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Workshop on TFT's at Northwestern
Notes, errors, and incompleteness by Alex Hoffnung

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DRAFT VERSION ONLY

1 Lecture 1: Axiomatics of TQFT's

In these lectures, manifolds are smooth, compact, and oriented, often with boundary.

$Cob(n)$ has

objects: closed $(n-1)$-manifolds

morphisms: bordisms of $(n-1)$-manifolds mod diffeomorphisms

This is a symmetric monoidal catgory with $\otimes$ given by disjoint union of manifolds.

Definition 1. *(Atiyah) An* **n-dimensional TQFT** *is a* $\otimes$*-functor*

$$Z: Cob(n) \to \mathbb{C} - \mathrm{Vect}$$

This means $Z(M \coprod N) \cong Z(M) \otimes Z(N)$.

Examples (n=2)

- David explained that 2-dimensional TQFT's are equivalent to commutative Frobenius algebras

  $Z(S^1) = A$ (vector space)

  This is enough to know what happens on m circles.

  Z on a pair of pants gives a commutative associative product on A making it an algebra

  This multiplication has a unit and a counit.

  The counit will be called the trace and the claim is that this is non-degenerate. So we can identify A with its dual.

M a closed n-manifold, then

$$Z(M): \mathbb{C} \to \mathbb{C}$$

This is multiplication by a number $Z(M)$. These are diffeomorphism invariants by definition.

Say Z is associated to a 2-dimensional TQFT: What is $Z(\Sigma_g)$?
Cut Σ_g into pieces which look like Frobenius pieces.
Examples:

$$Z(S^2) = tr(1)$$

$$Z(\text{torus}) = dim A$$

This analysis in dimension 2 may not help us for higher dimensions.

After triangulation of a manifold we recover the manifold by gluing along pieces of boundary of simplices (i.e. manifolds with boundary, corners)

Sketch of definition:

Definition 2. *An* **extended** *n-dimensional TQFT is a rule which assigns data to manifolds of dimension less than or equal to n.*

$$\text{n-manifolds} \to \text{complex number}$$

$$\text{(n-1)-manifolds} \to \mathbb{C} - \text{vector space}$$

$$\text{bordisms of (n-1)-manifolds} \to \text{linear maps}$$

$$\text{(n-2)-manifolds} \to \mathbb{C} - \text{linear categories}$$

$$\text{bordisms of (n-2)-manifolds} \to \mathbb{C} - \text{linear functors}$$

$$\vdots$$

We want to say these invariants "glue nicely". We do this using the language of category theory.

Another formulation: An extended n-dimensional TQFT is a $\otimes$-functor between n-categories.

Baez-Dolan Cobordism hypothesis:

"Extended TQFT's are "easy" to describe/construct/classify."

Example: n=1

Suppose Z is a TQFT

$$Z \colon Cob(1) \to \text{Vect}_{\mathbb{C}}$$

Objects are finite collections of oriented points.

$$Z(*^+) = X \qquad Z(*^-) = Y$$

$$Z(unit) \colon X \otimes Y \xrightarrow{ev} \mathbb{C}$$

$$Z(counit) \colon \mathbb{C} \to X \otimes Y$$

These exhibit X and Y as duals. This forces X and Y to be finite dimensional. So knowing X determines Y up to isomorphism.

Since Z is an $\otimes$-functor, knowing it on a point gives value on all objects. What about morphisms?

$$Z(\rightarrow) = id_X$$
$$Z(\leftarrow) = id_{X^\vee}$$
$$Z(unit) = ev$$
$$Z(counit) = coev$$

$Z(S^1) =$ a map $\mathbb{C} \to \mathbb{C}$ the trace of the identity endomorphism, i.e. the dimension of X

This was easier than the 2-dimensional case.

Upshot: A 1-dimensional TQFT Z is uniquely determined by $Z(*) \in \mathrm{Vect}_{\mathbb{C}}$.

Question: What is the n-dimensional analogue?

$nBord$ is an n-category with:

- objects: oriented 0-manifolds
- 1-morphisms: bordisms of 0-manifolds
- 2-morphisms: bordisms between bordisms of 0-manifolds

$\vdots$

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- n-morphisms: n-manifolds with corners mod diffeomorphisms

This is a symmetric monoidal n-category.
Incorrect guess: Let $\mathcal{C}$ be any symmetric monoidal n-category.

$$Fun^{\otimes}(nBord, \mathcal{C}) \to \text{objects of } \mathcal{C}$$
$$Z \mapsto Z(*)$$

Why is this wrong?

- A) wrong if $n = 1, \mathcal{C} = \mathrm{Vect}_{\mathbb{C}}$, only finite dimensional vector spaces appear as $Z(*)$ So we need to impose some finiteness conditions.

  So we are going to use fully dualizable objects of $\mathcal{C}$.

- B) wrong if $n > 1$, this is because of non-trivial tangent bundles fix $nBord$ with "framed" version $nBord^{fr}$

  If M is a manifold of dimension $m \leq n$, an **n-framing** of M is an isomorphism
  $$T_M \oplus \mathbb{R}^{n-m} \cong \mathbb{R}^n$$
  $nBord^{fr}$ is defined like $nBord$ but using n-framed manifolds

Correct guess: Let $\mathcal{C}$ be any symmetric monoidal n-category.

$$Fun^{\otimes}(nBord^{fr}, \mathcal{C}) \to \text{fully dualizable objects of } \mathcal{C}$$
$$Z \mapsto Z(*)$$

Question: What does this mean?

Question: What happens if we allow non-framed manifolds?

Question: What are good choices for $\mathcal{C}$ and what are some fully dualizable objecs?

Question: What does the theorem give?

Plan: Sketch the definition of n-category and answer questions.