Jacob Lurie
Workshop on TFT's at Northwestern
Notes, errors, and incompleteness by Alex Hoffnung

May 19, 2009

DRAFT VERSION ONLY

1 Lecture 2: Overview of higher category theory

Definition 1. *A* **category** $\mathcal{C}$ *consists of*

- *a collection of objects* $X, Y, Z, \ldots$
- *for pairs of objects* $X, Y \in \mathcal{C}$*, a set* $\hom_{\mathcal{C}}(X, Y)$
- *composition maps*

$$\hom_{\mathcal{C}}(X, Y) \times \hom_{\mathcal{C}}(Y, Z) \to \hom_{\mathcal{C}}(X, Z)$$

- *associativity, units, etc.. . .*

Yesterday we saw the category of cobordisms. Here the objects and morphisms look a lot alike. This is a good candidate for what we will call a 2-category.
Bad definition:
A **strict 2-category** $\mathcal{C}$ consists of

- a collection of objects $X, Y, Z, \ldots$
- for pairs of objects $X, Y \in \mathcal{C}$, a category $\hom_{\mathcal{C}}(X, Y)$
- composition functors

$$\hom_{\mathcal{C}}(X, Y) \times \hom_{\mathcal{C}}(Y, Z) \to \hom_{\mathcal{C}}(X, Z)$$

- associativity, units, etc.. . .

Let X be a space, $\pi_1(X, x)$ is a group whose elements are homotopy classes of loops. If we did not want to coose a basepoint, we could form a category: $\pi_{\leq 1} X$ where the objects are points $x \in X$ and morphisms are paths in X mod homotopy.

Here we also have 2-morphisms from $p\colon x \to y$ to $q\colon x \to y$ which are homotopies from p to q mod homotopy.
This gives the 2-category $\pi_{\leq 2}X$.

Now we look at composition of paths. Composition works because we can reparametrize. But when we are composing three paths reparametrization gets us into trouble regarding associativity. This is not a problem for the fundamental group or the fundamental groupoid since we only consider paths up to homotopy. So the fundamental 2-groupoid is a weak 2-category, which we will just call a 2-category.

We can get around the associativity problem, but uses an ad hoc tric which does not generalize to higher dimensions.
Bad definition:
A **strict n-category** consists of

- a collection of objects $X, Y, Z, \ldots$
- for pairs of objects $X, Y \in \mathcal{C}$, a strict $(n-1)$-category $\hom_{\mathcal{C}}(X, Y)$
- composition functors

$$\hom_{\mathcal{C}}(X, Y) \times \hom_{\mathcal{C}}(Y, Z) \to \hom_{\mathcal{C}}(X, Z)$$

- strict associativity, units, etc....

Example:
Let X be a space,
$\pi_{\leq n}X$ is a possible example of an n-category where

- the objects are points $x \in X$;
- morphisms are paths in X;
- 2-morphisms homotopies between paths;
- 3-morphisms are homotopies of homotopies;
- $\vdots$
- n-morphisms are higher homotopies mod homotopy

So a new sketch of a definition will drop strictness everywhere:

An **n-category** consists of

- a collection of objects $X, Y, Z, \ldots$
- for pairs of objects $X, Y \in \mathcal{C}$, an $(n-1)$-category $\hom_{\mathcal{C}}(X, Y)$

- composition functors

$$\hom_{\mathcal{C}}(X,Y) \times \hom_{\mathcal{C}}(Y,Z) \to \hom_{\mathcal{C}}(X,Z)$$

- associativity up to coherent isomorphism, units, etc..…

For any space X, $\pi_{\leq n}X$ is an n-groupoid.

Thesis:
Every n-groupoid should arise in this way.

Better:
There should be a theory of ∞-groupoids

More precisely:
For any space X, $\pi_{\leq\infty}X$ is an ∞-groupoid.

Thesis:
Every ∞-groupoid should arise in this way, for a space X which is unique up to weak homotopy equivalence.

$$X \mapsto \pi_{\leq\infty}X$$

spaces mod homotopy go to ∞-groupoids mod equivalence.

One should regard this thesis as a criterion tat a definition must pass.

Definition 2. *An* ∞-**groupoid** *is a topological space (or maybe a simplicial set).*

Definition 3. *An* n-**groupoid** *is a space* X *such that its homotopy groups are trivial* $\pi_k X \cong *$ *for* $k > n$.

So it is easier to define ∞-groupoids than n-groupoids.
Sketch of definition
An (∞, n)-**category** is a higher category in which all k-morphism are invertible for $k > n$.

Example: $(n = 0)$
$(\infty$,0)-category $\cong$ ∞-groupoid $\cong$ topological space

Idea:
An (∞, n)-category is

- a collection of objects $X, Y, Z, \ldots$
- for pairs of objects $X, Y \in \mathcal{C}$, an $(\infty, n-1)$-category $\hom_{\mathcal{C}}(X,Y)$

- composition law
- associativity, units, etc.. . .

Example: (n=1)
An $(\infty, 1)$-catgory is a topological category (or a simplicial category, an $\mathcal{S}$-category.

Our definition is correct but often inconvenient. We want to think about strategies for making sense of associativity weakly.

In a Segal category, we have

$$\begin{tikzcd} & \hom(X,Y,Z) \arrow[dl, "\sim"'] \arrow[dr] & \\ \hom(X,Y) \times \hom(Y,Z) & & \hom(X,Z) \end{tikzcd}$$

So instead of a composition law, we have something that induces a composition law up to coherent homotopy.

$Cob(n)$ is an ordinary category.
$nBord$ is an n-categoy (a fancy version of $Cob(n)$).
$\widetilde{Cob(n)}$ is an $(\infty, 1)$-category (a fancy version of $Cob(n)$).

Definition 4. *$\widetilde{Cob(n)}$ has*

- *objects which are $(n-1)$-manifolds*
- *morphisms are bordisms of $(n-1)$-manifolds*
- *2-morphisms are diffeomorphisms of bordisms rel ∂*
- *3-morphisms are isotopies of diffeomorphims*
- $\vdots$

This is an example of an $(\infty, 1)$-category.

We want $\hom(M, N)$, a classifying space for bordisms from M to N.

This space exists and is uniquely determined up to homotopy.

By universality we get a map called composition which is determined up to homotopy, so we can only check associativity up to homotopy.

Describe: $\widetilde{Cob(m)}$ as a Segal category.

We want: a simplicial space X with X_0 discrete.

$$X_0 = \text{set of closed (m-1)-manifolds}$$

$$X_n = \coprod_{\text{objects } M_0, M_1, \ldots, M_n} \hom(M_0, M_1, \ldots, M_n)$$

where

$$\hom(M_0, M_1, \ldots, M_n) = \{(t_0 \leq t_1 \leq \cdots \leq t_n) \in \mathbb{R} m\text{-dimensional proper submanifolds}$$
$$B \subset [t_0, t_n] \times \mathbb{R}^\infty \text{ such that } B \text{ is transverse to } B \cap (t_i \times \mathbb{R}^\infty) \cong M_i\}$$

Similarly, we can describe more elaborate bordism categories using a more elaborate version of the theory of Segal categories/Segal spaces. Using these ideas, one can define an object $\widetilde{nBord}$.

Here the objects are 0-manifolds, morphisms are bordisms,...,n-morphisms are n-manifolds with corners, and $(n+1)$-morphisms are diffeomorphisms.