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Workshop on TFT's at Northwestern
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DRAFT VERSION ONLY

1 Lecture 3: Overview of higher category theory

In first talk we introduced the notion of ext. TQFT and explained that these should be viewed as functors between two n-categories. In the second lecture we covered the notion of an n-categry and more generally an (∞, n)-category. We will shift notation and denote $\widetilde{nBord}$ as $nBord$. which is an (∞, n)-category with objects 0-manifolds, 1-morphisms are bordisms of 0-morphisms, ... n-morphisms are n-manifolds with corner and $(N+1)$-morphisms are diffeomorphisms,.... Remember this is a $\otimes$-category and functors should respect that.
Goal: Understand $nBord$.

There are restrictions on what kind of things we can assign. In the first lecture, we had a 2-dimensional theory $1Bord \to \mathcal{C}$. Here we had $Z(*^+) = X$ a vector space. We needed the object to have nice finiteness properties. Here it was that X should have a dual.

Definition 1. *Let $\mathcal{C}$ be a $\otimes$-category with $X \in \mathcal{C}$ an object. A* **dual** *of X is*

- *an object $\check{X}$*
- $e\colon X \otimes \check{X} \to 1 \quad (Z(rightelbow))$
- $c\colon 1 \to \check{X} \otimes X$
- *Zorro's lemma:*

$$X \xrightarrow{c} X \otimes \check{X} \otimes X \xrightarrow{e} X \text{ is } id_X$$
$$\check{X} \xrightarrow{c} \check{X} \otimes X \otimes \check{X} \xrightarrow{e} \check{X} \text{ is } id_{\check{X}}$$

Theorem 2. *Let $\mathcal{C}$ be an $\otimes(\infty, 1)$-category.*

$$Fun^{\otimes}(1Bord, \mathcal{C}) \cong \text{dualizable objects of } \mathcal{C}$$

$$Z \mapsto Z(*) = X \in \mathcal{C}$$

This is nontrivial even in dimension 1.

Note: $Z\colon 1Bord \to \mathcal{C}$, $\hom_{1Bord}(\emptyset,\emptyset) \cong \coprod_n($ classifying spaces for 1-manifolds of form $S^1 \coprod \cdots \coprod S^1) \supset \mathbb{CP}^\infty \cong BDiff^+(S^1) \cong BS^1$.

$$Z(S^1) = dimX \quad 1 \overset{coev}{X} \otimes \check{X} \xrightarrow{ev} 1$$

So our calculation told us what to assign but not how the symmetry group acts. The cobordism hypothesis gives us a unique circle action.

What if $n > 1$?

There is a functor

$$1Bord^{fr} \to 2Bord^{fr}$$

if we use te $infty$ versions of these. We turn diffeomorphisms into bordisms.

Since this is not an equivalence of categories, we will need stronger conditions than dualizability.

First, a digression: consider the 2-category Cat with categories, functors and natural transformations. So category theory is the study of this particular 2-category. A similar statement can be made for group theory and the category of groups. Moreover, category theory gives a language to think about constructions that are ubiquitous in mathematics and study them in a general setting.

We can take a similar philosophy for higher categories and think about adjoint functors.

Adjoint functors:

Let $\mathcal{C}$ and $\mathcal{D}$ b categories. Recall: A pair of functors $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$ are **adjoint** if

$$\hom_{\mathcal{C}}(C, GD) \cong \hom_{\mathcal{D}}(FC, D), \forall C, D \text{ functorially.}$$

We can do a similar thing for 2-categories.

$C = GD$

$$id_{GD} \ni \hom_{\mathcal{C}}(C, GD) \to \hom_{\mathcal{F}}(FC, D)$$
$$(F \circ G)(D) \to D$$
$$v\colon F \circ G \to id_{\mathcal{D}}$$

$$\hom_{\mathcal{C}}(C, GD) \to \hom(FC, (F \circ G)D) \to \hom(FC, D)$$

To get a map in the other diection,

$$u\colon id_{\mathcal{C}} \to G \circ F$$

u and v must be related:

$$F \xrightarrow{u} F \circ G \circ F \xrightarrow{v} F \quad id_F$$

$$G \xrightarrow{u} G \circ F \circ G \xrightarrow{v} G \quad id_G$$

Definition 3. *If $\mathcal{C}$ is any 2-category $f\colon x \to y$ is a morphism in $\mathcal{C}$, a* **right adjoint** *to f is a morphism $g\colon y \to x$, $u\colon id_x \to g \circ f$, $v\colon f \circ g \to id_y$ such that Zorro's lemma holds.*

Definition 4. *An $\infty, n)$-category $\mathcal{C}$* **has adjoints** *if*

- *every k-morphism in $\mathcal{C}$ has left and right adjoints when $0 \leq k \leq n$.*
  OR:
- *If $n = 1$, or if $n > 1$ and $\hom_{\mathcal{C}}(X, Y)$ has adjoints $\forall X, Y$ and every 1-morphism in $\mathcal{C}$ has left and right adjoints.*

Theorem 5. *Assume $\mathcal{C}$ has duals*

$$Fun^{\otimes}(nBord^{fr}, \mathcal{C}) \cong \textit{objects of } \mathcal{C}$$

Note: For any $\mathcal{C}$, there exists "largest" subcategory $\mathcal{C}^{f.d} \subset \mathcal{C}$ such that $\mathcal{C}^{f.d}$ has duals.

There is nothing to be lost since any TQFT from *nBord* factors through this full subcategory.

Theorem 6. *Assume $\mathcal{C}$ is any $\otimes$ (∞, n)-categories*

$$Fun^{\otimes}(nBord^{fr}, \mathcal{C}) \cong \textit{objects of } \mathcal{C}^{f.d,\sim}$$

NEED TO INCLUDE SOME STUFF ON ACTIONS OF O(n)

Say G is any topological group with a map $G \to O(n)$
Then we have a theory of "G-manifolds".

AND MORE HERE...

What is the analog of the cobordism hypothesis...

Assume $\mathcal{C}$ has duals:

$$Fun^{\otimes}(nBord^{G}, \mathcal{C}) = (\mathcal{C}^{\sim})^{hG} \text{with a } G \to O(n) \text{ action}$$

Example: $(n = 1, \mathcal{C} = \text{Vect}, G = O(1))$

$$Fun^{\otimes}(nBord^{un}, \text{Vect}) \cong (\text{Vect}^{f.d})^{hO(1)}$$