a functor $\beta_k : \mathcal{S}_* \to \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$. We will refer to $\mathbb{E}_k$-monoid objects of $\mathcal{S}$ simply as *$\mathbb{E}_k$-spaces*, and $\mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ as the *∞-category of $\mathbb{E}_k$-spaces*. The functor β_k implements the observation that for every pointed space X, the k-fold loop space of X is an $\mathbb{E}_k$-space. This observation has a converse: the functor β is *almost* an equivalence of ∞-categories. However, it fails to be an equivalence for two reasons:

- (a) If X is a pointed space, then the k-fold loop space $\Omega^k X$ contains no information about the homotopy groups $\pi_i X$ for $i < k$. More precisely, if $k > 0$ and $f : X \to Y$ is a map of pointed spaces which induces isomorphisms $\pi_i X \to \pi_i Y$ for $i \geq k$, then the induced map $\Omega^k X \to \Omega^k Y$ is a weak homotopy equivalence of spaces. Consequently, the functor $\beta_k : \mathcal{S}_* \to \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ fails to be conservative. To correct this problem, we need to restrict our attention to *k-connective* spaces: that is, pointed spaces X such that $\pi_i X \simeq *$ for $i < k$; for such spaces, there is no information about low-dimensional homotopy groups to be lost.

- (b) Suppose that $k > 0$ and let $Y \in \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$; we will abuse notation by identifying Y with the space $Y(\langle 1 \rangle)$. Then Y carries an action of the ∞-operad $\mathbb{E}_1$: in particular, there is a multiplication map $Y \times Y \to Y$ which is unital and associative up to homotopy. This multiplication endows the set of connected components $\pi_0 Y$ with the structure of a monoid (which is commutative if $k > 1$). If $Y \simeq \Omega^k X$ lies in the image of the functor β, then we have a canonical isomorphism $\pi_0 Y \simeq \pi_k X$ (compatible with the monoid structures on each side). In particular, we deduce that the monoid $\pi_0 Y$ is actually a group (that is, Y is *grouplike* in the sense of Definition 5.2.6.6 below).

Remark 5.2.6.1. In the case $k = 0$, issues (a) and (b) do not arise: in fact, we have canonical equivalences of ∞-categories

$$\mathcal{S}_* \simeq \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{S}) \simeq \mathrm{Mon}_{\mathbb{E}_0}(\mathcal{S})$$

(here we regard $\mathcal{S}$ as endowed with the Cartesian monoidal structure). The first equivalence results from Proposition 2.1.3.9, and the second from Proposition 2.4.2.5; the composition of these equivalences agrees with the map β defined above. For this reason, we will confine our attention to the case $k > 0$ in what follows.

We first introduce some terminology which is motivated by the above discussion.

Definition 5.2.6.2. Let $\mathcal{C}$ be an ∞-category which admits finite products and let G be an associative monoid object of $\mathcal{C}$. Let $m : G \times G \to G$ the multiplication maps, and let $p_1, p_2 : G \times G \to G$ be the projection maps onto the first and second factors, respectively. We will say that G is *grouplike* if the maps

$$(p_1, m) : G \times G \to G \times G \qquad (m, p_2) : G \times G \to G \times G$$

are equivalences. We let $\mathrm{Mon}^{\mathrm{gp}}_{\mathrm{Assoc}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Mon}_{\mathrm{Assoc}}(\mathcal{C})$ spanned by the grouplike monoid objects of $\mathcal{C}$.

Example 5.2.6.3. Let G be a monoid, which we can regard as an associative monoid object of the ∞-category $\mathrm{N}(\mathcal{S}\mathrm{et})$ of sets. Then G is grouplike if and only if the constructions

$$(x, y) \mapsto (x, xy) \qquad (x, y) \mapsto (xy, y)$$

determine bijections from G to itself. It is not hard to see that this condition holds if and only if G is a group.

Example 5.2.6.4. Let G be an associative monoid object of the ∞-category $\mathcal{S}$ of spaces. If G is grouplike, then $\pi_0 G$ is a grouplike associative monoid object of the ∞-category $\mathrm{N}(\mathcal{S}\mathrm{et})$ and therefore a group. Conversely, suppose that $\pi_0 G$ is a group. For each point $x \in G$ determining a connected component $[x] \in \pi_0 G$, we can choose another point $y \in G$ satisfying $[y] = [x]^{-1}$. Then left multiplication by x induces a homotopy equivalence $G \to G$ (with homotopy inverse given by left multiplication by y). Since this condition holds for every $x \in G$, it follows that the map $(p_1, m) : G \times G \to G \times G$ is also a homotopy equivalence (since we can check on homotopy fibers of p_1). A similar argument shows that the map $(m, p_2) : G \times G \to G \times G$ is a homotopy equivalence.

