NOTES ON CRYSTALS AND ALGEBRAIC $\mathcal{D}$-MODULES

Let X be a smooth manifold, and let V be a vector bundle on X equipped with a flat connection

$$\nabla : V \to V \otimes \Omega_X.$$

Then the flat sections of V determine a local system L on X. For every point $x \in X$, the fiber of the local system L_x can be identified with the fiber V_x. Given a path $p : [0,1]$ from $x = p(0)$ to $y = p(1)$, there is a map $p_! : L_x \to L_y$ is given by parallel transport along p, using the connection ∇; moreover the map $p_!$ depends only on the homotopy class of the path p. This construction is entirely reversible: the local system L determines the vector bundle V and its connection ∇ up to canonical isomorphism. In other words, the category of vector bundles with flat connection on X is equivalent to the category of local systems of vector spaces on X.

Now suppose that X is a smooth algebraic variety over a field k of characteristic zero (fixed through the remainder of this lecture). There is a purely algebraic notion of a vector bundle with flat connection on X: that is, an algebraic vector bundle V on X equipped with a map of sheaves

$$\nabla : V \to V \otimes \Omega_X$$

which satisfies the Leibniz rule. If k is the field of complex numbers, then the set of k-valued points $X(k)$ is endowed with the structure of a smooth (complex) manifold, so that V determines a local system on $X(k)$ as above. However, the relationship between vector bundles with connection to local systems is essentially transcendental. There is no algebraic notion of a path from a point $x \in X$ to another point $y \in X$, and hence no algebraic theory of parallel transport along paths.

Let us return for the moment to a case of a general manifold X. Every point $x \in X$ has a neighborhood U which is homeomorphic to a Euclidean space $\mathbb{R}^n$. Consequently, for every point y which is sufficiently close to x (so that $y \in U$), we can choose a path from x to y which is contained in U: moreover, this path is uniquely determined up to homotopy. Consequently, parallel transport along some connection from x to y does not depend on a choice of path, provided that path lies in U. We can summarize this informally as follows: if x and y are nearby points of X and V is a vector bundle with connection on X, then we get a canonical isomorphism $V_x \simeq V_y$.

If X is an algebraic variety, then it typically does not have a basis consisting of "contractible" Zariski-open subsets (for example, if X is a smooth curve of genus > 0, then it has no simply-connected open subsets at all). However, Grothendieck's theory of schemes provides a good substitute: namely, the notion of infinitesimally close points.

Definition 0.1. Let X be a scheme over k, let R be a k-algebra. We let $X(R) = \mathrm{Hom}(\mathrm{Spec}\, R, X)$ be the set of R-valued points of X. Let I denote the nilradical of R: that is, the ideal in R consisting of nilpotent elements. We say that two R-valued points $x, y \in X(R)$ are *infinitesimally close* if x and y have the same image under the map $X(R) \to X(R/I)$.

Remark 0.2. Note that if $x, y : \mathrm{Spec}\, R \to X$ are infinitesimally close points, then they induce the same map of topological spaces from $\mathrm{Spec}\, R$ into X: the only difference is what happens with sheaves of functions. This is one sense in which x and y really can be regarded as "close".

Using this notion of "infinitesimally close" points, we can formulate what it means for a sheaf $\mathcal{F}$ on a scheme X to have a good theory of "parallel transport along short distances":

Definition 0.3. [Grothendieck] Let X be a smooth scheme over k. A *crystal of quasi-coherent sheaves on* X consists of the following data:

(1) A quasi-coherent sheaf $\mathcal{F}$ on X. For every R-valued point $x : \operatorname{Spec} R \to X$, the pullback $x^*(\mathcal{F})$ can be regarded as a quasi-coherent sheaf on $\operatorname{Spec} R$: that is, as an R-module. We will denote this R-module by $\mathcal{F}_x$.