Remark 5.2.6.5. Let $\mathcal{C}$ be an ∞-category which admits finite products. Composition with the functor $\mathrm{Cut} : \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathrm{Assoc}^{\otimes}$ of Construction 4.1.2.9 determines a functor $\theta : \mathrm{Mon}_{\mathrm{Assoc}}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{C})$. An associative monoid object $G \in \mathrm{Mon}_{\mathrm{Assoc}}(\mathcal{C})$ is grouplike if and only if the simplicial object $\theta(G)$ is a groupoid object of $\mathcal{C}$, in the sense of §HTT.6.1.2 .

Definition 5.2.6.6. Let $\mathcal{C}$ be an ∞-category which admits finite products. We will say that an $\mathbb{E}_1$-monoid object $G : \mathbb{E}_1^{\otimes} \to \mathcal{C}$ is *grouplike* it belongs to the essential image of $\mathrm{Mon}^{\mathrm{gp}}_{\mathrm{Assoc}}(\mathcal{X})$ under the equivalence of ∞-categories $\mathrm{Mon}_{\mathrm{Assoc}}(\mathcal{X}) \to \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{X})$ determined by the equivalence $\mathbb{E}_1^{\otimes} \to \mathrm{Assoc}^{\otimes}$. We let $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{X}) \subseteq \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{X})$ denote the full subcategory spanned by the grouplike $\mathbb{E}_1$-monoid objects of $\mathcal{X}$.

If $0 < k \leq \infty$, then we will say that an $\mathbb{E}_k$-monoid object $G : \mathbb{E}_k^{\otimes} \to \mathcal{X}$ is *grouplike* if the composite map $\mathbb{E}_1^{\otimes} \hookrightarrow \mathbb{E}_k^{\otimes} \overset{G}{\to} \mathcal{X}$ is an grouplike $\mathbb{E}_1$-monoid object of $\mathcal{X}$. We let $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{X}) \subseteq \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X})$ denote the full subcategory spanned by the grouplike $\mathbb{E}_k$-monoid objects.

Remark 5.2.6.7. Let $\mathcal{C}$ be an ∞-category which admits finite products and let G be an $\mathbb{E}_k$-monoid object of $\mathcal{C}$ for $2 \leq k \leq \infty$. Then the multiplication map $m : G \times G \to G$ is commutative up to homotopy. Consequently, G is grouplike if and only if the map $(m, p_2) : G \times G \to G \times G$ is an equivalence in $\mathcal{C}$.

Remark 5.2.6.8. In the situation of Definition 5.2.6.6, the condition that X is grouplike does not depend on which of the natural embeddings $\mathbb{E}_1^{\otimes} \hookrightarrow \mathbb{E}_k^{\otimes}$ is chosen.

Remark 5.2.6.9. Let $1 \leq k \leq \infty$. Then the full subcategory $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S}) \subseteq \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ is closed under small colimits. To prove this, we note that $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S})$ is given by the inverse image of $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathrm{N}(\mathcal{S}\mathrm{et}))$ under the colimit-preserving functor

$$\pi_0 : \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S}) \to \mathrm{Mon}_{\mathbb{E}_k}(\mathrm{N}(\mathcal{S}\mathrm{et})).$$

It will therefore suffice to show that the category of (abelian) groups is closed under small colimits in the category of (commutative) monoids, which is clear.

The main goal of this section is to prove the following:

Theorem 5.2.6.10 (Boardman-Vogt, May). *Let $0 < k < \infty$, and let $\mathcal{S}_*^{\geq k}$ denote the full subcategory of $\mathcal{S}_*$ spanned by the k-connective spaces. Then:*

1. *The functor $\beta_k : \mathcal{S}_* \to \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ is fully faithful when restricted to $\mathcal{S}_*^{\geq k}$.*
2. *The essential image of $\beta_k|_{\mathcal{S}_*^{\geq k}}$ is the full subcategory $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S}) \subseteq \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ spanned by the grouplike $\mathbb{E}_k$-spaces.*
3. *The equivalence $\mathcal{S}_*^{\geq k} \to \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S})$ admits an explicit homotopy inverse, given by the k-fold bar construction of §5.2.3.*

We will prove Theorem 5.2.6.10 in two steps. First, we show that for any ∞-topos $\mathcal{X}$, the k-fold bar construction induces an equivalence from grouplike $\mathbb{E}_k$-monoid objects of $\mathcal{X}$ to pointed k-connective objects of $\mathcal{X}$ (Theorem 5.2.6.15). We then show that in the special case $\mathcal{X} = \mathcal{S}$, the bar construction is homotopy inverse to β_k. We begin with some general considerations.

Notation 5.2.6.11. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{X}_*$ be the ∞-category of pointed objects of $\mathcal{X}$. We will regard $\mathcal{X}$ and $\mathcal{X}_*$ as equipped with the Cartesian symmetric monoidal structures. Let $k > 0$ be an integer, and observe that the forgetful functors

$$\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{X}_*) \to \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{X}) \qquad \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{X}_*^{\mathrm{op}}) \to \mathcal{X}_*^{\mathrm{op}}$$

are equivalences of ∞-categories. Since $\mathcal{X}$ admits geometric realizations of simplicial objects, we can consider the k-fold bar construction

$$\mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X}) \simeq \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{X}_*) \overset{\mathrm{Bar}^{(k)}}{\to} \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{X}_*^{\mathrm{op}})^{\mathrm{op}} \simeq \mathcal{X}_* .$$

Since $\mathcal{X}$ admits totalizations of cosimplicial objects, the k-fold bar construction admits a right adjoint, given the k-fold cobar construction

$$\mathcal{X}_* \simeq \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{X}_*^{\mathrm{op}})^{\mathrm{op}} \overset{\mathrm{Cobar}^{(k)}}{\to} \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{X}_*) \simeq \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X}).$$

In what follows, we will abuse notation by denoting these adjoint functors by

$$\mathrm{Bar}^{(k)} : \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X}) \to \mathcal{X}_* \qquad \mathrm{Cobar}^{(k)} : \mathcal{X}_* \to \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X}).$$

Remark 5.2.6.12. Let $\mathcal{X}$ be an ∞-topos and let $0 < k < \infty$. Using Examples 5.2.2.4 and 5.2.3.14, we see that the the composite functor

$$\mathcal{X}_* \overset{\mathrm{Cobar}^{(k)}}{\to} \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X}) \to \mathcal{X}$$

is given by $X \mapsto \Omega^k X$.

Example 5.2.6.13. When $k = 1$, we can identify $\mathrm{Mon}_{\mathbb{E}_1}(\mathcal{X})$ with the ∞-category $\mathrm{Mon}(\mathcal{X}) \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{X})$ of monoid objects of $\mathcal{X}$. Under this equivalence, the bar construction $\mathrm{Bar}^{(1)}$ carries a monoid $G_\bullet$ to its geometric realization $|G_\bullet|$, regarded as a pointed object of $\mathcal{X}$ via the augmentation map $G_0 \to |G_\bullet|$. Since $\mathcal{X}$ is an ∞-topos, this construction determines a fully faithful embedding from grouplike monoid objects of $\mathcal{X}$ (see Remark 5.2.6.5) to the full subcategory of $\mathcal{X}_*$ spanned by those pointed objects X for which the map $* \to X$ is an effective epimorphism.

Example 5.2.6.14. Let G be a discrete group, regarded as a grouplike $\mathbb{E}_1$-monoid object of $\mathrm{N}(\mathcal{S}\mathrm{et}) \subseteq \mathcal{S}$. Then $\mathrm{Bar}^{(1)}(G)$ can be identified with the usual classifying space BG (regarded as a pointed space).

If $\mathcal{X}$ is an ∞-topos, we let $\mathcal{X}_*$ denote the ∞-category of pointed objects of $\mathcal{X}$. For each integer $a \geq 0$, we let $\mathcal{X}_*^{\geq a}$ denote the full subcategory of $\mathcal{X}_*$ spanned by the a-connective pointed objects. We regard $\mathcal{X}_*^{\geq a}$ as endowed with the Cartesian symmetric monoidal structure. Note that the unit object of $\mathcal{X}_*^{\geq a}$ is its final object and that $\mathcal{X}_*^{\geq a}$ admits geometric realizations of simplicial objects, so that the formalism of §5.2.3 can be applied.

Theorem 5.2.6.15. *Let $\mathcal{X}$ be an ∞-topos. For each $k > 0$, the k-fold cobar construction*

$$\mathrm{Cobar}^{(k)} : \mathcal{X}_* \to \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X})$$

restricts to an equivalence of ∞-categories $\mathcal{X}_*^{\geq k} \to \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{X})$.