(2) For every pair of infinitesimally close points $x, y \in X(R)$, an isomorphism of R-modules $\eta_{x,y} : \mathcal{F}(x) \to \mathcal{F}(y)$. These isomorphisms are required to be functorial in the following sense: let $R \to R'$ be any map of commutative rings, so that x and y have images $x', y' \in X(R')$. Then
$$\eta_{x',y'} : \mathcal{F}(x') \simeq \mathcal{F}(x) \otimes_R R' \to \mathcal{F}(y) \otimes_R R' \simeq \mathcal{F}(y')$$
is obtained from $\eta_{x,y}$ by tensoring with R'.

(3) Let $x, y, z \in X(R)$. If x is infinitesimally close to y and y is infinitesimally close to z, then x is infinitesimally close to z; we require that $\eta_{x,z} \simeq \eta_{y,z} \circ \eta_{x,y}$. In particular (taking $x = y = z$), we see that $\eta_{x,x}$ is the identity on $\mathcal{F}(x)$, and (taking $x = z$) that $\eta_{x,y}$ is inverse to $\eta_{y,x}$.

There is another way to formulate Definition 0.3. Let X be an arbitrary functor from commutative rings to sets, not necessarily a functor which is representable by a scheme. A *quasi-coherent sheaf* $\mathcal{F}$ on X consists of a specification, for every R-point $x \in X(R)$, of an R-module $\mathcal{F}(x)$, which is compatible with base change in the following sense:

(*a*) If $R \to R'$ is a map of commutative rings and $x' \in X(R')$ is the image of X, we are given an isomorphism $\alpha_{x,x'} : \mathcal{F}(x') \simeq \mathcal{F}(x) \otimes_R R'$.

(*b*) Given a pair of maps $R \to R' \to R''$ and a point $x \in X(R)$ having images $x' \in X(R')$ and $x'' \in X(R'')$, the map $\alpha_{x,x''}$ is given by the composition
$$\begin{aligned} \mathcal{F}(x) \otimes_R R'' &\to (\mathcal{F}(x) \otimes_R R') \otimes_{R'} R'' \\ &\overset{\alpha_{x,x'}}{\to} \mathcal{F}(x') \otimes_{R'} R'' \\ &\overset{\alpha_{x',x''}}{\to} \mathcal{F}(x''). \end{aligned}$$

If X is a scheme, then this definition recovers the usual notion of a quasi-coherent sheaf on X. We define X^{dr}, the *deRham stack* of X, to be the functor given by the formula $X^{dr}(R) = X(R/I)$, where I is the nilradical of R. If X is a smooth scheme, then the map $X(R) \to X(R/I)$ is surjective, so that $X^{dr}(R)$ can be described as the quotient of $X(R)$ by the relation of "infinitesimal closeness". Unwinding the definitions, we see that a crystal of quasi-coherent sheaves on X is essentially the same thing as a quasi-coherent sheaf on X^{dr}.

The main point of introducing these definitions is the following result:

Theorem 0.4. *Let X be a smooth scheme over k. Then the category of crystals of quasi-coherent sheaves on X is equivalent to the category of quasi-coherent $\mathcal{D}_X$-modules.*

The equivalence of Theorem 0.4 is compatible with the forgetful functor to quasi-coherent sheaves. In other words, we are asserting that if $\mathcal{F}$ is a quasi-coherent sheaf on X, then equipping $\mathcal{F}$ with a flat connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega_X$ is equivalent to endowing $\mathcal{F}$ with the structure of a crystal. This can be regarded as an algebro-geometric version of the equivalence of categories mentioned at the beginning of this lecture.

We now sketch the proof of Theorem 0.4. Fix a quasi-coherent sheaf $\mathcal{F}$ on X. We would like to understand, in more concrete terms, how to endow $\mathcal{F}$ with the structure (2) described in Definition 0.3. To this end, we note that a pair of R-points $x, y \in X(R)$ can be regarded as an R-point of the product $X \times X$. The points x and y are infinitesimally close if and only if the map $\operatorname{Spec} R/I \to \operatorname{Spec} R \to X \times X$ factors through the diagonal. This is equivalent to the requirement that the map $\operatorname{Spec} R \to X \times X$ factor set-theoretically through the diagonal. In other words, it is equivalent to the requirement that $(x, y) : \operatorname{Spec} R \to X \times X$ factors through $(X \times X)^\vee$, where $(X \times X)^\vee$ denotes the formal completion of $X \times X$ along the diagonal.