Lemma 5.2.6.16. *Let $\mathcal{X}$ be an ∞-topos and let G be a monoid object of $\mathcal{X}$. If G is 1-connective, then G is grouplike.*

Proof. Since $\mathcal{X}$ is an ∞-topos, we can choose a small ∞-category $\mathcal{C}$ and a fully faithful embedding $f_* : \mathcal{X} \to \mathcal{P}(\mathcal{C})$ with a left exact left adjoint $f^* : \mathcal{P}(\mathcal{C}) \to \mathcal{X}$. Then $f_* G$ is a monoid object of $\mathcal{P}(\mathcal{C})$, so that $\tau_{\leq 0} f_* G$ inherits the structure of a monoid object of $\tau_{\leq 0}\,\mathcal{P}(\mathcal{C})$. Let H denote the fiber of the truncation map $(f_* G) \to \tau_{\leq 0} f_* G$. Then $f^* H$ can be identified with the fiber of the truncation map

$$G \simeq f^* f_* G \to f^* \tau_{\leq 0} f_* G \simeq \tau_{\leq 0} f^* f_* G \simeq \tau_{\leq 0} G \simeq \mathbf{1},$$

and is therefore equivalent to G as a monoid object of $\mathcal{X}$. To prove that G is grouplike, it will suffice to prove that H is grouplike. We may therefore replace $\mathcal{X}$ by $\mathcal{P}(\mathcal{C})$ (and G by H) and thereby reduce to the case where $\mathcal{X}$ is an ∞-topos of presheaves. In this case, the monoid object G is grouplike if and only if for each object $C \in \mathcal{C}$, the evaluation $G(C)$ is grouplike when regarded as a monoid object of $\mathcal{S}$. We may therefore assume without loss of generality that $\mathcal{X} = \mathcal{S}$, in which case the desired result follows immediately from the characterization given in Example 5.2.6.4. □

Remark 5.2.6.17. Let k and k' be positive integers. It follows from Remark 5.2.6.12 that an object $X \in \mathcal{X}_*^{\geq k}$ belongs to $\mathcal{X}_*^{\geq k+k'}$ if and only if $\mathrm{Cobar}^{(k)}(X)$ belongs to the full subcategory $\mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X}_*^{\geq k'}) \subseteq \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X}_*) \simeq \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X})$. Consequently, if Theorem 5.2.6.15 holds for the integer k, then the cobar construction $\mathrm{Cobar}^{(k)}$ induces an equivalence of ∞-categories

$$\mathcal{X}_*^{\geq k+k'} \to \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X}_*^{\geq k'}).$$

Proof of Theorem 5.2.6.15. In the case $k = 1$, the desired result follows from Example 5.2.6.13. We treat the general case using induction on k. If $k \geq 2$, we can write $k = a + b$ for $a, b \geq 1$. Using Remark 5.2.3.14, we see that the k-fold cobar construction $\mathrm{Cobar}^{(k)}$ factors as a composition

$$\begin{array}{ccl} \mathcal{X}_* & \overset{\mathrm{Cobar}^{(a)}}{\to} & \mathrm{Mon}_{\mathbb{E}_a}(\mathcal{X}_*) \\ & \overset{\mathrm{Cobar}^{(b)}}{\to} & \mathrm{Mon}_{\mathbb{E}_a}(\mathrm{Mon}_{\mathbb{E}_b}(\mathcal{X}_*)) \\ & \simeq & \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X}_*) \\ & \simeq & \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{X}). \end{array}$$

The desired result now follows from the inductive hypothesis together with Remark 5.2.6.17. □

Let us now extract some consequences of Theorem 5.2.6.15.

Corollary 5.2.6.18. *For any ∞-topos $\mathcal{X}$, the loop functor The loop functor $\Omega^k : \mathcal{S}_*^{\geq k} \rightarrow \mathcal{S}$ is conservative and preserves sifted colimits.*

Proof. Using Theorem 5.2.6.15 and Remark 5.2.6.12, we may reduce to the problem of showing that the forgetful functor $\theta : \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathcal{S}) \rightarrow \mathcal{S}$ is conservative and preserves sifted colimits. Since $\mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}(\mathcal{S})$ is stable under colimits in $\mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$, it suffices to show that the forgetful functor $\mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S}) \rightarrow \mathcal{S}$ is conservative and preserves sifted colimits. This follows from Proposition 2.4.2.5, Proposition 3.2.3.1, and Lemma 3.2.2.6. □

We note that the loop functor $\Omega : \mathcal{S}_*^{\geq 1} \rightarrow \mathcal{S}$ is corepresentable by the 1-sphere $S^1 \in \mathcal{S}_*^{\geq 1}$. It follows from Corollary 5.2.6.18 that S^1 is a compact projective object of $\mathcal{S}_*^{\geq 1}$. Since the collection of compact projective objects of $\mathcal{S}_*^{\geq 1}$ is stable under finite coproducts, we deduce the following:

Corollary 5.2.6.19. *Let F be a finitely generated free group, and BF its classifying space. Then BF is a compact projective object of $\mathcal{S}_*^{\geq 1}$.*

For each $n \geq 0$, let $F(n)$ denote the free group on n generators, and $BF(n)$ a classifying space for $F(n)$. Let $\mathcal{F}$ denote the full subcategory of the category of groups spanned by the objects $\{F(n)\}_{n\geq 0}$. We observe that the construction $F(n) \mapsto BF(n)$ determines a fully faithful embedding $i : \mathrm{N}(\mathcal{F}) \rightarrow \mathcal{S}_*^{\geq 1}$. Let $\mathcal{P}_{\Sigma}(\mathrm{N}(\mathcal{F}))$ be defined as in §HTT.5.5.8 (that is, $\mathcal{P}_{\Sigma}(\mathrm{N}(\mathcal{F}))$ is the ∞-category freely generated by $\mathrm{N}(\mathcal{F})$ under sifted colimits).

Remark 5.2.6.20. According to Corollary HTT.5.5.9.3 , the ∞-category $\mathcal{P}_{\Sigma}(\mathrm{N}(\mathcal{F}))$ is equivalent to the underlying ∞-category of the simplicial model category $\mathbf{A}$ of simplicial groups.

It follows from Proposition HTT.5.5.8.15 that the fully faithful embedding i is equivalent to a composition

$$\mathrm{N}(\mathcal{F}) \xrightarrow{j} \mathcal{P}_{\Sigma}(\mathrm{N}(\mathcal{F})) \xrightarrow{F} \mathcal{S}_*^{\geq 1},$$

where F is a functor which preserves sifted colimits (moreover, the functor F is essentially unique).

Corollary 5.2.6.21. *The functor $F : \mathcal{P}_{\Sigma}(\mathrm{N}(\mathcal{F})) \rightarrow \mathcal{S}_*^{\geq 1}$ is an equivalence of ∞-categories.*

Remark 5.2.6.22. Combining Corollary 5.2.6.21 and Remark 5.2.6.20, we recover the following classical fact: the homotopy theory of pointed connected spaces is equivalent to the homotopy theory of simplicial groups. See [58] for an explicit combinatorial version of this equivalence.

Proof of Corollary 5.2.6.21. Since $i : \mathrm{N}(\mathcal{F}) \rightarrow \mathcal{S}_*^{\geq 1}$ is fully faithful and its essential image consists of compact projective objects (Corollary 5.2.6.19), Proposition HTT.5.5.8.22 implies that F is fully faithful. We observe that the functor i preserves finite coproducts, so that F preserves small colimits by virtue of Proposition HTT.5.5.8.15 . Using Corollary HTT.5.5.2.9 , we deduce that F admits a right adjoint G. Since F is fully faithful, G is a colocalization functor; to complete the proof, it will suffice to show that G is conservative.

Let $f : X \to Y$ be a morphism in $\mathcal{S}_*^{\geq 1}$ such that $G(f)$ is an equivalence; we wish to prove that f is an equivalence. Let $\mathbf{Z}$ be the free group on one generator, and $j\,\mathbf{Z}$ its image in $\mathcal{P}_\Sigma(\mathrm{N}(\mathcal{F}))$. Then f induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{S}_*^{\geq 1}}(S^1, X) \simeq \mathrm{Map}_{\mathcal{P}_\Sigma(\mathrm{N}(\mathcal{F}))}(j\,\mathbf{Z}, GX) \to \mathrm{Map}_{\mathcal{P}_\Sigma(\mathrm{N}(\mathcal{F}))}(j\,\mathbf{Z}, GY) \simeq \mathrm{Map}_{\mathcal{S}_*^{\geq 1}}(S^1, Y).$$

It follows that $\Omega(f) : \Omega X \to \Omega Y$ is a homotopy equivalence, so that f is a homotopy equivalence by virtue of Corollary 5.2.6.18. □

Theorem 5.2.6.10 is an immediate consequence of Theorem 5.2.6.15 together with the following:

Proposition 5.2.6.23. *For each integer $k > 0$, the functors $\beta_k, \mathrm{Cobar}^{(k)} : \mathcal{S}_* \to \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})$ are equivalent to one another.*

Remark 5.2.6.24. Suppose we are given a pair of nonnegative integers k and k'. The bifunctor of ∞-operads $\mathbb{E}_k^{\otimes} \times \mathbb{E}_{k'}^{\otimes} \to \mathbb{E}_{k+k'}^{\otimes}$ determines a map

$$\rho : \mathrm{Mon}_{\mathbb{E}_{k+k'}}(\mathcal{S}) \to \mathrm{Mon}_{\mathbb{E}_k}(\mathrm{Mon}_{\mathbb{E}_{k'}}(\mathcal{S})),$$

which is an equivalence of ∞-categories by virtue of Theorem 5.1.2.2. Let K be a pointed Kan complex. Then the counit map