More concretely, let $\mathcal{J}$ denote the ideal sheaf of the diagonal closed immersion $X \to X \times X$. For each $n \geq 0$, we let $\mathcal{J}^{n+1}$ denote the $(n+1)$st power of the ideal sheaf $\mathcal{J}$, and $X^{(n)} \subseteq X \times X$ the corresponding closed subscheme. Then $(X \times X)^\vee$ is defined to be the Ind-scheme $\varinjlim X^{(n)}$. At the level of points, this means that $(X \times X)^\vee(R) \simeq \varinjlim X^{(n)}(R)$. This is because given an R-point $(x, y) : \operatorname{Spec} R \to X \times X$, the points $x, y \in X(R)$ are infinitesimally close if and only if the ideal generated by $(x, y)^*\mathcal{J}$ is contained in the nilradical of R, which is equivalent to the requirement that $(x, y)^*\mathcal{J}^n$ has trivial image in R for $n \gg 0$.

Consequently, to supply the data described in (2), we need to give an isomorphism $\pi_1^*\mathcal{F} \simeq \pi_2^*\mathcal{F}$, where $\pi_1, \pi_2 : (X \times X)^\vee \to X$ denote the two projections. Let $\pi_i^{(n)}$ denote the restriction of π_i to $X^{(n)}$; we need to give a compatible family of maps $(\pi_1^{(n)})^*\mathcal{F} \to (\pi_2^{(n)})^*\mathcal{F}$ of quasi-coherent sheaves on $X^{(n)}$. This is equivalent to giving a map of sheaves

$$\mathcal{F} \to (\pi_1^{(n)})_*(\pi_2^{(n)})^*\mathcal{F}$$

on X. To understand this data, we need to understand the functor $(\pi_1^{(n)})_*(\pi_2^{(n)})^*$ from the category of quasi-coherent sheaves on X to itself.

Note that the underlying topological space of $X^{(n)}$ coincides with the underlying topological space of X. We may therefore view the structure sheaf $\mathcal{O}_{X^{(n)}}$ as a sheaf on X; the projection maps $\pi_1^{(n)}$ and $\pi_2^{(n)}$ endow $\mathcal{O}_{X^{(n)}}$ with two (different!) $\mathcal{O}_X$-module structures. The functor $(\pi_1^{(n)})_*(\pi_2^{(n)})^*$ is given by the relative tensor product

$$\mathcal{F} \mapsto \mathcal{O}_{X^{(n)}} \otimes_{\mathcal{O}_X} \mathcal{F}.$$

Let $\mathcal{D}_X^{\leq n}$ denote the sheaf of algebraic differential operators on X of order $\leq n$. There is a canonical pairing

$$\langle , \rangle : \mathcal{D}_X^{\leq n} \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{(n)}},$$

which can be described as follows. Think of sections of $\mathcal{O}_X$ as functions $f(x)$, and sections of $\mathcal{O}_{X^{(n)}}$ as functions $g(x, y)$ of two variables, defined modulo the $(n+1)$th power of $\mathcal{J}$. Given a differential operator D on X, we can regard $g(x, y)$ as a function of x (keeping y constant) to obtain a new function Dg of two variables. We now define $\langle D, g\rangle(x) = (Dg)(x, x)$. If D has order $\leq n$, then D carries $\mathcal{J}^{n+1}$ into $\mathcal{J}$, so that the resulting function on X is independent of the choice of g.

The pairing defined above is actually perfect: it identifies $\mathcal{O}_{X^{(n)}}$ with the $\mathcal{O}_X$-linear dual of $\mathcal{D}_X^{\leq n}$. We will check this in the special case where X is the affine line; the general case follows by the same reasoning, with more complicated notation. We can identify $\mathcal{O}_X$ with the polynomial ring $k[x]$ and $\mathcal{O}_{X^{(n)}}$ with the algebra $k[x, y]/(x - y)^{n+1}$. As a module over $k[x]$, it is free on a basis $\{(x, y)^k\}_{0 \leq k \leq n}$. On the other hand, we can identify $\mathcal{D}_X^{\leq n}$ with the free $\mathcal{O}_X$-module generated by symbols $\{\frac{1}{k!}(\frac{\partial}{\partial x})^k\}_{0 \leq k \leq n}$. A simple calculation shows that these bases are dual to one another under the pairing $\langle , \rangle$.

It follows that giving a map $\mathcal{F} \to \mathcal{O}_{X^{(n)}} \otimes_{\mathcal{O}_X} \mathcal{F}$ is equivalent to giving a map $\mathcal{D}_X^{\leq n} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$. Giving a compatible family of such maps for each n is equivalent to giving a map $\alpha : \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$. Any such map determines parallel transport morphisms $\eta_{x,y} : \mathcal{F}(x) \to \mathcal{F}(y)$ for an arbitrary pair of infinitesimally close points $x, y \in X(R)$.

To complete the analysis, we should spell out the meaning of condition (3) in Definition 0.3: under what conditions do we have $\eta_{x,z} \simeq \eta_{y,z} \circ \eta_{x,y}$? The translation amounts to the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\alpha} & \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \\ \downarrow \beta & & \downarrow \alpha \\ \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\alpha} & \mathcal{F}, \end{array}$$

where β is induced by the multiplication on $\mathcal{D}_X$. Similarly, the condition that $\eta_{x,x} = \mathrm{id}$ is equivalent to the requirement that the unit $1 \in \mathcal{D}_X$ act by the identity on $\mathcal{F}$ (together with transitivity, this guarantees that $\eta_{x,y}$ is inverse to $\eta_{y,x}$, so thateach $\eta_{x,y}$ is invertible). This proves Theorem 0.4: endowing $\mathcal{F}$ with the structure of a crystal is equivalent to endowing $\mathcal{F}$ with the structure of a $\mathcal{D}_X$-module, compatible with the existing $\mathcal{O}_X$-module structure on $\mathcal{F}$.

Theorem 0.4 provides us with two different ways to look at the same kind of structure. Each has its advantages:

- (*a*) The definition of a crystal of quasi-coherent sheaves was somewhat abstract. The theory of $\mathcal{D}_X$-modules provides a much more concrete approach to the same objects, and enables us to make use of a battery of tools (such as noncommutative algebra) in their study.
- (*b*) Definition 0.3 provides a very conceptual way of thinking about $\mathcal{D}_X$-modules. Given a quasi-coherent sheaf $\mathcal{F}$ which is described in some functorial way, it might be difficult to explicitly identify a connection ∇ or a $\mathcal{D}_X$ action on $\mathcal{F}$. However, Definition 0.3 is easy to apply if we understand $\mathcal{F}$ as a functor.
- (*c*) The theory of crystals has quite a bit of flexibility. For example, differential operators are badly behaved if the variety X is not smooth. However, we can still contemplate quasi-coherent sheaves on the deRham stack X^{dr}. This turns out to behave badly in general, but it behaves well if we work with complexes of sheaves rather than sheaves (it recovers the usual derived category of quasi-coherent $\mathcal{D}$-modules on X, which can be obtained more concretely by embedding X in some smooth variety).

Another advantage of Definition 0.3 is that it adapts easily to nonlinear settings. For example, we have the following:

Definition 0.5. Let S be a smooth scheme over k. A *crystal of schemes on S* consists of the following data:

1. An S-scheme $X \to S$. For every R-valued point $x : \operatorname{Spec} R \to S$, we will denote the pullback $X \times_S \operatorname{Spec} R$ by $x^* X$.
2. For every pair of infinitesimally close points $x, y \in S(R)$, an isomorphism of R-schemes $\eta_{x,y} : x^* X \simeq y^* X$. (As in Definition 0.3, we require that these isomorphisms be compatible with base change in R).
3. Let $x, y, z \in S(R)$. If x is infinitesimally close to y and y is infinitesimally close to z, then x is infinitesimally close to z; we require that $\eta_{x,z} \simeq \eta_{y,z} \circ \eta_{x,y}$.