$$\mathrm{Sing}_\bullet(\Omega^k|\,\mathrm{Sing}_\bullet(\Omega^{k'}|K|)|) \to \mathrm{Sing}_\bullet(\Omega^{k+k'}|K|)$$

underlies an equivalence in the ∞-category $\mathrm{Mon}_{\mathbb{E}_k}(\mathrm{Mon}_{\mathbb{E}_{k'}}(\mathcal{S}))$, which depends functorially on K. In other words, the diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{S}_* \arrow[r, "\beta_k"] \arrow[d, "\beta_{k+k'}"] & \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S}_*) \arrow[d, "\beta_{k'}"] \\
\mathrm{Mon}_{\mathbb{E}_{k+k'}}(\mathcal{S}_*) \arrow[r, "\rho"] & \mathrm{Mon}_{\mathbb{E}_k}(\mathrm{Mon}_{\mathbb{E}_{k'}}(\mathcal{S}_*))
\end{tikzcd}$$

commutes up to homotopy.

Proof of Proposition 5.2.6.23. Using Remarks 5.2.6.24 and 5.2.3.14, we can reduce to the case $k = 1$. Let $G : \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{S}) \to \mathcal{S}$ denote the forgetful functor, so that $G \circ \beta_1$ and $G \circ \mathrm{Cobar}^{(1)}$ are both equivalent to the functor $\Omega : \mathcal{S}_* \to \mathcal{S}$. It follows that for any pointed space X with base point component X°, the canonical maps

$$\beta_1(X^\circ) \to \beta_1 X \qquad \mathrm{Cobar}^{(1)} X^\circ \to \mathrm{Cobar}^{(1)} X$$

are equivalences. It will therefore suffice to show that the functors $\beta_1|_{\mathcal{S}_*^{\geq 1}}$ and $\mathrm{Cobar}^{(1)}|_{\mathcal{S}_*^{\geq 1}}$ are equivalent. Using Corollary 5.2.6.18, we see that the functors $\beta_1|_{\mathcal{S}_*^{\geq 1}}$ and $\mathrm{Cobar}^{(1)}|_{\mathcal{S}_*^{\geq 1}}$ preserve sifted colimits. It will therefore suffice to show that the functors β_1 and $\mathrm{Cobar}^{(1)}$ are equivalent when restricted to spaces of the form BF, where F is a finitely generated free group. We have canonical equivalences

$$\begin{aligned} \beta_1(BF) &\simeq \pi_1(BF) \\ &\simeq F \\ &\simeq \mathrm{Cobar}^{(1)} \mathrm{Bar}^{(1)} F \\ &\simeq \mathrm{Cobar}^{(1)} BF. \end{aligned}$$

since the bar construction $\mathrm{Bar}^{(1)}(F)$ can be functorially identified with the classifying space BF (Example 5.2.6.14). □

Remark 5.2.6.25. Let K be a pointed space. Then the k-fold loop space $\Omega^k K$ can be endowed with two *a priori* different $\mathbb{E}_k$-structures: one coming from the geometric structure of the little cubes operads, and the second coming from the identification of $\Omega^k K$ with the k-fold cobar construction of §5.2.3. Proposition 5.2.6.23 asserts that these two structures are equivalent to one another. However, our proof is somewhat unsatisfying: rather than directly exhibiting an equivalence of $\mathbb{E}_k$-spaces, we instead argued that they cannot help but to be equivalent by virtue of all of the naturality properties that both constructions enjoy. Let us briefly sketch a construction which relates the functors β_k and $\mathrm{Cobar}^{(k)}$ more directly.

Fix a pointed topological space X. Let $C = \square^k$ be an open k-dimensional cube and let C^+ denote its one-point compactification. We let $\mathrm{Map}(C, X)$ denote the space of all maps from C to X (which is homotopy equivalent to X), and $\mathrm{Map}_*(C^+, X)$ the space of all pointed maps from C^+ into X (which is homotopy equivalent to the k-fold loop space $\Omega^k(X)$). There is an evident inclusion map $\rho_C : \mathrm{Map}_*(C^+, X) \hookrightarrow \mathrm{Map}(C, X)$. Given a collection of rectilinear embeddings

$$\gamma_1 : C_1 \hookrightarrow C \qquad \cdots \qquad \gamma_n : C_n \hookrightarrow C$$

having disjoint images, we can associate a map of pointed spaces $\delta : C^+ \to C_1^+ \vee C_2^+ \vee \cdots \vee C_n^+$. We have a commutative diagram

$$\begin{tikzcd}
\prod_{1 \leq i \leq n} \mathrm{Map}_*(C_i^+, X) \arrow[r, "\prod \rho_{C_i}"] \arrow[d] & \prod_{1 \leq i \leq n} \mathrm{Map}(C_i, X) \\
\mathrm{Map}_*(C, X) \arrow[r, "\rho_C"] & \mathrm{Map}(C, X). \arrow[u]
\end{tikzcd}$$