Let us now make the connection between Definition 0.5 and the theory of $\mathcal{D}$-schemes described earlier in the seminar. Let $\pi : X \to S$ be a crystal of schemes over S, and assume that π is affine. Then $\pi_* \mathcal{O}_X$ is a crystal of quasi-coherent sheaves on S, which we can identify with a quasi-coherent $\mathcal{D}_S$-module $\mathcal{A}$. However, it has more structure: namely, there is a multiplication $\pi_* \mathcal{O}_X \otimes_{\mathcal{O}_S} \pi_* \mathcal{O}_X \to \pi_* \mathcal{O}_X$. This multiplication is a map of crystals, and translates (under the equivalence of categories of Theorem 0.4) to a map of $\mathcal{D}_S$-modules $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} \to \mathcal{A}$. This map endows $\mathcal{A}$ with the structure of a quasi-coherent $\mathcal{D}_S$-algebra. As in Theorem 0.4, no information is lost in the passage from $\pi : X \to S$ to $\mathcal{A}$: we can recover X as the relative spectrum of $\mathcal{A}$, and the $\mathcal{D}_S$-module structure of $\mathcal{A}$ exhibits X as a crystal of schemes on S. We can summarize our discussion as follows:

Theorem 0.6. *Let S be a smooth scheme over k. Then the category of commutative quasi-coherent $\mathcal{D}_S$-algebras is equivalent to the category of crystals of schemes $\pi : X \to S$ such that π is affine.*

Remark 0.7. Theorem 0.6 provides a concrete understanding of crystals of schemes in the affine case. However, it can be used to understand crystals of schemes in general. Assume for simplicity that the base S is separated, and suppose that $\pi : X \to S$ is a crystal of schemes over S. Let $U \subseteq X$ be an affine open subset. We claim that $U \to S$ is also a crystal of schemes. To prove this, we need to give a canonical isomorphism $x^*U \simeq y^*U$ for every pair of infinitesimally close moprhisms $x, y : \operatorname{Spec} R \to S$. Note that x^*U and y^*U can be identified with open subsets of the R-schemes x^*X and y^*X, which are identified by virtue of our assumption that $X \to S$ is a crystal of schemes. We claim that this identification restricts to an isomorphism $x^*U \simeq y^*U$. This is a purely topological question. We may therefore replace R by the quotient R/I, where I is the nilradical of R. After this maneuver, we have $x = y$ and the result is obvious.

Since S is separated, for every affine open subset $U \subseteq X$ the map $\pi|U$ is an affine map from U to S, so that $(\pi|U)_*\mathcal{O}_U$ is a sheaf of quasi-coherent $\mathcal{O}_S$-algebras which we will denote by $\mathcal{A}_U$. The above reasoning shows that, if X is a crystal of schemes over S, then each $\mathcal{A}_U$ has the structure of $\mathcal{D}_S$-algebra; moreover, this structure depends functorially on U.

Conversely, suppose we are given a compatible family of $\mathcal{D}_S$-algebra structures on $\mathcal{A}_U$, for all open affines $U \subseteq X$. Then each affine $U \subseteq X$ has the structure of a crystal of schemes over S. We claim that X then inherits the structure of a crystal of schemes over S. To prove this, we need to exhibit an isomorphism $\eta_{x,y} : x^*X \to y^*X$ for every pair of infinitesimally close points $x, y \in S(R)$. The underlying map of topological spaces of $\eta_{x,y}$ is clear (since dividing out by the nilradical of R does not change these topological spaces). The problem of promoting this map of topological spaces to a map of schemes is then local: it therefore suffices to give such a map over an open covering of x^*X, and such a covering is given by $\{x^*U\}$ where U ranges over the affine open sets in X.

As in the case of quasi-coherent sheaves, we can phrase the definition of crystal in terms of deRham stacks. More precisely, let S be any functor from the category of commutative k-algebras to sets. We define an *S-scheme* to be another functor X from commutative k-algebras to sets, equipped with a map $\pi : X \to S$, which is *relatively representable* in the following sense: for any R-point $s \in S(R)$, the fiber product $X \times_S \{s\}$ (another functor from commutative k-algebras to sets) is representable by an R-scheme. If S is itself representable by a k-scheme, this recovers the usual notion of a scheme X with a map to S. If S is a smooth k-scheme, then an S^{dr}-scheme is the same thing as a crystal of schemes over S.

Let $\pi : S' \to S$ be a map of functors. If X is an S-scheme, then the fiber product $S' \times_S X$ is an S'-scheme, which we will denote by π^*S. The construction π^* has a right adjoint π_*, at least at the level of functors. Namely, let $X' \to S'$ be a morphism in the category of functors from commutative k-algebras to sets. We define π_*X' to be the set of pairs (s, ϕ), where $s \in S(R)$ and ϕ belongs to the inverse limit $\varprojlim_{s'} X'_{s'}(R')$, taken over all pairs (R', s') where R' is a commutative R-algebra and $s' \in S'(R')$ lifts the image of s in $S(R')$. The functor π_*X' is called the *Weil restriction* of X' along π. In general, it need not be an S-scheme, even if we assume that X' is an S'-scheme.

Example 0.8. Let S be a separated smooth k-scheme, and let $\pi : X \to S$ be an arbitrary map of schemes. For each $n \geq 0$, let $S^{(n)}$ denote the nth order neighborhood of the diagonal in $S \times S$. We can mimic the constructions appearing in the proof of Theorem 0.4 at the level of schemes: namely, we can pull X back to $S^{(n)}$ along the first projection, and then push it forward along the second projection, by means of the Weil restriction. More concretely, we define $J^{(n)}(X)$ to be an S-scheme with the following universal property: for every S-scheme Y, we have a bijection $\mathrm{Hom}_S(Y, J^{(n)}(X)) \simeq \mathrm{Hom}_S(Y \times_S S^{(n)}, X)$. A point of $J^{(n)}(X)$ consists of a point $x \in X$ together with an order n jet of a section of π passing through x.

We have forgetful maps $J^{(n+1)}(X) \to J^{(n)}(X)$ for $n \geq 0$. These maps are affine, so that the inverse limit $J(X) = \varprojlim J^{(n)}(X)$ is well-defined. We call $J(X)$ the jet-scheme of the projection π. By construction, for every R-valued point $x \in S(R)$, the pullback $x^* J(X)$ can be identified with the scheme which parametrizes sections of π over a formal neighborhood of x in $S \times \operatorname{Spec} R$. If $x, y \in S(R)$ are infinitesimally close, then their formal neighborhoods coincide in $S \times \operatorname{Spec} R$, so we get a canonical isomorphism of R-schemes $x^* J(X) \simeq y^* J(Y)$. These isomorphsims exhibit $J(X)$ as a crystal of schemes over S.

One can give another more abstract argument that $J(X)$ should have the structure of a crystal of schemes over S. Namely, we claim that $J(X)$ is given by the Weil restriction of X along the quotient map $\pi : S \to S^{dr}$. More precisely, $J(X)$ is the underlying S-scheme of this Weil restriction: that is, it is given by $\pi^* \pi_* X$. To prove this, we observe that there is a pullback diagram

$$\begin{array}{ccc} (S \times S)^\vee & \xrightarrow{\pi_1} & S \\ \Big\downarrow{\scriptstyle \pi_2} & & \Big\downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\pi} & S^{dr}. \end{array}$$

There is a natural transformation of functors

$$(\pi^* \pi_* X) \simeq (\pi_2)_* \pi_1^* X,$$

which can be shown to be an isomorphism in this case. Note that $(S \times S)^\vee \simeq \varinjlim S^{(n)}$, so that $(\pi_2)_*(\pi_1^* X)$ is the inverse limit of the Weil restrictions of the fiber products $X \times_S S^{(n)}$. By construction, this inverse limit is given by $J(X) = \varprojlim J^{(n)}(X)$.