where the vertical maps are given by composition with δ and the γ_i. One can show that these commutative diagrams exhibit ρ_C as an $\mathbb{E}_k$-algebra in the twisted arrow category of spaces. Taking $X = |K|$ for some Kan complex K, we obtain an object of $\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{TwArr}(\mathcal{S}))$ whose image in $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S})$ agrees with $\beta_k K$ and whose image in $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S}^{\mathrm{op}}) \simeq \mathcal{S}$ agrees with K. This determines a canonical map $\beta_k K \to \mathrm{Cobar}^{(k)} K$, which can be shown to be an equivalence.

Remark 5.2.6.26. For every integer k, the diagram

$$\begin{tikzcd}
\mathcal{S}_* \arrow[r, "\Omega"] \arrow[d, "\beta_{k+1}"] & \mathcal{S}_* \arrow[d, "\beta_k"] \\
\mathrm{Mon}_{\mathbb{E}_{k+1}}(\mathcal{S}) \arrow[r] & \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})
\end{tikzcd}$$

commutes up to homotopy, where the lower vertical map is induced by the inclusion of ∞-operads $\mathbb{E}_k \hookrightarrow \mathbb{E}_{k+1}$. Theorem 5.2.6.10 therefore supplies an identification of the ∞-category $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathcal{S}) \simeq \varprojlim \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_k}(\mathcal{S})$ with the homotopy limit of the tower of ∞-categories

$$\cdots \rightarrow \mathcal{S}_*^{\geq 2} \xrightarrow{\Omega} \mathcal{S}_*^{\geq 1} \xrightarrow{\Omega} \mathcal{S}_*^{\geq 0},$$

which is the ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ of connective spectra.

Corollary 5.2.6.27. *Let* $\mathrm{Sp}^{\mathrm{cn}}$ *denote the* ∞*-category of connective spectra. Then the functor* $\Omega^\infty : \mathrm{Sp}^{\mathrm{cn}} \rightarrow \mathcal{S}$ *is conservative and preserves sifted colimits.*

Proof. Using Remark 5.2.6.26, we are reduced to proving that the forgetful functor $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathcal{S}) \rightarrow \mathcal{S}$ is conservative and preserves sifted colimits. Since $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathcal{S})$ is closed under colimits in $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})$, we are reduced to proving that the forgetful functor $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}) \rightarrow \mathcal{S}$ is conservative and preserves sifted colimits. This follows from Proposition 2.4.2.5, Proposition 3.2.3.1, and Lemma 3.2.2.6. $\square$

Remark 5.2.6.28. Let $\mathcal{X}$ be an ∞-topos, and regard $\mathcal{X}$ as endowed with the Cartesian symmetric monoidal structure. Theorem 5.2.6.15 guarantees the existence of an equivalence $\theta : \mathcal{X}_*^{\geq 1} \simeq \mathrm{Mon}^{\mathrm{gp}}(\mathcal{X}) \simeq \mathrm{Alg}^{\mathrm{gp}}(\mathcal{X})$, where $\mathrm{Alg}^{\mathrm{gp}}(\mathcal{X})$ denotes the essential image of $\mathrm{Mon}^{\mathrm{gp}}(\mathcal{X})$ under the equivalence of ∞-categories $\mathrm{Mon}(\mathcal{X}) \simeq \mathrm{Alg}(\mathcal{X})$ supplied by Propositions 4.1.2.10 and 2.4.2.5. This equivalence fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, \mathcal{X}) \times_{\mathrm{Fun}(\{1\}, \mathcal{X})} \mathcal{X}_*^{\geq 1} \arrow[r, "\overline{\theta}"] \arrow[d] & \mathrm{LMod}^{\mathrm{gp}}(\mathcal{X}) \arrow[d] \\
\mathcal{X}_*^{\geq 1} \arrow[r, "\theta"] & \mathrm{Alg}^{\mathrm{gp}}(\mathcal{X}),
\end{tikzcd}$$

where $\mathrm{LMod}^{\mathrm{gp}}(\mathcal{X})$ denotes the fiber product $\mathrm{LMod}(\mathcal{X}) \times_{\mathrm{Alg}(\mathcal{X})} \mathrm{Alg}^{\mathrm{gp}}(\mathcal{X})$ and $\overline{\theta}$ is an equivalence of ∞-categories. In other words, if $X \in \mathcal{X}$ is a pointed connected object, then there is a canonical equivalence between the ∞-topos $\mathcal{X}_{/X}$ and the ∞-category $\mathrm{LMod}_{\theta(X)}(\mathcal{X})$ of $\theta(X)$-module objects of $\mathcal{X}$.

To prove this, we let $\mathcal{D}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}}, \mathcal{X})$ spanned by those functors F with the following properties:

- (i) The functor F is a right Kan extension of its restriction to the full subcategory $\mathcal{K} \subseteq \Delta^1 \times \mathrm{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}}$ spanned by the objects $(0, [-1])$, $(1, [-1])$, and $(1, [0])$.

(*ii*) The object $F(1,[0]) \in \mathcal{X}$ is final.

(*iii*) The augmentation map $F(1,[0]) \to F(1,[-1])$ is an effective epimorphism (equivalently, the object $F(1,[-1]) \in \mathcal{X}$ is 1-connective).

It follows from Proposition HTT.4.3.2.15 that the restriction map $F \mapsto F|\mathcal{K}$ determines a trivial Kan fibration $\mathcal{D} \to \mathrm{Fun}(\Delta^1, \mathcal{X}) \times_{\mathrm{Fun}(\{1\},\mathcal{X})} \mathcal{X}_*^{\geq 1}$. Recall that we have a canonical equivalence $\mathrm{LMod}(\mathcal{X}) \simeq \mathrm{LMon}(\mathcal{X})$ (Propositions 4.2.2.9 and 2.4.2.5). To construct the functor $\overline{\theta}$, it will suffice to show that the restriction functor $F \mapsto F|(\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}})$ is a trivial Kan fibration from $\mathcal{D}$ onto $\mathrm{LMon}(\mathcal{X}) \times_{\mathrm{Mon}(\mathcal{X})} \mathrm{Mon}^{\mathrm{gp}}(\mathcal{X})$, where $\mathrm{LMon}(\mathcal{X})$ is defined as in Definition 4.2.2.2. Using Proposition HTT.4.3.2.8 , we see that (*i*) is equivalent to the following pair of assertions:

(i_0) The restriction $F|(\{1\} \times \mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}})$ is a right Kan extension of its restriction to $\{1\} \times \mathrm{N}(\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}}$.

(i_1) The functor F determines a Cartesian natural transformation from $F_0 = F|(\{0\} \times \mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}})$ to $F_1 = F|(\{1\} \times \mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}})$.

Assertions (i_0), (*ii*), and (*iii*) are equivalent to requirement that the functor F_1 belongs to the full subcategory $\mathcal{C} \subseteq \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}}, \mathcal{X})$ appearing in the proof of Theorem 5.2.6.15. In particular, these conditions guarantee that F_1 is a colimit diagram. Combining this observation with Theorem HTT.6.1.3.9 allows us to replace (i_1) by the following pair of conditions:

(i_1') The functor F_0 is a colimit diagram.

(i_1'') The restriction $F|(\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}})$ is a Cartesian transformation from $F_0|\,\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}$ to $F_1|\,\mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}$.

It follows that $\mathcal{Y}$ can be identified with the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathbf{\Delta}_+)^{\mathrm{op}})$ spanned by those functors F such that $F' = F|(\Delta^1 \times \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}})$ belongs to $\mathrm{LMon}(\mathcal{X}) \times_{\mathrm{Mon}(\mathcal{X})} \mathrm{Mon}^{\mathrm{gp}}(\mathcal{X})$ and F is a left Kan extension of F'. The desired result now follows from Proposition HTT.4.3.2.15 .

Remark 5.2.6.29. In the situation of Remark 5.2.6.28, let X be a pointed 1-connective object of the ∞-topos $\mathcal{X}$. Under the equivalence $\mathcal{X}_{/X} \simeq \mathrm{LMod}_{\theta(X)}(\mathcal{X})$, the forgetful functor $\mathrm{LMod}_{\theta X}(\mathcal{X}) \to \mathcal{X}$ corresponds to the functor $(Y \to X) \mapsto (Y \times_X \mathbf{1})$ given by passing to the fiber over the base point $\eta : \mathbf{1} \to X$ (here $\mathbf{1}$ denotes the final object of $\mathcal{X}$). It follows that the free module functor $\mathcal{X} \to \mathrm{LMod}_{\theta(X)}(\mathcal{X})$ corresponds to the functor $\mathcal{X} \simeq \mathcal{X}_{/\mathbf{1}} \to \mathcal{X}_{/X}$ given by composition with η.

5.3 Centers and Centralizers

Let A be an associative algebra over a field k with multiplication m. Then the cyclic bar complex

$$\ldots \longrightarrow A \otimes_k A \otimes_k A \xrightarrow{m \otimes \mathrm{id} - \mathrm{id} \otimes m} A \otimes_k A$$