The argument sketched above has an additional virtue: it establishes a universal property enjoyed by the construction $X \mapsto J(X)$. Namely, we have proven the following:

Proposition 0.9. *Let S be a smooth separated k-scheme. Then the construction $X \mapsto J(X)$ is right adjoint to the forgetful functor from crystals of S-schemes to S-schemes. In other words, for any crystal of S-schemes Y, composition with the projection map $J(X) \to X$ induces a bijection between the set $\mathrm{Hom}_{S^{dr}}(Y, J(X))$ of maps of crystals to the set $\mathrm{Hom}_S(Y, X)$ of maps of S-schemes.*

We now introduce a more specific example which is relevant to our study in this seminar:

Example 0.10. Let X be an algebraic curve over k and G a reductive algebraic group, and let $\pi : \mathrm{Gr}^1 \to X$ denote the Beilinson-Drinfeld Grassmannian. More precisely, an R-valued point of Gr^1 is given by a triple $(x, \mathcal{P}, \eta)$, where $x \in X(R)$ is a point of X, $\mathcal{P}$ is a G-bundle on $X \times \operatorname{Spec} R$, and η is a section of $\mathcal{P}$ over the open set $(X \times \operatorname{Spec} R) - x(\operatorname{Spec} R)$. Then π exhibits Gr^1 as a crystal (of Ind-schemes) over X. To see this, it suffices to observe that if $x, y \in X(R)$ are infinitesimally close, then the open sets $(X \times \operatorname{Spec} R) - x(\operatorname{Spec} R)$ and $(X \times \operatorname{Spec} R) - y(\operatorname{Spec} R)$ coincide.

Example 0.11. Let X be an algebraic curve. Given an R-point $x \in X(R)$, let $\mathcal{O}^\vee_{X,x}$ denote the ring of functions on the formal scheme given by completing $X \times \operatorname{Spec} R$ along x. Then the ordinary scheme $\operatorname{Spec} \mathcal{O}^{\vee}_{X,x}$ contains $\operatorname{Spec} R$ as a divisor; we will denote the difference $\operatorname{Spec} \mathcal{O}^{\vee}_{X,x} - \operatorname{Spec} R$ by D°_x, and refer to it as the *punctured formal disk around* x. (If R is a field, or more generally a local ring, then D°_x is noncanonically isomorphic to the spectrum of a Laurent power series ring $R((t))$.)

Let Y be a scheme. We define a *relative loop space* LY as follows: an R-valued point of LY is given by a pair (x, ϕ), where $x \in X(R)$ and $\phi : D^{\circ}_x \to Y$ is a map of schemes. If Y is affine, then LY is an Ind-scheme, and we have an obvious projection $LY \to X$. This map exhibits LY as a crystal of Ind-schemes over X. To see this, it suffices to observe that if $x, y \in X(R)$ are infinitesimally close, then the formal completions of $X \times \operatorname{Spec} R$ along x and y coincide. We therefore have an isomorphism of rings $\mathcal{O}^{\vee}_{X,x} \simeq \mathcal{O}^{\vee}_{X,y}$ and hence an isomorphism of affine schemes $\operatorname{Spec} \mathcal{O}^{\vee}_{X,x} \simeq \operatorname{Spec} \mathcal{O}^{\vee}_{X,y}$, which restricts to an isomorphism between the open subschemes $D^{\circ}_x \simeq D^{\circ}_y$.

In the special case where Y is a reductive algebraic group G, the map $LG \to X$ has fibers over a rational point $x \in X(k)$ given by $G(\mathcal{K}_x)$, $\mathcal{K}_x$ is denotes the field of Laurent series corresponding to $x \in X$. In this case, LG is a group stack over X, and has a natural action $LG \times_X \mathrm{Gr}^1 \to \mathrm{Gr}^1$. It is not difficult to see that this action is horizontal: that is, the preceding map is a map of crystals.